Online Appendix for “School Admissions Reform in Chicago and England: Comparing Mechanisms by their Vulnerability to Manipulation”

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This online appendix includes proofs of all mathematical results in the main text and a documentation appendix for Table 1.

THEOREM 1: Suppose each student has a complete rank ordering and \( k > 1 \). The old Chicago Public Schools mechanism \((\text{Chi}^k)\) is at least as manipulable as any weakly stable mechanism.

PROOF:
Fix a problem \( P \) and let \( \varphi \) be an arbitrary mechanism that is weakly stable. Suppose that \( \text{Chi}^k \) is not manipulable for problem \( P \).

Claim 1: Any student assigned under \( \text{Chi}^k(P) \) receives her top choice.
Proof. If not, since each student has a complete rank order list, \( |I| > Q \), \( k > 1 \), there must be a student that is assigned to a school \( s \) he has not ranked first. Consider the highest composite score student \( i \) who is unassigned. Student \( i \) can rank school \( s \) first and will be assigned a seat there in the first round of \( \text{Chi}^k \) mechanism instead of some student who has not ranked school \( s \) first. That contradicts \( \text{Chi}^k \) is not manipulable for problem \( P \).

Claim 2. The set of students who are assigned a seat under \( \text{Chi}^k(P) \) is equal to the set of top \( Q \) composite score students.
Proof. If not, there is a school seat assigned to a student \( j \) who does not have a top \( Q \) score. Let student \( i \) be the highest scoring top \( Q \) student who is not assigned. Since student \( i \) has a complete rank order list, she can manipulate \( \text{Chi}^k \) by ranking student \( j \)’s assignment as her top choice again contradicting \( \text{Chi}^k \) is not manipulable for problem \( P \).

Since each of the top \( Q \) students is matched to her top choice in matching \( \text{Chi}^k(P) \), all other students are unassigned.

Claim 3. In problem \( P \), matching \( \text{Chi}^k(P) \) is the unique weakly stable matching.
Proof. By Claims 1 and 2 it is possible to assign each one of the top \( Q \) students a seat at their top choice school under \( P \) and \( \text{Chi}^k(P) \) picks that matching. Let \( \mu \neq \text{Chi}^k(P) \). That means under \( \mu \) there exists a top \( Q \) student \( i \) who is not
assigned to her top choice \( s \). Pick the highest composite score such student \( i \). Since all higher score students are assigned to their top choices, either there is a vacant seat at her top choice \( s \) or it admitted a student with lower composite score. In either case the pair \((i, s)\) strongly blocks matching \( \mu \). Hence \( \text{Chi}^k(P) \) is the unique weakly stable matching under \( P \).

We are now ready to complete the proof. By Claim 3, \( \varphi(P) = \text{Chi}^k(P) \) and hence mechanism \( \varphi \) assigns all top \( Q \) students a seat at their top choices. None of the top \( Q \) students has an incentive to manipulate \( \varphi \) since each receives her top choice. Moreover no other student can manipulate \( \varphi \) because regardless of their stated preferences, \( \varphi(P) = \text{Chi}^k(P) \) remains the unique weakly stable matching and hence \( \varphi \) picks the same matching for the manipulated economy. Hence, any other weakly stable mechanism is also not manipulable under \( P \).

**PROPOSITION 1:** Suppose there are at least \( k \) schools and let \( k > 1 \). The old Chicago mechanism (\( \text{Chi}^k \)) is more manipulable than truncated serial-dictatorship (\( \text{SD}^k \)) CPS adopted in 2009.

**PROOF:**

\( \text{Chi}^k \) is a special case of the Fpf\(^k\) mechanism where all schools are first preference first with an identical priority ranking. Similarly \( \text{SD}^k \) is a special case of GS\(^k\) where all schools have an identical priority ranking. Therefore \( \text{Chi}^k \) being as manipulable as \( \text{SD}^k \) directly follows from Proposition \ref{prop:manipulability}.

We complete the proof by giving an example where \( \text{Chi}^k \) is manipulable even though \( \text{SD}^k \) is not.

There are three students and three schools each with one seat. The student preferences and the uniform school priorities are:

- \( R_{i_1} : s_1, s_2, s_3, i_1 \)
- \( R_{i_2} : s_1, s_2, s_3, i_2 \)
- \( R_{i_3} : s_2, s_3, s_1, i_3 \)

\( \pi_{s_1} : i_1, i_2, i_3 \)
\( \pi_{s_2} : i_1, i_2, i_3 \)
\( \pi_{s_3} : i_1, i_2, i_3 \)

The outcomes of \( \text{Chi}^2 \) and \( \text{SD}^2 \) are:

\[
\text{Chi}^2(R) = \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_2 \end{pmatrix} \quad \text{and} \quad \text{SD}^2(R) = \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix}.
\]

Since no student remains unmatched under \( \text{SD}^2 \), strategy-proofness of \( \text{SD} \) implies that no student can manipulate \( \text{SD}^2 \) under profile \( R \). In contrast

\[
\text{Chi}^2(R_{i_2}, R'_{i_2}) = \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix}
\]

where \( R'_{i_2} \) is any preference relation student \( i_2 \) ranks school \( s_2 \) as his first choice,
and therefore
\[ \text{Chi}^2_i(R_{-i_2}, R'_{i_2}) P_{i_2} \text{ Chi}^2_i(R) \]
implies that \( \text{Chi}^2 \) is vulnerable under profile \( R \). Hence \( \text{Chi}^2 \) is more manipulable than \( \text{SD}^2 \). It is straightforward to extend this example to show that \( \text{Chi}^k \) is more manipulable than \( \text{SD}^k \) for \( k > 2 \).

