Incomplete Markets and Aggregate Demand∗

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I study the relationship between aggregate consumption and interest rates when markets are incomplete. I first provide a generalized Euler relation involving the real interest rate, current and future aggregate consumption under extreme illiquidity (no borrowing and no outside assets). This provides a tractable way of incorporating incomplete markets into macroeconomic models. When household income risk is acyclical I show that this relation coincides with that of a representative agent, although time-varying discount factors may potentially act as aggregate demand shocks. The same representation extends to the case with positive liquidity as long as liquidity relative to output is acyclical. A corollary of these ‘as if’ results is that forward guidance policies are as powerful as in representative agent models. Away from the ‘as if’ benchmark, I show that aggregate consumption becomes more sensitive to interest rates, especially future ones, when idiosyncratic income risk is countercyclical or when liquidity is procyclical. Finally, I also apply my analysis to a Real Business Cycle model, providing an exact analytical aggregation result that complements existing numerical findings.

1 Introduction

Macroeconomic models are minimalist at heart. For example, the basic New Keynesian model can be reduced to two fundamental components,

1. the Intertemporal Euler equation or “Demand block”;

2. the Phillips curve or “Supply block”;

the picture may be completed by describing monetary policy, providing a third equation. A similar description can be applied to models without nominal rigidities, such as the Real Business Cycle model, which also features the Euler equation at center stage and adopts the flexible price limit as its supply block.

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Microfoundations are available for both blocks, but at best these should be viewed as extremely simplified approximations of a deeper reality. In particular, underlying the standard Euler equation is the assumption of either complete markets or the adoption of a representative agent. Similarly, underlying the New Keynesian Phillips curve is the simplifying assumption that opportunities to change prices arrive at random, i.e. Calvo pricing. These assumptions can be easily rejected at face value, but, at the same time, it is hard to deny their usefulness as tractable starting points.

This paper is concerned with the ‘demand block’ and asks: What are the effects of market incompleteness on aggregate demand? How should incomplete markets be incorporated into macroeconomic models and what can we expect from doing so? How is it relevant for business cycles and policy analyses? Given the well-documented importance of idiosyncratic uncertainty and the limits to insurance at the household level, these questions hardly require motivation. Less formally, or introspectively, nobody living in the real world can possibly feel well represented by the representative agent lurking in our models! Indeed, anyone outside our models would quickly point out that idiosyncratic uncertainty looms large relative to the business cycle, that the risk of unemployment is the main concern with recessions, that current or future borrowing constraints limit the smoothing of consumption, etcetera. Intuitively, a more realistic model could potentially make a difference and this is the motivation for a large and important literature on incomplete markets. Despite recent progress, the literature still finds itself far from offering a comprehensive and conclusive answer to the questions posed above.

To address these questions, I study a Bewley-Huggett-Aiyagari incomplete-market economy: households save and borrow, but lack insurance for the idiosyncratic uncertainty they face. A key feature of my formulation is to postulate that each household earns income that depends on an idiosyncratic shock (the microeconomic component) as well as on aggregate spending (the macroeconomic component). This allows me to impose the general-equilibrium condition that aggregate consumption equal aggregate income. With this setup in place, the main goal is to describe the paths for aggregate consumption and interest rates that are consistent with each another. In the complete market or representative agent case this boils down to the standard intertemporal Euler equation, i.e. \( U'(C_t) = \beta_t R_t U'(C_{t+1}) \) where the discount factor \( \beta_t \) is often taken to be constant.

Crucially, this exercise goes beyond aggregating under partial equilibrium, that is, for fixed household income processes and arbitrary interest rates. Indeed, for a given interest rate path household income must be endogenously determined using the requirement that aggregate income equal aggregate consumption, so that the asset market is cleared. It is also important to note that my setup captures the aggregate demand block for a wide
class of models, with or without nominal rigidities, avoiding the need to fully specify the supply block. I will illustrate how this structure serves as a component for the New Keynesian and the Real Business Cycle models.

A necessary condition for incomplete markets to matter at the aggregate level is for it make a difference at the household level. The capacity of the market to self insure agents depends on the amount of liquidity—the value of available assets and the amount of borrowing permitted. I first explore the case of extreme illiquidity, with no borrowing and no outside assets. This natural benchmark captures the greatest deviation from complete markets; it is also relatively tractable, thanks to the fact that, in equilibrium, no intertemporal trade is possible. In this context I provide a simple, yet general and insightful, result, deriving an equilibrium relation involving the real interest rate, current and future aggregate consumption, i.e. \( g_t(R_t, C_t, C_{t+1}) = 0 \). According to this generalized Euler relation, current consumption may be more or less sensitive to the current interest rate relative to the intertemporal elasticity of substitution given by preferences; likewise, consumption may respond more or less than one-for-one with changes in future consumption—upsetting the standard consumption smoothing property. In short, the generalized Euler equation may depart from the standard Euler equation. Indeed, it offers a convenient platform to study such departures and I shall return to this below.

However, I first show that for a benchmark case, with constant relative risk aversion utility functions and household income that depends proportionally on aggregate income, the general Euler relation reduces to a standard representative-agent Euler equation, i.e. \( U'(C_t) = \beta_t R_t U'(C_{t+1}) \). This provides a useful ‘as if’ result: as far as the response of aggregate consumption to interest rate changes is concerned, the economy can be summarized by an Euler equation relation as if markets were complete or populated by a representative agent. The assumption that household income varies proportionally with aggregate income is equivalent to assuming that the distribution of relative income is unaffected by aggregate income, so that household income risk is acyclical.

I also consider cases with a positive supply of liquidity, when an outside asset is available or when households can borrow from one another. In general, this situation does not lend itself as easily to aggregation, since the allocation no longer coincides with financial autarky and may depend in complex ways on the path for interest rates. Despite these difficulties, I show that when utility is logarithmic and borrowing constraints are proportional to aggregate income a similar ‘as if’ result applies.

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1 However, despite the lack of borrowing and saving, consumption is still endogenously determined since aggregate and household income are endogenous—the general-equilibrium feedback discussed above.
The ‘as if’ result provides an important benchmark, as well as a useful launchpad to investigate other cases, which I will turn to further below. Before doing so it is worth interpreting the result with care, understanding what it does and does not say.

The ‘as if’ result does not say that incomplete markets are irrelevant for aggregates. Idiosyncratic uncertainty, lack of insurance and borrowing constraints all have an impact on the level of aggregate consumption: given interest rates, uncertainty and scarce liquidity depress consumption. What the ‘as if’ result does say is that the sensitivity or elasticity of aggregate demand to interest rates and future consumption is unaffected by the incompleteness of markets: aggregate consumption reacts to changes in the path of interest rates just as in a representative agent model.

