Networks, Shocks, and Systemic Risk*

Daron Acemoglu[†] Asuman Ozdaglar[‡] Alireza Tahbaz-Salehi[§]

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Abstract

This chapter develops a unified framework for the study of how network interactions can function as a mechanism for propagation and amplification of microeconomic shocks. The framework nests various classes of games over networks, models of macroeconomic risk originating from microeconomic shocks, and models of financial interactions. Under the assumption that shocks are small, we provide a fairly complete characterization of the structure of equilibrium, clarifying the role of network interactions in translating microeconomic shocks into macroeconomic outcomes. This characterization enables us to rank different networks in terms of their aggregate performance. It also sheds light on several seemingly contradictory results in the prior literature on the role of network linkages in fostering systemic risk.

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[†]Department of Economics, Massachusetts Institute of Technology.
[‡]Laboratory for Information and Decision Systems, Massachusetts Institute of Technology.
[§]Columbia Business School, Columbia University.

1 Introduction

The recent financial crisis, often attributed in part to contagion emanating from pervasive entanglements among financial institutions, has rekindled interest in the role of complex economic, financial or social interlinkages as channels for propagation and amplification of shocks. In the words of Charles Plosser, the president of the Federal Reserve Bank of Philadelphia:

> "due to the complexity and interconnectivity of today's financial markets the failure of a major counterparty has the potential to severely disrupt many other financial institutions, their customers, and other markets" (Plosser, 2009).

Similar ideas on the role of interconnections and the possibility of cascades have also surfaced in a variety of other contexts. For instance, Acemoglu et al. (2012, 2014b) and Jones (2013) have argued that idiosyncratic shocks at the firm or sectoral level can propagate over input-output linkages within the economy, with potentially significant implications for macroeconomic volatility and economic growth, while Caplin and Leahy (1993) and Chamley and Gale (1994) have emphasized the spread of economic shocks across firms due to learning and imitation.

Though the domains studied by these and other related papers are often different, their underlying approaches share important economic and mathematical parallels. Most importantly, in each case, the problem is one of a set of interacting agents who influence each other, thus opening the way for shocks to one agent to propagate to the rest of the economy. Furthermore, on the methodological side, almost all these papers rely on a network model to capture the pattern and extent of interactions between agents. Despite these parallels, there is a bewildering array of different (and sometimes even contradictory) results, often presented and developed with little linkage to other findings in the literature.

The disparity in the predictions and results of different studies in the literature can be best illustrated by focusing on a concrete setting, namely, that of financial interactions. The models of financial interactions studied in a variety of papers (such as Allen and Gale (2000), Giesecke and Weber (2006), Blume et al. (2011), Battiston et al. (2012), Elliott, Golub, and Jackson (2014), Cabrales, Gottardi, and Vega-Redondo (2014) and Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015b)) are, at least on the surface, very similar. In each case, a financial institution's "state", which for example captures its health or ability to meet its obligations, depends on the state of other financial institutions to which it is connected.[1] Consequently, shocks to a given institution can propagate to other institutions within the economy, potentially snowballing into a systemic crisis. Despite such commonalities, the predictions of many of the papers in this literature are quite different or sometimes even contradictory. For example, in the models of Allen and Gale (2000) and Freixas, Parigi, and Rochet (2000), denser interconnections mitigate systemic risk, whereas several other papers, such

[1] For instance, in the context of counterparty relationships considered by Acemoglu et al. (2015b), the connections capture the extent of prior interbank lending and borrowing and each bank's state captures its ability to meet those obligations. As highlighted in Cabrales, Gale, and Gottardi (2015), other forms of interlinkages operate in a similar fashion.

as Vivier-Lirimont (2006) and Blume et al. (2011), have suggested that such dense interconnections can act as a destabilizing force.

Our aim in this chapter is to unify and improve the understanding of the key economic and mathematical mechanisms in much of the literature on the effects of network interactions on the economy's aggregate performance. We start with a general reduced-form model in which n agents interact with one another. Each agent is assigned a real-valued variable, known as its *state* which, depending on the context, may capture her choice of actions (e.g., output or investment) or some other economic variable of interest. Our reduced-form model consists of three key ingredients: (i) a fairly general *interaction function* that links each agent's state to a summary measure of the states of other agents; (ii) an (*interaction*) *network* that specifies how these summary measures are determined as a function of other agents' states; and (iii) an *aggregation function* that describes how agent-level states collectively shape the macroeconomic variable of interest.

We first show that our general framework nests a wide variety of problems studied in the literature, including those mentioned above. We also show that under fairly general conditions on the interaction function, an equilibrium — defined as a mutually consistent set of states for all agents in the network — always exists and is generically unique. We then use our framework to study how the nature of inter-agent interactions shape various measures of aggregate performance. Our analysis not only nests the main results obtained in several papers in the literature, but also clarifies where the sources of differences lie.

In order to obtain sharp and analytical predictions for the role of network interactions in shaping economic outcomes, we focus on an economy in which agent-level shocks are small. This assumption enables us to approximate the equilibrium state of each agent and the economy's macroeconomic state by the first few terms of their Taylor expansions. Our results show that the impact of network structure depends on the properties of the economy's *Leontief matrix* corresponding to the underlying interaction network. This matrix, which is defined in a manner analogous to the same concept used in the literature on input-output economies, accounts for all possible direct and indirect effects of interactions between any pair of agents. Using this characterization, we show that the curvatures of the interaction and aggregation functions play a central role in how the economy's underlying network translates microeconomic shocks into macroeconomic outcomes.

As our first characterization result, we show that as long as the interaction and aggregation functions are linear, the economy exhibits a "*certainty equivalence*" property from an *ex ante* perspective, in the sense that the expected value of the economy's macro state is equal to its unperturbed value when no shocks are present. This observation means that, in a linear world, the economy's aggregate performance, in expectation, does not depend on the intricate details of its underlying interaction network.

Our next set of results illustrates that this certainty equivalence property may no longer hold if either the aggregation or interaction function is non-linear. Rather, in the presence of a non-linear interaction or aggregation function, the exact nature of these non-linearities are central to determining how the economy's underlying interaction network affects its *ex ante* performance.

We show that with a non-linear aggregation function, the economy's *ex ante* performance depends on the heterogeneity in the extent to which agents interact with one another. In particular, if the aggregation function is concave — for example, to capture the idea that volatility is detrimental to the economy's aggregate performance — a more uniform distribution of inter-agent interactions increases macroeconomic performance in expectation. An important corollary to this result establishes that with a concave aggregation function, *regular* economies (in which the overall influence of each agent on the rest of the agents is identical across the network) outperform all other economies. These results are consistent with, and in some ways generalize, those of Acemoglu et al. (2012), who, in the context of input-output economies, show that the volatility of the economy's aggregate output increases in the extent of heterogeneity in the role of different firms as input-suppliers. Our results thus clarify that it is the concavity of economy's aggregation function — resulting from the focus on volatility — that lies at the heart of the results in Acemoglu et al. (2012).

We then focus on understanding how non-linearities in the interaction function shape the economy's *ex ante* performance. Our results illustrate that when the interaction function is concave, economies with denser interconnections outperform those whose interaction networks are more sparse. In particular, the complete network, in which interlinkages are maximally dense, outperforms all other (symmetric) economies. Furthermore, we show that with a convex interaction function, this performance ordering flips entirely, making the complete network the worst performing economy. This flip in the comparative statics of aggregate performance with respect to the network structure parallels the findings in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015b), who show that, in the context of financial interactions, whether the complete network fosters stability or instability depends on the size and number of shocks: with a few small shocks, the complete network is the most stable of all economies, whereas when shocks are numerous or large, there is a phase transition, making the complete network the least stable financial arrangement. Our results here clarify that the findings of Acemoglu et al. (2015b) are essentially due to the fact that increasing the size or the number of shocks corresponds to a shift from a concave to a convex region of the interaction function, thus reversing the role of interbank connections in curtailing or causing systemic risk. They also highlight that similar phase transitions transforming the role of network interconnections in shaping aggregate performance can emerge in other settings with non-linear interactions.

Overall, our results highlight that the relationship between the economy's aggregate performance and its underlying network structure depends on two important economic variables: (i) the nature of economic interactions, as captured by our interaction function; and (ii) the properties of the aggregate performance metric, as captured by the notion of aggregation function in our model.

We also use our framework to provide a characterization of how the nature of interactions determine the agents' relative importance in shaping aggregate outcomes. As long as agent-level interactions are linear, the well-known notion of *Bonacich centrality* serves as a sufficient statistic for agents' "systemic importance": negative shocks to an agent with a higher Bonacich centrality leads to a larger drop in the economy's macro state. We also demonstrate that, in the presence of small enough shocks, this result generalizes to economies with non-linear interactions, but with one important caveat: even though a strictly larger Bonacich centrality means that the agent has a more pronounced impact on the economy's macro state, two agents with identical Bonacich centralities are not necessarily equally important. This is due to the fact that Bonacich centrality only provides a first-order approximation to the agents' impact on aggregate variables. Therefore, a meaningful comparison of systemic importance of two agents with identical Bonacich centralities (as in a regular network) requires that we also take their higher-order effects into account. As our final result, we provide such a characterization of agents' systemic importance in regular economies. We show that the second-order impact of an agent on the economy's macro state is summarized via a novel notion of centrality, called *concentration centrality*, which captures the concentration of an agent's influence on the rest of the agents (as opposed to its overall influence captured via Bonacich centrality).

These characterization results thus highlight that relying on standard and off-the-shelf notions of network centrality (such as Bonacich, eigenvector, or betweenness centralities) for the purpose of identifying systemically important agents may be misleading. Rather, the proper network statistic has to be informed by the nature of microeconomic interactions between different agents.

Related Literature As already indicated, this chapter relates to several strands of literature on social and economic networks, such as the literature on network games, various models of systemic risk, and the literature that studies microeconomic foundations of macroeconomic fluctuations. Many of the papers related to our setup are discussed in the next section when we describe how different models are nested within our general framework. Here, we provide a brief overview of the literature and some of the key references.

The critical building block of our general framework is an interaction network, whereby each player's "state" is a function of the state of its neighbors in a directed, weighted network. These interlinked states could be thought of as best responses of each player to the actions of her neighbors. As such, our setup builds on various different contributions on the network games literature, such as Calvó-Armengol and Zenou (2004), Ballester, Calvó-Armengol, and Zenou (2006), Candogan, Bimpikis, and Ozdaglar (2012), Allouch (2012), Badev (2013), Bramoullé, Kranton, and D'Amours (2014) and Elliott and Golub (2014), several of which can be cast as special cases of our general framework.[2] Several papers consider applications of network games to various specific domains. For example, Calvó-Armengol, Patacchini, and Zenou (2009) study peer effect and education decisions in social networks; Calvó-Armengol and Jackson (2004) study the role of referral networks in the labor market; and Galeotti and Rogers (2013), Acemoglu, Malekian, and Ozdaglar (2013) and Dziubiński and Goyal (2014) consider a network of interlinked players making endogenous security investments against an infection or an attack. Jackson and Zenou (2015) and Bramoullé and Kranton (2015) provide thorough surveys of the network games literature.