PROPOSITION 2: Let \( \ell > k > 0 \) and suppose there are at least \( \ell \) schools. Then \( GS^k \) is more manipulable than \( GS^\ell \).

PROOF:
Suppose there is a student \( i \) and preference \( \hat{P}_i \) such that
\[ GS^\ell_i(\hat{P}_i, P_{-i}) P_i GS^\ell_i(P) \ldots \]
For any student \( j \), let \( P^\ell_j \) be the truncation of \( P_j \) after the \( \ell \)th choice. This means that in \( P^\ell \) any choice after the top \( \ell \) in \( P_j \) are unacceptable, and choices among the top \( \ell \) are ordered according to \( P_j \). Observe that relation (1) implies that
\[ GS_i(\hat{P}^\ell_i, P^\ell_{-i}) P_i GS_i(P^\ell) \ldots \]
Since \( GS \) is strategy-proof, relation (2) implies that student \( i \) does not receive one of her top \( \ell \) choices from the \( GS \) mechanism under profile \( P^\ell \). Hence, \( GS_i(P^\ell) = GS^\ell_i(P) = i \).

For \( k < \ell \), there are two cases to consider.

Case 1: \( GS^k_i(P) = i \).

Let \( GS^k_i(\hat{P}_i, P_{-i}) = s \) and let \( \hat{P}_i \) be such that \( s \) is the only acceptable school.

Claim: \( GS^k_i(\hat{P}_i, P_{-i}) = s \).

Proof: First note that \( GS^k_i(\hat{P}_i, P_{-i}) = s \). Moreover, by definition
\[ GS^\ell_i(\hat{P}_i, P_{-i}) = GS(\hat{P}_i, P^\ell_{-i}) \quad \text{and} \quad GS^k(\hat{P}_i, P_{-i}) = GS(\hat{P}_i, P^k_{-i}) \ldots \]
Gale and Sotomayor (1985) (see also Theorem 5.34 of Roth and Sotomayor 1990) implies that
\[ GS_i(\hat{P}_i, P^k_{-i}) R_i GS_i(\hat{P}_i, P_{-i}) \ldots \]
Substituting the definitions,
\[ GS^k_i(\hat{P}_i, P_{-i}) R_i GS^\ell_i(\hat{P}_i, P_{-i}) \ldots \]
Since \( c \) is the only acceptable school in \( \hat{P}_i \), the claim follows. \( \diamond \)
Thus, in the first case, student $i$ can manipulate $GS^k$:

$$\underbrace{GS^k_i(\tilde{P}_i, P_{-i})}_{=s} \underbrace{P_i}_{=i} \underbrace{GS^k_i(P)}_{=i}.$$

Case 2: $GS^k_i(P) \neq i$.

**Claim 1**: $\exists j \in I$ such that $GS^k_j(P) = j$ although $GS^k_i(P) \neq j$.

**Proof**: Suppose not. Then, since $GS^k_i(P) = i$ and $GS^k_i(P) \neq i$, there is a school that is assigned strictly more students under $GS^k_i(P)$ than $GS^\ell_i(P)$. This is a contradiction to Gale and Sotomayor (1985), which requires that each school is weakly worse off under $GS^k$ (since profile $P^k$ is a truncation of profile $P^\ell$). ♦

Pick any $j \in I$ such that $GS^k_j(P) = j$ although $GS^k_j(P) \neq j$. Let $GS^\ell_j(P) = s$ and let $\tilde{P}_j$ be such that $s$ is the only acceptable school.

**Claim 2**: $GS^k_j(\tilde{P}_j, P_{-j}) = s$.

**Proof**: Since $GS^\ell_j(P) = s$, we have $GS^\ell_j(\tilde{P}_j, P_{-j}) = s$ as well. Moreover, by definition

$$GS^\ell(\tilde{P}_j, P_{-j}) = GS(\tilde{P}_j, P_{-j}) \quad \text{and} \quad GS^k(\tilde{P}_j, P_{-j}) = GS(\tilde{P}_j, P_{-j}).$$

Gale and Sotomayor (1985) implies that

$$GS^\ell_j(\tilde{P}_j, P_{-j}) R_j GS^\ell_j(\tilde{P}_j, P_{-j}).$$

Substituting the definitions,

$$GS^\ell_j(\tilde{P}_j, P_{-j}) R_j GS^\ell_j(\tilde{P}_j, P_{-j}) = s.$$

Since $s$ is the only acceptable school in $\tilde{P}_j$,

$$GS^\ell_j(\tilde{P}_j, P_{-j}) = s,$$

which establishes the claim. ♦

Thus, for the second case, student $j$ can manipulate $GS^k$:

$$\underbrace{GS^k_j(\tilde{P}_j, P_{-j})}_{=s} \underbrace{P_j}_{=j} \underbrace{GS^k_j(P)}_{=j}.$$