The level effects show up in the subjective discount factors, $\beta_t$, of the ‘as if’ representative agent formulation. Indeed, in the background household consumption may involve rich dynamics, as in the deleveraging episodes in Eggertsson and Krugman (2012) and Guerrieri and Lorenzoni (2011). While as far as aggregates are concerned, the effects are captured by time varying discounting. This provides a foundation for working a representative-agent formulation augmented with aggregate demand shocks.\(^2\)

The ‘as if’ result also has some important practical implications. An immediate one is for “forward guidance”, defined as the commitment by a central bank to keep future interest rates low. These policies have been advocated as a way to stimulate current aggregate demand during a “liquidity trap”—when the zero lower bound on interest rates is binding. Whenever the ‘as if’ result is applicable, forward guidance works exactly as it does in the representative agent setting, just as powerfully. The same is true of other related policies that work by committing future policy in other ways, such as price level or nominal GDP targeting. A related implication is for the potency of “news shocks”, since these also work by affecting expectations of future consumption and affecting current demand through the consumption-smoothing channel. Under the ‘as if’ result, these shocks are just as effective as in representative agent models.

Although the ‘as if’ result does not imply that market incompleteness is irrelevant, it may seem surprising at first. How can the response of aggregate consumption to interest rates not be affected by market incompleteness? After all, some households may find themselves up against, or near, a borrowing constraint and are likely to be insensitive to interest rates. This seems especially relevant for future interest rate changes, since borrowing constraints may not bind today but will bind at some point in the future, frus-

\(^2\)Interestingly, the literature on “liquidity traps” has often taken the shortcut of assuming a representative agent with temporarily high discount factors, to push the economy towards zero interest rates. My results provide a microfoundation for this approach.
trating consumption smoothing and, thus, breaking the transmission chain running from future interest rates to present consumption.

My result does not rely on denying any of these considerations. Quite the contrary, it embraces them since it applies when borrowing constraints are binding or may become binding in the future. The problem lies with the notion that these factors necessarily mute the response of consumption to interest rates, since this argument is based on partial equilibrium logic only. General equilibrium effects turn out to be important, especially the feedback loop from consumption to income. When interest rates fall, households that are not liquidity constrained substitute intertemporally to increase spending. They would do so even if their income were to remain unchanged—a partial equilibrium response. However, in equilibrium, their income and that of others increases as a result of this increase in their spending. Liquidity constrained households, in contrast, cannot substitute intertemporally, but they are especially sensitive to the increase in income. As it turns out, both constrained and unconstrained households respond proportionally in equilibrium, but they do so for different reasons. This general equilibrium feedback between consumption and spending ensures that the ‘as if’ result obtains despite partial equilibrium forces to the contrary. It is worth noting that this is the very same feedback that amplifies the effects of fiscal policy with liquidity constrained consumers (Gali et al., 2007; Farhi and Werning, 2012). A greater multiplier arises when the high propensity to spend by constrained consumers is coupled with the feedback between their spending and their income—the “Keynesian cross”. Thus, not only is the mechanism underlying my results not counterintuitive as it may first appear, but it should be recognized as a staple in the macroeconomics toolbox.

I also explore departures from the ‘as if’ result. With zero liquidity the ‘as if’ result holds when household income varies proportionally with aggregate income, which amounts to assuming household income risk is acyclical. Instead, when uncertainty is countercyclical, i.e. rising in recessions, I show that aggregate demand becomes more sensitive to interest rates, current and future (the reverse is true when uncertainty is procyclical). Countercyclical risk is widely used in the asset pricing literature to explain the equity premium puzzle, at least since Mankiw (1986), and has received empirical support (Storesletten et al., 2004; Guvenen et al., 2014).

To see the effects from countercyclical risk clearly, I consider a simple setting with an

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3A large literature in asset pricing investigates the implications of incomplete markets for asset pricing, as compared to a representative agent model. A partial list of papers includes Heaton and Lucas (1996), Constantinides and Duffie (1996), Krusell and Smith (1997), Alvarez and Jermann (2001), Krueger and Lustig (2010), Krusell et al. (2011). Some of these papers approach the problem numerically or analytically by studying benchmarks, such as cases where the equilibrium involves autarky.
extensive margin for employment. Aggregate income continues to affects income along the intensive margin, but it now also affects the probability of full employment. I show that this makes aggregate demand more sensitive to current interest rates, and even more sensitive to future interest rates. Intuitively, when interest rates are lowered, aggregate income rises. This increases the probability of full employment, lowering income risk. In earlier periods, this diminishes the desire for precautionary savings, stimulating consumption further. This result relies on the fact that the probability of full employment varies with aggregate income. The greater the adjustment along this extensive margin, the greater the sensitivity of aggregate demand. Indeed, in the limit where all the adjustment is along this extensive margin, without an intensive margin, aggregate demand becomes infinitely elastic to interest rates.

With positive liquidity, I isolate an important role for the cyclicality of liquidity relative to output. Working with a three period economy with acyclical income risk, I show that the ‘as if’ result holds as long as liquidity is acyclical, when borrowing limits and asset values expand proportionally with aggregate income. Instead, aggregate demand becomes more sensitive to interest rates whenever liquidity is procyclical, when borrowing limits and asset values expand more than proportionally with aggregate income. Conversely, aggregate demand becomes less sensitive when liquidity is countercyclical.

To understand this result, consider the case where a lower interest rate increases aggregate income, but increases the value of assets by more. This extra liquidity allows household to buffer idiosyncratic income shocks more effectively, lowering consumption risk. Once again, in earlier periods, this diminishes the desire for precautionary savings, stimulating consumption further. Interestingly, this result does not rely on binding liquidity constraints. However, I also show that aggregate demand becomes even more sensitive if some households are constrained.

The cyclicality of liquidity is partly endogenous, since equilibrium asset prices are endogenous. With log utility the ratio of asset prices relative to aggregate income turns out to be constant, explaining why the ‘as if’ result holds in this case. In contrast, with more curvature (when the intertemporal elasticity of substitution is less than one) equilibrium asset prices fluctuate more than income, so that liquidity is procyclical. Borrowing constraints can also be modeled endogenously. One common specification assumes that borrowing is limited by the present value of pledgeable income. When interest rates fall, borrowing limits expand more than aggregate income when, once again, the intertemporal elasticity of substitution is less than one. Both results suggest that the case with procyclical liquidity is plausible in the model.