Even though the literature on network games does not generally consider the propagation of id-

[2]Network games of incomplete information are studied in Galeotti et al. (2010).

iosyncratic shocks, our results highlight that, depending on the specific economic question at hand, the interaction models at the heart of this literature could be used for the study of such propagation.

A related literature has directly originated from the study of cascades. Various models have been developed in the computer science and network science literatures, including the widely-used threshold models (Granovetter, 1978) and percolation models (Watts, 2002). A few works have applied these ideas to various economic settings, including Durlauf (1993) and Bak et al. (1993) in the context of economic fluctuations; Morris (2000) in the context of contagion of different types of strategies in coordination games; and more recently, Gai and Kapadia (2010) and Blume et al. (2011) in the context of spread of an epidemic-like financial contagion.

The framework developed in this chapter is also closely linked to a small literature in macroeconomics that studies the propagation of microeconomic shocks over input-output linkages. This literature, which builds on the seminal paper by Long and Plosser (1983), has witnessed a recent theoretical and empirical revival. On the theoretical side, Acemoglu et al. (2012, 2014b) and Jones (2013) argue that the propagation of idiosyncratic shocks and distortions over input-output linkages can have potentially significant implications for macroeconomic volatility and economic growth.[3] On the empirical side, Foerster, Sarte, and Watson (2011), Carvalho (2014), di Giovanni, Levchenko, and Méjean (2014), Acemoglu, Autor, Dorn, Hanson, and Price (2015a) and Carvalho, Nirei, Saito, and Tahbaz-Salehi (2015) provide evidence for the relevance of such propagation mechanisms in different countries.

As mentioned earlier, this chapter is also closely related to the growing literature on the spread of financial shocks over a network of interconnected financial institutions. The seminal papers of Allen and Gale (2000) and Freixas, Parigi, and Rochet (2000) developed some of the first formal models of contagion over financial networks. The recent financial crisis resulted in further attention to this line of work. Some of the more recent examples include Gai, Haldane, and Kapadia (2011), Battiston et al. (2012), Alvarez and Barlevy (2014) and Glasserman and Young (2015).

Within this literature, four recent papers deserve further discussion. The first, which is our own work (Acemoglu, Ozdaglar, and Tahbaz-Salehi, 2015b), considers a network of banks linked through unsecured debt obligations and studies the emergence of financial cascades resulting from counterparty risk. This paper, which in turn builds on and extends Eisenberg and Noe (2001)'s seminal framework of financial interlinkages, is explicitly treated as a special case of our general framework here. The second is the related paper by Elliott, Golub, and Jackson (2014), which also considers financial contagion in a network, though based on microfoundations linked to cross-shareholdings across institutions rather than the counterparty risk as in our own previous work. The third is Cabrales, Gottardi, and Vega-Redondo (2014), which is closely connected to Elliott et al. (2014) and in addition considers the endogenous formation of the financial network.[4] Finally, Cabrales, Gale, and Gottardi (2015) provide a unified treatment of the previous three papers, highlighting various

[3]Relatedly, Gabaix (2011) argues that microeconomic shocks can lead to aggregate fluctuations if the firm-size distribution within the economy exhibits a heavy enough tail, even in the absence of input-output linkages.

[4]Other papers that study network formation in related contexts include Bala and Goyal (2000), Babus (2014), Zawadowski (2013), Acemoglu, Ozdaglar, and Tahbaz-Salehi (2014c), Farboodi (2014), and Erol and Vohra (2014).

commonalities as well as some important differences between them. The key distinction between their unified treatment and ours is that they start with the fixed point equation resulting from the interactions in the various financial network models, whereas we develop a more general framework starting from the best response equations or the equations linking each agent's state to her neighbors'. This formulation enables us to nest not only existing models of financial networks but a wider array of network interactions, use first and second-order approximations to provide a sharper characterization of the structure of equilibrium, and clarify the role of interaction and aggregation functions in transforming small, agent-level shocks into differences in aggregate performance or volatility.

Outline The rest of this chapter is organized as follows. In Section 2, we provide our general framework for the study of network interactions and present a few examples of how our setup maps to different applications. In Section 3, we provide a second-order approximation to the macro state of the economy in terms of the economy's underlying interaction network. Section 4 uses these results to characterize how the nature of interactions between different agents impacts the macro state of the economy from an *ex ante* perspective, whereas Section 5 provides a characterization of the systemic importance of different agents. Section 6 concludes.

2 General Framework

Consider an economy consisting of n agents indexed by $N = \{1, \dots, n\}$. Of key interest to our analysis is each agent i's *state*, $x_i \in \mathbb{R}$, which captures the agent's choice of action (e.g., output or investment) or some other economic variable of interest (such as the solvency of a financial institution). In the next three subsections we will provide concrete examples clarifying the interpretation of these states. For the time being, however, we find it convenient to work with a general, reduced-form setup without taking a specific position on how to interpret the agents or their states.

The key feature of the environment is that the states of different agents are interlinked. Such interdependencies may arise due to strategic considerations, contractual agreements, or some exogenous (e.g., technological) constraints on the agents. Formally, the state of any given agent i depends on the states of other agents via the relationship

$$x_i = f\left(\sum_{j=1}^{n} w_{ij} x_j + \epsilon_i\right), \tag{1}$$

where f is a continuous and increasing function, which we refer to as the economy's *interaction function*. As the name suggests, this function represents the nature of interactions between the agents in the economy. The variable ϵ_i is an "agent-level" shock, which captures stochastic disturbances to i's state. We assume that these shocks are independently and identically distributed (so that they correspond to "idiosyncratic" shocks) and have mean zero and variance σ^2.

The constant $w_{ij} \geq 0$ in (1) captures the extent of interaction between agents i and j. In particular, a higher w_{ij} means that the state of agent i is more sensitive to the state of agent j, whereas

$w_{ij} = 0$ implies that agent j does not have a direct impact on i's state. Without much loss of generality, we assume that $\sum_{j=1}^{n} w_{ij} = 1$, which guarantees that the extent to which the state of each agent depends on the rest of the agents is constant. We say the economy is *symmetric* if $w_{ij} = w_{ji}$ for all pairs of agents i and j.

For a given f, the interactions between agents can be also represented by a weighted, directed graph on n vertices, which we refer to as the economy's *interaction network*. Each vertex in this network corresponds to an agent and a directed edge from vertex j to vertex i is present if $w_{ij} > 0$, that is, if the state of agent i is directly affected by the state of agent j.

Finally, we define the *macro state* of the economy as

$$y = g\left(h(x_1) + \cdots + h(x_n)\right), \tag{2}$$

where $g, h : \mathbb{R} \to \mathbb{R}$. As we will clarify in what follows, y represents some macroeconomic outcome of interest that is obtained by aggregating the individual states of all agents. Throughout the paper, we refer to g as the economy's *aggregation function*.

An equilibrium in this economy is defined in the usual fashion by requiring each agent's state to be consistent with those of others. Formally:

Definition 1. Given the realization of the shocks $(\epsilon_1, \ldots, \epsilon_n)$, an *equilibrium* of the economy is a collection of states $(x_1, \ldots, x_n)$ such that equation (1) holds for all agents i simultaneously.

As the above definition clarifies, our solution concept is an *ex post* equilibrium notion, in the sense that agents' states are determined after the shocks are realized. This notion enables us to study how the equilibrium varies as a function of the shock realizations.

Throughout the paper, we assume that $f(0) = g(0) = h(0) = 0$. This normalization guarantees that, in the absence of shocks, the equilibrium state of all agents and the economy's macro state are equal to zero.

We next show how a wide variety of different applications can be cast as special cases of the general framework developed above.

2.1 Example: Network Games

Our framework nests a general class of network games as a special case. Consider, for example, an n-player, complete information game, in which the utility function of agent i is given by

$$u_i(x_1, \ldots, x_n) = -\frac{1}{2}x_i^2 + x_i f\left(\sum_{j=1}^{n} w_{ij} x_j + \epsilon_i\right),$$

where x_i denotes the action of player i and ϵ_i is realization of some shock to her payoffs. That is, the payoff of player i depends not only on her own action, but also on those of her neighbors via the interaction function f. In this context, the underlying network, encoded in terms of coefficients w_{ij}, captures the pattern and strength of strategic interactions between various players in the game.

It is immediate to verify that as long as the interaction function f satisfies certain regularity conditions — essentially to ensure that one can use the first-order conditions — and that $w_{ii} = 0$ for all i, the best-response of player i as a function of the actions of other players is given by equation (1). Consequently, the collection $(x_1, \ldots, x_n)$ that solves the system of equations (1) corresponds to the Nash equilibrium of the game.

The game described above nests a wide variety of models studied in the literature. Note that since f is increasing, the players face a game of strategic complements over the network: the benefit of taking a higher action to player i increases the higher the actions of her neighbors are. Examples of such network games include research collaboration among firms (Goyal and Moraga-González, 2001), crime networks (Ballester, Calvó-Armengol, and Zenou, 2006), peer effect and education decisions in social networks (Calvó-Armengol, Patacchini, and Zenou, 2009), and local consumption externalities (Candogan, Bimpikis, and Ozdaglar, 2012). On the other hand, had we assumed that the interaction function f is decreasing, the players would have faced a network game of strategic substitutes, as in Bramoullé and Kranton (2007) who study information sharing and the provision of local public goods.[5]

An important subclass of network games is the case in which players' payoff functions are quadratic,

$$u_i(x_1, \ldots, x_n) = -\frac{1}{2}x_i^2 + \alpha \sum_{j=1}^{n} w_{ij} x_i x_j + \alpha x_i \epsilon_i, \tag{3}$$

where $\alpha \in (0,1)$ is some constant.[6] Under such a specification, the corresponding interaction function is given by $f(z) = \alpha z$, hence, implying that the equilibrium of the game can be characterized as a solution to a system of linear equations.

We end our discussion by pointing out two natural candidates for the economy's macro state in this context. The first is the sum (or the average) of the agents' equilibrium actions,

$$y_{\text{agg}} = x_1 + \cdots + x_n,$$

representing the aggregate level of activity in the economy. In our general framework, this corresponds to the assumption that $g(z) = h(z) = z$. The second is the total or average utility (or equivalently total social surplus) in the equilibrium, given by $y_{\text{sw}} = \sum_{i=1}^{n} u_i$. Although summing both sides of equation (3) over all players i shows that social surplus depends not only on the agents' states, but also on weights w_{ij} and the realizations of the shocks ϵ_i, using the fact that equilibrium actions satisfy (1) enables us to write y_{sw} in the form of equation (2) as

$$y_{\text{sw}} = \frac{1}{2} \sum_{i=1}^{n} x_i^2,$$

which corresponds to $g(z) = z$ and $h(z) = z^2/2$ in our general framework.

[5] Allowing both for strategic complementarities and substitutabilities, Acemoglu, Garcia-Jimeno, and Robinson (2014a) develop an application of these models in the context of local municipalities' state capacity choices, and estimate the model's parameters using Colombian data.

[6] See Zenou (2015) for a discussion and a variety of extensions of the baseline network game with quadratic payoffs.