Finally, we describe a problem where $GS^\ell$ is not manipulable by any student, but $GS^k$ is manipulable by some student for $\ell > k > 0$. Suppose there are two students, $i_1$ and $i_2$, and two schools, $s_1$ and $s_2$, each with one seat. The students have identical preferences which rank $s_1$ ahead of $s_2$ and both schools
have identical priority rankings where student $i_1$ has higher priority than student $i_2$. Under $GS^2$, no student can manipulate because each one is assigned a school and $GS$ is strategy-proof. In contrast, student $i_2$ is unassigned under $GS^1$, and he can benefit from ranking $s_2$ as his top choice. This example can be generalized to the case of $GS^k$ and $GS^\ell$ for any $\ell > k > 0$. Since all schools have the same priority ranking in this example, it also proves that $Sd^k$ is more manipulable than $Sd^\ell$ for any $\ell > k > 0$. This completes the proof.\(^1\)

PROPOSITION 3: Suppose there are at least $k$ schools where $k > 1$. Then $FPF^k$ is more manipulable than $GS^k$.

PROOF:

For any student $j$, let $P^k_j$ be the truncation of $P_j$ after the $k^{th}$ choice. By definition,

$$FPF^k(P) = FPF(P^k) \quad \text{and} \quad GS^k(P) = GS(P^k).$$

Suppose that no student can manipulate $FPF^k$. We will show that no student can manipulate $GS^k$ either. Consider two cases:

Case 1: $FPF^k(P) = FPF(P^k)$ is stable under profile $P$.

Since $FPF(P^k)$ is stable under $P$, it is stable under $P^k$ as well. Moreover, $GS(P^k)$ is stable for $P^k$ by definition. Since the set of unmatched students across stable matchings is the same (McVitie and Wilson 1970), for all students $i$,

$$GS_i(P^k) = i \iff FPF_i(P^k) = i. \quad (3)$$

Pick some student $i$. If $GS^k_i(P^k) \neq i$, then student $i$ receives one of her top $k$ choices. This implies that $i$ receives one of her top $k$ choices under $GS$. Since $GS$ is strategy-proof, student $i$ cannot manipulate $GS^k$.

Suppose $GS^k_i(P^k) = i$ and student $i$ can manipulate. We derive a contradiction. Since $i$ can manipulate, there exists some school $s$ and preference $\hat{P}_i$ such that

$$GS^k_i(\hat{P}_i, P^k_{-i}) = i \quad \text{satisfies} \quad FPF_i(P^k).$$

Observe that $s$ is not one of the top $k$ choices of student $i$ under $P_i$ for otherwise student $i$ could manipulate $GS$. Construct $\hat{P}_i$ which lists $s$ as the only acceptable school.

\(^1\)It is also possible to provide an alternative, indirect proof of this result using the equilibrium interpretation of the definition of weakly more manipulable than together with the characterization of the set of Nash equilibria in the preference revelation game induced by $GS^k$ in Theorem 6.5 of Haeringer and Klijn (2009).
Matching $GS_k(\hat{P}_i, P^-_{k-i})$ remains stable under $(\hat{P}_i, P^-_{k-i})$ and therefore

$$GS_k(\hat{P}_i, P^-_{k-i}) = s.$$  

Since $GS(P^k)$ is stable under $P^k$ and $GS_k^i(P^k) = i$ by assumption, relation (3) implies

$$FPF_i(P^k) = i.$$  

By Roth (1984), matching $FPF_i(P^k)$ is not stable under $(\tilde{P}_i, P^-_{k-i})$ since student $i$ remains single under $FPF_i(P^k)$ although not under stable matching $GS_k(\hat{P}_i, P^-_{k-i})$. Since matching $FPF_i(P^k)$ is not stable under $(\hat{P}_i, P^-_{k-i})$, but it is stable for $P^k$, the only possible blocking pair of $FPF_i(P^k)$ in $(\tilde{P}_i, P^-_{k-i})$ is $(i, s)$. But since $FPF_i(P^k) = i$, this implies that $(i, s)$ also blocks $FPF_i(P^k)$ under $P^k$, which is the desired contradiction. Thus, in case 1, no student can manipulate $GS$.  

Case 2: $FPF_i(P^k)$ is not stable for profile $P$.

In this case, a student $i$ along with a first preference first school $s$ block $FPF_i(P^k)$: That is, there exists $j \in FPF_s(P^k)$ such that not only $i$ has higher base priority than $j$ at school $s$, but also $s P_i FPF_i(P^k)$.

Construct $\tilde{P}_i$ so that school $s$ is the only acceptable school for student $i$. Since $j \in FPF_s(P^k)$ and student $i$ has higher base priority than student $j$ at school $s$, we must have $i \in FPF_s(\tilde{P}_i, P^-_{k-i})$. But this means that

$$\underbrace{FPF_i(\tilde{P}_i, P^-_{k-i})}_{s} P_i FPF_i(P^k),$$

contradicting the assumption that no student can manipulate $FPF$ at $P^k$.

