Wages and Employment Persistence with Multi-worker Firms

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This version: September 19, 2010
First version: July 23, 2006

Abstract

We present a generalization of the standard random-search model of unemployment in which firms hire multiple workers and in which the hiring process is time-consuming as well as costly. We follow Stole and Zwiebel (1996a,b) and assume that wages are determined by continuous bargaining between the firm and its employees. This generates a non-trivial dispersion of firm sizes; when firms' production technologies exhibit decreasing returns to labor, it also generates wage dispersion, even though all firms and all workers are ex ante identical. We characterize the steady-state equilibrium of the model; some important special cases are characterized in closed form. We also characterize the out-of-steady state dynamics of employment and wages in response to productivity shocks. Firms can respond to shocks on both an intensive margin (a change in the intensity of vacancy posting of incumbent firms) and extensive margin (a change in the number of active firms); we show that both margins, as well as whether there are decreasing returns to labor at the firm level, are important for the qualitative behavior of the unemployment rate and of the distribution of employment and wages across firms. When there are decreasing returns to labor and free entry of firms, the responses of unemployment and of the vacancy to unemployment ratio to a shock to labor productivity are significantly more persistent than in the Mortensen-Pissarides benchmark. The persistence is caused by a novel mechanism arising from an interaction of two key elements of our model: new entrant firms are small for some time because hiring is time-consuming and they pay high wages because of bargaining; this drives up wages for other firms and slows down job creation.

JEL Codes: E24, E32, J31, J64

Keywords: bargaining, labor market dynamics, persistence, search, wages
1 Introduction

In this paper, we study a model in which the process by which firms hire workers in a frictional labor market is both costly and time-consuming. Firms wish to hire many workers in our model, and time-consuming hiring means there is endogenous dispersion in the distribution of firm sizes, as new firms are born and cannot grow immediately to their desired number of workers. Since we allow for firms’ production functions to exhibit decreasing returns to labor, this heterogeneity in firm sizes means there is also dispersion in the marginal product of labor across firms. Since we model wage determination by bargaining — more precisely, since we use the generalization of Nash bargaining first investigated by Stole and Zwiebel (1996a,b) — this then generates dispersion in wages, even though, ex ante, all firms and all workers are identical.

A key contribution of our paper is a novel mechanism which generates persistent responses of unemployment and of labor market tightness to productivity shocks. In response to a positive shock to total factor productivity, new firms enter and are initially small since hiring is time-consuming. Under our bargaining assumption, these firms pay high wages while they remain small. This drives up wages for workers in other firms by driving up the value of their outside option, unemployment. Accordingly, these incumbent firms reduce their recruitment effort, offsetting the boost to labor market tightness caused by the new entrant firms. (In fact, in our calibrated example, market tightness even falls slightly and unemployment rises slightly at the arrival of a positive productivity shock.) As the new entrant firms progressively increase their size, their wage payments fall, and therefore so does the value of unemployment and the wages paid by other firms. Market tightness gradually increases and unemployment gradually decreases, but in our calibrated example, ten quarters after the initial productivity shock, unemployment has still only undergone around 30% of the transition to its new steady-state value. In a comparably-parameterized Mortensen-Pissarides model, for example Shimer’s (2005) calibration of the Pissarides (1985) model, the comparable number is greater than 99.999%.

We first study a steady-state economy. In Section 3 we solve for a steady-state equilibrium, prove existence, and examine the comparative statics in response to parameter changes. We also show how a slight variant of our model — that is less tractable for computational work outside the steady state and therefore not the basic model of the paper — can be essentially solved in closed form.

We then move to studying the dynamics of the economy in response to productivity shocks. The general analysis is contained in Section 4. We calibrate the model in Section 5, and show that the dynamics of adjustment following a productivity shock have the interesting features already mentioned. As already discussed, wages jump up on impact of the shock since entrant firms pay high wages and this causes hiring to respond only sluggishly. In order better to understand the dynamics we observe in the benchmark model in which there are decreasing returns to labor, we then proceed in Section 5.4 to study the
analytically-simpler case of constant returns to scale in production. This environment is very similar to the benchmark Mortensen-Pissarides model, as all jobs are now identical. Not surprisingly, there is now no cross-sectional wage dispersion, the persistence mechanism already mentioned does not operate, and the dynamics of endogenous variables such as unemployment and labor market tightness in this case are identical to those from the Mortensen-Pissarides model. The contrast between the dynamics in the benchmark and in the case of constant returns to scale, and with the case of linear vacancy-posting costs as in the work of Elsby and Michaels (2010) discussed below, makes it clear that our persistence mechanism arises only from the combination of decreasing returns to labor, bargained wages, and time-consuming hiring.

Our work is related to several literatures. First, we work in the tradition of Mortensen and Pissarides (1994) and Pissarides (2000) in modeling search frictions using undirected search and bargained wages.\(^1\) We differ from this literature in studying large firms with decreasing returns to labor, a topic of increasing interest following the availability of improved establishment-level data on hiring and vacancy creation (Davis, Faberman, and Haltiwanger, 2006).\(^2\)

The model of firm-worker bargaining with diminishing returns at the firm level that we use was first introduced in the case without labor market frictions by Stole and Zwiebel (1996a,b).\(^3\) Smith (1999) studied the efficiency implications. The Stole and Zwiebel wage bargaining assumption is by now the canonical bargaining solution for wage determination in models where firms wish to employ multiple workers. Applications have recently also included work studying contracting and technological adoption (Acemoglu, Antr`as, and Helpman, 2007), wage determination (Roys, 2010), the interaction of product market regulation and the labor market (Felbermayr and Prat, 2007; Delacroix and Samaniego, 2009; Ebell and Haefke, 2009), trade (Co¸ sar, Guner, and Tybout, 2010; Helpman and Itskhoiki, 2010), as well as several others. Hawkins (2010) studies whether the Stole and Zwiebel bargaining assumption can be tested empirically.

Three papers deserve particular note since they are closely related to our work in studying a dynamic Stole and Zwiebel-style bargaining problem between workers and firms in the presence of search frictions and decreasing returns. The important paper by Wolinsky (2000) was the first to study such an environment, but Wolinsky’s analysis is essentially partial equilibrium, since the arrival of new workers to firms is assumed to be exogenous. Consequently, Wolinsky’s model does not endogenize the unemployment rate and cannot

\(^{1}\)In particular, we follow Shimer (2005), Hall (2005), and Hagedorn and Manovskii (2008) in performing a quantitative analysis of a calibrated model. See also Merz (1995), Andolfatto (1996), Costain and Reiter (2008), and Shimer (2010) for other notable quantitative investigations.

\(^{2}\)Moscarini and Postel-Vinay (2008, 2010) develop a framework based on Burdett and Mortensen’s search model to study labor market fluctuations and emphasize the differential behavior of small and large firms (even though there are no decreasing returns to labor in their framework).

\(^{3}\)Preceding investigations of models with dynamic firm-worker bargaining and decreasing returns include Bertola and Caballero (1994) and Bertola and Garibaldi (2001); these authors use Nash bargaining which is harder to justify in such an environment.
be used for equilibrium analysis in the labor market. Equilibrium models of bargaining in such an environment are presented in Cahuc, Marque, and Wasmer (2008) and by Elsby and Michaels (2010). However, in both these papers, the possibility of firms being away from their target size is assumed away, so that neither the cross-sectional dispersion of firm sizes and of worker productivity and wages nor the persistence of the response to shocks that we study arise. The assumption that firms are always at their target size is explicit in the case of Cahuc, Marque, and Wasmer (2008), and is made for the sake of tractability, since their emphasis is on hold-up problems whose study would be intractable if they assumed time-consuming hiring as we do. Elsby and Michaels (2010) focus explicitly on cross-sectional dispersion and on aggregate fluctuations, but assume that the cost of posting additional vacancies exhibits constant returns to scale. Under this assumption, the optimal vacancy posting policy of a firm takes a ‘bang-bang’ form, in the sense that a new entrant firm posts an enormous number of vacancies for a vanishingly short period of time, growing immediately to its desired size, and then remaining there until the arrival of an idiosyncratic or aggregate shock. The goal of this paper is to investigate explicitly the dispersion that arises from not allowing this degenerate firm growth pattern to occur. The persistence mechanism that we identify in Section 5 is absent in their work.

Finally, although the Stole and Zwiebel bargaining assumption we make is the usual one where firms with decreasing returns to labor attempt to hire workers in a setting of random search, it is worth noting there is also a new and small literature using the alternative assumption of directed search to study such environments. Hawkins (2006) was the first to study a related environment, and Kaas and Kircher (2010) study cyclical fluctuations in such an environment. Further research is needed to determine whether an analog to the persistence mechanism we identify here is present also in such an environment.

2 Model

There is a unit measure of risk-neutral workers in the economy and a large measure of risk-neutral firms. Time is continuous; workers and firms discount the future at rate $r \geq 0$. Firms are either inactive or active. At any moment, any inactive firm can elect to become active by paying an entry cost of $k$ units (all production and costs are measured in units of the single good produced in this economy). Active firms have the ability to operate a production technology which uses labor as the only input; the flow output of the final good produced by an active firm together with $n$ workers is denoted $y(n)$. We assume that $y(n)$ is strictly increasing, strictly concave, and continuously differentiable in $n$, and we normalize

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4Hawkins (2010) also makes this assumption, but the focus of that paper is on whether the nature of bargaining between firms and workers can be identified empirically.

\( y(0) = 0 \). We also assume that \( y(\cdot) \) satisfies the standard Inada conditions

\[
\lim_{n \to 0^+} y'(n) = +\infty \quad \text{and} \quad \lim_{n \to \infty} y'(n) = 0.
\]

(1)

The labor market is frictional; in order to hire workers, firms must post vacancies. We assume that to post \( v \geq 0 \) vacancies, firms pay a strictly increasing, strictly convex, continuously differentiable vacancy posting cost of \( c(v) \) which satisfies the Inada conditions

\[
\lim_{v \to 0^+} c'(v) = 0 \quad \text{and} \quad \lim_{v \to \infty} c'(v) = +\infty.
\]

(2)

An aggregate matching function \( M(u, \bar{v}) \) that determines the flow rate of new meetings between firms and workers that are generated, as a function of the measure of unemployed workers, \( u \) and the total measure of vacancies posted by active firms, \( \bar{v} \). Each firm meets a worker at Poisson rate proportional to the number of vacancies it posts. Each worker meets a firm at a Poisson rate that is identical across workers. Which unemployed worker meets which vacancy is randomly determined, and does not depend on any other characteristics of worker or firm (except, as already mentioned, for the number of vacancies posted by the firm). We assume that \( M \) exhibits constant returns to scale in \((u, \bar{v})\), and decreasing returns to scale in \( u \) or in \( \bar{v} \) separately. Denote by \( \theta \) the vacancy-to-unemployment ratio \( \bar{v}/u \); then the Poisson rate at which a firm that posts \( v \) vacancies meets a firm is \( vq(\theta) \), where \( q(\theta) = M(u, \bar{v})/\bar{v} \). The rate with which an unemployed worker meets some firm is \( \theta q(\theta) = M(u, \bar{v})/u \). For notational convenience, we will often omit the dependence of \( q(\theta) \) of \( \theta \) and simply write \( q \).