2.2 Example: Production Networks

Our general setup also nests a class of models that focus on the propagation of shocks in the real economy. In this subsection, we provide an example of one such model along the lines of Long and Plosser (1983) and Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012), and show that it can be cast as a special case of our general framework.

Consider an economy consisting of n competitive firms (or sectors) denoted by $\{1, 2, \ldots, n\}$, each of which producing a distinct product.[7] Each product can be either consumed by a mass of consumers or used as an input for production of other goods. All firms employ Cobb-Douglas production technologies with constant returns to scale that transform labor and intermediate goods to final products. Production is subject to some idiosyncratic technology shock. More specifically, the output of firm i, which we denote by X_i, is equal to

$$X_i = b_i A_i^{\alpha} l_i^{1-\alpha} \left(\prod_{j=1}^{n} X_{ij}^{w_{ij}} \right)^{\alpha}, \tag{4}$$

where A_i is the corresponding productivity shock; l_i is the amount of labor hired by firm i; X_{ij} is the amount of good j used for production of good i; b_i is a constant; and $\alpha \in (0,1)$ is the share of intermediate goods in production. The exponent $w_{ij} \geq 0$ in (4) captures the share of good j in the production technology of good i: a higher w_{ij} means that good j is more important in producing i, whereas $w_{ij} = 0$ implies that good j is not a required input for i's production technology. The assumption that firms employ constant returns to scale technologies implies that $\sum_{j=1}^{n} w_{ij} = 1$ for all i.

The economy also contains a unit mass of identical consumers. Each consumer is endowed with one unit of labor which can be hired by the firms for the purpose of production. We assume that the representative consumer has symmetric Cobb-Douglas preferences over the n goods produced in the economy. In particular,

$$u(c_1, \ldots, c_n) = \tilde{b} \prod_{i=1}^{n} c_i^{1/n},$$

where c_i is the amount of good i consumed and $\tilde{b}$ is some positive constant.

One can naturally recast the interactions between different firms in such an economy in terms of a network, with each vertex corresponding to a firm and the factor shares w_{ij} capturing the intensity of interactions between them. Furthermore, given the log-linear nature of Cobb-Douglas production technologies, the equilibrium (log) output of each firm can be written in the form of equation (1), linking it to the outputs of its input suppliers and the productivity shocks in the economy.

To see this, consider the first-order conditions corresponding to firm i's problem:

$$X_{ij} = \alpha w_{ij} p_i X_i / p_j \tag{5}$$

$$l_i = (1-\alpha) p_i X_i / \omega, \tag{6}$$

[7] Since each one of these firms is supposed to act competitively, they can also be interpreted as "representative firms" standing in for a set of competitive firms within each of the n sectors.

where ω denotes the market wage and p_i is the price of good i. The market clearing condition for good i, given by $X_i = c_i + \sum_{j=1}^n X_{ji}$, implies that

$$s_i = \omega/n + \alpha \sum_{j=1}^n w_{ji} s_j,$$

where $s_i = p_i X_i$ is the equilibrium sales of firm i. Note that in deriving the above expression, we are using the fact that the first-order condition of the consumer's problem requires that $c_i = \omega/np_i$. Given that the above equality defines a linear system of equations in terms of the equilibrium sales of different firms, it is straightforward to show that $s_i = p_i X_i = \zeta_i \omega$ for some constant ζ_i.[8] Therefore, replacing for equilibrium price p_i in equations (5) and (6) in terms of the output of firm i yields $X_{ij} = \alpha w_{ij} \zeta_i X_j / \zeta_j$ and $l_i = (1-\alpha)\zeta_i$. Plugging these quantities back into the production function of firm i leads to

$$X_i = b_i \zeta_i (1-\alpha)^{1-\alpha} A_i^\alpha \prod_{j=1}^n \left(\alpha w_{ij} X_i / \zeta_j\right)^{\alpha w_{ij}}.$$

Now it is immediate that with the proper choice of constants b_i, the log output of firm i, denoted by $x_i = \log(X_i)$, satisfies

$$x_i = \alpha \sum_{j=1}^n w_{ij} x_j + \alpha \epsilon_i, \tag{7}$$

where $\epsilon_i = \log(A_i)$ is the log productivity shock to firm i. In other words, the interactions between different firms can be cast as a special case of our general framework in equation (1) with linear interaction function $f(z) = \alpha z$.

We end our discussion by remarking that the logarithm of real value added in the economy, which is the natural candidate for the economy's macro state y, can also be expressed in terms of our general formulation in (2). Because of the constant returns to scale assumption, firms make zero profits in equilibrium, all the surplus in the economy goes to the consumers, and as a consequence, value added is simply equal to the market wage ω. Choosing the ideal price index as the numeraire, that is, $\frac{n}{\tilde{b}}(p_1 \dots p_n)^{1/n} = 1$, and using the fact that $p_i = \zeta_i \omega / X_i$, we obtain that the log real value added in the economy is equal to

$$\log(\omega) = \frac{1}{n} \sum_{i=1}^n \log(X_i) - \frac{1}{n} \sum_{i=1}^n \log(\zeta_i) + \log(\tilde{b}/n).$$

Therefore, with the appropriate choice of $\tilde{b}$, we can rewrite $\log(\text{GDP})$ as

$$y = \log(\text{GDP}) = \frac{1}{n} \sum_{i=1}^n x_i, \tag{8}$$

as in (2) in our general framework with $g(z) = z/n$.

[8] To be more precise, $\zeta_i = v_i/n$, where v_i is the i-th column sum of matrix $(I - \alpha W)^{-1}$. In Section 3, we show that this quantity coincides with the notion of Bonacich centrality of firm i in the economy.

2.3 Example: Financial Contagion

As a final example, we show that our general framework also nests models of financial contagion over networks. As a concrete example, we focus on a variant of a model along the lines of Eisenberg and Noe (2001) and Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015b), who study how the patterns of interbank liabilities determine the extent of financial contagion.

Consider an economy consisting of n financial institutions (or banks), which are linked to one another via unsecured debt contracts of equal seniority. Each bank i has a claim of size $\xi_{ij} = w_{ij}\xi$ on bank j, where we assume that $\sum_{j=1}^{n} w_{ij} = \sum_{j=1}^{n} w_{ji} = 1$, thus guaranteeing that all banks have identical total claims (of size ξ) on the rest of the banking system. In addition to its interbank claims and liabilities, bank i has an outside asset of net value a and is subject to some liquidity shock ϵ_i.

Following the realizations of these liquidity shocks, banks need to repay their creditors. If a bank cannot meet its liabilities in full, it defaults and repays its creditors on a pro rata basis. Let z_{im} denote the repayment of bank m on its debt to bank i. The cash flow of bank i is thus equal to $c_i = a+\epsilon_i+\sum_{m=1}^{n} z_{im}$. Therefore, as long as $c_i \geq \xi$, bank i can meet its liabilities in full, guaranteeing that $z_{ji} = w_{ji}\xi$ for all banks j. If, on the other hand, $c_i \in (0,\xi)$, the bank defaults and its creditors are repaid in proportion to the face value of their contracts, i.e., $z_{ji} = w_{ji}c_i$. Finally, if $c_i \leq 0$, bank i's creditors receive nothing, that is, $z_{ji} = 0$. Putting the above together implies that the repayment of bank i on its debt to a given bank j is equal to

$$z_{ji} = \max\left\{\min\left\{w_{ji}\left(a+\epsilon_i+\sum_{m=1}^{n} z_{im}\right), w_{ji}\xi\right\}, 0\right\}.$$

Summing both sides of the above equation over the set of banks j and letting $x_i = \sum_{j=1}^{n} z_{ji}$ denote the total out-payment of bank i to its creditors imply

$$x_i = \max\left\{\min\left\{\sum_{m=1}^{n} w_{im}x_m + a + \epsilon_i, \xi\right\}, 0\right\}, \tag{9}$$

where we are using the fact that $z_{im} = w_{im}x_m$.

It is then straightforward to see that the interactions between different banks can be represented as a network, with each vertex corresponding to a bank and the size of bank i's obligation to bank j representing the intensity of interactions between the two. Furthermore, the specific nature of interbank repayments can be cast as a special case of our general model (1) with interaction function $f(z) = \max\{\min\{z+a,\xi\},0\}$.

Note that unlike the examples presented in Subsections 2.1 and 2.2, this interaction function does not satisfy the normalization assumption $f(0) = 0$ if $a > 0$. Nevertheless, this is not of major consequence, as a simple change of variables would restore the original normalization: redefining the state of agent i as $\hat{x}_i = x_i - \xi$ leads to the modified interaction function $\hat{f}(z) = \max\{\min\{z+a,0\},-\xi\}$, which satisfies $\hat{f}(0) = 0$ whenever $a > 0$. Given that all our results and their corresponding economic insights are robust to the choice of normalization, we find it easier to work with the original model.

Finally, assuming that each default results in a deadweight loss of size A (for example, because of the cost of early liquidation of long-term projects as in our previous work), the social surplus in the economy is equal to

$$y = A\sum_{i=1}^{n}\mathbf{1}\{x_i \geq \xi\},$$

corresponding to $g(z) = z$ and $h(z) = A\mathbf{1}\{z \geq \xi\}$ in our general framework.

2.4 Existence and Uniqueness of Equilibrium

We now return to the general framework introduced above and establish the existence and (generic) uniqueness of equilibrium. In general, the set of equilibria not only depends on the economy's interaction network, but also on the properties of the interaction function. We impose the following regularity assumption on f:

Assumption 1. There exists $\beta \leq 1$ such that $|f(z) - f(\tilde{z})| \leq \beta|z - \tilde{z}|$ for all $z, \tilde{z} \in \mathbb{R}$. Furthermore, if $\beta = 1$, then there exists $\delta > 0$ such that $|f(z)| < \delta$ for all $z \in \mathbb{R}$.

This assumption, which is satisfied in each of the economies discussed in Subsections 2.1–2.3 as well as in most other natural applications of this framework, guarantees that the economy's interaction function is either (i) a contraction with Lipschitz constant $\beta < 1$; or alternatively, (ii) a bounded non-expansive mapping. Either way, it is easy to establish that an equilibrium always exists. In particular, when $\beta < 1$, the contraction mapping theorem implies that (1) always has a fixed point, whereas if f is bounded, the existence of equilibrium is guaranteed by the Brouwer fixed point theorem.

Our first formal result shows that beyond existence, Assumption 1 is also sufficient to guarantee that the equilibrium is uniquely determined over a generic set of shock realizations.

Theorem 1. *Suppose that Assumption 1 is satisfied. Then, an equilibrium always exists and is generically unique.*

A formal proof of the above result is provided in the Appendix. Intuitively, when $\beta < 1$, the contraction mapping theorem ensures that the economy has a unique equilibrium. The economy may have multiple equilibria, however, when $\beta = 1$ (for example, as in the financial contagion example in Subsection 2.3). Nevertheless, Theorem 1 guarantees that the equilibrium is generically unique, in the sense that the economy has multiple equilibria only for a measure zero set of realizations of agents-level shocks.