Finally we give an example where $FPF^2$ is manipulable but $GS^2$ is not. It is straightforward to extend the example for any $k > 2$. There are three students and three first preference first schools each with one seat. Since all schools are first preference first, $FPF$ mechanism reduces to the special case of the Boston mechanism in this example. The student preferences and the (uniform) school priorities are:

\[
\begin{align*}
R_{i_1} &: s_1, s_2, s_3, i_1 \\
R_{i_2} &: s_1, s_2, s_3, i_2 \\
R_{i_3} &: s_2, s_3, s_1, i_3
\end{align*}
\]

\[
\begin{align*}
\pi_{s_1} &: i_1, i_2, i_3 \\
\pi_{s_2} &: i_1, i_2, i_3 \\
\pi_{s_3} &: i_1, i_2, i_3
\end{align*}
\]

The outcomes of $FPF^2$ and $GS^2$ are:

\[
\begin{align*}
FPF^2(R) &= \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix} & \text{and} & \quad GS^2(R) &= \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix}.
\end{align*}
\]

Since no student remains unmatched under $GS^2$, strategy-proofness of $GS$ implies
that no student can manipulate $GS^2$ under profile $R$. In contrast,

$$\text{FPP}^2(R_{-i_2}, R'_{i_2}) = \left( \begin{array}{ccc} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{array} \right)$$

where $R'_{i_2}$ is any preference relation student $i_2$ ranks school $s_2$ as his first choice, and therefore

$$\text{FPP}^2(R_{-i_2}, R'_{i_2}) P_{i_2} \text{FPP}^2(R)$$

implies that $\text{FPP}^2$ is vulnerable under profile $R$. Hence $\text{FPP}^2$ is more manipulable than $GS^2$.

**LEMMA 1:** Fix a set of agents $I' \subset J \cup C$. Let $\varphi, \psi$ be two stable mechanisms such that, for any preference profile $P$, and any agent $i \in I'$,

$$\varphi_i(P) R_i \psi_i(P).$$

Then mechanism $\psi$ is as strongly manipulable as mechanism $\varphi$ for members of $I'$.

**PROOF:**

Let $I' \subset J \cup C$ and mechanisms $\varphi, \psi$ be as in the statement of the Lemma. Let preference profile $P$, agent $i \in I'$, and preference relation $\hat{P}_i$ be such that

$$(4) \quad \varphi_i(\hat{P}_i, P_{-i}) P_i \varphi_i(P).$$

We want to show that there exists a preference relation $\tilde{P}_i$ such that

$$\psi_i(\tilde{P}_i, P_{-i}) P_i \psi_i(P).$$

By assumption

$$(5) \quad \varphi_i(P) R_i \psi_i(P).$$

Let the preference relation $\tilde{P}_i$ be such that only agents in $\varphi_i(\hat{P}_i, P_{-i})$ are acceptable to agent $i$ under $\tilde{P}_i$. Since matching $\varphi(\hat{P}_i, P_{-i})$ is stable under profile $(\hat{P}_i, P_{-i})$, it is also stable under profile $(\tilde{P}_i, P_{-i})$. Moreover by Roth (1984), agent $i$ is matched with the same number of agents on the other side of the market at any stable matching under any given preference profile, and in particular under profile $(\tilde{P}_i, P_{-i})$. Therefore, since only agents in $\varphi_i(\hat{P}_i, P_{-i})$ are acceptable to agent $i$ under $\tilde{P}_i$, stability of matching $\varphi(\hat{P}_i, P_{-i})$ under $(\tilde{P}_i, P_{-i})$ implies

$$(6) \quad \psi_i(\tilde{P}_i, P_{-i}) = \varphi_i(\tilde{P}_i, P_{-i}).$$
Hence, by (4), (5), and (6), we have
\[
\psi_i(\tilde{P}_i, P_{-i}) = \varphi_i(P) R_i \psi_i(P),
\]
which shows that agent \(i\) can manipulate mechanism \(\psi\) by reporting \(\tilde{P}_i\). This completes the proof.

**Proposition 4:** \(GS^J\) is strongly more manipulable than \(GS^C\) for colleges.

**Proof:**
Given any problem, the college-optimal stable matching is weakly preferred to student-optimal stable matching by any college (Gale and Shapley 1962). Therefore, Lemma ?? implies \(GS^J\) is as strongly manipulable as \(GS^C\) for colleges.

Next, we give a problem where \(GS^C\) is not manipulable by any college, while some college can manipulate \(GS^J\). Suppose there are two students, \(j_1\) and \(j_2\), and two colleges, \(c_1\) and \(c_2\), each with one seat. The student and college preferences are

\[
\begin{align*}
R_{j_1} : c_1, c_2, j_1 & \quad R_{c_1} : \{j_2\}, \{j_1\}, \emptyset \\
R_{j_2} : c_2, c_1, j_2 & \quad R_{c_2} : \{j_1\}, \{j_2\}, \emptyset.
\end{align*}
\]

The outcomes of \(GS^C\) and \(GS^J\) are:

\[
\begin{align*}
GS^C(R) = [j_1 \ j_2 \\
\quad c_2 \ c_1]
\end{align*}
\quad \text{and} \quad \begin{align*}
GS^J(R) = [j_1 \ j_2 \\
\quad c_1 \ c_2]
\end{align*}
\]

Since each college obtains its top choice under \(GS^C\), no college can manipulate. However, if college \(c_1\) declares that only \(j_2\) is acceptable, it can manipulate \(GS^J\). This completes the proof.