Taking stock, one contribution of this paper is to develop a mapping from assumptions
to results, with the hope that it may be used to explore the territory and provide a better understanding of the different possibilities, as well as the economic mechanisms at work. Table 1 provides a schematic representation of the mapping suggested by my results, highlighting the central role for the cyclicity of idiosyncratic income risk and liquidity. The ‘as if’ result holds when both income risk and liquidity are acyclical. Aggregate demand becomes more sensitive to interest rates, especially future interest rates, when income risk is countercyclical and liquidity is procyclical; the reverse is true if instead income risk is procyclical and liquidity is countercyclical.

Although immediate applicable in contexts with nominal rigidities, where monetary policy affects real interest rates, my methods and results transcend such settings. Indeed, my demand block framework sidesteps modeling the supply side and avoids making assumptions regarding nominal rigidities. To highlight this point, I close the paper studying a Real Business Cycle model featuring capital, labor and productivity shocks. This model is also of interest because the supply of liquidity, given by capital, is then endogenous.

I show that the ‘as if’ result holds for the Brock-Mirman specification, with log utility and full depreciation. The saving rate is then constant, which implies that the value of assets is proportional to output. In other words, liquidity is acyclical and the ‘as if’ result follows just as before. Admittedly, full depreciation is not plausible, so the value of this exact analytical aggregation result is mainly conceptual, rather than practical. However, it complements existing numerical results, such as Krusell and Smith (1998) which showed that approximate aggregation holds when households are able to smooth their consumption effectively. In contrast, my exact ‘as if’ result holds regardless of the size and persistence of idiosyncratic shocks, so consumption smoothing may be greatly impeded. This points to a new rationale for aggregation, distinct from those cases where households manage to smooth consumption relatively well.

**Related Literature.** This paper belongs to a vast literature in macroeconomics exploring the implications of relaxing the representative agent assumption and adopting an incomplete markets model. A seminal contribution was Krusell and Smith (1998) in the context of a Real Business Cycle model. Useful overviews can be found in Heathcote et al. (2009).
and Guvenen (2011). More recent work includes Guerrieri and Lorenzoni (2011), Kaplan and Violante (2011), Ravn and Sterk (2012), Eggertsson and Krugman (2012), McKay and Reis (2013), Sterk and Tenreyro (2013), Sheedy (2014), Auclert (2015) and McKay et al. (2015). These contributions focus on various different aspects (e.g. the effects of deleveraging episodes and the zero lower bound, the role of illiquid assets in high marginal propensities to consume, the redistributional effects from inflation or interest rate exposures, etc.) and have uncovered important implications of incomplete markets in different contexts.

A distinguishing feature of my paper is the approach of focusing on the aggregate demand block, avoiding specifying a full model with a particular supply side. This allows for an overarching treatment of the effects of incomplete markets on aggregate demand, under relatively general conditions, isolating the key roles played by the cyclicality of risk and liquidity. This is accomplished by deriving a general Euler relation expressed in terms of aggregates, providing conditions under which the ‘as if’ result holds and exploring departures from this result.

Although not my focus in this paper, my results have implications for forward guidance that may be contrasted with McKay et al. (2015), which suggests that incomplete markets diminish the power of such policies. My results clarify that it is not market incompleteness, per se, that determines the power of forward guidance, but its interaction with auxiliary assumptions like the cyclicality of risk and liquidity. Two important modeling assumptions in their benchmark New Keynesian model tend to make income risk procyclical and relative liquidity countercyclical—placing them in the bottom row of Table 1. First, monopolistic profits are distributed evenly across workers, so they are disproportionately important for low earners; since profits relative to output are countercyclical, this makes income risk countercyclical. Second, the only available asset is a short-term real bond whose absolute real value is kept constant by the government; thus, relative liquidity is assumed countercyclical.

\(^4\)In the benchmark model, as in many basic New Keynesian models, wages are flexible, prices are rigid, desired markups are constant, real marginal costs are constant and there are no fixed costs. This leads to countercyclical markups and, thus, profits relative to output (and potentially even profits) inherit this countercyclicality, as wages rise more than prices. Models with increasing marginal costs, or fixed costs, or other features such as wage rigidity and labor hoarding could make monopolistic profits relative to output procyclical. Another important component of firm profits, not necessarily related to markups, is the returns to capital and the cyclicality of its return is likely to depend on a number of other assumptions.

\(^5\)An alternative model in their appendix considers a case with zero liquidity and two idiosyncratic states. In the unemployment state household income is fixed; all earnings and profits accrue in the employment state. Since the probability of unemployment is assumed constant, household income risk increases with aggregate income. This assumption may be contrasted with my varying extensive margin model in 3.4 and Appendix A.
My results can also help interpret the findings in Ravn and Sterk (2012), which argues that incomplete markets and imperfections in labor markets lead to deeper recessions when interest rates do not adjust, e.g. at the zero lower bound. The main mechanism in their model is that household income risk is countercyclical, with the probability of unemployment rising in a recession (I develop a similar two state specification in Section 3.4). This places their model in the top row of Table 1.

Countercyclical income risk is also at the heart of the feedback loop between aggregate demand, idiosyncratic risk and precautionary saving in the work by Chamley (2013) and Beaudry et al. (2014).

2 A General Incomplete Market Setting

This section introduces the incomplete market model where households make consumption and savings decisions. The setup is fairly general and will be specialized in different directions later. The focus will be on households, with firms and the government largely relegated to the background. To complete the model, this “demand block” could be combined with a “supply block”, together with specifications for monetary and fiscal policy. For my purposes, it is best to avoid committing to particular ways of doing this and focus on the “demand block” only.

The main goal is to understand how market incompleteness affects macroeconomic aggregates. Of particular interest is the relationship between the paths for household spending and interest rates. Ideally, one aims to obtain something like the standard representative Euler equation for the incomplete-market model. This turns out to be possible in some interesting cases.

2.1 Economic Environment

The framework I develop is based on standard Bewley-Huggett-Aiyagari incomplete market models. The time horizon is infinite, with discrete periods $t = 0, 1, \ldots$ Each period, there is a single final good. The economy is populated by a unit measure of infinitely-lived households that are subject to idiosyncratic shocks to their income and spending needs. Insurance against these shocks is absent and credit may be limited.

To simplify, I abstract from aggregate uncertainty, focusing exclusively on idiosyncratic uncertainty. Aggregate shocks could be added at some expense, but they do not seem essential to understanding how aggregate demand is affected by uninsurable idiosyncratic uncertainty.
**Household heterogeneity.** There is a finite set of household types indexed by $i \in I$. Household type $i$ represents a fraction $\mu^i > 0$ of the population. Types may differ with respect to their preferences, including discounting. They may also have different labor earnings process and different degrees of access to credit, i.e. borrowing constraints.