3 Smooth Economies

In the remainder of this chapter, we study how the economy's underlying network structure as well as different properties of the aggregation and interaction functions, shape economic outcomes. In particular, we are interested in characterizing how these features determine the extent of propagation and amplification of shocks within the economy.

To achieve this objective, we impose two further assumptions on our model. First, we assume that the underlying economy is *smooth*, in the sense that functions f, g and h are continuous and at least twice differentiable. The class of smooth economies nests many of the standard models studied in the literature, such as variants of the network games and the production economy presented in Subsections 2.1 and 2.2. On the other hand, the model of financial interactions presented in Subsection 2.3 is not nested within this class, as the corresponding interaction function is not differentiable everywhere. Nevertheless, this non-smoothness is not of major consequence, as the interaction function f can be arbitrarily closely approximated by a smooth function $\tilde{f}$ in such a way that economic implications of the model under this smooth approximation are identical to those of the original model.[9]

As our second assumption, we focus on the case where agent-level shocks are small. This assumption enables us to approximate the equilibrium state of each agent and the economy's macro state by the first few terms of their Taylor expansions. Even though it may appear restrictive, our following results highlight that such a "small-shock analysis" can lead to fairly general and robust insights on how different network interactions shape economic outcomes.

3.1 First-Order Approximation

We start our analysis by providing a first-order (that is, linear) approximation to the agents' equilibrium states around the point where $\epsilon_i = 0$ for all i. If the size of the agent-level shocks are small, such an approximation captures the dominant effects of how shocks shape the economy's macro state.

Let us first use the implicit function theorem to differentiate both sides of the interaction equation (1) with respect to the shock to agent r:

$$\frac{\partial x_i}{\partial \epsilon_r} = f'\left(\sum_{m=1}^{n} w_{im} x_m + \epsilon_i\right)\left(\sum_{m=1}^{n} w_{im}\frac{\partial x_m}{\partial \epsilon_r} + \mathbf{1}\{r=i\}\right). \tag{10}$$

Evaluating the above equation at the point $\epsilon = (\epsilon_1, \dots, \epsilon_n) = 0$ yields

$$\left.\frac{\partial x_i}{\partial \epsilon_r}\right|_{\epsilon=0} = f'(0)\left(\sum_{m=1}^{n} w_{im} \left.\frac{\partial x_r}{\partial \epsilon_i}\right|_{\epsilon=0} + \mathbf{1}\{r=i\}\right),$$

where we are using the fact that in the absence of shocks $x_m = 0$ for all m. This equation can be rewritten in matrix form as $\partial x/\partial \epsilon_r|_{\epsilon=0} = f'(0) W \partial x/\partial \epsilon_r|_{\epsilon=0} + f'(0) e_r$, where $x = (x_1, \dots, x_n)'$ is the vector of agents' states and e_r represents the r-th unit vector. It is therefore immediate that the derivative of the agents' states with respect to the shock to agent r is given by

$$\left.\frac{\partial x}{\partial \epsilon_r}\right|_{\epsilon=0} = f'(0)\left[I - f'(0)W\right]^{-1} e_r. \tag{11}$$

[9] More specifically, it is sufficient for $\tilde{f}$ to satisfy Assumption 1 and, as f, be initially concave and then convex.

Note that, as long as $f'(0) < 1$, the matrix $I - f'(0)W$ is invertible, implying that the right-hand side of (11) is well-defined. We find it useful to define the following concept:

Definition 2. The *Leontief matrix* of the economy with parameter $\alpha \in [0, 1)$ is $L = (I - \alpha W)^{-1}$, where $W = [w_{ij}]$ is the economy's interaction matrix.

In view of the above definition, we can rewrite equation (11) as

$$\left.\frac{\partial x_i}{\partial \epsilon_r}\right|_{\epsilon=0} = \alpha \ell_{ir}, \tag{12}$$

where $\alpha = f'(0)$ and ℓ_{ir} is the (i, r) element of the economy's Leontief matrix with parameter α. The equilibrium state of agent i around the point $\epsilon = 0$ can then be linearly approximated as

$$x_i = \alpha \sum_{r=1}^{n} \ell_{ir} \epsilon_r. \tag{13}$$

In other words, when the agent-level shocks are small (so that we can rely on a linear approximation), the economy's Leontief matrix serves as a sufficient statistic for the network's role in determining the state of agent i. More specifically, the impact of a shock to agent r on the equilibrium state of agent i is simply captured by ℓ_{ir}.

Before continuing with our derivations, a few remarks are in order. First, note that Definition 2 generalizes the well-known concept of the Leontief input-output matrix to an economy with a general form of interaction among agents. In particular, the (i, r) element of the matrix, not only captures the direct interaction between agents i and r, but also accounts for all possible indirect interactions between the two. To see this, note that ℓ_{ir} can be rewritten as

$$\ell_{ir} = 1 + \alpha w_{ir} + \alpha^2 \sum_{k=1}^{n} w_{ik} w_{kr} + \ldots, \tag{14}$$

where the higher-order terms account for the possibility of indirect interactions between i and r. Thus, essentially, equation (14) shows that a shock to agent r impacts agent i not only through their direct interaction term w_{ir}, but also via indirect interactions with the rest of the agents: such a shock may impact the state of some agent k and then indirectly propagate to agent i. However, note that the impact of a shock to agent r on i's state is deflated by a factor $\alpha < 1$ whenever the length of the indirect interaction chain between the two agents is increased by one.

In view of the interpretation that ℓ_{mi} captures the equilibrium impact of agent i on the state of agent m, it is natural to interpret $\sum_{m=1}^{n} \ell_{mi}$ as the extent of agent i's overall influence on the rest of the agents in the economy. We define the following concept, which is well-known in the study of social and economic networks:

Definition 3. For a given parameter $\alpha \in [0, 1)$, the *Bonacich centrality* of agent i is $v_i = \sum_{m=1}^{n} \ell_{mi}$, where $L = [\ell_{ij}]$ is the corresponding Leontief matrix of the economy.

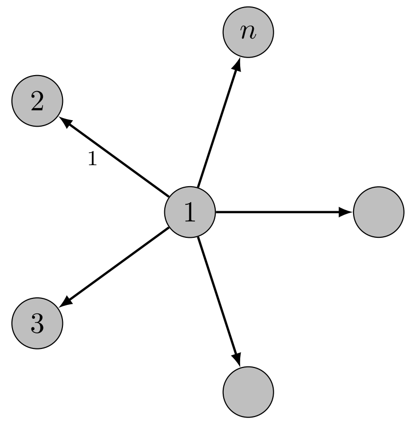


Figure 1. The star interaction network.

To see how the above concept captures an intuitive notion of network centrality as well as the overall extent of agents' influence on one another, consider the *star interaction network* depicted in Figure 1. As the figure suggests, a shock to agent 1, which takes a more central position in the network, should have a larger impact on other agents' states compared to a shock to agent $i \neq 1$. Indeed, it is easy to verify that the Bonacich centrality of agent 1 is equal to $v_1 = 1 + \alpha n/(1-\alpha)$, whereas $v_i = 1$ for $i \neq 1$.

More generally, in any given interaction network, agent i's Bonacich centrality can be written recursively in terms of the centralities of the rest of the agents in the economy:

$$v_i = 1 + \alpha \sum_{j=1}^{n} v_j w_{ji}. \tag{15}$$

This expression shows that i has a higher centrality (and hence a more pronounced impact on the rest of the agents) if it interacts strongly with agents that are themselves central.

Returning to our derivations, we next provide a linear approximation to the economy's macro state y in the presence of small shocks. Differentiating (2) with respect to ϵ_r yields

$$\frac{\partial y}{\partial \epsilon_r} = g'(h(x_1) + \cdots + h(x_n)) \sum_{m=1}^{n} h'(x_m) \frac{\partial x_m}{\partial \epsilon_r}. \tag{16}$$

Evaluating this expression at $\epsilon = 0$ and replacing for the derivative of agent i's state from (12), we obtain

$$\left. \frac{\partial y}{\partial \epsilon_r} \right|_{\epsilon=0} = \alpha g'(0) h'(0) \sum_{m=1}^{n} \ell_{mr}, \tag{17}$$

where we have again used that fact that, in the absence of shocks, $x_m = 0$ for all m and that $h(0) = 0$. Putting Definition 3 and equation (17) together leads to the following linear approximation to the economy's macro state as a function of its underlying interaction network, the interaction and aggregation functions, and the agent-level shocks:

Theorem 2. *Suppose that $f'(0) < 1$. Then, the first-order approximation to the macro state of the economy is*

$$y^{\text{1st}} = f'(0)g'(0)h'(0)\sum_{i=1}^{n} v_i\epsilon_i, \tag{18}$$

where v_i is the Bonacich centrality of agent i with parameter $f'(0)$.

The above result highlights that, as long as one is concerned with the first-order effects, the agents' Bonacich centralities serve as sufficient statistics for how shocks impact the economy's macro state. In particular, shocks to agents who take more central roles in the economy's interaction network have a more pronounced influence on economy's macro state. The intuition underlying this result can be understood in terms of the recursive definition of agents' centralities in (15): a shock to an agent with a higher Bonacich centrality impacts the states of other relatively central agents, which in turn propagate the shock further to other agents, and so on, eventually leading to a larger aggregate impact.

We end our discussion by remarking that when the interaction and aggregation functions are linear, Theorem 2 provides an exact characterization of — as opposed to a linear approximation to — the macro state of the economy in terms of the agent-level shocks. For example, recall the special case of network games with quadratic utilities studied in Subsection 2.1. By Theorem 2, the aggregate level of activity in such an economy is proportional to a convex combination of agent–level shocks, with weights given by each agent's Bonacich centrality in the network:

$$y_{\text{agg}} = \alpha \sum_{j=1}^{n} v_j\epsilon_j.$$

This result coincides with those of Ballester et al. (2006) and Calvó-Armengol et al. (2009), to cite two examples. Similarly, in the context of production economies with Cobb-Douglas production functions studied in Subsection 2.2, recall from (7) that the log output of any firm i is a linear function of the log-output of its suppliers. Using the log-value added, defined in (8), as the macro state of the economy, Theorem 2 implies that

$$\log(\text{GDP}) = \frac{\alpha}{n}\sum_{j=1}^{n} v_j\epsilon_j,$$

where ϵ_j is the log productivity shock to firm j, confirming a representation used in Acemoglu et al. (2012).

3.2 Second-Order Approximation

The linear approximation provided in the previous subsection characterizes how, in the presence of small shocks, the nature and strength of interactions between agents shape the economy's macro state. An important limitation of such an approximation is that the solution exhibits a *certainty equivalence* property, in the sense that the expected value of the economy's macro state is equal to its unperturbed value when no shocks are present. More specifically, as Corollary 1 below will show, $\mathbb{E}[y^{1\text{st}}]=0$ regardless of the economy's interaction network or the shape of the interaction and aggregation functions.[10] Consequently, even though potentially useful from an *ex post* perspective, the first-order approximation provided in Theorem 2 is not particularly informative about how the economy's interaction network shapes aggregate outcomes from an *ex ante* point of view. In order to go beyond this certainty equivalence property, we next provide a second-order approximation to the economy's macro state. As our results in the following sections will show, taking the second-order effects into account provides a more refined characterization of how agent-level shocks shape economic outcomes.