Wages paid by firms to workers are determined following Aumann and Shapley (1974) and Stole and Zwiebel (1996a,b) by assuming that firms and workers bargain over the marginal surplus generated by their employment relationship. To formulate this, denote the Hamilton-Jacobi-Bellman value of a firm with \( n \) employees at date \( t \) by \( J(n, t) \), and the value of a worker employed at such a firm by \( V(n, t) \). Denote the value of an unemployed worker by \( V^u(t) \). Then we assume that wages are determined in such a way that

\[
\phi J_n(n, t) = (1 - \phi) [V(n, t) - V^u(t)].
\]

(3)

Here the subscript \( J_n(n, t) \) denotes partial differentiation with respect to the first argument. Note that by symmetry, a firm will pay the same wage to all its workers; denote by \( w(n, t) \) the wage paid by a firm with \( n \) workers at time \( t \).

Employment relationships are subject to two types of shocks. At Poisson rate \( \delta > 0 \), an active firm is destroyed; in this case, all its workers are returned to unemployment, and the firm is removed from the economy with zero scrapping value. At Poisson rate \( s > 0 \), each

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\( ^6 \)In an earlier version of this paper, Acemoglu and Hawkins (2007), we solved a version of the model with workers of positive size \( \varepsilon \) and derived the form of the Hamilton-Jacobi-Bellman equation more formally by taking limits as \( \varepsilon \to 0 \). The reader is referred to that version of the paper for more detail.
worker employed by the firm is separated from the firm; in this case, the firm continues in existence with all its other incumbent workers.

Unemployed workers generate unemployment income $b > 0$.

3 Steady-state equilibria

In this section, we characterize steady-state equilibria, so that all notation for time can be dropped. We will solve only for equilibria in which the wage payments of firms depend only on the current number of employees of the firm, $n$; thus the wage payments of a firm can be denoted $w(n)$.

3.1 Value functions and definitions

Assume for now that the value function of a firm as a function of its employment level $n$, $J(n)$, is strictly concave and twice continuously differentiable in $n$. In this case, the Hamilton-Jacobi-Bellman equation for a firm may be written as

$$(r + \delta)J(n) = y(n) - nw(n) - snJ'(n) + \max_{v \geq 0} [-c(v) + qvJ'(n)].$$

Intuitively, current flow output is $y(n)$, and total flow wage payments $nw(n)$. In addition, the firm loses a flow of measure $sn$ workers per unit time to the separation shock, with a flow capital loss per unit measure of workers lost of $J'(n)$. Next, the firm chooses its vacancy posting strategy, denoted by $v$, to maximize the difference between the flow capital gains of $qvJ'(n)$ arising from the hiring flow of $qv$ workers per unit time and the flow cost of vacancy posting, $c(v)$.

Define $v(n)$ to be the optimal choice of vacancy posting by a firm with $n$ workers; that is,

$$v(n) = \arg \max_{v \geq 0} [-c(v) + qvJ'(n)].$$

The first-order condition characterizing the optimal choice of vacancy posting by the firm is

$$c'(v(n)) = qJ'(n);$$

denote the solution by $v(n)$. The solution is unique conditional on $J(\cdot)$ and does in fact define the optimal vacancy-posting policy; this follows by the convexity of $c(\cdot)$. Using the optimized value, of course it follows that the firm’s Hamilton-Jacobi-Bellman (HJB) equation can also be written as

$$(r + \delta)J(n) = y(n) - nw(n) - snJ'(n) - c(v(n)) - qv(n)J'(n).$$
or

$$(r + \delta)J(n) = y(n) - nw(n) - snJ'(n) - c\left([c']^{-1}(qJ'(n))\right) + qJ'(n)\left[c']^{-1}(qJ'(n))\right). \quad (8)$$

Analogously to the HJB equation for firms, the HJB equation for workers can now be written as

$$rV(n) = w(n) + (s + \delta)\left[V(n) - V^u\right] + (qv(n) - sn)V'(n). \quad (9)$$

The Stole and Zwiebel bargaining assumption (3) in this stationary environment can be more simply written as

$$\phi J'(n) = (1 - \phi)\left[V(n) - V^u\right] \quad (10)$$

The final HJB equation is that for an unemployed worker. To write this equation, it is necessary to introduce additional notation for the steady-state distribution of firms as a function of the number of workers already employed by the firm. Denote the steady-state firm-size distribution by $G(n)$. We can now write the HJB equation for an unemployed worker, as follows.

$$rV^u = b + \theta \int_0^{n^*} \frac{v(n) dG(n)}{\int_0^{n^*} v(n) dG(n)}; \quad (11)$$

Because search is random and workers meet firms in proportion to the number of vacancies posted, the second term on the right side arises from averaging the capital gain that a worker obtains when leaving unemployment to become employed by a firm of size $n$, weighting by the density of firms of this size, $dG(n)$, and by the intensity of vacancy posting by such firms, $v(n)$. It is convenient to eliminate $V(n) - V^u$ in terms of $J(n)$ using the bargaining equation (3), so that (11) can also be expressed as

$$rV^u = b + \frac{\phi}{1 - \phi} \theta q \int_0^{n^*} \frac{J'(n)v(n) dG(n)}{\int_0^{n^*} v(n) dG(n)}. \quad (12)$$

For the firm size distribution $G(\cdot)$ to be consistent with steady-state equilibrium, it must satisfy a flow balance condition. To write this condition, suppose that there exists a steady-state for $G(\cdot)$ such that the density function $g(n)$ is continuous on $(0, n^*)$. The distribution $G(\cdot)$ is a steady-state distribution if $g(\cdot)$ satisfies the steady state accounting equation that, for each $n \in (0, n^*)$ and each $\varepsilon > 0$ sufficiently small, the flow of firms into and out of the interval $(n - \varepsilon/2, n + \varepsilon/2)$ are equal:

$$\left(\frac{qv(n)}{\varepsilon} + \frac{sn}{\varepsilon} + \delta\right)g(n) = \frac{qv(n - \varepsilon)}{\varepsilon}g(n - \varepsilon) + \frac{s(n + \varepsilon)}{\varepsilon}g(n + \varepsilon) + O(\varepsilon). \quad (13)$$

Taking limits as $\varepsilon \to 0$, it follows that any differentiable solution to this equation must
satisfy the following differential equation for $g(n)$:

$$
\frac{g'(n)}{g(n)} = \frac{s - \delta - qv'(n)}{qv(n) - sn}.
$$

(14)

It follows that

$$
g(n) = C \exp \left( \int_0^n \frac{s - \delta - qv'(\nu)}{qv(\nu) - s\nu} \, d\nu \right),
$$

(15)

where the constant of integration $C$ is chosen so that $g(\cdot)$ integrates to 1 on the region of integration $[0, n^*]$. Finally, observe that since the stochastic process for a firm’s size satisfies an ergodicity condition, the steady state distribution $G(\cdot)$ is unique, so there was no loss of generality in solving only for a distribution in which $g(\cdot)$ is continuously differentiable on $(0, n^*)$.

The final condition needed to characterize steady-state equilibrium can now be specified. This is the free entry condition, which takes the form

$$
J(0) \leq k \text{ and } \theta \geq 0, \text{ with complementary slackness.}
$$

(16)

This requires that in order for there to be positive activity in equilibrium, the value of $k$ needs to be sufficiently low for firms to find it attractive to gain access to the production technology.

The equations introduced so far are sufficient to allow us to characterize completely any equilibrium of the model in the class we are considering (those that satisfy the differentiability and symmetry assumptions). We record this as the following definition.

**Definition 1.** A tuple $\langle \theta, q, V^u, G(\cdot), g(\cdot), J(\cdot), V(\cdot), v(\cdot), w(\cdot) \rangle$ is a steady-state equilibrium if

- $q$ and $\theta$ are related according to the matching function, so that $q = q(\theta)$;
- $J(\cdot), V(\cdot)$, and $V^u$ satisfy the Hamilton-Jacobi-Bellman equations (4), (9), and (11) and the bargaining equation (10);
- vacancy posting is optimal, so that $v(\cdot)$ satisfies (5);
- $G(\cdot)$ has density $g(\cdot)$ satisfying the flow balance condition (15); and
- there is free entry in the sense that equation (16) holds.

Notice that the steady-state equilibrium did not specify the unemployment rate $u$. This is because, as in the standard Mortensen-Pissarides model, the unemployment rate can be determined after the other endogenous variables. In particular, in steady state, a standard...
accounting argument implies that the mass $u$ of unemployed workers will be matched and thus hired at the flow rate $\theta q(\theta)$. On the other side, workers lose their job because of separations at the flow rate $s$ and because of firm shutdowns at the flow rate $\delta$. Consequently, the steady-state unemployment rate is given by equating flows into unemployment, $(1 - u)(s + \delta)$, with flows out of unemployment, $u\theta q(\theta)$, thus

$$u = \frac{s + \delta}{s + \delta + \theta q(\theta)}.$$ \hspace{1cm} (17)

It is straightforward to verify that, as in the standard Mortensen-Pissarides model, $u$ is a monotonically decreasing function of $\theta$: steady-state unemployment is lower when the labor market is tighter.

### 3.2 Wages

In this section we use the bargaining equations and value functions to study the pattern of wages that arise in equilibrium. It turns out that by manipulating the HJB equations for firms and workers (4) and (9), together with the bargaining equation (10), we can establish that a simple expression for the wage function $w(\cdot)$ holds. This is the subject of the following Lemma.