**Preferences.** Household of type $i \in I$ has preferences over consumption given by the utility function

$$\sum_{t=0}^{\infty} \beta^i_t E_0[u^i_t(c(s^i_t), s_t)]$$

where $c_t$ denotes consumption of the single final good, $s_t \in S^i_t$ denotes an idiosyncratic state of nature that follows a stochastic process, discussed further below, and $s^i_t$ is the history of $s_t$. Shocks to utility are included for both generality and realism. They may capture important lifetime events, such as health shocks or family size changes, that affect the relative desirability of current spending.

Because of the focus on consumption and savings choices, I first directly postulate a labor income process directly, instead of deriving it from labor supply choices. This also keeps us closer to the incomplete markets literature, which studies consumption and savings while taking the income process as given. With this in mind, there is no need at this stage to describe preferences over leisure or labor.

**Budget Constraints.** Households of type $i$ face the budget constraints

$$c(s^i_t) + q_t \cdot a(s^i_t) + b(s^i_t) \leq y^i_t(s_t) + (q_t + d_t)a(s^{t-1}_t) + R_{t-1} \cdot b(s^{t-1}_t)$$

for all $t = 0, 1, \ldots$ and histories $s^i_t \in S^{t+1}_t$; here $a(s^i_t)$ and $b(s^i_t)$ denote savings in the outside asset and riskless one-period bonds, respectively; $q_t$ is the price of the outside asset and $d_t$ is its dividend; $R_t$ denotes the interest rate on riskless bonds, in real terms; $y^i_t$ is income from labor.

Labor income depends on the household state $s_t$ and aggregate income according to

$$y^i_t(s_t) = \gamma^i_t(s_t, Y_t),$$

for some function $\gamma^i_t$. The nature of the relationship between household and aggregate income encapsulated by $\gamma^i_t$, will turn out to be crucial.
Borrowing Constraints. Households of type $i \in I$ are also subject to borrowing constraints
\begin{equation}
    b(s^t) + q_t \cdot a(s^t) \geq -B^i(s_t, Y_t),
\end{equation}
which limits how negative their wealth is allowed to become. Here $B^i$ is a nonnegative borrowing limit, determined as a function of the current household state and aggregate income.\footnote{The constraint on borrowing is specified in terms of total wealth; alternatively, one can impose separate constraints for assets and bonds, $b(s^t) \geq -B^i(s_t, Y_t)$ and $a(s^t) \geq 0$, and the results would be similar.}

Idiosyncratic Uncertainty. Uncertainty is purely idiosyncratic: the realization of states are independent across agents and for each type $i \in I$ the probability of a certain set of histories $s^t$ equals the fraction of agents in the cross section experiencing this history. For each household type $i \in I$, the exogenous state $\{s_t\}$ follows a stochastic process. An important case is when $s_t$ follows a Markov process, although this assumption is not required for most of my analysis. For each household type $i \in I$, the stochastic process for household states $\{s_t\}$ is independent of the path for aggregate income $\{Y_t\}$. This assumption is essentially a normalization, since we have not placed restrictions on the functions $\gamma_t^i$ and $B_t^i$.

Initial Conditions. At $t = 0$ the economy inherits, for each household type $i \in I$, a joint distribution over initial states and initial asset and bonds $\Lambda^i_0(s_0, a_0, b_0)$. The stochastic process then induces a joint distribution of histories $s^t$ and $(a_0, b_0)$ denoted by $\Lambda^i_t(s^t, a_0, b_0)$.

Outside Asset. The outside asset is in fixed supply, normalized to unity, providing a dividend stream that is a function of current aggregate income,
\begin{equation}
    d_t = D_t(Y_t).
\end{equation}
To capture the case with zero outside assets simply set dividends to zero: $D_t(Y_t) = 0$ for all $t$.\footnote{When $D = 0$ there always exists an equilibrium where the asset price is zero. In some cases there may be other equilibria, akin to monetary equilibria where fiat money has value, I shall not consider these equilibria.} The assumption that the outside asset is in fixed supply is relaxed in Section 6, which extends the analysis to a model with capital and investment.

To ensure that $Y_t$ can be interpreted as aggregate income from labor and capital, we
require that the functions $\gamma_t$ and $D_t$ satisfy the identity

$$\sum_{i \in I} \mu^i \int \gamma^i_t(s_t, Y_t) d\Lambda^i_t + D_t(Y_t) = Y_t,$$

(6)

for all $Y_t$. Note that given $\{\gamma^i_t\}$ the dividend function $D_t(Y_t)$ can be backed out from this identity.

### 2.2 Equilibrium

I now introduce a natural equilibrium concept for this framework and provide a simple characterization, reducing the conditions to a few equations.

**Equilibrium Definition.** An equilibrium specifies interest rates and consumption decisions that are required to be optimal as well as consistent with aggregate income. Formally, given initial conditions $R_{-1}$ and $\Lambda^i_0$, an equilibrium is a path for aggregates

$$\{C_t, Y_t, A_t, B_t, R_t, q_t\},$$

and household choices, conditional on initial conditions,

$$\{c^i(s^t; a_0, b_0), a^i(s^t; a_0, b_0), b^i(s^t; a_0, b_0)\},$$

satisfying the following:

1. **household optimization:** taking as given the path for aggregate income and interest rates $\{Y_t, R_t\}$, household choices maximize utility (1) subject to (2), (3) (4) and (5);

2. **market clearing:** for all $t = 0, 1, \ldots$ the good, asset and bond markets clear,

   $$C_t = Y_t,$$
   $$A_t = 1,$$
   $$B_t = 0;$$
3. **aggregation**: the aggregate quantities are consistent with household quantities,

\[
C_t = \sum_{i \in I} \mu_i^i \int c^i(s^i; a_0, b_0) \, d\Lambda^i_t(s^i, a_0, b_0),
\]

\[
A_t = \sum_{i \in I} \mu_i^i \int a^i(s^i, a_0, b_0) \, d\Lambda^i_t(s^i, a_0, b_0),
\]

\[
B_t = \sum_{i \in I} \mu_i^i \int b^i(s^i, a_0, b_0) \, d\Lambda^i_t(s^i, a_0, b_0).
\]