We start by differentiating both sides of equation (10) with respect to the shock to agent j:

$$\begin{aligned}\frac{\partial^2 x_i}{\partial\epsilon_r\epsilon_j} &= f''\left(\sum_{m=1}^n w_{im}x_m+\epsilon_i\right)\left(\sum_{m=1}^n w_{im}\frac{\partial x_m}{\partial\epsilon_r}+\mathbf{1}\{i=r\}\right)\left(\sum_{m=1}^n w_{im}\frac{\partial x_m}{\partial\epsilon_j}+\mathbf{1}\{i=j\}\right)\\ &\quad+f'\left(\sum_{m=1}^n w_{im}x_m+\epsilon_i\right)\left(\sum_{m=1}^n w_{im}\frac{\partial^2 x_m}{\partial\epsilon_r\partial\epsilon_j}\right).\end{aligned}$$

Evaluating this expression at $\epsilon=0$ implies

$$\left.\frac{\partial^2 x_i}{\partial\epsilon_r\epsilon_j}\right|_{\epsilon=0}=f''(0)\left(\alpha\sum_{m=1}^n w_{im}\ell_{mr}+\mathbf{1}\{i=r\}\right)\left(\alpha\sum_{m=1}^n w_{im}\ell_{mj}+\mathbf{1}\{r=j\}\right)+\alpha\sum_{m=1}^n w_{im}\left.\frac{\partial^2 x_m}{\partial\epsilon_r\partial\epsilon_j}\right|_{\epsilon=0},$$

where, once again, we are using the fact that $x_m=0$ for all m and that the first derivative of the agents' states with respect to the shocks can be written in terms of the economy's Leontief matrix, as given by (12). On the other hand, one can show that $\alpha\sum_{m=1}^n w_{im}\ell_{mr}=\ell_{ir}-\mathbf{1}\{i=r\}$.[11] Therefore, the previous expression can be simplified to

$$\left.\frac{\partial^2 x_i}{\partial\epsilon_r\epsilon_j}\right|_{\epsilon=0}=f''(0)\ell_{ir}\ell_{ij}+\alpha\sum_{m=1}^n w_{im}\left.\frac{\partial^2 x_m}{\partial\epsilon_r\partial\epsilon_j}\right|_{\epsilon=0},$$

leading to

$$\left.\frac{\partial^2 x_i}{\partial\epsilon_r\epsilon_j}\right|_{\epsilon=0}=f''(0)\sum_{m=1}^n \ell_{im}\ell_{mr}\ell_{mj}, \tag{19}$$

where we are using the definition of the Leontief matrix. The above equation thus provides the second-order derivates of agents' equilibrium states as a function of the interaction function and the Leontief matrix of the economy.

To obtain a second-order approximation to the macro state of the economy, we need to also differentiate (16) with respect to ϵ_j:

$$\begin{aligned}\frac{\partial^2 y}{\partial\epsilon_r\partial\epsilon_j} &= g''(h(x_1)+\dots+h(x_n))\sum_{m=1}^n\sum_{i=1}^n h'(x_m)h'(x_i)\left(\frac{\partial x_m}{\partial\epsilon_r}\right)\left(\frac{\partial x_i}{\partial\epsilon_j}\right)\\ &\quad+g'(h(x_1)+\dots+h(x_n))\sum_{m=1}^n\left[h'(x_m)\frac{\partial^2 x_m}{\partial\epsilon_r\epsilon_j}+h''(x_m)\frac{\partial x_m}{\partial\epsilon_r}\frac{\partial x_m}{\partial\epsilon_j}\right].\end{aligned}$$

[10] See Schmitt-Grohé and Uribe (2004) for a similar argument in the context of a general class of discrete-time rational expectations models.

[11] To see this, recall that the Leontief matrix can be rewritten as $L=\sum_{k=0}^\infty\alpha^kW^k$, which implies that $\alpha WL=L-I$.

Replacing for the first-order and second-order derivates of agents' equilibrium states from (13) and (19), respectively, leads to

$$\left.\frac{\partial^2 y}{\partial \epsilon_r \partial \epsilon_j}\right|_{\epsilon=0} = \alpha^2 g''(0)\left[h'(0)\right]^2 \sum_{m=1}^{n}\sum_{i=1}^{n} \ell_{mr}\ell_{ij} + g'(0)\sum_{m=1}^{n}\left[h'(0)f''(0)\sum_{k=1}^{n}\ell_{mk}\ell_{kr}\ell_{kj} + \alpha^2 h''(0)\ell_{mr}\ell_{mj}\right],$$

which can be further simplified to

$$\left.\frac{\partial^2 y}{\partial \epsilon_r \partial \epsilon_j}\right|_{\epsilon=0} = g'(0)h'(0)f''(0)\sum_{m=1}^{n} v_m \ell_{mr}\ell_{mj} + \alpha^2 g''(0)\left[h'(0)\right]^2 v_r v_j + \alpha^2 g'(0)h''(0)\sum_{m=1}^{n}\ell_{mi}\ell_{mj},$$

where v_m is the Bonacich centrality of agent m with parameter α. Combining the above with (17) leads to the following result:

Theorem 3. *Suppose that $f'(0) < 1$. Then, the second-order approximation to the macro state of the economy is given by*

$$\begin{aligned} y^{\text{2nd}} &= f'(0)g'(0)h'(0)\sum_{i=1}^{n} v_i \epsilon_i \\ &+ \frac{1}{2}g''(0)\left[f'(0)h'(0)\right]^2 \sum_{i=1}^{n}\sum_{j=1}^{n} v_i v_j \epsilon_i \epsilon_j \\ &+ \frac{1}{2}g'(0)\sum_{i=1}^{n}\sum_{j=1}^{n}\left(h'(0)f''(0)\sum_{m=1}^{n} v_m \ell_{mi}\ell_{mj} + \left[f'(0)\right]^2 h''(0)\sum_{m=1}^{n}\ell_{mi}\ell_{mj}\right)\epsilon_i\epsilon_j, \end{aligned} \tag{20}$$

where $L = [\ell_{ij}]$ is the economy's Leontief matrix with parameter $\alpha = f'(0)$ and v_i is the corresponding Bonacich centrality of agent i.

This result thus refines Theorem 2 by providing a second-order approximation to the role of agent-level shocks in shaping the economy's macro state. Note that the first line of (20) is simply the first-order approximation, y^{1st}, characterized in (18). The rest of the terms, which depend on the curvatures of the interaction and aggregation functions, capture the second-order aggregate effects. The second line, in particular, corresponds to additional terms resulting from the non-linearity of the aggregation function, g. Note that these terms depend simply on Bonacich centralities, the v_i terms. This is due to the fact that as long as the interaction function f is linear, the total influence of agent i on the rest of the agents in the economy is given by the Bonacich centrality of agent i, $v_i = \sum_{m=1}^{n} \ell_{mi}$. The third line, on the other hand, shows that if either the interaction function f or the h function is non-linear, the centrality measures are no longer sufficient statistics for the shocks' second-order effects. Rather, other network statistics — in particular, $\sum_{m=1}^{n} \ell_{mi}\ell_{mj}$ and $\sum_{m=1}^{n} v_m \ell_{mi}\ell_{mj}$ — also play a key role in how shocks propagate throughout the economy.

It is also worth noting that as long as shocks are small enough and the linear approximation is non-trivial, the second-order terms in (20) are dominated by the effect of the first-order terms. However, as our following results will show, in many applications the linear terms are equal to zero (reflecting the above-mentioned certainty equivalence property), and hence are uninformative about the nature of the economy's macro state, making the second-order approximation essential for a meaningful characterization of the aggregate impact of microeconomic shocks.

4 *Ex Ante* Aggregate Performance

In the remainder of this chapter, we use Theorems 2 and 3 to characterize how network interactions translate small, agent-level shocks into aggregate effects measured by the economy's macro state.

This section provides a comparative study of the role of the economy's underlying network structure — as well as its interaction and aggregation functions — in shaping y from an *ex ante* perspective, by interpreting the expectation of the macro state y as the economy's "performance metric". Formally:

Definition 4. An economy *outperforms* another if $\mathbb{E}[y]$ is larger in the former than the latter.

A natural first step to obtain a comparison between the performance of different economies in the presence of small shocks is to compare their first-order approximations. Recall from Theorem 2 that the first-order approximation of an economy's macro state is equal to a linear combination of agent-level shocks with the corresponding weights given by the agents' Bonacich centralities, i.e., $y^{\text{1st}} = f'(0)g'(0)h'(0)\sum_{i=1}^{n} v_i\epsilon_i$, leading to the following immediate corollary:

Corollary 1. $\mathbb{E}[y^{\text{1st}}] = 0$.

This simple corollary shows that the economy exhibits a *certainty equivalence property* from an *ex ante* perspective up to a first-order approximation: the expected value of the economy's macro state is equal to its unperturbed value when no shocks are present, regardless of the nature of pairwise interactions or the shape of the interaction and aggregation functions. The more important implication, however, is that the linear approximation provided in Theorem 2 is not informative about the comparative performance of different economies, even in the presence of small shocks. Rather, a meaningful comparison between the *ex ante* performance of two economies requires that we also take the higher-order terms into account.

Thus, a natural next step is to use the second-order approximation provided in Theorem 3. Equation (20) shows that once second-order terms are taken into account, the *ex ante* performance of the economy, $\mathbb{E}[y^{\text{2nd}}]$, depends on the curvatures of the interaction and aggregation functions. In order to tease out these effects in a transparent manner, in the remainder of this section, we focus on how non-linearities in each of these functions shape the economy's macro state, while assuming that the rest of the functions are linear.

4.1 Non-Linear Aggregation: Volatility

We first consider an economy with a general, potentially non-linear aggregation function g, while assuming that f and h are increasing, linear functions. In this case, the *ex ante* performance of the economy is given by

$$\mathbb{E}[y] = \mathbb{E}g(x_1 + \cdots + x_n).$$

This observation highlights that the curvature of g essentially captures the extent to which society cares about volatility, for instance because of risk-aversion at the aggregate level. To see this, suppose that g is concave. In this case, the economy's performance is reduced the more correlated agents' states are with one another. In fact, if $g(z) = -z^2$, the economy's *ex ante* performance simply captures the volatility of $x_1 + \cdots + x_n$. On the other hand, a convex g corresponds to the scenario in which performance increases with volatility.

In either case, Theorem 3 implies that the expected value of the economy's macro state, up to a second-order approximation, is given by

$$\mathbb{E}[y^{\text{2nd}}] = \frac{1}{2}\sigma^2 g''(0) \left[f'(0)h'(0)\right]^2 \sum_{i=1}^{n} v_i^2, \tag{21}$$

where we are using the assumption that all shocks are independent with mean zero and variance σ^2 and the assumption that functions f and h are linear. Equation (21) shows that, in contrast to Corollary 1, not all economies have identical performances once second-order terms are taken into account. Rather, the economy's *ex ante* performance depends on $\sum_{i=1}^{n} v_i^2$, which in turn, can be rewritten as

$$\sum_{i=1}^{n} v_i^2 = n \cdot \operatorname{var}(v_1, \ldots, v_n) + \frac{n}{(1-\alpha)^2},$$

where $\alpha = f'(0)$, thus leading to the following result:

Proposition 4. *Suppose that the aggregation function g is concave (convex). An economy's ex ante performance decreases (increases) in* $\operatorname{var}(v_1, \ldots, v_n)$.