**Theorem 2:** Let \(\varphi\) be an arbitrary stable mechanism. Then

a) \(\varphi\) is as strongly manipulable as \(GS^C\) for colleges,

b) \(GS^J\) is as strongly manipulable as \(\varphi\) for colleges, and

c) \(GS^C\) is as strongly manipulable as \(\varphi\) for students.

**Proof:**
Let \(\varphi\) be any stable mechanism and \(P\) be any preference profile. Then

a) \(GS^C(P) R_c \varphi(P)\) for any \(c \in C\),
b) \( \varphi_c(P) \) \( R_c GS^\mathcal{D}_c(P) \) for any \( c \in C \), and
c) \( \varphi_j(P) \) \( R_j GS^\mathcal{C}_j(P) \) for any \( j \in J \)

by Gale and Shapley (1962). Therefore Lemma ?? implies the desired result.

**PROPOSITION 5:** The discriminatory auction is intensely and strongly more manipulable than the uniform-price auction.

**PROOF:**

Let \( \delta \) denote the discriminatory auction and \( \Upsilon \) denote the uniform-price auction. Fix \( \epsilon > 0 \) and a bidder \( i \). Let \( t_{-i} \) be the type profile of all other bidders. The type of each bidder is the vector of his valuations. Given \( t_{-i} \), order the \( k(|I| - 1) \) valuations of all bidders in \( I \setminus \{i\} \) from highest to lowest. Let \( b_1 \) be the highest valuation, \( b_2 \) be the next highest valuation, and so on. That is, \( b_1 \geq b_2 \geq \cdots \geq b_{k(|I|-1)} > 0 \).

Let \( t_i = (v_1^i, \ldots, v_k^i) \) be the type of bidder \( i \). We will consider two cases. For the first case bidder \( i \) will not be able to manipulate the uniform-price auction. For the second case he potentially can but whenever that happens he will have an at least as profitable deviation under the discriminatory auction.

**Case 1:** \( v_1^i < b_k \). For this case bidder \( i \)'s highest valuation is less than \( b_k \). Therefore if he reports his true values under the uniform-price auction, he will not receive any object and will not make any payment. Hence \( u_i(\Upsilon(t)) = 0 \). In order to have a profitable manipulation, bidder \( i \) will need to receive an object. However, since \( v_1^i < b_k \), that will require bidder \( i \) to pay a unit price that is higher than his highest valuation. Hence \( u_i(\Upsilon(t'_i, t_{-i})) - u_i(\Upsilon(t)) \leq 0 \) for any \( t'_i \in T_i \), showing there exists no profitable manipulation of the uniform-price auction for Case 1.

**Case 2:** \( v_1^i \geq b_k \). Let bidder \( i \) receive \( m \) units under the uniform price auction when he reports his true type \( t_i = (v_1^i, \ldots, v_k^i) \). That means \( v_i^m \geq b_{k-m+1} \) and the market-clearing-price for profile \( t \) is

\[
p^* = \begin{cases} 
\max\{v_i^{m+1}, b_{k-m+1}\} & \text{if } m < k \\
b_{k-m+1} & \text{if } m = k
\end{cases}
\]

which in turn implies

\[
(7) \quad u_i(\Upsilon(t)) = (v_i^1 + \cdots + v_i^m) - mp^* \geq 0.
\]

Let the potential manipulation \( \hat{t}_i = (\hat{v}_1^i, \ldots, \hat{v}_k^i) \) be such that bidder \( i \) receives \( n \) units under \( \Upsilon(\hat{t}_i, t_{-i}) \). Then the market-clearing price for profile \( (\hat{t}_i, t_{-i}) \) is

\[
\hat{p} = \begin{cases} 
\max\{v_i^{n+1}, b_{k-n+1}\} & \text{if } n < k \\
b_{k-n+1} & \text{if } n = k
\end{cases}
\]
and hence

\[ u_i(\hat{\Upsilon}(\hat{t}_i, t_{-i})) = (v_i^1 + \cdots + v_i^n) - n\hat{p}. \]

Observe that,

\[ \hat{p} \geq b_{k-n+1}. \]

Suppose

\[ u_i(\hat{\Upsilon}(\hat{t}_i, t_{-i})) - u_i(\Upsilon(t)) = (v_i^1 + \cdots + v_i^n - n\hat{p}) - (v_i^1 + \cdots + v_i^m - mp^*) > 0 \]

and thus bidder \( i \) can manipulate the uniform-price auction at profile \( t \). We will construct \( \tilde{t}_i \in T_i \) such that

\[ u_i(\delta(\tilde{t}_i, t_{-i})) - u_i(\delta(t)) > u_i(\hat{\Upsilon}(\hat{t}_i, t_{-i})) - u_i(\Upsilon(t)) - \epsilon. \]