Given a path for interest rates, one may seek a path for aggregate consumption that forms part of an equilibrium. It is important to stress that this goes beyond the pure aggregation of household consumption choices, for given income processes. Indeed, the equilibrium notion used here also incorporates the general equilibrium feedback from consumption to income.\(^8\)

**Implementability Conditions.** The equilibrium requirements can be reduced to a small set of conditions as follows. First, the two riskless assets must satisfy a no-arbitrage condition equating returns,\(^9\)

\[
\frac{q_{t+1} + d_{t+1}}{q_t} = R_t.
\]

This no-arbitrage condition is implied by imposing that the asset price equal the present value of its dividends,

\[
q_t = \sum_{s=0}^{\infty} \frac{1}{R_t R_{t+1} \cdots R_{t+s}} D_{t+s}(Y_{t+s}). \tag{7}
\]

Household optimality requires budget constraints to hold with equality. Defining total wealth \(\hat{a}^i(s^{i-1}; a_0, b_0) \equiv q_t \cdot a^i(s^i) + b^i(s^i)\), this requires for \(t \geq 1\)

\[
c^i(s^i; a_0, b_0) + \hat{a}^i(s^i; a_0, b_0) = \gamma^i_t(s_t, Y_t) + R_{t-1} \cdot \hat{a}^i(s^{i-1}; a_0, b_0), \tag{8a}
\]

Similarly, the budget constraint at \(t = 0\) requires

\[
c^i(s_0; a_0, b_0) + \hat{a}^i(s_0; a_0, b_0) = \gamma^i_0(s_0, Y_0) + (q_0 + D_0(Y_0))a^i_0 + R_{-1} \cdot b^i_0, \tag{8b}
\]

\(^8\)This perspective can be contrasted some well-known aggregation exercises. For example, Huggett (1993) and Aiyagari (1994) aggregate consumption and savings for given interest rates, but taking the income process as given. They then employ this aggregate relationship graphically to determine an equilibrium that clears the market.

\(^9\)Strictly speaking, when borrowing is completely ruled out, so that \(B^i = 0\), an equilibrium requires only that \(R^a_t \geq R_t\). However, in such cases, the equilibrium with \(R^a_t > R_t\) is not robust to the introduction of vanishingly small amounts of borrowing.
Wealth must satisfy the borrowing constraints
\[
\hat{a}^i(s^t; a_0, b_0) \geq -B_i^i(s_t, Y_t).
\] (9)

Household optimization reduces to the Euler condition,
\[
u_{c,t}^i(c^i(s^t; a_0, b_0), s_t) \geq \beta^i R_t \mathbb{E}_t[u_{c,t+1}^i(c^i(s^{t+1}; a_0, b_0), s_{t+1})],
\] (10)

with the complementary slackness requirement that this condition hold with equality in period \( t \) whenever the borrowing constraint (9) in period \( t \) holds with strict inequality. Finally, we impose the market clearing condition \( C_t = Y_t \) and the aggregation condition for \( C_t \). All the other market clearing conditions are then implied.

To summarize, an equilibrium can be reduced to aggregates
\[
\{C_t, R_t\}
\]
and household consumption and wealth \( \{c^i(s^t; a_0, b_0), \hat{a}^i(s^t; a_0, b_0)\} \) satisfying aggregation \( C_t = \sum_{i \in I} \mu_i \int c^i(s^t; a_0, b_0) d\Lambda^i(s^t, a_0, b_0) \), the budget constraints (8), borrowing constraints (9) and Euler condition (10) with complementary slackness. In these conditions we obtain \( q_0 \) by (7) and \( Y_t = C_t \).

When these conditions hold, one can find the remaining equilibrium objects as follows. The asset price in all periods is given by (7). Asset and bond holdings, however, are indeterminate: any portfolio split satisfying \( \hat{a}^i(s^{t-1}; a_0, b_0) = q_t \cdot a^i(s^t) + b^i(s^t) \) constitutes an equilibrium.

### 2.3 Pitfalls of Aggregation under Partial Equilibrium

Before stating my aggregation results, it is worth briefly reviewing common pitfalls of aggregation under partial equilibrium. This discussion will help underscore how my results rely on general equilibrium considerations.

The Euler condition is nonlinear, but absent uncertainty, absent borrowing constraints and assuming homogeneous discounting and constant relative risk aversion utility functions \( u(c) = c^{1-\sigma} / (1 - \sigma) \) it can be transformed into a linear relationship and aggregated. To see this, note that the household Euler equation implies \( c^i(s^t) = (\beta R_t)^{-\frac{1}{\sigma}} c^i(s^{t+1}) \) and thus, aggregating,
\[
C(s^t) = (\beta R_t)^{-\frac{1}{\sigma}} C(s^{t+1})
\] (11)

Unfortunately, this aggregation is delicate and easily upset by idiosyncratic uncertainty
and borrowing constraints, among other things.

In the presence of idiosyncratic uncertainty and possibly binding borrowing constraints, we start from $c^i(s^t) \leq (\beta R_t)^{-\frac{1}{\beta}} \cdot (\mathbb{E}_t[c^i(s^{t+1}) - \sigma])^{-\frac{1}{\sigma}}$, so that aggregating and using Jensen’s inequality gives

$$C(s^t) < (\beta R_t)^{-\frac{1}{\beta}} C(s^{t+1}).$$

as long as some households are constrained or face uncertainty. The magnitude of the departure from (11) varies with the amount of uncertainty and the extent to which borrowing constraints bind. This makes it difficult to expect an equation such as (11) to hold, even for a modified discount factor. The problem is that the departures from (11) depend on precautionary effects and binding borrowing constraints, which are not stable and endogenous to the distribution of wealth and the path of future interest rates.

These difficulties in aggregation are overcome below, but by taking a different route: rather than simply aggregating Euler equations, I use additional information by imposing another equilibrium condition, using it to pinpoint an exact aggregate equilibrium relation. In particular, an equilibrium requires the equality of aggregate income and consumption. Imposing this condition turns out to be crucial. Indeed, I do not uncover any novel aggregation result for consumption when taking the income process as given. I find tractable aggregate relations only when imposing the equality of aggregate consumption and income. The aggregate relations I obtain in this way do not always take the form in (11) and even in the cases that it does admit such a representation the discount factor $\beta$ is different and possibly time varying.