**Lemma 1.** In any steady-state threshold equilibrium, wages satisfy

$$w(n) = (1 - \phi) r V^u + n^{-\frac{1}{\phi}} \int_0^n \frac{1}{\nu} y'(\nu) d\nu.$$ \hspace{1cm} (18)

Note that this wage equation takes the same form as in other papers using the Stole-Zwiebel framework, such as Cahuc, Marque, and Wasmer (2008), and Elsby and Michaels (2010). An alternative formulation of this equation that may be more intuitive is that

$$w(n) = (1 - \phi) r V^u + \phi \frac{\int_0^n \frac{1}{\nu} y'(\nu) d\nu}{\int_0^n \frac{1}{\nu} d\nu};$$ \hspace{1cm} (19)

that is, the wage of a worker at a firm employing $n$ workers is a weighted average of the flow outside option, $r V^u$, and a term that is itself a weighted average of all the inframarginal products, $y'(\nu)$, for $\nu \in (0, n)$.\footnote{Note that in this last weighted average, greater weight is placed on inframarginal products relatively near the current employment level $n$ – to see this, observe that $\nu^{-\frac{1}{\phi}}$ is increasing in $\nu$.}

Generically equation (18) cannot be further simplified to give a closed form solution. Two notable cases where this is nonetheless possible are: (1) if $y(n) = An^\alpha$ is Cobb-Douglas, then the wage takes the form

$$w(n) = (1 - \phi) r V^u + \frac{\alpha \phi}{1 - \phi + \alpha \phi} An^{\alpha - 1},$$ \hspace{1cm} (20)
(2) if alternatively $\phi = \frac{1}{2}$, integration by parts establishes that

$$w(n) = \frac{1}{2} rV^u + \frac{1}{n} \left[ y(n) - \frac{1}{n} \int_0^n y(\nu) \, d\nu \right].$$

(21)

The lemma shows that despite the additional general equilibrium interactions, wages in this model take a form identical to those in Stole and Zwiebel (1996a,b) and Wolinsky (2000). The first term in (18) is the contribution of the (flow value of the) outside option of the worker to his wage. The second term is the worker’s share of his contribution to the value of the firm, taking into account that if the worker were to quit, this would also influence the wages of other employees of the firm.

This explicit form characterization of wages will be important in further characterization and proving the existence of a steady-state equilibrium. A graphical representation of the dependence of wages on the number of workers employed at the firm is indicated in Figure 1. (This figure shows the wage function arising in the example of a Cobb-Douglas production function used in the quantitative evaluation in Section 5 below.) Also shown are a horizontal line indicating the flow value of the unemployed, $rV^u$, and the marginal product function, $y'(n)$.

[Figure 1 about here.]

Our next results show that, as depicted in Figure 1, wages and flow profits satisfy convenient boundary conditions.

**Lemma 2.** In a steady-state equilibrium, wages are strictly positive, strictly decreasing with firm size, and, satisfy

$$\lim_{n \to 0^+} w(n) = +\infty \text{ and } \lim_{n \to \infty} w(n) = (1 - \phi)rV^u.$$  

Moreover, the flow profit $\pi(n) = y(n) - nw(n)$ is strictly concave, maximized at a unique $n \in (0, \infty)$, and satisfies

$$\lim_{n \to 0^+} \pi(n) = 0 \text{ and } \lim_{n \to \infty} \pi(n) = -\infty.$$  

**Proof.** See Appendix A. \qed

The fact that wages at very small firms become very large is a consequence of the Inada condition on the firm’s production function, since the marginal product also becomes arbitrarily large as $n$ decreases to 0.

### 3.3 Equilibrium characterization

We next characterize steady-state equilibria in detail.
Denote by $n^*$ the smallest value of $n$ such that $qv(n) = sn$.\footnote{Such an $n < \infty$ must exist since according to the wage equation, $w(n) = rV^u + \frac{\phi}{1-\phi}\pi'(n)$. Because $rV^u > b$, wages are bounded away from zero. According to the Inada condition (1), it follows that for $n$ large enough, $\pi(n) = y(n) - nw(n) < 0$. It cannot be optimal to hire beyond this point.} At $n = n^*$, the firm posts just enough vacancies to keep its size constant, by hiring exactly the same number of workers that it loses to separation. A firm with this size remains there until it is exogenously destroyed, earning a constant flow profit of $\pi(n^*)$ and paying a constant flow vacancy-posting cost of $c(v(n^*))$. It follows immediately that the value $J(n^*)$ satisfies

$$(r + \delta)J(n^*) = \pi(n^*) - c(v(n^*)).$$

We now look for an equation characterizing $n^*$ further. To obtain this equation, first evaluate the first-order condition for $v(n)$, equation (6), at $n = n^*$ to see that

$$J'(n^*) = \frac{1}{q}c'(v(n^*)) = \frac{1}{q}c' \left( \frac{sn^*}{q} \right).$$

On the other hand, differentiating the HJB equation for the firm, (7), with respect to $n$ and substituting from the first-order condition for $v(n)$ and from the definition of $n^*$ to eliminate terms in $qJ'(n^*) - c'(v(n^*))$ and $qv(n^*) - sn^*$, which both equal zero, we obtain

$$(r + \delta + s)J'(n^*) = \pi'(n^*).$$

Eliminating $J'(n^*)$ from equations (22) and (23) then establishes that

$$\frac{r + \delta + s}{q}c' \left( \frac{sn^*}{q} \right) = \pi'(n^*).$$

According to Lemma 2, $\pi'(0)$ is strictly positive and $\pi(\cdot)$ is strictly concave, so that $\pi'(\cdot)$ is strictly decreasing. $c'(\cdot)$ is strictly increasing by assumption and satisfies the Inada conditions (2), so that (24) has a unique solution for $n^*$ (conditional on the values of the endogenous variables $q$ and $rV^u$). Having solved for $n^*$, the differential equation (8) (which satisfies the usual Lipschitz condition on $[0, n^*]$) together with the boundary condition (22) now gives an initial value problem which can be solved uniquely for $J(\cdot)$ on $[0, n^*]$.

The characterization of $n^*$ given by (24) can be understood intuitively when expressed in terms of wages. Differentiate the definition of profits, $\pi(n) = y(n) - nw(n)$, substitute from the closed form solution for wages, (18), and rearrange to obtain that for all $n$, $w(n) = rV^u + \frac{\phi}{1-\phi}\pi'(n)$. This equation holds for all $n > 0$; substitute $n = n^*$ and use (24) to see that

$$w(n^*) = rV^u + (r + \delta + s) \frac{\phi}{1-\phi} c'(v(n^*)).$$

Equation (25) is very intuitive and states that the firm continues to hire until the wage it pays equals the outside option of the worker, $rV^u$, plus a term that is proportional to the
severity of the labor market friction (measured by the marginal flow cost of the last vacancy required to maintain a workforce of size $n^*$, which is $\frac{1}{q}c'(v(n^*))$). This wage equation is therefore comparable to the result obtained in a static setting by Stole and Zwiebel (1996a) (see their Corollary 1 on page 396) and generalized to a dynamic setting by Wolinsky (2000). In these previous analyses, since there is no hiring margin (and no frictions), the second term is absent. Consequently, those models always imply “over-hiring” relative to a hypothetical competitive benchmark; firms will hire more than this competitive benchmark in order to reduce the marginal product of workers and thus their bargaining power according to the Shapley bargaining protocol (see Stole and Zwiebel, 1996a). Our analysis shows that this over-hiring result may or may not apply in general equilibrium; when $\gamma$ is small, it will, but it could fail to do so when $\gamma$ is large (so that $c'(\cdot)$ is high, ceteris paribus) and $q$ is relatively small. Equation (25) also corresponds to equations obtained by Cahuc, Marque, and Wasmer (2008) and Elsby and Michaels (2010), who considered cases where firms doing positive amounts of hiring are always at their target hiring level $n^*$.

We summarize the analysis in this section as the following proposition (proof in the text):

**Proposition 1.** Let $q = q(\theta) > 0$ and $rV^u > 0$ be given. Then there is a steady-state equilibrium allocation where firms contact workers at rate $q(\theta)$ per unit measure of posted vacancies and where the value of an unemployed worker is given by $rV^u$ if and only if

- where $n^*$ is the unique solution to (24), $J(\cdot)$ is the unique solution to the initial value problem given by the differential equation (8) with boundary condition (22);
- $rV^u$ satisfies (12) where $G(\cdot)$ is given (15) with $v(n)$ given by (6)
- the free entry condition (16) is satisfied.

We are now in a position to establish the existence of a steady-state equilibrium.

**Theorem 1.** An steady-state equilibrium with cutoff hiring strategies exists.

*Proof. See Appendix A.*

The proof of the Theorem consists of showing that there exist $(q, rV^u)$ satisfying the hypothesis of Proposition 1. Here, we present a diagrammatic exposition, emphasizing the intuition. The proof of Theorem 1 establishes that an equilibrium with positive activity exists if

$$k < \frac{1}{r + \delta} \max_{n>0} \left\{ g(n) - n^{-1-\phi} \int_0^n \nu^{-\phi} y'(\nu) d\nu - n(1-\phi)b \right\},$$

where the existence of the maximum on the right side follows as in the proof of Lemma 2. In this case, Figure 2 shows a diagram depicting in $(q, rV^u)$-space the two curves described in the discussion preceding Proposition 1. The upward-sloping curve represents the locus of $(q, rV^u)$ pairs consistent with the free-entry condition of firms, equation (16); intuitively, it is
upward-sloping since, all else equal, an increase in $rV^u$ must be compensated by an increase in $q$. This is because a higher $rV^u$ translates into higher wages, so that the profit margins of firms decline. Starting from a point $(q, rV^u)$ consistent with the free entry condition, if $rV^u$ increases, then to keep the value of $J(0) = k$ unchanged, the firm needs to be able to hire more rapidly. This requires that the cost of hiring must decline, which is achieved by an increase in $q$. The downward-sloping curve is the HJB equation for unemployed workers. Intuitively, this relation corresponds to a downward-sloping locus in $(q,rV^u)$-space since an increase in $rV^u$ on the right side of (11) corresponds to an increase in wages; to keep the flow value of an unemployed worker satisfying this equation, it must be that hiring is more rapid (that is, $q$ is larger), so that when hired, the worker spends less time earning the high wages paid by smaller firms as firms expand more rapidly. (We are not, however, generally able to show that this second curve is everywhere downward-sloping, although this has been true in all calibrated examples we have investigated.) The proof of Theorem 1 given in Appendix A relies on using a continuity argument to establish that an intersection of these two curves must exist.