First observe that \( u_i(\delta(t)) = 0 \), since bidder \( i \) pays her reported valuation for each unit she wins under the discriminatory auction. Let \( \tilde{t}_i = (\tilde{v}_i^1 \ldots \tilde{v}_i^k) \) be such that

\[ \tilde{v}_i^\ell = \begin{cases} b_{k-n+1} + \frac{\epsilon}{2n} & \text{if } \ell \leq n \\ 0.5b_{k-n+1} & \text{if } \ell > n \end{cases} \]

Given \( t_{-i} \), bidder \( i \) wins \( n \) units and pays \( b_{k-n+1} + \frac{\epsilon}{2n} \) for each unit upon reporting \( \tilde{t}_i \). Therefore inequalities 7 and 8 imply

\[ u_i(\delta(\tilde{t}_i, t_{-i})) - u_i(\delta(t)) = (v_i^1 + \cdots + v_i^n - n(b_{k-n+1} + \frac{\epsilon}{2n})) - 0 \]
\[ = (v_i^1 + \cdots + v_i^n - nb_{k-n+1}) - \frac{\epsilon}{2} \]
\[ > (v_i^1 + \cdots + v_i^n - n\hat{p}) - (v_i^1 + \cdots + v_i^m - mp^*) - \epsilon \]
\[ = u_i(\hat{\Upsilon}(\hat{t}_i, t_{-i})) - u_i(\Upsilon(t)) - \epsilon \]

showing that bidder \( i \) has an at least as profitable manipulation, subject to an upper bound of \( \epsilon \) deviation, under the discriminatory auction for Case 2.

This covers all cases, so to complete the proof, we describe an example where some bidders can manipulate \( \delta \), but not \( \Upsilon \). Suppose that all bidders other than bidder 1 have the same value \( \bar{v} \) for all of the units. Bidder 1’s value for the first unit is strictly greater than \( \bar{v} \), while her value for each of the remaining units is strictly less than \( \bar{v} \). Under the uniform price auction, when bidders are truthful, every bidder wins one unit. Bidder 1 cannot manipulate to win more units because she would have to pay \( \bar{v} \) for the additional units. She does not want to manipulate to win fewer units because she obtains strictly positive utility by reporting the truth and she cannot manipulate to change the price she pays. No other bidder would find it strictly profitable to manipulate because each would still have to pay at least \( \bar{v} \) for that unit, and none can change the price paid. Hence, no bidder can manipulate the uniform-price auction. Under the discriminatory price auction,
when each bidder reports truthfully, every bidder wins one unit. However, bidder 1 would prefer to under-report her valuation for the first unit to pay less for it. Hence, for this example, bidder 1 can manipulate $\delta$, but not $\Upsilon$.

**PROPOSITION 6:** The Generalized First Price Auction is intensely and strongly more manipulable than the Generalized Second Price Auction.

**PROOF:**
Given a type profile $t$, let $\text{Gsp}(t)$ denote the outcome of GSP auction and $\text{Gfp}(t)$ denote the outcome of GFP auction. Fix $\epsilon > 0$ and a bidder $i$. Let $t_{-i}$ be the type profile of all other bidders. Recall that the type of each bidder is his valuation per click. Given $t_{-i}$, order the $|I| - 1$ valuations of all bidders in $I \setminus \{i\}$ from highest to lowest. Let $b_1$ be the highest valuation, $b_2$ be the next highest valuation, and so on. That is, $b_1 \geq b_2 \geq \cdots \geq b_{|I|-1} > 0$.

Let $t_i = v_i$ be the type of bidder $i$. We will consider two cases with four sub-cases for the second case. For all cases except Case 2d, bidder $i$ will not be able to manipulate the GSP auction. For Case 2d, he potentially can but whenever that happens he will have an at least as profitable deviation under the GFP auction.

**Case 1:** $v_i \leq b_k$.
In this case $u_i(\text{Gsp}(t)) = 0$ either because bidder $i$ does not receive a slot, or because she receives a slot at 0 utility.² Let $t'_i = v_i'$ be a potential manipulation. For this manipulation to be profitable, bidder $i$ shall receive a slot. Let this slot be slot $\ell$. Then

$$b_{\ell-1} \geq v_i' \geq b_{\ell} \geq b_k \geq v_i \quad \text{and therefore,}$$

$$u_i(\text{Gsp}(t',t_{-i})) = \alpha_{\ell}v_i - \alpha_{\ell}b_{\ell} = \alpha_{\ell}(v_i - b_{\ell}) \leq 0.$$  

Hence bidder $i$ does not have a profitable manipulation of GSP for Case 1.

**Case 2:** $v_i > b_k$.
Let bidder $i$ receive slot $m$ under GSP when he reveals his type truthfully. Then $b_{m-1} \geq v_i \geq b_m$ and

$$u_i(\text{Gsp}(t)) = \alpha_m v_i - \alpha_m b_m \geq 0. \quad (9)$$

Let $t'_i = v'_i$ be a potential manipulation and suppose bidder $i$ receives slot $\ell$ under $t'_i = v'_i$. This implies $v'_i \geq b_{\ell}$. We have four sub-cases to consider.