To sum up, my results do not rely on cleverly overcoming standard microeconomic aggregation problems. Instead, I solve for aggregate consumption by exploiting its general equilibrium relation with aggregate income.\(^\text{10}\)

3 Zero Liquidity

If the incompleteness of markets is to matter at the aggregate level, it must first make a difference at the household level, generating a significant departure from complete markets. Risk sharing is impaired and the perfect insurance outcome characteristic of complete markets is generally impossible. However, as has been widely noted, due to the households’ incentive to smooth consumption and their precautionary motives to accumulate

\(^{10}\)By the same token, my results are designed for comparative static exercises, e.g. changing the interest rate path. They may not resolve the empirical problem faced by microeconomic tests (e.g. Attanasio and Weber, 1993).
assets, the outcome with incomplete markets may achieve significant improvements over autarky and in some cases approximate the complete market outcome.

The extent to which this is true depends on the capacity of the market to self insure agents. This in turn depends on the amount of liquidity, the value of available assets and the amount of borrowing permitted. When liquidity is plentiful, the outcome is closer to complete markets; when liquidity is scarce, the allocation may be greatly affected by the imperfection in financial markets.

The simplest results obtain under the assumption of absolute illiquidity, with liquidity completely absent: there is no borrowing and there is no outside asset in positive net supply. This situation is best thought of as a limiting case of extreme scarcity in liquidity, with very limited borrowing and small asset values. While extreme, economies with zero or vanishing liquidity lie on the opposite side of the spectrum from complete markets. A priori, they are the poster child for studying the effects that financial market imperfections have on aggregate demand.

### 3.1 A Generalized Euler Relation

To study situations with zero liquidity assume that

\[
D_t(Y_t) = 0, \\
B^i_t(s_t, Y_t) = 0,
\]

and the initial conditions \( \hat{a}_{0}^i = b_{0}^i = 0 \) for all agents.

Since savers can only save by lending to borrowers, and borrowing is ruled out, it follows that in equilibrium no intertemporal trade is possible and the allocation coincides with autarky,

\[
c^i(s^t) = y^i(s_t) = \gamma^i_t(s_t, C_t),
\]

where I have substituted the equilibrium condition \( C_t = Y_t \). The equilibrium interest rate path that sustains this equilibrium must, in each period, ensure that no household has an incentive to save, which requires

\[
R_t \leq R^*_t = \min_{i \in I, s^t} \left( \beta^i_t \mathbb{E} \left[ \frac{u_{c,t+1}(\gamma^i_{t+1}(s_{t+1}, C_{t+1}), s_{t+1})}{u_c(s^t, C_t, s^t)} \mid s^t \right] \right)^{-1}. \tag{12}
\]

Equilibrium interest rates are not uniquely determined: interest rates below \( R^*_t \) imply that agents find the corner solution, with \( b^i_t = 0 \), optimal. However, interest rates strictly
below $R^*_t$ are not robust to the introduction of small amounts of liquidity. Positive but vanishing levels of liquidity require $R_t = R^*_t$, since the Euler equation must hold with equality for some agents when $D_t(Y) > 0$ and $B^i_t(s, Y) > 0$. Given to this refinement, which I henceforth adopt, the equilibrium is unique. The next proposition summarizes this characterization.

**Proposition 1.** A path for aggregate consumption and interest rates $\{C_t, R_t\}$ is part of an equilibrium with vanishing liquidity if and only if

$$g_t(R_t, C_t, C_{t+1}) = 0,$$

where the function $g_t$ is given by

$$g_t(R, C, C') \equiv \log R + \log \left( \max_{i \in I, s} \beta^i_t \mathbb{E} \left[ \frac{u^i_{c,t+1}(\gamma^i_{t+1}(s_{t+1}, C'), s_{t+1})}{u^i_{c,t}(\gamma^i_t(s_t, C), s_t)} \mid s_t \right] \right).$$

This proposition provides a simple and condensed way of exploring the aggregate consequences incomplete market. It all boils down to a single relation involving the same variables as in the standard representative-agent Euler equation; namely, the current interest rate, $R_t$, present and future consumption, $C_t$ and $C_{t+1}$. The familiar complete-market representative-agent Euler equation, with time-invariant discounting $\beta$ and utility function $U(c)$, is a special case where

$$g(R, C, C') = \log R + \log \beta + \log U'(C') - \log U'(C).$$

While the more general Euler condition (13) does not necessarily take this particular functional form, it is nevertheless a simple condition involving only aggregates, and in that sense, much like the standard Euler condition. Due to this property, despite incomplete markets, the aggregate equilibrium conditions are tractable. In particular, a relation like (13) can be handled, alongside other equilibrium conditions, as easily as the standard Euler equation in positive or normative equilibrium analyses.\(^\text{11}\)

In general the function $g_t$ varies over time. This is not surprising, since no stationarity assumptions have been placed on any primitives, such as utility functions, discounting, the stochastic process, the income function $\gamma^i_t$. As we discuss further below in the context of our next result, this may help capture interesting situations, such as temporary

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\(^\text{11}\)In particular, computing an equilibrium does not require carrying a large endogenous state or confronting the curse of dimensionality, as in Krusell and Smith (1998). Indeed, the simplifying assumption of zero liquidity ensures that wealth is zero for all agents, at all times, so that there is no wealth distribution to keep track of.
episodes with heightened idiosyncratic uncertainty. However, even when primitives are stationary, the function $g_t$ may vary simply because the cross-section of states $s_t$ is not at an invariant steady-state distribution. This time dependence, however, vanishes in the long run if the Markov process is ergodic, since the cross section of states then converges to its invariant distribution in the long run.

3.2 ‘As If’ Representative Agent Result

I now apply the general characterization obtained above to an important benchmark specification. In the benchmark, utility functions are power function and taste shocks are multiplicative,

$$u^i_t(c,s) = \theta^i_t(s) \cdot U(c) \quad \text{with} \quad U^i(c) = \frac{c^{1-\sigma}}{1-\sigma},$$

(14)

for $\sigma > 0$; household income is proportional to aggregate income,

$$\gamma^i_t(s,Y) = \tilde{\gamma}^i_t(s)Y,$$

(15)

for some function $\tilde{\gamma}^i_t$.

The next result shows that aggregate consumption and interest rates are related by a standard Euler equation, just as in complete-market or representative-agent economies.