Comparative statics of the response of the endogenous variables $q(\theta)$ and $rV^u$ can now be obtained from the diagrammatic representation of the equilibrium. In the sequel, we study the response of the economy to productivity shocks extensively, so we focus here first on this. Intuitively, the effect of a productivity increase is clear: it should lead to more entry, a tighter labor market, lower unemployment, and higher wages, just as in the benchmark Mortensen-Pissarides model. An example where this is the case is shown in Figure 3. In the figure, the dashed lines indicate the movement of the curves after a Hicks-neutral increase in productivity, indicating the typical case we find in calibrated examples. However, the effect of a productivity shock on market tightness and on the value of an unemployed worker are ambiguous in theory. It is unambiguous that the curve corresponding to the firm’s free entry condition, (16), moves upwards: for a given $(q, rV^u)$, greater productivity increases flow profits for all firms and so increases the implied value of entry, $J(0)$; to keep the free entry condition satisfied, $rV^u$ must increase for each $q$. However, the movement of the curve corresponding to the worker’s HJB equation, (11), is ambiguous. First, for a given $(q, rV^u)$, the wages paid at a firm with any fixed number $n$ of workers, $w(n)$, increase; however, the increase in $n^*$ means that more workers are employed at larger firms, which, all else equal, pay lower wages. In calibrated examples, the first effect dominates, so that the curve moves upwards as one might expect, but we do not have a proof that it always does so. If both curves move upwards, then it is clear from the graph that the effect of a productivity shock is positive for the flow value of the unemployed worker, but the effect on market tightness is not clear. Nevertheless, in many calibrated examples, including the example shown in Figure 3, $q(\theta)$ decreases in response to the increase in productivity, so that workers’ job-finding rate rises and steady-state unemployment falls. Another interesting feature of this
example is that \( n^* \) also falls in response to the positive productivity shock. This implies that in the new steady state firms are, on average, smaller. Consequently, much of the adjustment to the new steady-state takes place at the extensive margin, that is, by the entry of new firms (while existing firms in fact decline in size). This is a pattern we find consistently in quantitative examples, and underlines the importance of separating the intensive and extensive margins of employment creation.

[Figure 3 about here.]

Other comparative statics can also be analyzed. The effect of an increase in the entry cost, \( k \), is particularly clear. Similarly to the logic given in the case of a productivity shock, if \( k \) decreases, the curve corresponding to the free-entry condition, (16), moves upwards in response. Since a change in \( k \) does not affect the other curve, it has unambiguous effects on the steady-state equilibrium in this situation presented in the diagram, in which the worker’s Bellman equation is downward-sloping. Also, a decrease in \( k \) leads to an increase in \( rV^u \) and a decrease in \( q \). This also corresponds to a decline in \( \theta \) and therefore, from (17), to an increase in the steady-state unemployment rate.

We conclude the analysis in this section with two final remarks. First, the model studied in this section does have one important limitation. Although the model endogenously generates dispersion in firm sizes, the steady-state firm size distribution, as indicated by equations (14) and (15), is difficult to match to data. The distribution of firm sizes is right-skewed. This is the realistic case to consider since the vast majority of separations do not occur at establishment shutdown. Allowing for heterogeneity in firm productivity would be required in order to generate a tighter match between the firm size distribution generated by the model and its empirical counterpart. Other authors also face analogous issues (Burdett and Mortensen, 1998). We do not pursue this modification here.

Finally, we note that although the version of the model studied thus far is very tractable, it does not have the advantage of being solvable in closed form. A slight modification of the assumption on the vacancy-posting technology brings the model closer to that point and is presented in the next subsection.

\[ \text{A sufficient condition to ensure that this is true in the model is that } s > \delta; \text{ in this case (15) describes a right-skewed distribution even if } v(n) \text{ were constant in } n. \text{ Since } v(n) \text{ decreases with } n, \text{ the tendency of } G(\cdot) \text{ to be right-skewed is greater.} \]

\[ \text{For example, Davis, Haltiwanger, and Schuh (1996, Figure 2.3, page 29) report for U.S. manufacturing plants that in quarterly data, 11.6\% of job destruction occurs in plant shutdowns, while in annual data, 22.9\% of job destruction occurs in plant shutdowns. The reasons for the discrepancy are that closure takes time and that transitory plant-level employment changes are more important in higher frequency data (fn. 9, p. 27).} \]
3.4 Closed-form solution

In this subsection, we briefly present a variant of our baseline model that allows closed-form solutions.\footnote{This version of the model was presented in much more detail in a previous version of the model, Acemoglu and Hawkins (2007), and the reader is referred there for a more formal treatment.} Suppose that the function \( c(v) \) takes the form \( c(v) = \gamma v \) for \( v \in [0, 1] \) and \( c(v) = +\infty \) for \( v > 1 \). Even though this functional form does not satisfy the strict convexity assumption made on \( c(\cdot) \), it remains tractable. It is possible to show that there now exist an equilibrium of a threshold type, in which there exists an \( n^* > 0 \) such that firms set \( v(n) = 1 \) for \( n < n^* \) and \( v(n) = 0 \) for \( n > n^* \). In this case, the HJB equation for the firm, equation (4), can be written

\[
(r + \delta)J(n) = \begin{cases} 
  y(n) - nw(n) - \gamma + (q - sn)J'(n) & n < n^* \\
  y(n) - nw(n) - snJ'(n) & n > n^*. 
\end{cases} 
\]

(26)

An analogous argument to that used to prove Lemma 2 establishes that (18) still characterizes wages in this case. Substituting this into (26) gives a differential equation which can be solved in closed form by use of the appropriate integrating factor; the solution is

\[
J(n) = (q - sn)^{-\frac{r + \delta}{s}} \left[ (q - s n^*)^{\frac{r + \delta}{s}} J(n^*) + \int_{n}^{n^*} (q - s \nu)^{\frac{r + \delta}{s} - 1} (y(\nu) - \nu w(\nu) - \gamma) \, d\nu \right]. 
\]

(27)

It is also possible to solve the integral on the right side of the HJB equation for the unemployed worker, equation (11), in closed form. However, solutions for the values of \( q \) and \( r V^u \) consistent with equilibrium still need to be established numerically.

Despite the slightly greater analytical tractability, this version of the model has two important disadvantages, which lead us not to consider it further in the sequel. The less important of the two reasons is that the maximum firm size possible with this vacancy-posting technology is \( q/s \) (since a firm with that many workers loses workers to the separation shock as rapidly as it hires). Since \( 1/q \) is the expected duration of a vacancy in the model, this suggests a discrepancy between the size of the largest establishments in the economy and the observed vacancy filling time, which is of the order of a couple of months (van Ours and Ridder, 1992). For example, if expected vacancy duration is 2 months, the worker separation rate to unemployment is 0.10 at quarterly frequencies (Shimer, 2005), of which (an upper bound) 25% arise from firm destruction, then \( q = 1.5 \) and \( s = 0.075 \) at quarterly frequencies, giving a maximum firm size of 20.

The more important of the two drawbacks of the version of the model with this vacancy-posting cost assumption is that it is less tractable for numerical work. Because firms are exactly indifferent between posting any number of vacancies between 0 and 1 when at their target size \( n^* \), numerical inaccuracies arising when approximating value functions can make solutions for equilibrium variables sensitive to approximation error in the steady state and
intractable for the algorithm we use for computing transitional dynamics. For this reason in our numerical work, we do not study this version of the model further.

4 Out-of-steady state dynamics

In this section, we briefly indicate how to modify the steady-state version of our model so as to be able to discuss dynamics out of steady state. Recall from Section 2 that in an environment that changes over time, we need to add a time argument to all endogenous variables (for example, \( J(n,t) \) rather than \( J(n) \)), and we denote partial derivatives by subscripts (thus, \( J_{nn}(n,t) \) denotes the second partial derivative of \( J(n,t) \) with respect to \( n \)).

A dynamic equilibrium is defined similarly to a steady-state equilibrium, but is naturally more involved, since all objects are time-varying. Before defining such an equilibrium more formally, let us develop the equivalent of the steady-state Hamilton-Jacobi-Bellman equations. A standard derivation gives the firm’s HJB equation, which requires that the value of a firm with \( n \) workers at time \( t \) satisfy:

\[
(r + \delta)J(n,t) = y(n,t) - nw(n,t) - snJ_n(n,t) + \max_{v \geq 0} \{-c(v) + q(\theta(t))vJ_n(n,t)\} + J_t(n,t),
\]  

\((28)\)

where \( \theta(t) \) is labor market tightness at time \( t \), \( y(n,t) \) is the output of a firm with \( n \) employees at time \( t \), and \( w(n,t) \) is the wage function at time \( t \), which will again be determined in equilibrium. All the terms have similar interpretation to the Bellman equation (4) above, except that there is also the time derivative of the value function, \( J_t(n,t) \) on the right hand side. Denote by \( v(n,t) \) the optimal vacancy-posting strategy of the firm, which satisfies

\[
v(n,t) = \arg \max_{v \geq 0} -c(v) + q(\theta(t))vJ_n(n,t).
\]  

\((29)\)

In addition, the HJB equation for the workers implies that the value function \( V(n,t) \) must satisfy

\[(r + \delta + s)V(n,t) = w(n,t) + [q(\theta(t))v(n,t) - sn]V_n(n,t) + (\delta + s)V^n(t) + V_t(n,t),\]

\((30)\)

which again only differs from the steady-state equation (9) because of the time derivative \( V_t(n,t) \) on the right hand side.

Free entry still holds at all dates, so we also need to have

\[
J(0,t) \leq k,
\]  

\((31)\)

with equality whenever there is entry.

Wages are again determined by bargaining à la Stole and Zwiebel, as in (3).
The value of an unemployed worker is now time varying and is denoted \( V^u(t) \). We have for each \( t \),

\[
rV^u(t) = b + \theta(t) q(\theta(t)) \int_0^\infty \left[ V(\nu, t) - V^u(t) \right] v(\nu, t) \, dG(\nu, t) \int_0^\infty v(\nu, t) \, dG(\nu, t) + V_t^u(t). \tag{32}
\]

To complete the description of the environment, we need to specify the distribution of firm sizes over time, represented by \( g(n, t) \). To derive this distribution, let us reason as in the previous section and start with the case in which each worker is of size \( \varepsilon > 0 \). In this case, away from steady state, the rate of change in \( g(n, t) \) over time, \( g_t(n, t) \), is given by the difference between flows in and flows out of firms into the “state” of having \( n \) employees.

Denote by \( n^*(t) \) the maximum firm size at time \( t \). For \( n < n^*(t) \),

\[
g_t(n, t) = -\left( \frac{q(\theta(t))v(n, t)}{\varepsilon} + \frac{sn}{\varepsilon} + \delta \right) g(n, t) + \frac{q(\theta(t))v(n - \varepsilon, t)}{\varepsilon} g(n - \varepsilon, t) + \frac{s(n + \varepsilon)}{\varepsilon} g(n + \varepsilon, t) + O(\varepsilon).
\]

Now taking the limit \( \varepsilon \to 0 \), we obtain the partial differential equation for \( n < n^*(t) \):\(^{13}\)

\[
g_t(n, t) = - (\delta - s) g(n, t) + sng_n(n, t) - q(\theta(t)) \frac{\partial}{\partial n} [v(n)g(n)](n, t). \tag{33}
\]

Given these equations, we can define a dynamic equilibrium as follows:

**Definition 2.** A tuple \( \langle \theta(t), q(t), V^u(t), G(n, t), g(n, t), J(n, t), V(n, t), v(n, t), w(n, t) \rangle \) is a **dynamic equilibrium** if for all \( t \),

- \( q(t) \) and \( \theta(t) \) are related according to the matching function, so that \( q(t) = q(\theta(t)) \);
- \( J(\cdot), V(\cdot), \) and \( V^u(\cdot) \) satisfy the HJB equations (28), (30), and (32), as well as the bargaining equation (3);
- there is free entry in the sense that equation (31) holds;
- vacancy posting is optimal, so that \( v(\cdot) \) satisfies (29); and
- \( G(\cdot) \) has density \( g(\cdot) \) satisfying the flow balance condition (33).