**Case 2a:** $v'_i > b_{m-1}$.
For this case, $\ell \leq m - 1$ and hence $b_{\ell} \geq b_{m-1} \geq v_i$. Therefore

$$u_i(\text{Gsp}(t'_i,t_{-i})) = \alpha_{\ell}v_i - \alpha_{\ell}b_{\ell} = \alpha_{\ell}(v_i - b_{\ell}) \leq 0$$

²The latter can happen only if $v_i = b_k$. 
and thus, bidder \( i \) does not have a profitable manipulation of GSP for Case 2a.

**Case 2b: \( v'_i = b_{m-1} \).**

For this case there is a tie and bidder \( i \) either receives slot \( m - 1 \) at a cost of \( \alpha_{m-1}b_{m-1} \) or slot \( m \) at a cost of \( \alpha_{m}b_{m} \). If the former happens,

\[
u_i(GSP(t'_i, t_{-i})) = \alpha_{m-1}v_i - \alpha_{m-1}b_{m-1} = \alpha_{m-1}(v_i - b_{m-1}) \leq 0.
\]

If the latter happens,

\[
u_i(GSP(t'_i, t_{-i})) = \alpha_{m}v_i - \alpha_{m}b_{m} = u_i(GSP(t)).
\]

In either case, bidder \( i \) does not have a profitable manipulation of GSP.

**Case 2c:** Either \( b_{m-1} > v'_i > b_{m} \) or \( v'_i = b_{m} \) and bidder \( i \) receives slot \( m \) with tie-breaker.

In this case bidder \( i \) receives slot \( m \) at a cost of \( \alpha_{m}b_{m} \). Therefore,

\[
u_i(GSP(t'_i, t_{-i})) = \alpha_{m}v_i - \alpha_{m}b_{m} = u_i(GSP(t)),
\]

and hence bidder \( i \) does not have a profitable manipulation of GSP.

**Case 2d: \( v'_i \leq b_{m} \) and bidder \( i \) receives a slot \( \ell \) with \( \ell > m \).**

In this case

\[
v_i \geq b_{m} \geq b_{\ell}
\]

and

\[
u_i(GSP(t'_i, t_{-i})) = \alpha_{\ell}v_i - \alpha_{\ell}b_{\ell} = \alpha_{\ell}(v_i - b_{\ell}) \geq 0.
\]

Suppose \( u_i(GSP(t'_i, t_{-i})) > u_i(GSP(t)) \) so that bidder \( i \) can manipulate GSP at profile \( t \). We will construct \( \tilde{t}_i \in T_i \) such that,

\[
u_i(GFP(\tilde{t}_i, t_{-i})) - u_i(GFP(t)) > u_i(GSP(t'_i, t_{-i})) - u_i(GSP(t)) - \epsilon.
\]

First observe that,

\[
u_i(GFP(t)) = 0.
\]

Let \( \tilde{t}_i = \tilde{v}_i = b_{\ell} + \frac{\epsilon}{2\alpha_{\ell}} \). Given \( t_{-i} \), bidder \( i \) either wins slot \( \ell \) at a cost of \( \alpha_{\ell}(b_{\ell} + \frac{\epsilon}{2\alpha_{\ell}}) \) or a better slot \( n \) (with \( \alpha_n > \alpha_{\ell} \)) at a cost of \( \alpha_n(b_{\ell} + \frac{\epsilon}{2\alpha_{n}}) \). If the former happens,

\[
u_i(GFP(\tilde{t}_i, t_{-i})) = \alpha_{\ell}v_i - \alpha_{\ell}(b_{\ell} + \frac{\epsilon}{2\alpha_{\ell}}) = \alpha_{\ell}(v_i - b_{\ell}) - \frac{\epsilon}{2}
\]
and if the latter happens,

$$u_i(GFP(\tilde{t}_i, t_{-i})) = \alpha_n v_i - \alpha_n \left( b_\ell + \frac{\epsilon}{2\alpha_\ell} \right) = \alpha_n (v_i - b_\ell) - \frac{\alpha_n \epsilon}{2\alpha_\ell} > \alpha_\ell (v_i - b_\ell) - \frac{\epsilon}{2}$$

where the last inequality holds by inequality 10 and $\alpha_n > \alpha_\ell$. Therefore,

(13)  
$$u_i(GFP(\tilde{t}_i, t_{-i})) \geq \alpha_\ell (v_i - b_\ell) - \frac{\epsilon}{2}.$$  

We are ready to finalize Case 2d. Relations 9, 11, 12, and 13 imply

$$\frac{u_i(GFP(\tilde{t}_i, t_{-i})) - u_i(GFP(t))}{\geq \alpha_\ell (v_i - b_\ell) - \frac{\epsilon}{2}} = 0 = \frac{u_i(GSP(t')) - u_i(GSP(t))}{\alpha_\ell (v_i - b_\ell)} = \alpha_m v_i - \alpha_m b_m \geq 0$$

showing that bidder $i$ has an at least profitable manipulation, subject to an upper bound of $\epsilon$ deviation, under GFP auction for Case 2d.