Proposition 2. Suppose utilities satisfy (14) and household income satisfy (15). Then a sequence $\{C_t, R_t\}$ is part of an equilibrium with vanishing liquidity if and only if

$$U'(C_t) = \beta_t R_t U'(C_{t+1}),$$

with the discount factors given by

$$\beta_t = \max_{i \in I, s^t} \beta^i_t \cdot \mathbb{E} \left[ \frac{\theta^i_{t+1}(s_{t+1})}{\theta^i_t(s_t)} U' \left( \frac{\tilde{\gamma}^i_{t+1}(s_{t+1})}{\tilde{\gamma}^i_t(s_t)} \right) \mid s^t \right].$$

(16)

According to this proposition, aggregate demand is determined as if the economy were populated by a single representative agent with discount factor $\beta_t$. By implication, changes in the path for interest rates have the same effect on the aggregate consumption path as they do in the representative-agent benchmark. In this sense, the response of consumption to interest rates, is not affected by incomplete markets.\textsuperscript{12}

\textsuperscript{12}If household are heterogeneous with $U^i(c) = C^{1-\sigma^i} / (1 - \sigma^i)$ then this result extends, and one obtains

$$U_t(C_t) = \beta_t U_{t+1}(C_{t+1})$$
Are Incomplete Markets Irrelevant? No, not at all. In fact, Proposition 2 not only implies that market incompleteness is not irrelevant, it identifies very clearly the influence. Due to market incompleteness the discount factor $\beta_t$ is a function of idiosyncratic uncertainty, as shown in (16). For instance, due to the convexity of the marginal utility function $U'(c)$, in periods with greater uncertainty or downward tail risk (for the growth rate of household income) we can expect the discount factor to be higher; given interest rate and future consumption, $R_t$ and $C_{t+1}$, this implies lower current consumption, $C_t$. In contrast, with complete markets the aggregate Euler equation holds with a discount factor that does not depend on idiosyncratic uncertainty (just as in (11)) implying that idiosyncratic uncertainty has no effect on current consumption.

This discussion underscores that incomplete markets matters, affecting the level of demand, even if according to Proposition it does not affect the responsiveness of demand to current and future interest rates.

3.3 Departures from ‘As If’ Result: Cyclical Income Risk

It is useful to recast Proposition 2 as providing conditions for

$$g_t(R_t, C_t, C_{t+1}) = \log R_t + \log \beta_t + \sigma \log C_t - \sigma \log C_{t+1},$$

so that $g_t$ is exactly log linear, with equal coefficients in absolute value on $\log C_t$ and $\log C_{t+1}$ given by $\sigma$, the reciprocal of the intertemporal elasticity of substitution. This implies the exact relationship for the log changes,

$$d \log R_t + \sigma d \log C_t - \sigma d \log C_{t+1} = 0.$$

Below I investigate departures from this representation by characterizing the analog first-order expansion.

I now depart from this baseline and characterize $g_t$ by considering the coefficients in the expansion

$$d \log R_t + \alpha_{C,t} d \log C_t - \alpha_{C',t} d \log C_{t+1} = 0.$$

I will not study this as an approximation, but rather as an exact relation characterizing the derivatives of $g_t$; thus, the coefficients $\alpha_{C',t}$ depend on the position where they are evaluated.

where $\beta_t = \max_{i \in I,i} \beta_i \cdot \mathbb{E}[U'(\frac{\gamma_i(s_{t+1})\theta_i(s_{t+1})}{\gamma_i(s_t)\theta_i(s_t)}) | s_t]$ and the utility function $U_t$ in period $t$ equals the utility function of the household $i \in I$ that attains the maximum in the definition of $\beta_i$. 

19
Define the elasticity of $\gamma^i_t$ by

$$\epsilon^i_t(s, Y) \equiv \frac{\partial}{\partial Y} \gamma^i_t(s, Y).$$

This elasticity measures the responsiveness of household income to aggregate income. Then next result shows that the coefficients depend can be expressed in terms of this elasticity.

**Proposition 3.** Suppose utilities satisfy (14) without taste shocks, i.e. $\theta^i_t(s) = 1$. Then

$$\alpha_{C,t} = Cg_{C,t}(R, C, C') = \sigma \epsilon^i_t,$$

$$\alpha_{C',t} = -C'g_{C',t}(R, C, C') = \sigma \frac{\mathbb{E}_t[u^i_{C,t+1} \epsilon^i_{t+1}]}{\mathbb{E}_t[u^i_{C,t+1}]}.$$

for the household type $i \in I$ and history $s^i_t$ that attains the maximum in (16).

Note that

$$\frac{d \log C_t}{d \log C_{t+1}} \bigg|_{d \log R_t = 0} = \frac{\alpha_{C',t}}{\alpha_{C,t}} = \mathbb{E}_t \left[ \frac{u^i_{C,t+1}}{\mathbb{E}_t[u^i_{C,t+1}]} \cdot \frac{\epsilon^i_{t+1}}{\epsilon^i_t} \right]$$

$$= \mathbb{E}_t \left[ \frac{\epsilon^i_{t+1}}{\epsilon^i_t} \right] + \text{Cov}_t \left[ \frac{\lambda^i_{t+1}}{\mathbb{E}[\lambda^i_{t+1} | s_t]} \cdot \frac{\epsilon^i_{t+1}}{\epsilon^i_t} \right].$$

Under the conditions for Proposition 2 this ratio equals 1, so that current consumption, $C_t$, varies proportionally with future consumption, $C_{t+1}$. One can see this by observing that these conditions ensure that $\epsilon^i_t = 1$, so that the first term is 1 while the covariance term is zero.

**Implications for Forward Guidance.** Away from this neutrality case, what are the plausible possibilities? Consider a simple, but telling, example. Assume $\gamma^i_t$ is independent of the period $t$ and $s_t$ is a Markov chain, with finitely many values for $s_t$, that is mean reversion. Suppose further that $\gamma^i_t$ is an increasing function of $s_t$, so that a higher state $s_t$ is associated with a higher income level. Suppose, as is plausible, that the agent most willing to save, the agent pinning down the interest rate, the household type $i \in I$ and history $s^i_t$ that attains the maximum in (16), has the highest value of $s_t$. This implies that $s_{t+1}$ is expected to be lower, $s_{t+1} < s_t$, due to mean reversion.

Now suppose $\epsilon^i_{t+1}$ is a decreasing function. Then the first term must be be greater than 1, due to the mean reversion. The second, covariance term, is positive. Together,
this implies that the ratio is strictly greater than 1. The converse is also true. Thus, in this example, when income at the bottom is more sensitive to aggregate income, we find that current consumption is more sensitive to future consumption than one-for-one.

This implies that changes in interest rates into the future have stronger effects on current consumption than changes in current interest rates. Thus, in this sense, forward guidance is more powerful.