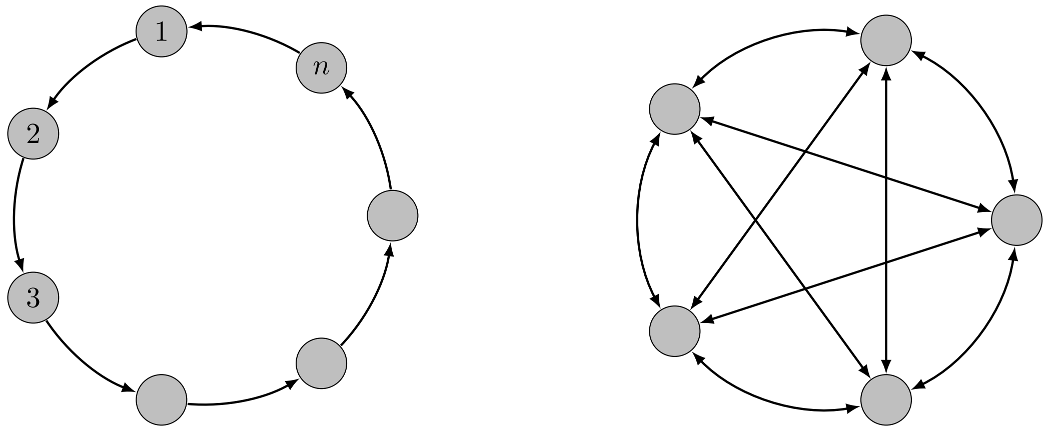
This proposition implies that, if g is concave, networks in which agents exhibit a less heterogenous distribution of Bonacich centralities outperform those with a more unequal distribution. This is due to the fact that a more equal distribution of Bonacich centralities means that shocks to different agents have a more homogenous impact on the economy's macro state, and thus wash each other out more effectively at the aggregate level. On the other hand, a more unequal distribution of centralities implies that shocks to some agents play a disproportionally larger role in shaping y and as a result, are not canceled out by the rest of the agent-level shocks, increasing the overall volatility and reducing the value of $\mathbb{E}[y]$ whenever g is concave.

To see the implications of Proposition 4, consider an economy with the underlying star interaction network depicted in Figure 1. As already mentioned, the Bonacich centralities of agents in such an economy are highly unequal as agent 1 has a disproportionally large impact on the states of the rest of the agents. In fact, it is easy to show that $\sum_{i=1}^{n} v_i^2$ is maximized for the star interaction network. This implies that when g is concave, the star network has the least *ex ante* performance (and hence, the highest level of volatility) among all economies.[12]

At the other end of the spectrum are regular economies in which the extent of interaction of each agent with the rest of the agents is constant. More formally,

Definition 5. An economy is *regular* if $\sum_{j=1}^{n} w_{ji} = 1$ for all agents i.

[12]Note that by Hölder's inequality, $\sum_i v_i^2 \leq (\max_i v_i)(\sum_i v_i) = n \max_i v_i/(1-\alpha) \leq n(1-\alpha+\alpha n)/(1-\alpha)^2$, regardless of the economy's interaction network, where recall that $\alpha = f'(0)$. This inequality is tight for the star network, implying that $\sum_i v_i^2$ obtains its maximal value.



(a) The ring financial network

(b) The complete financial network

Figure 2. Two regular economies

Figures 2(a) and 2(b) depict two regular networks, known as the *ring* and *complete* interaction networks, respectively. Because they are symmetric, all agents in both economies should have identical Bonacich centralities. In fact, summing both sides of (14) over i in an arbitrary regular economy implies that

$$v_r = 1 + \alpha + \alpha^2 + \cdots = 1/(1-\alpha)$$

for all r, where recall that $\alpha = f'(0)$. This implies the following result:

Lemma 1. *In any regular economy, all agents have identical Bonacich centralities.*

Therefore, $\mathrm{var}(v_1, \ldots, v_n)$ is minimized for all regular economies, implying that with a concave g, they outperform all other economies from an *ex ante* perspective: all agent-level shocks in such an economy take symmetric roles in determining the macro state, and minimize the overall volatility of $x_1 + \cdots + x_n$ and thus increase $\mathbb{E}[y]$. This implies the following corollary to Proposition 4:

Corollary 2. *Suppose that aggregation function g is concave (convex). Any regular economy outperforms (underperforms) all other economies, whereas the economy with the star interaction network underperforms (outperforms) all others.*

This corollary and Proposition 4 are closely connected to the results in Acemoglu et al. (2012), who show that in the context of the production economies presented in Subsection 2.2, aggregate output volatility is increasing in the extent of heterogeneity in the firms' centralities and is maximized (minimized) for the star (regular) network. This parallel can be better appreciated by noting that the logarithm of output of a given firm i satisfies linear equation (7) and that $\log(\mathrm{GDP}) = (1/n)\sum_{i=1}^{n} x_i$. Therefore, the volatility of log value added is simply

$$\mathrm{var}\left(\log(\mathrm{GDP})\right) = \frac{1}{n^2}\mathbb{E}(x_1 + \cdots + x_n)^2.$$

Setting $g(z) = -(z/n)^2$ implies that economies that have a higher *ex ante* performance in the sense of Definition 4 are less volatile at the aggregate level. Hence, Proposition 4 and Corollary 2 guarantee that any economy in which firms exhibit more heterogeneity in terms of their roles as input-suppliers exhibits higher levels of aggregate (log) output volatility due to idiosyncratic firm-level shocks. Our results thus show that it is the concavity of economy's aggregation function that lies at the heart of the findings of Acemoglu et al. (2012).

4.2 Non-Linear Interactions

We now focus on the role of non-linear interactions in shaping the economy's *ex ante* performance. To illustrate this role in a transparent manner, we consider an economy with a general, non-linear interaction function f, while assuming that g and h are increasing, linear functions.[13] The *ex ante* performance of such an economy is given by

$$\mathbb{E}[y] = \sum_{i=1}^{n} \mathbb{E}[x_i] = \sum_{i=1}^{n} \mathbb{E} f \left(\sum_{j=1}^{n} w_{ij} x_j + \epsilon_i \right).$$

The above equation highlights that the curvature of the interaction function f captures the extent of "risk-aversion" at the micro-level.

To understand the role of interlinkages in affecting economic performance, we focus on the set of symmetric, regular economies.[14] Recall from Theorem 3 that, in the presence of small shocks, the expected value of the economy's macro state can be approximated by

$$\mathbb{E}[y^{\text{2nd}}] = \frac{1}{2}\sigma^2 g'(0) h'(0) f''(0) \sum_{i=1}^{n} \sum_{m=1}^{n} v_m \ell_{mi}^2. \tag{22}$$

Note that as before, we need to rely on a second-order approximation, as the first-order terms are not informative about the comparative performance of different economies; that is $\mathbb{E}[y^{\text{1st}}] = 0$ regardless of the shape of f or the economy's interaction network. Given that all agents in a regular network have identical Bonacich centralities, Equation (22) shows that the economy's performance depends on the value of $\sum_{i,m} \ell_{mi}^2$. On the other hand, it is easy to verify that

$$\sum_{m=1}^{n} \ell_{mi}^2 = v^2/n + \text{var}(\ell_{1i}, \dots, \ell_{ni}), \tag{23}$$

where $v = \sum_{m=1}^{n} \ell_{mi} = 1/(1-\alpha)$ is the agents' (common) Bonacich centrality, thus suggesting that the term $\sum_{i=1}^{n} \sum_{m=1}^{n} \ell_{mi}^2$ decreases if inter-agent influences ℓ_{mi} are more evenly distributed. The following result, which is proved in the Appendix, captures this idea formally:

Corollary 3. *Suppose that there are no self-interaction terms, that is, $w_{ii} = 0$. If the interaction function f is concave (convex), then the complete network outperforms (underperforms) all other symmetric economies.*

[13] The results, and in fact the expressions, are essentially identical when h is also non-linear.
[14] Recall that an economy is said to be symmetric if $w_{ij} = w_{ji}$ for all $i \neq j$.

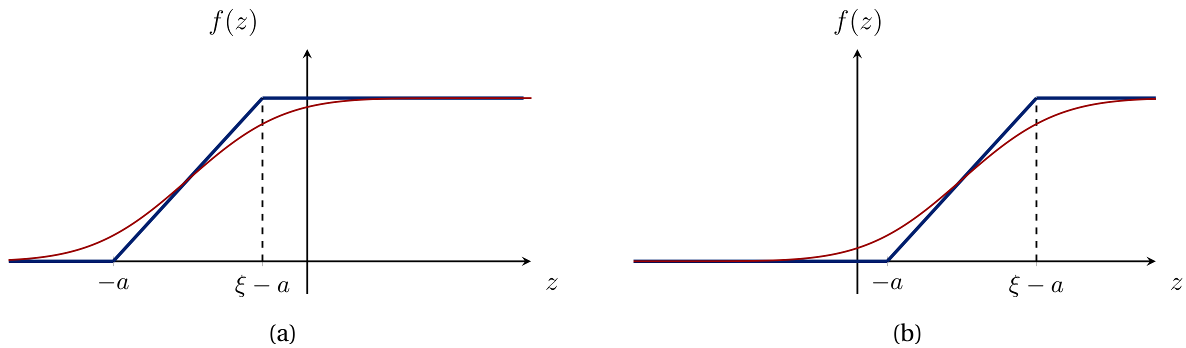


Figure 3. The interaction function $f(z) = \max\{\min\{z + a, \xi\}, 0\}$ corresponding to the model of financial interactions in Subsection 2.3. Panels (a) and (b) plot the function for the case that $a > \xi$ and $a < 0$, respectively. The thin red line in each panel depicts a smooth approximation to the interaction function.

This corollary is related to the findings of Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015b), who, in the context of the model of financial interactions presented in Subsection 2.3, show that the complete financial network exhibits a "phase transition": when the total net asset value of the financial system is large enough, the complete network is the financial network with the least number of defaults. However, as the net asset value of the financial system is reduced, beyond a certain point, the complete network flips to be the economy with the maximal number of bank failures.[15]

To see the connection between their results and Corollary 3, recall that the corresponding interaction function in such an economy is given by $f(z) = \max\{\min\{z + a, \xi\}, 0\}$. As depicted in Figure 3(a), for large enough values of a (in particular, when $a > \xi$), this interaction function is concave in the neighborhood of 0. Therefore, Corollary 3 implies that the complete network outperforms all other economies.[16] In contrast, once the banks' net asset value a become small enough, the interaction function is locally convex around 0, as depicted in Figure 3(b). In stark contrast to the former case, Corollary 3 now implies that all other economies would outperform the complete network. Thus, our characterization results clarify that the findings of Acemoglu et al. (2015b) are due to the fact that reducing the banks' net asset values (for example, due to some exogenous shocks) essentially corresponds to a shift from the concave to the convex region of the interaction function, thus reversing the role of interbank connections.