We will largely proceed to study the out-of-steady state dynamics of our model numerically. However, it is worth pointing out that the analog to the wage equation (18) does still hold even out of steady state. A very similar argument to that used to establish that equation now proves that

\[
w(n) = (1 - \phi) [rV^u(t) - V_t^u(t)] + n^{-\frac{1}{\delta}} \int_0^n \nu^{1-\frac{\alpha}{\delta}} y(\nu) \, d\nu. \tag{34}
\]

\(^{13}\)For \( n > n^*(t) \), we can derive a similar partial differential equation. Nevertheless, if there are initially no firms above \( n^*(t) \) and provided that \( n^*_t(t) > -sn^*(t) \) at all \( t \), there will never be any such firms, so that this partial differential equation is of limited interest for our focus.
The only difference between this equation and \((18)\), which applied in the steady-state analysis, is the presence of \(V_u(t)\) in the first term, which represents that the effect of the change in the outside option of the worker on his current wage.

5 Quantitative investigation

In this section, we parameterize our model so that key endogenous variables in the steady state correspond closely to those of the U.S. economy. We investigate the comparative static effect across steady states of changing productivity. We then study the dynamics of the transition between steady states. To foreshadow our main results, we find that our model generates little amplification of productivity shocks, but substantial persistence. Finally, in Section 5.4, we modify our model by removing the assumption of decreasing returns to labor in production, and find that under constant returns, our persistence mechanism is not present.

5.1 Targets

We first describe our quantitative methodology and parameter targets. We assume the production function takes the Cobb-Douglas form, \(y(n) = An^\alpha\). (This functional form is convenient since \((20)\) then gives a closed-form solution for the wage.) We also need a functional form for the vacancy-posting cost. We therefore assume that \(c(\cdot)\) takes a power functional form, so that, with \(p \geq 0\),

\[
c(v) = \frac{\gamma v^{1+p}}{1+p}.
\]  

Finally, we assume that the matching function \(M(u,\bar{v})\) takes the form \(M(u,\bar{v}) = Zu^\eta \bar{v}^{1-\eta}\) where \(Z > 0\) is a constant; this implies that \(q(\theta) = Z\theta^{-\eta}\). This Cobb-Douglas form is consistent with evidence from Petrongolo and Pissarides (2001) and Shimer (2005).

Table 1 gives the parameterization for the pre-shock steady state of the model. Some values require some comment. We choose the unit of time to be quarterly. For the sake of comparability with the literature, we borrow parameters as closely as possible from the benchmark paper in the quantitative study of fluctuations in the Mortensen-Pissarides family of models, which is Shimer (2005).

First, we set the ratio of the unemployment income to match output, given by \(b/A\), to 0.4, following Shimer. As is by now familiar from the work of Hagedorn and Manovskii (2008), increasing this value tends to amplify the effects of productivity shocks in the Mortensen-Pissarides model and its kindred; since we do not have anything substantive to add to the literature on this, we also consider the effect of a changing this value to 0.95. We set the scale parameter \(Z\) in the matching function to 1.355 and the elasticity \(\eta\) to 0.72, following Shimer, to match the mean job-finding rate and the slope of the Beveridge curve conditional on an
initial steady-state value for $\theta$ of 1. This is sensible since vacancies are relatively poorly measured in the U.S., while the model admits a normalization: multiplying $Z$ by $x^{\eta-1}$ and $\gamma$ by $x^{-(1+p)\eta}$ changes the equilibrium only by multiplying $\theta$ by $x$, $q$ by $x^{-\eta}$, and $v(\cdot)$ by $x^\eta$, so that $qv(\cdot)$ and $\theta q$, the matching rates for firms and workers, are unchanged. We set $\phi = 0.72$ for comparability with Shimer. We set $s + \delta = 0.10$ to match Shimer, and follow the evidence of Davis, Haltiwanger, and Schuh (1996) in assuming that one-sixth of job destruction is attributable to firm shut-down.

We somewhat arbitrarily impose in our baseline quantitative example that $p = 1$, so that the vacancy cost function is quadratic. (We therefore also consider the case of much more convex vacancy-posting costs, by investigating also the case $p = 4$.) We also set the Cobb-Douglas scale parameter $\alpha = 0.67$; this is on the low end of reasonable targets (it is reasonable if the source of decreasing returns to scale is that all factors other than labor are fixed at the firm level over the relevant time scale).

We take from Davis, Haltiwanger, Jarmin, and Miranda (2006) that the average employment of U.S. firms (both publicly- and privately-held) is 23.8, and we choose the level parameter $\gamma$ in the vacancy-posting cost function so that the mean firm size matches this figure. We choose the scale parameter in the matching function, $Z$, to be consistent with a steady-state unemployment rate of 6.87%, again following Shimer. (Note that together with a steady-state unemployment rate of 6.87%, this requires that the number of firms in the economy be $(1 - 0.0687)/23.8 = 0.0391$. To be consistent with a vacancy-unemployment ratio $\theta = 1$, this requires that the average number of vacancies posted per firm be around 1.76.) We choose the free-entry cost $k$ so that the free entry condition, equation (16) is satisfied. Finally, we choose the level of productivity, $A$, so as to normalize the value of an unemployed worker in steady-state equilibrium, $r V^u$, to 1.

To put these derived parameters in context, note that at the mean level of vacancy-posting, the flow cost incurred by a firm in posting vacancies, $c(v) = \gamma v^p/(1 + p)$, is around 0.18, or around 22% of the marginal product of a worker of a firm at the mean firm size. It takes in expectation $(qv)^{-1} \approx 0.42$ quarters for a worker to be hired. That is, generating a new hire costs around 9.2% of one quarter’s production by a single worker. However, since wages are a large fraction of output, this is more than 43% of the average flow profit per worker at a firm with the mean number of employees. The free-entry cost corresponds to around 66 quarters of the flow output of the marginal worker or just under 13 quarters of the total flow profit of such a firm.

[Table 1 about here.]

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14Shimer’s justification for this choice, the Hosios condition, does not apply here in the case of a decreasing returns to scale production function, though it would apply if $y(\cdot)$ were linear, as we consider in Section 5.4 below.

15In Section 5.4 below we will investigate the effects of the other extreme case, where $\alpha = 1$ and the production function exhibits constant returns to labor.
5.2 Comparative statics of steady states

The method of solution of the steady-state version of the model is intuitive given the discussion in Proposition 1. We guess values of the two key endogenous variables, $\theta$ and $rV^u$, and set $q = q(\theta)$. We then solve the initial value problem implied by (8) with the boundary condition that at the value of $n^*$ satisfying (24), equation (22) is satisfied. This allows us to determine the vacancy posting policy of the firm via (6), and this in turn allows us to construct the firm size distribution according to (15). We can then check whether the free entry condition (16) and the Bellman equation for the unemployed worker (11), neither of which were used so far, hold; if not, we alter our initial guesses for $\theta$ and $rV^u$ and repeat the process. We always find a unique solution for $\theta$ and $rV^u$, although this was not guaranteed theoretically (Theorem 1 establishes existence but not uniqueness).

Table 2 shows the long-run response of key endogenous variables of the economy to a positive productivity shock. The three panels of the table report the comparative statics in the case of three different parameterizations. The first is the baseline quantitative example, as described in Table 1. Also reported are two alternatives, first where unemployment income $b$ comprises a much higher fraction of the flow value of an unemployed worker, and second, where the vacancy posting cost is much more convex. In each alternative, we modify productivity $A$, the size of the vacancy-posting cost function, and the free entry cost so as to match our targets that market tightness, $\theta$, and the flow value of an unemployed worker, $rV^u$ both equal unity and the average employment per firm equals 23.80 in the initial steady state. The required values for these parameters are reported in each case.

[Table 2 about here.]

In each of the three panels of Table 2, Column 1 reports the initial steady-state. The only difference between the three alternatives is that when $p$ is higher, firm sizes are slightly more dispersed: the largest firms are larger in the third panel. This is because with highly convex vacancy-posting costs, small firms with high marginal products do not increase their vacancy posting relative to larger firms by as much as they would if vacancy-posting costs increased less rapidly.

Column 2 of Table 2 reports the steady state which results after a 1% increase in productivity, $A$. The qualitative responses of all the key endogenous variables are similar across all three panels, and are consistent with the discussion given in Section 3 above. Market tightness increases, slightly more than in the Shimer (2005) calibration of the Mortensen-Pissarides model, and unemployment decreases slightly. The flow value of an unemployed worker (and hence, average wages) increase more than one-for-one with the productivity shock. The average firm size decreases; with the increase in productivity, it is profitable for a greater measure of firms to pay the entry cost. In addition, with the higher value of $rV^u$ and hence of wages, firms stop growing at a lower maximum size. The only important difference between the three panels is that in the high-$b$ alternative of the second panel,
market tightness and unemployment (although not, interestingly, firm size) respond much more elastically to the productivity shock.

Column 3 of Table 2 reports the results of allowing no entry after the productivity shock. That is, we assume that the number of firms remains fixed at the level of the original steady state, and the free entry condition no longer holds after productivity changes. That is, we shut down the extensive margin of adjustment to the shock (entry by new firms) and allow only the intensive margin (changes in the size of incumbent firms) to operate. Across all three panels, it is noticeable that the response of unemployment, market tightness, and wages are less elastic than when the intensive margin is operative. Average and maximum firm sizes increase in each case, although except in the high-b alternative, the effect is tiny: the increase in wages means that firms scarcely wish to hire more workers in the new steady state than they did in the old. The difference in results between Column 2 and Column 3 shows that the combination of the extensive and intensive margins available in our benchmark model is crucial for the quantitative and qualitative response of the economy across steady states.