This covers all cases, so to complete the proof, we describe an example where some bidder can manipulate GFP, but no bidder can manipulate the GSP. Suppose that $v_1 > v_2 = \ldots = v_S = v_{S+1} > v_{S+2} > \ldots > v_N$. Under the GSP, when all bidders are truthful, the highest value bidder’s payoff is $\alpha_1 (v_1 - v_2) > 0$. She cannot change her payoff unless she reports a bid of $v_2$ or lower. If she reports her value to be $v_2$, she obtains a zero payoff. If she reports her value to be less than $v_2$, she does not win a slot and obtains a zero payoff. Hence, she cannot manipulate. Any bidder with value equal to $v_2$ who obtains a slot cannot manipulate. Reporting a value greater than $v_1$ will give the first slot, but this is not profitable. Reporting a value between $v_1$ and $v_2$ does not change her payoff. Reporting a value below $v_2$ prevents her from obtaining a slot. Finally, no bidder with value less than $v_2$ can manipulate because the only way to change the outcome is to report a value greater than or equal to $v_2$, which is unprofitable. Hence, with this value distribution, no bidders can manipulate the GSP. In the GFP, if every bidder reports the truth, the outcome is the same as the GSP, but each bidder obtains a zero payoff. If bidder 1 reports a value less than $v_1$, but greater than $v_2$, she wins the first slot, but pays a lower price than had she reported the truth. Hence, bidder 1 can manipulate the GFP, but not GSP.

### Table 1. School Admissions Reforms: Documentation Web Appendix

<table>
<thead>
<tr>
<th>Allocation System</th>
<th>Year</th>
<th>From</th>
<th>To</th>
<th>Manipulable (More or Less?)</th>
<th>Source</th>
<th>References</th>
</tr>
</thead>
</table>
Bolton 2007* FPF³ GS³ Less A,D

Bradford 2007* FPF³ GS³ Less A,D

Brighton and Hove 2007 Boston³ GS³ Less A,C,D,E

Calderdale 2006 FPF³ GS³ Less A,C

Cumbria 2007* FPF³ GS³ Less D

Darlington 2007* FPF³ GS³ Less A,D

Derby 2005* FPF⁴ GS⁴ Less A,D

Devon 2006* FPF³ GS³ Less A,D

Durham 2007 FPF³ GS³ Less A,D

Ealing 2006* FPF³ GS⁶ Less A,D

East Sussex 2007 Boston³ GS³ Less A,D

Gateshead 2007* FPF³ GS³ Less D

Halton 2007* FPF³ GS³ Less A,D

Hampshire 2007 FPF³ GS³ Less A,D

Hartlepool 2007 FPF³ GS³ Less A,D

Isle of Wight 2007* FPF³ GS³ Less D
Kent

2007  Boston³  GS³  Less  A,D

Lincolnshire

2007*  FPF³  GS³  Less  A,D

Lancashire

2007*  FPF³  GS³  Less  A,D

Luton

2007*  FPF³  GS³  Less  D

Manchester

2007*  FPF³  GS³  Less  A,D

Merton

2007  FPF³  GS³  Less  A,D

North Lincolnshire

2007*  FPF³  GS³  Less  A,D

North Somerset

2007*  FPF³  GS³  Less  A,D

North Tyneside

2007*  FPF³  GS³  Less  A,D

Oldham

2007*  FPF³  GS³  Less  A,D

Peterborough

2007*  FPF³  GS³  Less  A,D

Plymouth 2007* FPF1 GS1 Less A,D
Poole 2007* FPF1 GS1 Less A,D
Portsmouth 2007* FPF1 GS1 Less A,D
Richmond 2005 FPF1 GS1 Less D
Sefton primary 2007 Boston1 GS1 Less A,D
Sefton secondary 2007 FPF1 GS1 Less A,D
Slough 2006* FPF1 GS1 Less D
Somerset 2007* FPF1 GS1 Less A,D
South Gloucestershire 2007* FPF1 GS1 Less A,D
South Tyneside 2007* FPF1 GS1 Less D
Southampton 2007* FPF1 GS1 Less D
Stockton 2007* FPF1 GS1 Less A,D
Stoke-on-Trent 2007* FPF1 GS1 Less D
Suffolk 2007* FPF1 GS1 Less D
Sunderland 2007* FPF1 GS1 Less D
Surrey 2007 FPF1 GS1 Less A,D
2010 GS1 GS1 Less A
Sutton 2006 FPF1 GS1 Less A,D
Swindon 2007* FPF1 GS1 Less D
Tameside 2007* FPF1 GS1 Less D


<table>
<thead>
<tr>
<th>Area</th>
<th>Year</th>
<th>Type</th>
<th>Grade</th>
<th>Less</th>
<th>Bureaucrat</th>
</tr>
</thead>
</table>

Wrexham County Borough 2011 FPF3 GS3 Less A


Notes. * For changes in the 2007 code, an asterisk indicates that we assume that the number of choices allowed has not changed. A - Documentation from schools (brochures) or official policy minutes; B - Direct communication with school officials; C - Documentation from press clippings; D - Coldron report; E - Other academic papers; F - Other online materials. In some cases, we do not know the exact year the mechanism changed, the years correspond to the last possible year.

Other choice plans referenced in text

Cambridge


Charlotte-Mecklenburg


Denver


Miami-Dade


Providence