In addition to providing a different perspective on the results of Acemoglu et al. (2015b), Corollary 3 presents a partial answer to the question posed in the Introduction, related to the sometimes contradictory claims on the role of dense network interconnections in creating systemic risk and instability. It shows that when economic (financial) interactions correspond to a concave f, denser interconnections are stabilizing (as in Allen and Gale (2000)), whereas they play the role of generat-

[15]To be more precise, Acemoglu et al. (2015b) state their results in terms of whether exogenous shocks that hit financial institutions are small or large. Nevertheless, given that such shocks simply impact the net asset value of the banks, their results can be equivalently stated in terms of the size of the net asset value of the banks, a.

[16]As already noted, even though the corresponding interaction function is not smooth, it can be arbitrarily closely approximated by a smooth function in such a way that the economic implications of the model under this smooth approximation are identical to those of the original model. Figure 3 depicts one such smooth approximation.

ing systemic risk when these interactions correspond to a convex f function. Finally, our result that more densely interconnected networks are more unstable in the presence of convex interactions is akin to similar results in the epidemic-like cascade models (such as Blume et al. (2011)), in which a bank fails once the number of its defaulting counterparties passes a certain threshold.

5 Systemically Important Agents

A central concern in many analyses of economic and social networks is the identification of "key players" or "systemically important agents"(e.g., Ballester et al. (2006) and Zenou (2015)). Loosely speaking, these are entities that have a disproportionally high impact on some aggregate statistic of interest. For example, Banerjee, Chandrasekhar, Duflo, and Jackson (2013, 2014) study how the social network position of the first individual to receive information about a new product within a village can increase the extent of information diffusion within that community. Similarly, in the context of multi-agent contracting in the presence of externalities, Bernstein and Winter (2012) are interested in obtaining an ordering of agents who when subsidized induce the maximal level of participation by other agents. Relatedly, in the context of the example presented in Subsection 2.3, Acemoglu et al. (2015b) characterize the set of systemically important institutions in a financial network, a shock to whom would lead to a large cascade of defaults.

In this section, we utilize Theorems 2 and 3 to study how different features of the environment determine the impact of each agent on the macro state of the economy and provide a characterization of the set of agents that are more important from a systemic perspective. We start by defining this concept formally:

Definition 6. Agent i is said to be *systemically more important* than agent j if $y_{(i)} < y_{(j)}$, where $y_{(i)}$ denotes the macro state of the economy when agent i is hit with a negative shock.

In other words, agent i is systemically more important than agent j if a shock to i leads to a larger drop in the economy's macro state. Note that in general, the relative systemic importance of an agent may depend on the size of the negative shock. Nevertheless, we can use our results in Section 3 to provide a characterization of the systemic importance of different agents for small enough shocks.

We should also remark that our notion of a systemically important agents is related to, but distinct from the notion of "key players" studied by Ballester et al. (2006) and Zenou (2015). Whereas our focus is on how a shock to a given agent impacts some macroeconomic variable of interest, these papers study the impact of the removal of an agent from the network.

5.1 Linear Interactions

We start by focusing on economies where the interaction and aggregation functions are linear. This enables us to highlight, in a transparent manner, how the presence of non-linearities can shape equilibrium outcomes.

Recall that when the interaction and aggregation functions are linear, Theorem 2 provides an exact characterization of the economy's macro state in equilibrium. More specifically, it shows that y is a linear combination of the idiosyncratic, agent-level shocks, with the weights proportional to the Bonacich centralities of the corresponding agents, leading to the following result:

Proposition 5. *Suppose that the economy's interaction function is linear. Then agent i is more systemically important than agent j if $v_i > v_j$, where v_i is the Bonacich centrality of agent i.*

In other words, in an economy with linear interactions, a negative shock to the agent with the highest Bonacich centrality leads to the largest drop in the economy's macro state. The intuition underlying this result is simple and well-known in the literature: shocks to more central agents propagate more extensively over the network and as a result have larger impacts on the economy's macro state.

To see the implications of the above result, consider the economies depicted in Figures 1 and 2. Given that the ring and complete networks depicted in Figure 2 are regular, Proposition 5 suggests that in the presence of linear interactions, all agents in such economies are equally systemically important. In contrast, in the economy depicted in Figure 1, agent 1 takes a more central position with respect to the rest of the agents, leading to the intuitive result that it is the most systemically important agent within the economy.

Proposition 5 also has sharp predictions for the set of systemically important agents in the class of network games with quadratic payoffs discussed in Subsection 2.1. Recall that the first-order conditions in such games can be represented in the form of a linear interaction function. Thus, by Proposition 5, the player with the highest Bonacich centrality would be the most influential player in the game. This is indeed in line with the observations of Candogan, Bimpikis, and Ozdaglar (2012), who argue that subsidizing players with the highest centrality would induce the largest increase in the level of aggregate activity in the economy.

Similarly, in the context of production economies with Cobb-Douglas (and hence, log-linear) production technologies discussed in Subsection 2.2, Acemoglu et al. (2012) show that productivity shocks to firms with higher centralities have a larger impact on the economy's aggregate output, an observation consistent with the predictions of Proposition 5. More specifically in line with the examples we discussed above, they also argue that, compared to a shock of equal size to one of the more peripheral firms, a shock to firm 1 in the star network depicted in Figure 1 would have a much larger impact on the log value added of the economy.

Finally, Proposition 5 also echoes some of the results in the literature on social learning that studies the long-run implications of different learning rules. In particular, Golub and Jackson (2010) show that if agents update their beliefs as a linear combination of their neighbors' opinions (what is commonly known as DeGroot-style learning), the information available to those with higher centralities plays a more prominent role in the eventual beliefs in the society. Relatedly, Jadbabaie et al. (2012, 2013) show that the rate of information aggregation in a social network is more sensitive to the quality of the signals observed by the more central agents.[17]

5.2 Non-Linear Interactions

Our previous results show that if the economy's interaction function is linear, Bonacich centrality provides a comprehensive measure for agents' systemic importance. This observation also means that, as long as agent-level shocks are small enough, more central agents would play a more prominent role in shaping the economy's macro state, even if the interactions are non-linear. This is due to the fact that by Theorem 2, the economy's macro state can be linearly approximated by

$$y_{(i)}^{\text{1st}} = f'(0)g'(0)h'(0)v_i\epsilon_i,$$

leading to the following result:

Corollary 4. *If $v_i > v_j$, then agent i is systemically more important than agent j for all interaction functions f.*

This conclusion is subject to an important caveat: even though $v_i > v_j$ implies that i is more systemically important than j in the presence of small shocks, $v_i = v_j$ does not guarantee that the two agents are equally systemically important. Rather, in such a scenario, a meaningful comparison of the agents' systemic importance requires that we also take their higher-order effects into account. Thus, Corollary 4 is simply not applicable to regular economies, in which all agents have identical Bonacich centralities.

In order to obtain a meaningful measure for agents' systemic importance in a regular economy, a natural step would be to utilize Theorem 3 to compare the second-order effects of agent-level shocks on the economy's macro state. From (20), we have that, in any regular economy,

$$y_{(i)}^{\text{2nd}} = f'(0)g'(0)h'(0)v\epsilon + \frac{g''(0)}{2}\left(f'(0)h'(0)\right)^2 v^2\epsilon^2 + \frac{1}{2}g'(0)\left(vh'(0)f''(0) + \left[f'(0)\right]^2 h''(0)\right)\left(\sum_{m=1}^{n}\ell_{mi}^2\right)\epsilon^2,$$

where $v = 1/(1-\alpha)$ is the agents' (common) Bonacich centrality, thus implying that agent i's systemic importance is determined by the value of $\sum_{m=1}^{n}\ell_{mi}^2$. On the other hand, recall from (23) that $\sum_{m=1}^{n}\ell_{mi}^2$ essentially measures the variation in the extent to which agent i influences other agents in the economy. We define the following concept:

Definition 7. The *concentration centrality* of agent i is $d_i = \text{stdev}(\ell_{1i}, \ldots, \ell_{ni})$, where $L = [\ell_{ij}]$ is the economy's Leontief matrix.

Thus, a smaller d_i means that agent i's influence is more evenly distributed throughout the economy. In other words, whereas an agent's Bonacich centrality captures its overall influence, concentration centrality measures how evenly the agent's influence is distributed across the rest

[17] The main results in this literature are in terms of agents' eigenvector centralities, defined as a limiting case of Bonacich centrality. In particular, the eigenvector centrality of agent i satisfies $\hat{v}_i = \lim_{\alpha\to 1}(1-\alpha)v_i$. See Jackson (2008) for a discussion on other notions of centrality and their relationships to one another.

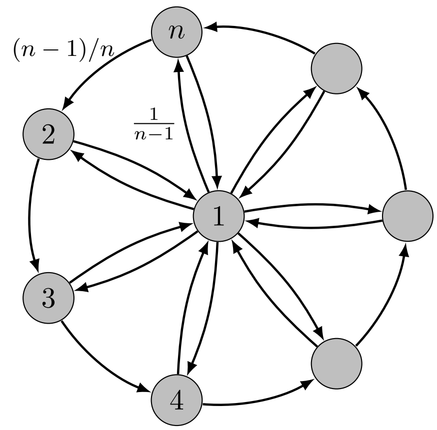


Figure 4. A regular economy where agents' have identical centralities, but differ in their concentration centralities.

of the agents. As an example, consider the economy depicted in Figure 4. It is easy to verify that the depicted network corresponds to a regular economy, implying that all agents have identical Bonacich centralities. However, the extent of dispersion is not identical across agents. Rather, for large enough values of n, $d_1 < d_i$ for all $i \neq 1$, as agent 1's interactions are more evenly distributed throughout the economy, whereas all other agents interact with only a handful of others.

This discussion is summarized in the next proposition.

Proposition 6. *Suppose that the economy's interaction network is regular.*

(a) If f is concave, then i is systemically more important than j if and only if $d_i > d_j$.

(b) If f is convex, then i is systemically more important than j if and only if $d_i < d_j$.

Taken together, Proposition 6 and Corollary 4 suggest that while Bonacich centralities summarize the first-order effects of agent-level shocks on aggregate outcomes, the second-order effects are captured by the agents' concentration centralities. These second-order effects become critical in a regular network, where first-order terms are simply uninformative about agents' systemic importance.

Proposition 6 also reenforces an observation made by Acemoglu et al. (2015b) that relying on standard and off-the-shelf notions of network centrality (such eigenvector or betweenness centralities) for the purpose of identifying systemically important agents may be misleading. As Proposition 6 suggests, the proper notion of network centrality has to be informed by the nature of microeconomic interactions between different agents.