In summary, the steady-state response of our model to productivity shocks does not demonstrate any very significant long-run interactions between the added structure that we investigate and the responses of the key endogenous labor market variables such as unemployment, market tightness, and wages. The most important lesson of the comparative static exercises reported here is that whether entry occurs on the intensive or extensive margin can make an important quantitative difference to the response of these variables; when entry of new jobs is only on the intensive margin, the new jobs created are relatively low in productivity, and so the response of the economy to the shock is smaller than when entry is on the extensive margin.

5.3 Transitional dynamics

In order to solve for the transitional dynamics following an unanticipated permanent productivity shock, analogously to the case considered in the case of a constant returns to labor production function in Section 5.4, a slight modification of the above procedure is required. We take a discrete-time approximation to the model, as well as approximating the value function on a discrete state space for $n$ (and using cubic splines to interpolate for other $n$). This is necessary since numerically differentiating an incorrect guess for $J(\cdot)$ induces errors which are intractable numerically. The numerical solution procedure we use is reported in Appendix B.

The solution generated has minimal error. At the reported allocation, relative error in $|\hat{V}(t)/V(t) - 1|$ and $|\hat{\theta}(t)/\theta(t) - 1|$ are less than $10^{-9}$ for each $t$, and relative error in the free entry condition, $|J(0,t)/k - 1|$ is less than $2 \times 10^{-7}$.

Figure 4 shows the behavior of the firm size distribution in the case of free entry. The three curves shown indicate the initial and final steady-state firm size distributions, together
with a distribution arising ten quarters into the transition. As can be seen, there is an initial burst of entry at the time of the shock; these new entrant firms take time to grow, so that even after ten quarters, they are still far from the maximum possible size. Most of the net change in the number of firms has occurred already by ten quarters of the shock, but the size distribution is still far from its final shape. The comparable dynamics of the firm size distribution in the case of no free entry are not reported, since the final steady-state distribution is almost indistinguishable from the initial one.

Figures 5 indicates the impulse responses of key endogenous variables to the productivity shock in the case where, after the shock, no new entry is allowed, so that all growth in the total number of jobs is on the intensive margin. First, note that key endogenous variables such as market tightness and the flow value of an unemployed worker do not immediately jump exactly to their new steady state values (as they would in the Mortensen-Pissarides model – see Pissarides, 2000, Section 1.7). The flow value of unemployment, \( rV_u - \dot{V}_u \), undergoes nearly 98% of its eventual adjustment in the period of the shock, and market tightness, \( \theta \), undergoes 87%. The unemployment rate is somewhat persistent: after ten quarters, unemployment has undergone just over 75% of its transition.

Under free entry, dynamics are qualitatively different. This is shown in Figure 6. (Note the different vertical scale in Figure 6 than in Figure 5.) First, on impact of the shock, wages jump up sharply as new firms enter and temporarily pay high wages. On impact, the flow value of an unemployed worker, \( rV_u - \dot{V}_u \), increases 6.8% in response to the 1% increase in productivity. In transition, wages gradually decline as the new firms grow and move down their marginal product schedules; after ten quarters, wages are 2.8% above their initial level, after twenty quarters, 2.0%, and in the long run, 1.5%, as already reported in Table 2. Market tightness actually decreases at the time of the shock, as the rise in wages due to new entrants causes incumbent firms to reduce their vacancy posting more than one-for-one. Thus, the effect on impact is a decline of 0.4 percent, or around \(-19\)% of the long-run increase of 2.4%. Ten quarters after the shock, 55% of the eventual adjustment is complete, and twenty quarters later, 80%. Entry still largely occurs at the time of the shock: 90% of the change in the total number of active firms occurs in the initial period, and more than 98.6% within the first year. Unemployment is now highly persistent. Due to the initial decrease in vacancy posting by incumbent firms, unemployment actually rises very slightly during the two quarters after the shock. After ten quarters, it has undergone 30% of its eventual adjustment, and after twenty quarters, 64%. The size of the largest firms declines monotonically to its new steady-state, and is also quite persistent.
In summary, the basic novel qualitative result of our model is that the responses of key labor market variables are highly persistent where they exhibited no dynamics, or only highly transitory ones in the benchmark Mortensen-Pissarides model. Moreover, this is markedly more so when there is free entry of new firms after the arrival of the productivity shock.

The key intuition for this contrast between the behavior of our model and of a standard Mortensen-Pissarides model is that the firm-size distribution is now a state variable of the model under our assumption of time-consuming hiring. Consider the case where there is entry of new firms after the productivity shock. Since hiring takes time, new entrant firms after the shock will remain small for some time. If there are decreasing returns to labor, this means a greater-than-usual number of high marginal product jobs will be temporarily available. Under the bargaining assumption we use, these jobs also pay high wages. This generates the possibility for the unemployed to earn high wages if they are lucky enough to match with such a firm, causing $rV^u$ to jump above its new steady-state value on impact. According to equations (18) and (34), this then causes wages to increase for all firms, including incumbents. This reduces vacancy-posting by incumbents, in the quantitative example by enough that market tightness actually falls on the impact of the shock. In addition, the fact that wages are temporarily higher than they will be in the long run at the new steady state renders it unprofitable for all the firms needed to accomplish the full transition to the new steady state to enter at once. The entry process of new firms therefore now continues for some time after the initial shock. For the same reason as already mentioned, these later entrants then pay high wages when they do enter, and this keeps wages higher for longer, further amplifying the persistence mechanism.

If the extensive margin is not operative, all increased vacancy posting and job creation in response to the shock must be done by incumbents, and the dynamics are different. In our quantitative examples, incumbent firms mostly have employment size quite near the average target of 23.8 workers per firm. In response to the shock, these firms must all grow slightly. As they increase vacancy-posting immediately after the shock, the probability of job-finding for an unemployed worker rises above the value it will take in the new steady-state. However, a greater fraction of vacancies than in the new steady state are in this case posted by relatively large firms, who offer relatively low wages. These effects almost exactly cancel each other in our quantitative evaluation, so that there are almost no dynamics in $rV^u$ after its initial jump on the arrival of the shock. Market tightness continues to increase slightly in transition as unemployment falls; without free entry on the extensive margin and with existing firms unwilling to increase vacancy posting too much due to the increasing marginal vacancy-posting costs, firms are unwilling to post enough vacancies at the time of impact for market tightness to increase all the way to its new steady-state values.

To summarize, none of the effects present in our benchmark would be so strong without all three contributing factors, time-consuming hiring, bargained wages, and decreasing
returns to scale production functions, so the basic mechanism is novel to our model.

5.4 Modification: constant returns in production

In order to understand the source of the persistence mechanism identified in the previous Section more fully, we now proceed to show that our assumption of decreasing returns to labor in the production function is crucial for the qualitative features of the dynamics. To do this, in this subsection we modify the underlying production technology of firms, replacing the decreasing returns to scale assumption of the benchmark model with constant returns.

Therefore, assume that the production function takes the linear form

\[ y(n) = An. \] (36)

Under the assumption in (36), our analysis simplifies significantly. The reason for the simplification is that the wage equation (34) now takes the simpler form

\[ \tilde{w}(n,t) \equiv w(t) = (1 - \phi) [rV^u(t) - V^u_{t}(t)] + \phi A, \] (37)

independent of \( n \). Because the size of the firm is now irrelevant for the wage it pays, the value \( V(n,t) \) of a worker hired by a firm with \( n \) employees is also independent of \( n \), and can be denoted \( V^e(t) \). This in turn means that the HJB equation for an unemployed worker takes the simple form

\[ rV^u(t) = b + \theta q \left[ V^e(t) - V^u(t) \right]. \]

Further, it also means that after a positive productivity shock, the evolution of the firm size distribution in the future is irrelevant for the equilibrium that obtains in the economy today. (That is, the model now has the same 'block recursivity' property as does the large-firm model of Pissarides (2000).) As in the Mortensen-Pissarides model, this implies that the transitional dynamics following such a shock are simple to characterize. On arrival of the unanticipated productivity shock, a positive measure of firms enters instantaneously, and the labor market tightness \( \theta \) jumps immediately to its new steady state level, as does the vacancy-posting policy \( v \) of each active firm, the wage \( w \) (which is independent of firm size), and the values of a firm with \( n \) workers, \( J(n,t) \), of an unemployed worker \( V^u \), and of an employed worker \( V^e \) (the latter is also independent of firm size).\(^{16}\) The same argument as in Section 1.7 of Pissarides (2000) establishes that in fact there are no other equilibria of the model.

Some variables do exhibit transitional dynamics, notably the unemployment rate and the size distribution of firms, but these variables do not enter in the equations that charac-

\(^{16}\) This would not be correct if we were considering an unanticipated decrease in productivity, following which the number of entrant firms would not suddenly decrease to the new steady state value — instead, it would decline slowly as firms exit due to exogenous destruction — so that it is important here that we are considering a positive shock to productivity.
terize the set of variables mentioned in the previous paragraph. The differential equation indicating the evolution of unemployment over time is identical to that in the Mortensen-Pissarides benchmark:

\[ u_t(t) = (s + \delta)(1 - u(t)) - \theta qu(t). \]

This has solution

\[ u(t) = u^1 + (u^0 - u^1) \exp(-(s + \delta + \theta q)t), \]

where \( u^0 \) and \( u^1 \) are respectively the pre- and post-shock steady state values of unemployment, given by (17) with the corresponding values of \( \theta \). The differential equation for the evolution of the size of any particular firm is

\[ n_t(t) = qv - sn(t); \]

analogously, this has solution

\[ n(t) = n^1 + (n^0 - n^1) \exp(-st), \]

where \( n^1 \) is the limiting value of the firm’s size, given by \( n^1 = qv/s \), and \( n^0 \) is its size at the time of the productivity shock. Note that the convergence of firms to the new steady state size is much slower than that of the unemployment rate (the ratio \( (s + \delta + \theta q)/s \) is around 17.5 in our quantitative exercise).

In order to illustrate how the transitional dynamics established above compare to those in the benchmark model characterized in Section 5.3, it is useful to choose the parameters in the model with linear production so that it corresponds as closely as possible to the parameter targets referred to in Table 1. We choose all parameters the same as in that model, with the exception that we must alter the values of productivity, the scale parameter in the vacancy-posting cost function, and the free-entry cost in order to hit our targets of an unemployment rate of 6.87%, a mean firm employment level of 23.8, and a flow value of unemployment of 1. This requires setting \( A = 1.0689 \), \( \gamma = 0.0985 \), and \( k = 5.300 \).

In Table 3, we report the steady-state effects of the arrival of a permanent 1% increase in \( A \). The comparative statics reported in Table 3 are qualitatively similar to those reported in Table 2. In response to the change in productivity, under free entry, unemployment falls slightly, market tightness rises slightly, wages rise slightly, and both average and maximum firm size rise slightly. The effects are all slightly less pronounced than in the model with decreasing returns to labor. With no free entry, the results are smaller (and in fact almost identical to those reported with decreasing returns to labor).\(^{18}\)

\(^{17}\)We also choose the scale parameter in the matching function to be \( Z = 1.355 \), since we can solve the model with a linear production function in continuous time, meaning the small time aggregation issues that arose in the numerical work in Section 5.3 in particular do not arise here.