6 Conclusion

This chapter presented a unified framework nesting a wide variety of network interaction models, such as various classes of network games, models of macroeconomic risk built up from microeconomic shocks, and models of financial interactions. Under the assumption that shocks are small (and the relevant interactions are smooth), our main results provide a fairly complete characterization of the equilibrium, highlighting the role of different types of network interactions in affecting the macroeconomic performance of the economy. Our characterization delineates how microeconomic interactions function as a channel for the propagation of shocks and enables us to provide a comparative study of the role of the economy's underlying network structure — as well as its interaction and aggregation functions — in shaping macroeconomic outcomes. In addition to clarifying the relationship between disparate models (for example, those focusing on input-output linkages, financial contagion and general cascades), our framework also highlights some of the reasons behind the apparently contradictory conclusions in the literature on to the role of network interactions in the emergence of systemic risk.

Our hope is that the framework provided here will be useful in future work on understanding network interactions in general and the study of network games, macroeconomic risk and financial contagion in particular. We believe that several important issues remain open to future research. First, our framework focuses on an environment in which shock realizations are common knowledge. Generalizing this setup to environments with incomplete and private information would enable us to understand the interplay between network interactions and information asymmetries. A second direction for future research would be to apply similar analyses to economies that exhibit richer strategic interactions (such, general imperfect competition rather than competitive or monopolistically competitive economies). Finally, a systematic investigation of endogenous network formation in the presence of rich propagation and cascade dynamics remains an important area for future research.

A Technical Appendix

Lemma 2. *Suppose that* $|f(\tilde{z})-f(z)| = |\tilde{z}-z|$ *for a pair of points* $\tilde{z} > z$*. Then, the interaction function* f *is linear in the interval* $[z,\tilde{z}]$ *with a unit slope.*

Proof. Pick an arbitrary point $\hat{z} \in [z,\tilde{z}]$. Given Assumption 1 and the monotonicity of the interaction function, it must be the case that

$$f(\tilde{z}) - f(\hat{z}) \leq \tilde{z} - \hat{z}$$
$$f(\hat{z}) - f(z) \leq \hat{z} - z.$$

Summing the above inequalities immediately implies that both inequalities have to be tight simultaneously. Therefore, for any $\hat{z}$ in the interval $[z,\tilde{z}]$, it must be the case that $f(\hat{z}) = \hat{z} + f(z) - z$. □

Lemma 3. *The interaction function* f *has at most countably many discontinuity points.*

Proof. Let D denote the set of points where f is discontinuous. For any $z \in D$, define

$$f(z^-) = \lim_{t\uparrow z} f(t)$$
$$f(z^+) = \lim_{t\downarrow z} f(t).$$

Given the fact that f is nondecreasing, it must be the case that $f(z^-) < f(z^+)$. Therefore, there exists a rational number $F(z) \in \mathbb{Q}$ such that

$$f(z^-) < F(z) < f(z^+).$$

Furthermore, for any pair of points $z, \tilde{z} \in D$ satisfying $z < \tilde{z}$, it is immediate that $F(z) < F(\tilde{z})$. Consequently, $F : D \to \mathbb{Q}$ has to be an injection, proving that D is at most countable. □

Proof of Theorem 1

We prove this result for two separate cases depending on whether (i) $\beta < 1$ or (ii) $\beta = 1$. Throughout, we assume that the economy's interaction network is *strongly connected* in the sense that there exists a directed path from each agent to any other agent in the economy. In case of a disconnected interaction network, the proof would apply to any connected component separately.

Case (i) First, suppose that $\beta < 1$. Define the mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\Phi_i(x_1,\dots,x_n) = f\left(\sum_{j=1}^n w_{ij}x_j + \epsilon_i\right). \tag{24}$$

For any $x, \tilde{x} \in \mathbb{R}^n$, we have

$$\begin{aligned}|\Phi_i(x_1,\dots,x_n) - \Phi_i(x_1,\dots,x_n)| &\leq \beta \left|\sum_{j=1}^n w_{ij}(x_j - \tilde{x}_j)\right| \\ &\leq \beta \sum_{j=1}^n w_{ij}\,|x_j - \tilde{x}_j|\,,\end{aligned}$$

where the first inequality is a consequence of Assumption 1 and the second inequality follows from a simple application of the triangle inequality. The fact that $\sum_{j=1}^{n} w_{ij} = 1$ implies that

$$|\Phi_i(x_1, \ldots, x_n) - \Phi_i(x_1, \ldots, x_n)| \leq \beta \max_j |x_j - \tilde{x}_j|,$$

and as a consequence,

$$\max_i |\Phi_i(x_1, \ldots, x_n) - \Phi_i(x_1, \ldots, x_n)| \leq \beta \max_j |x_j - \tilde{x}_j|.$$

In other words,

$$\|\Phi(x) - \Phi(\tilde{x})\|_\infty \leq \beta \|x - \tilde{x}\|_\infty .$$

Therefore, the mapping Φ is a contraction with respect to the infinity norm with a Lipschitz constant $\beta < 1$. The contraction mapping theorem then immediately implies that the mapping has a unique fixed point $x^* = \Phi(x^*)$, for all shock realizations $(\epsilon_1, \ldots, \epsilon_n) \in \mathbb{R}^n$. □

Case (ii) Next, suppose that $\beta = 1$. In this case, Assumption 1 guarantees that there exists $\delta > 0$ such that $|f(z)| < \delta$ for all z.

Recall mapping Φ from (24). By assumption, it is continuous and maps the compact and convex set $[-\delta, \delta]^n$ to itself. Therefore, by the Brouwer fixed point theorem, there exists $x^* \in [-\delta, \delta]^n$ such that $\Phi(x^*) = x^*$, thus proving the existence of an equilibrium.

Next, we prove that this equilibrium is generically unique. Suppose that the economy has two distinct equilibria, denoted by x and $\tilde{x}$. Let $e = |x - \tilde{x}| \in \mathbb{R}^n$ be the element-wise difference between the two equilibria, which by assumption, is a non-zero vector. By definition, for any given agent i, we have

$$\begin{aligned} e_i &= \Big| f\Big(\sum_{j=1}^{n} w_{ij} x_j + \epsilon_i\Big) - f\Big(\sum_{j=1}^{n} w_{ij} \tilde{x}_j + \epsilon_i\Big)\Big| \\ &\leq \Big|\sum_{j=1}^{n} w_{ij}(x_j - \tilde{x}_j)\Big| \qquad (25) \\ &\leq \sum_{j=1}^{n} w_{ij} e_j, \qquad (26) \end{aligned}$$

where the first inequality is a consequence of Assumption 1. We now show that both inequalities above are tight for all agents i.

Suppose that either inequality holds strictly for some agent i, implying that $e_i < \sum_{j=1}^{n} w_{ij} e_j$. Let $q \in \mathbb{R}^n$ denote the left eigenvector corresponding to the top eigenvalue of matrix W. By the Perron-Frobenius theorem, vector q is element-wise strictly positive.[18] Multiplying both sides by q_i and summing over all agents i implies that

$$\sum_{i=1}^{n} q_i e_i < \sum_{j=1}^{n} \sum_{i=1}^{n} q_i w_{ij} e_j = \sum_{j=1}^{n} q_j e_j,$$

[18] For more on the Perron-Frobenius theorem, see Chapter 2 of Berman and Plemmons (1979).

leading to a contradiction. Therefore, it is immediate that (25) and (26) hold as equalities, thus implying that $e_i = \sum_{j=1}^n w_{ij} e_j$ for all agents i, or in matrix notation, $We = e$.

Consequently, by the Perron-Frobenius theorem, e has to be proportional to the Perron vector of matrix W which is the vector of all ones. In other words, $e_i = c$ for all i and some strictly positive constant c. Furthermore, the fact that (26) holds as an equality implies that $x_i - x'_i$ has the same sign for all i. Assuming that $x_i < \tilde{x}_i$, it must be the case that $\tilde{x}_i = x_i + c$ for all agents i.

Summarizing the above implies that for all agent i,

$$x_i = f\left(\sum_{j=1}^n w_{ij} x_j + \epsilon_i\right)$$

and

$$x_i + c = f\left(\sum_{j=1}^n w_{ij} x_j + \epsilon_i + c\right).$$

Letting $z_i = \sum_{j=1}^n w_{ij} x_j + \epsilon_i$ and subtracting both sides of the above equalities lead to

$$|f(z_i + c) - f(z_i)| = c.$$

Thus, by Lemma 2, the interaction function f has to be linear with a unit slope within the interval $[z_i, z_i + c]$. Consequently, there exists some constant b_i such that $f(z) = z + b_i$ for all $z \in [z_i, z_i + c]$. Therefore,

$$x_i = \sum_{j=1}^n w_{ij} x_j + \epsilon_i + b_i$$

for all i. Multiplying both sides of the above equality by q_i and summing over all agents i lead to

$$\sum_{i=1}^n q_i \epsilon_i = -\sum_{i=1}^n q_i b_i, \tag{27}$$

where once again we are using the fact that $\sum_{i=1}^n q_i w_{ij} = q_j$. Therefore, the economy has two distinct equilibria if and only if the agent-level shocks satisfy (27). Now, Lemma 3 guarantees that there are at most countably many of such values b_i, as otherwise the interaction function f would have uncountably many points of discontinuity. In other words, for the economy to have multiple equilibria, the term $\sum_{i=1}^n q_i \epsilon_i$ has to belong to a countable set B. This coupled with the observation that $q_i > 0$ guarantees that the economy has a unique equilibrium for a generic set of shock realizations. □

Proof of Corollary 3

Suppose that f is concave. The proof for the case in which f is convex is identical. Recall from Equation (22) that the *ex ante* performance the economy is decreasing in $\sum_{i=1}^n \sum_{m=1}^n \ell_{mi}^2$, which can be rewritten as

$$\sum_{i=1}^n \sum_{m=1}^n \ell_{mi}^2 = \mathrm{trace}(L^2).$$

Denoting the k-th largest eigenvalue of a generic matrix X with $\lambda_k(X)$, we have:

$$\mathrm{trace}(L^2) = \sum_{k=1}^{n} \lambda_k^2(L) = \sum_{k=1}^{n} (1 - \alpha\lambda_k(W))^{-2},$$

where the second inequality is a consequence of the fact that $L = (I - \alpha W)^{-1}$.

On the other hand, the assumption that $w_{ii} = 0$ implies that $\mathrm{trace}(W) = \sum_{k=1}^{n} \lambda_k(W) = 0$, whereas $\sum_{j=1}^{n} w_{ij} = 1$ guarantees that $\lambda_1(W) = 1$. Putting these two observation together implies that $\sum_{k=2}^{n} \lambda_k(W) = -1$. Therefore,

$$\mathrm{trace}(L^2) = \frac{1}{(1-\alpha)^2} + \sum_{k=2}^{n} (1 - \alpha\lambda_k(W))^{-2} \tag{28}$$

$$\geq \frac{1}{(1-\alpha)^2} + (n-1)\left(1 + \frac{\alpha}{n-1}\right)^{-2}, \tag{29}$$

where the second equality is due to the fact that function $Q(z) = (1 - \alpha z)^{-2}$ is convex. On the other hand, it easy to show that for the complete network, $\lambda_k(W) = -1/(n-1)$ for all $k \neq 1$. Therefore, the complete network obtains the lower bound in (29), and hence, has maximal *ex ante* performance when the interaction function is concave. □

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