\(^{18}\)In Table 3, we report results only for the benchmark choices of parameters, where unemployment income \( b = 0.4 \) and the vacancy-posting costs are quadratic \( (p = 1) \); results for alternatives vary from the benchmark.
Figures 7 and 8 show the transitional dynamics graphically. Figure 7 shows the case where no entry is allowed following the shock, and Figure 8 the case where there is free entry. For comparability to the benchmark case of decreasing returns to be studied below, we also show the behavior of other variables which jump immediately to their new steady-state values on these figures, specifically, labor market tightness, the flow value of an unemployed worker, $rV^u - \dot{V}^u$, and the number of active firms.

Comparing Figures 7 and 8 with the corresponding graphs from the benchmark model with decreasing returns in production (Figures 5 and 6) make clear that decreasing returns in production are crucial for the dynamics in the benchmark model. The responses of all variables are far less persistent in the model with linear production. In the model with decreasing returns to scale, the unemployment rate underwent 75% of its transition to its new steady-state value in the first ten quarters after the arrival of the shock (if there was no free entry), or only 30% of its transition (if adjustment also took place on the extensive margin via the entry of new firms). In the model presented in this section, the corresponding statistic under either assumption on entry is greater than 99.999%. For some variables the contrast is even more striking: in the linear model, the value of an unemployed worker jumps immediately to its new steady-state value, while in the model with decreasing returns and free entry, this value overshot the final steady-state value significantly on the impact of the shock, and after twenty quarters was still 2.0% above the original steady-state value, significantly above the new steady-state value (1.5% above the former value).

6 Conclusion

This paper considered a generalization of the canonical Diamond-Mortensen-Pissarides search model to an environment in which firms employ multiple workers (with decreasing returns to labor) and hiring is explicitly time-consuming (because there is a convex cost of posting vacancies). We followed Stole and Zwiebel (1996a,b) and assumed that wages are determined by continuous bargaining between the firm and its employees. Our model introduces a meaningful distinction between the intensive and extensive margins of hiring – entry of new firms and hiring by existing firms, respectively. There is also an endogenous dispersion of firm sizes and wages (because small firms have higher marginal product of labor and pay higher wages). We proved the existence of a steady-state equilibrium, in ways analogous to those reported in Table 2 for the case of decreasing returns to labor.
characterized several properties of steady-state equilibria and provided numerical results on out-of-steady-state dynamics.

The most important contribution of our model is a new mechanism for labor market persistence. Because hiring is time-consuming, the distribution of firm sizes in the economy becomes a state variable, which adjusts only slowly after the arrival of a shock to labor demand. As a result, wages increase temporarily following a positive shock and decline towards their steady-state value only slowly. Under plausible parameters, this new mechanism is particularly pronounced because a large portion of the adjustment takes place by an increase in the extensive margin first (followed by a slow process of expansion of these new firms). Higher wages slow down hiring by all firms, significantly increasing labor market persistence.

We believe that the new source of persistence identified in this paper could be potentially important for understanding labor market dynamics, and future work could investigate this mechanism both theoretically and empirically. First, in our model all firms are homogeneous, except for their level of employment. This leads to unrealistic firm size and wage distributions. Incorporating heterogeneity both in worker and firm productivity could generate more realistic distributions and enable a meaningful investigation of how they change over the business cycle. Second, our model suggests that the temporarily higher wages that firms that are below their steady-state employment pay play an important role in business cycle dynamics. It would thus be interesting to investigate how wages and hiring change by firm size over the business cycle. Recent research by Moscarini and Postel-Vinay provides evidence on the important role that small firms pay in the early stages of expansions among which is potentially consistent with the dynamics implied by our model.

A Appendix: Omitted Proofs

Proof of Lemma 1. First specialize the Stole-Zwiebel bargaining equation (3) to this stationary case as

\[ \phi J'(n) = (1 - \phi) [V(n) - V^u], \] (38)

and differentiate this equation to note that

\[ \phi J''(n) = (1 - \phi)V'(n). \] (39)

Next, rearrange the worker HJB equation to observe that

\[ (r + \delta + s) [V(n) - V^u] = w(n) - rV^u + [qv(n) - sn] V'(n). \]
Use the bargaining equations (38) and (39) together with the firm’s HJB equation (4) to observe immediately that

\[
\phi \left[ y'(n) - w(n) - nw'(n) \right] = (1 - \phi) [w(n) - rV^u],
\]

or equivalently

\[
w(n) + \phi nw'(n) = \phi y'(n) + (1 - \phi) rV^u.
\]

Integrating this equation with respect to \( n \) implies that

\[
w(n) = (1 - \phi) rV^u + n^{-\frac{1}{\alpha}} \left[ c + \int_0^n \nu^{\frac{1-\phi}{\alpha}} y'(
u) d\nu \right], \tag{40}
\]

where \( c \) is a constant of integration. Assuming that the wage bill for a small firm is finite, so that \( nw(n) \) remains finite as \( n \to 0^+ \), it is immediate that the constant of integration \( c \) in (40) is zero, which completes the proof.

**Proof of Lemma 2.** Define

\[
\psi(n) = n^{-\frac{1}{\alpha}} \int_0^n \nu^{\frac{1-\phi}{\alpha}} y'(
u) d\nu,
\]

Then \( \psi(n) > 0, \psi'(n) < 0, \) and \( \lim_{n \to 0^+} \psi(n) = +\infty \) and \( \lim_{n \to \infty} \psi(n) = 0 \). The first of these claims is obvious; the second follows from writing

\[
\psi'(n) = -\frac{1}{\phi} n^{-\frac{1+\phi}{\alpha}} \int_0^n \nu^{\frac{1-\phi}{\alpha}} y'(
u) d\nu + n^{-1} y'(n)
\]

\[
= \frac{1}{\phi} n^{-\frac{1+\phi}{\alpha}} \int_0^n \nu^{\frac{1-\phi}{\alpha}} [y'(n) - y'(\nu)] d\nu
\]

which is strictly negative because of the strict concavity of \( y(\cdot) \). The third claim follows because, according to (19), \( \psi(n) \) is a weighted average of the values of \( y'(/\nu) \) on the interval \( \nu \in (0, n) \), and \( y(\cdot) \) satisfies an Inada condition. Finally, to see that \( \psi(n) \to 0 \) as \( n \to \infty \), integrate by parts to obtain that

\[
0 < \psi(n) = n^{-1} y(n) - \frac{1 - \phi}{\phi} n^{-\frac{1}{\alpha}} \int_0^n \nu^{\frac{1-2\phi}{\alpha}} y(\nu) d\nu < n^{-1} y(n).
\]

Since \( n^{-1} y(n) \to 0 \) as \( n \to \infty \), the result follows by the squeeze principle.

To obtain the results concerning the profit function, note that by definition

\[
\pi(n) = y(n) - n^{-\frac{1+\phi}{\alpha}} \int_0^n \nu^{\frac{1-\phi}{\alpha}} y'(
u) d\nu - n(1 - \phi) rV^u.
\]

\[\text{We assume that the integral on the right side of (40) is finite. In the case that } y(n) = An^\alpha \text{ is Cobb-Douglas, a necessary and sufficient condition to ensure this is that } \frac{1}{\alpha} + \alpha > 1, \text{ so that } \nu^{\frac{1-\phi}{\alpha}} y'(\nu) = An^{\frac{1}{\alpha} + \alpha - 2} \text{ is integrable near } \nu = 0. \text{ Since } \phi < 1 \text{ and } \alpha > 0, \text{ the condition is always satisfied in this case.}\]
It follows that \( \pi'(n) = \frac{1-\phi}{\phi} \psi(n) - (1-\phi) rV^u \), from which it is immediate that \( \pi(\cdot) \) is strictly concave because \( \psi(\cdot) \) is strictly decreasing. This shows that the maximizer of \( \pi(\cdot) \) is unique in \([0, \infty)\). In fact it is strictly positive since as \( n \to 0^+ \), \( \pi'(n) \to \frac{1-\phi}{\phi} \lim_{n \to 0^+} \psi(n) - (1-\phi) rV^u = +\infty \) according to a result already proved. To establish that \( \pi(n) \to 0 \) as \( n \to 0^+ \), use the last expression in the previous paragraph to note that

\[
\pi(n) = \frac{1 - \phi}{\phi} \int_0^n \nu \frac{1 - 2\phi}{\phi} y(\nu) d\nu - n(1 - \phi) rV^u
\]

and apply L'Hôpital's rule to see the limit therefore has the same value as that of \( \frac{(1-\phi)^2}{\phi^2} y(n) - n(1-\phi) rV^u \), which is zero since \( y(0) = 0 \). (Note that the limit of the integral in the numerator is indeed zero, so that L'Hôpital's rule can be applied, because of the Inada condition satisfied by \( y(\cdot) \).) To establish that \( \pi(n) \to \infty \) as \( n \to \infty \), note that

\[
\lim_{n \to \infty} \pi'(n) = \frac{1 - \phi}{\phi} \lim_{n \to \infty} \psi(n) - (1 - \phi) rV^u = -(1 - \phi) rV^u < 0.
\]

\[\square\]

**Proof of Theorem 1.** Define two functions \( \chi, \omega : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \) so that

- \( \chi(q, rV^u) \) is the value of \( J(0) \), where \( J(\cdot) \) is the unique solution to the differential equation (8) where, on the right side of (8), the wage function \( w(\cdot) \) is given by (18), and where the boundary condition for the initial value problem is \((n^*, J(n^*))\) with \( n^* \) defined as the unique solution to (24) and the value of \( J(n^*) \) given by (22).

- \( \omega(q, rV^u) = rV^u - \left[ b + \frac{\phi}{1-\phi} \theta q \int_{n^*}^n v(n) J'(n) dG(n) \right] \), where the distribution \( G(\cdot) \) is given by (15) and where \( J(\cdot) \) is as just defined.

That is, \( \chi(q, rV^u) \) represents the value of an entrant firm that takes \( q \) and \( rV^u \) as given, expects to pay wages \( w(\cdot) \) as given by (18), and chooses its vacancy-posting optimally; \( rV^u - \omega(q, rV^u) \) is the value of an unemployed worker in an economy populated by such firms.

As observed in the discussion preceding Proposition 1, \((q, rV^u)\) is part of an equilibrium allocation iff

\[
k = \chi(q, rV^u) \quad \text{ (41)}
\]

\[
0 = \omega(q, rV^u) \quad \text{ (42)}
\]

We will show that such an intersection exists by first observing that (41) defines a continuous 1-manifold in \( \mathbb{R}_+ \times \mathbb{R} \), then observing that \( \omega \) restricted to this manifold defines a continuous function, showing that it takes positive and negative values, and finally applying the intermediate value theorem.
To see that (41) defines a continuous 1-manifold, first note that \( \chi(q, rV^u) \) is nondecreasing in \( q \) and nonincreasing in \( rV^u \), with both relationships being strict if there is positive activity (that is, if it is optimal for a firm with zero workers to hire). This follows immediately from the definition of \( \chi(q, rV^u) \) as the maximal value of the problem for the firm as described. First, clearly if \( rV^u \) increases, then for any \( q \), \( \chi(q, rV^u) \) must decrease, since if the firm keeps the same hiring strategy as before, then it would increase the value of its program as \( w(n) \) decreases for each \( n \); reoptimizing the hiring strategy can only increase this effect. The increase in value is strict provided the firm ever hires a positive number of workers, and this is guaranteed by the assumed Inada conditions (1) and (2)). Second, if \( q \) increases to \( q' > q \), then the firm could increase the value of its program by replacing its former vacancy-posting strategy \( v(\cdot) \) by \( qv(\cdot)/q' \). This would lead to the same dynamics of its size and cost strictly less (again, the inequality is strict provided that the firm ever posts any vacancies).

Since \( \chi(\cdot) \) is continuous, it follows that (41) defines a continuous curve in \( \mathbb{R}^+ \times \mathbb{R} \). Call this curve \( C \). Since it’s clear that \( \omega|C \) is continuous (since \( \omega \) itself is continuous), an equilibrium will exist iff we can find points \( (q_1, v_1), (q_2, v_2) \in C \) such that \( \omega(q_1, v_1) \) and \( \omega(q_2, v_2) \) differ in sign.

To do this, first define \( \bar{v} \) to solve

\[
\frac{1}{r + \delta} \max_{n>0} \left[ y(n) - n^{-\frac{1}{\phi}} \int_0^n \nu^{\frac{1}{\phi}} y'(\nu) d\nu - n(1 - \phi)\bar{v} \right].
\]

A firm that pays wages given by \( w(n) = (1 - \phi)\bar{v} + n^{-\frac{1}{\phi}} \int_0^n \nu^{\frac{1}{\phi}} y'(\nu) d\nu \) to its employees will just break even if and only if it can reach that employment level \( n^* \) that maximizes the right side of (43) instantaneously on entry and at zero cost. It therefore follows that \( \lim_{q \to \infty} \chi(q, \bar{v}) = 0 \). Also, for \( v > \bar{v} \), \( \chi(q, v) < k \) by construction. Now, if \( q \to \infty \), then any firm will instantaneously hire \( n^* \); thus in the limit, \( J(n) = J(n^*) \) for all \( n \in [0, n^*] \). From the definition of \( \omega(\cdot) \), it follows that in this case, \( \omega(q, \bar{v}) = \bar{v} - b \).

In the other extreme, let \( \dot{q} > 0 \) satisfy \( \chi(\dot{q}, b) = 0 \). Such a \( \dot{q} \) will exist provided that \( \bar{v} > b \); assume this. By definition

\[
\omega(\dot{q}, b) = -\theta q \frac{\phi}{1 - \phi} \int J'(n) dG(n).
\]

If \( q \) is finite, then \( \theta q > 0 \), and \( J'(n) > 0 \) for each \( n \). Thus the only possibilities are that \( q = +\infty \) (which is impossible since \( \bar{v} \neq b \)) or that \( \omega(q, b) < 0 \). That is, if \( \bar{v} - b > 0 \), then \( C \) contains points at which \( \omega \) takes values of opposite signs, which completes the proof of the existence of an equilibrium via the intermediate value theorem.

If \( \bar{v} - b \leq 0 \) then it is trivial to prove that there is an equilibrium in which no firm ever enters. \( \square \)
Appendix: Numerical Approximate Solution Method

We use the following approximate solution procedure for the transitional dynamics of the model in Section 5.3.

- Select a time $T$ by which the transition will be largely complete, and impose that from time $T$ onwards, the economy will be in the steady state corresponding to the new, higher productivity level.

- Guess time paths for $\{\theta(t)\}_{t=0}^{T-1}$ and $\{rV^u(t)\}_{t=0}^{T-1}$. If entry is allowed, guess also a time path for firm entry, $\{e(t)\}_{t=0}^{T-1}$.

- Solve for the initial steady state firm size distribution, $G(\cdot, 0)$.

- Solve for the final steady state value function, $J(\cdot, T)$.

- Solve recursively for the functions $\{J(\cdot, t)\}_{t=0}^{T-1}$, iterating backwards in time, and using the assumed time paths for $\theta(t)$ and $rV^u(t)$. In this process, calculate the optimal vacancy posting policies of firms, $v(n, t)$ for $t \in 0, 1, 2, \ldots, T - 1$.

- Using the guessed time paths of $\theta(\cdot)$ and $e(\cdot)$ and the calculated vacancy posting policies $v(\cdot, \cdot)$, simulate the evolution of the firm size distribution $G(\cdot, t)$ and of the unemployment rate $u(t)$.

- Use these, together with the value function equation for the unemployed worker (32), to calculate the resulting time paths of $rV^u(t)$ and $\theta(t)$; denote these time paths by $r\hat{V}^u(t)$ and $\hat{\theta}(t)$. If the calculated time paths are sufficiently close to the guesses, stop. If not, update the guesses by selecting new guesses
  \[ V_{new}^u(t) = (1 - \lambda_V)V^u(t) + \lambda_V\hat{V}^u(t) \quad \text{and} \quad \theta_{new}(t) = (1 - \lambda_\theta)\theta(t) + \lambda_\theta\hat{\theta}(t), \]
  where $\lambda_V$ and $\lambda_\theta$ are constants chosen small enough that the procedure converges. (In the case of free entry, check also whether the free entry condition holds for all $t = 0, 1, \ldots, T - 1$, and if not, reduce (respectively, increase) entry slightly at times when the calculated value of entry, $J(0, t)$, is less than (respectively, greater than) $k$.)

- Verify that the distribution of firm sizes at time $T$ is sufficiently close to the steady-state distribution. If not, choose a larger $T$ and repeat the whole algorithm.

References


Figure 1: Wage, marginal product, and outside option
Figure 2: Equilibrium determination
Figure 3: Response to a positive productivity shock
Figure 4: Cumulative distributions of firm size: decreasing returns, free entry
Figure 5: Impulse response to productivity shock: decreasing returns, no entry
Figure 6: Impulse response to productivity shock: decreasing returns, free entry
Figure 7: Impulse response to productivity shock: constant returns, no entry
Figure 8: Impulse response to productivity shock: constant returns, free entry
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<th>Variable</th>
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<td>$\delta$</td>
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<td>1/6 of separations from firm closure</td>
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<td>$s$</td>
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<td>$Z$</td>
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<td>Shimer (2005); unemployment rate of 6.87%</td>
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<td>Shimer (2005); Hosios condition</td>
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Table 1: Parameterization, Decreasing Returns Model
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<tr>
<td>Market tightness, $\theta$</td>
<td>1</td>
<td>1.024</td>
<td>1.009</td>
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<tr>
<td>Unemployment, $u$</td>
<td>6.87%</td>
<td>6.83%</td>
<td>6.85%</td>
</tr>
<tr>
<td>Flow value of unemployed worker, $rV^u$</td>
<td>1</td>
<td>1.015</td>
<td>1.010</td>
</tr>
<tr>
<td>Vacancies per firm, $v$</td>
<td>1.756</td>
<td>1.759</td>
<td>1.768</td>
</tr>
<tr>
<td>Mean employment per firm</td>
<td>23.80</td>
<td>23.44</td>
<td>23.80</td>
</tr>
<tr>
<td>Maximum employment per firm</td>
<td>25.16</td>
<td>24.78</td>
<td>25.16</td>
</tr>
<tr>
<td>$b = 0.95, p = 1$</td>
<td>$A = 3.301, \gamma = 5.382 \times 10^{-3}, k = 117.1$</td>
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<td></td>
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<td>1.123</td>
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<tr>
<td>Unemployment, $u$</td>
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<td>6.53%</td>
<td>6.67%</td>
</tr>
<tr>
<td>Vacancies per firm, $v$</td>
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<td>1.900</td>
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<tr>
<td>Mean employment per firm</td>
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<td>23.43</td>
<td>23.85</td>
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<tr>
<td>Maximum employment per firm</td>
<td>24.12</td>
<td>23.79</td>
<td>24.20</td>
</tr>
<tr>
<td>$b = 0.40, p = 4$</td>
<td>$A = 3.533, \gamma = 1.885 \times 10^{-2}, k = 132.6$</td>
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<tr>
<td>Market tightness, $\theta$</td>
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<td>1.004</td>
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<tr>
<td>Unemployment, $u$</td>
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<td>6.83%</td>
<td>6.86%</td>
</tr>
<tr>
<td>Vacancies per firm, $v$</td>
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<td>1.757</td>
<td>1.761</td>
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<tr>
<td>Mean employment per firm</td>
<td>23.80</td>
<td>23.44</td>
<td>23.80</td>
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<tr>
<td>Maximum employment per firm</td>
<td>25.51</td>
<td>26.11</td>
<td>26.52</td>
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</tbody>
</table>

Table 2: Comparison of Steady States, Decreasing Returns Model
Baseline | Shock, free entry | Shock, no entry

\( b = 0.40, p = 1 \)

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Baseline</th>
<th>Shock, free entry</th>
<th>Shock, no entry</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market tightness, ( \theta )</td>
<td>1</td>
<td>1.017</td>
<td>1.010</td>
</tr>
<tr>
<td>Unemployment, ( u )</td>
<td>6.87%</td>
<td>6.84%</td>
<td>6.86%</td>
</tr>
<tr>
<td>Flow value of unemployed worker, ( rV^u )</td>
<td>1</td>
<td>1.010</td>
<td>1.010</td>
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<tr>
<td>Vacancies per firm, ( v )</td>
<td>1.757</td>
<td>1.757</td>
<td>1.769</td>
</tr>
<tr>
<td>Mean employment per firm</td>
<td>23.80</td>
<td>23.52</td>
<td>23.804</td>
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<tr>
<td>Maximum employment per firm</td>
<td>28.56</td>
<td>28.23</td>
<td>28.57</td>
</tr>
</tbody>
</table>

Table 3: Comparison of Steady States, Linear Model