**Income Growth Rate Perspective.** The characterization above relied on the elasticity of income $\gamma^i_t$. It adopted a perspective in terms of levels of income. Another useful perspective is that of growth rates in income. With CRRA utilities and absent taste shocks, it is the growth rate of income that enters household Euler equations. To pursue this perspective first define the growth rate

$$\Gamma^i_t(s, s', C, C') = \frac{\gamma^i_{t+1}(s', C')}{\gamma^i_t(s, C)}.$$

Then we obtain

$$\alpha^C_{C,t} = \sigma \frac{\mathbb{E} \left[ U'(\Gamma^i_t) \frac{\Gamma^i_t C}{\Gamma^i_t} s'_t \right]}{\mathbb{E} \left[ U'(\Gamma^i_t) \frac{\Gamma^i_t C}{\Gamma^i_t} s'_t \right]}$$

$$\alpha^C_{C',t} = -\sigma \frac{\mathbb{E} \left[ U'(\Gamma^i_t) \frac{\Gamma^i_t C'}{\Gamma^i_t} s'_t \right]}{\mathbb{E} \left[ U'(\Gamma^i_t) \frac{\Gamma^i_t C}{\Gamma^i_t} s'_t \right]}.$$

for household $i$, $s'_t$ that attains the maximum in (16).

By implication

$$\alpha^C_{C',t} - \alpha^C_{C,t} = -\sigma \mathbb{E} \left[ \frac{U'(\Gamma^i_t)}{\mathbb{E} U'(\Gamma^i_t) s'_t} \left( \frac{\Gamma^i_t C'}{\Gamma^i_t} C' + \frac{\Gamma^i_t C}{\Gamma^i_t} C \right) | s'_t \right].$$

According to this expression says the sum of the two elasticities of $\Gamma$ is crucial. To see why, suppose one increases consumption in the current and next period proportionally. How does this affect the growth rate in consumption for the households that prices the bond? The answer is precisely $\frac{\Gamma^i_t C'}{\Gamma^i_t} C' + \frac{\Gamma^i_t C}{\Gamma^i_t} C$. If higher aggregate income in both periods lowers the mean reversion, uncertainty and downside risk in consumption, as seems natural, then we should expect

$$\frac{d \log C_t}{d \log C_{t+1}} \bigg|_{d \log R_t=0} = \frac{\alpha^C_{C',t}}{\alpha^C_{C,t}} > 1,$$

21
so that, in proportional terms, current consumption reacts more than one-to-one to increases in future consumption.

### 3.4 Countercyclical Employment Risk

To illustrate the previous results, consider the following simple case where the income process is motivated by employment risk, with both intensive and extensive margins that vary with output. Importantly, the probability of full employment varies with the level of aggregate demand.

There is a single household type and all workers are identical ex ante. There are no taste shocks and utility is \( U(c) = c^{1-\sigma} / (1 - \sigma) \). Each period households may be fully employed or underemployed. When fully employed, they earn \( \bar{y} Y^\psi \); otherwise, they earn \( y Y^\psi < \bar{y} Y^\psi \). Underemployed households earn a positive amount, to avoid zero consumption. This can be motivated by supposing each household has many individual members and at least one of these is always able to work, or by extending the model to include an unemployment insurance benefit.\(^\text{13}\) The parameter \( \psi \) controls how much of the adjustment in income takes place along the intensive margin; if \( \psi = 1 \) then all the adjustment is along the intensive margin; if \( \psi = 0 \) then all adjustment is along the extensive margin.

Define the fraction of underemployed \( \lambda(Y) \) households so that we satisfy the income identity

\[
Y = (1 - \lambda(Y)) \bar{y} Y^\psi + \lambda(Y) y Y^\psi.
\]

We limit attention to values of \( Y \) that imply \( \lambda \in (0, 1) \). If \( \psi = 1 \) then \( \lambda \) is constant; as long as \( \psi < 1 \) then \( \lambda(Y) \) is a strictly decreasing function of \( Y \). We assume that \( \lambda(Y) \) also represents the probability each household faces of being unemployed.\(^\text{14}\)

The bond is priced by those currently employed, i.e. the optimum in (12) is attained by any household that is employed. Their Euler equation can be written as

\[
U'(\bar{y} Y^\psi_t) = \beta R \left( (1 - \lambda(Y_{t+1})) U'(\bar{y} Y^\psi_{t+1}) + \lambda(Y_{t+1}) U'(y Y^\psi_{t+1}) \right).
\]

This reflects the fact that their current consumption and income is \( \bar{y} Y^\psi_t \) while their consumption next period is either \( \bar{y} Y^\psi_{t+1} \), with probability \( 1 - \lambda(Y_{t+1}) \), or \( y Y^\psi_{t+1} \), with prob-

\(^{13}\)Indeed, in this zero liquidity environment introducing an unemployment insurance system requires balancing the budget: taxing employed workers and rebating the proceeds to unemployed workers. One can then consider a tax system where income of each unemployed is guaranteed to be a proportion of the income of each employed household.

\(^{14}\)To fit this into our notation, assume \( s \in [0, 1] \) is uniformly distributed and i.i.d. over time and across agents and that \( \gamma_i(Y_t) \) is a step function with an upward discontinuity at \( s = \lambda(Y) \).
ability $\lambda(Y_{t+1})$. Using the fact that $U'(c) = c^{-\sigma}$ this equilibrium condition can be represented as follows.

**Proposition 4.** In the intensive-extensive margin example economy with varying underemployment we have

$$\dot{U}'(C_t) = \hat{\beta}(C_{t+1})R\dot{U}'(C_{t+1}),$$

where $\dot{U}(c) = c^{1-\sigma\psi}/(1 - \sigma\psi)$ and the discount rate function $\hat{\beta}$ is decreasing and given by

$$\hat{\beta}(C) \equiv \beta \left(1 - \lambda(C) + \lambda(C)U'(y/\bar{y})\right).$$

Thus, $g_t(R_t, C_t, C_{t+1})$ is such that $\alpha_{C,t} = \sigma\alpha$ and $\alpha_{C',t} > \alpha_{C,t}$.

This proposition shows that the general Euler relation has two important differences from the standard Euler equation, $U'(C_t) = \beta RU'(C_{t+1})$. First, the utility function $\dot{U}$ has less curvature than $U$, with elasticity of substitution $\frac{1}{\sigma\psi}$ instead of $\frac{1}{\sigma}$. By implication, consumption is more elastic to the interest rate under incomplete markets, compared to complete markets or a representative agent. This elasticity becomes larger when $\psi$ is smaller, so that most of the adjustment takes place along the extensive margin, rather than the intensive margin; indeed, the elasticity becomes infinite as $\psi \to 0$. Second, the discount factor $\hat{\beta}$ is lower and strictly decreasing, rather than constant. This implies that current aggregate consumption $C_t$ reacts more than proportionally to changes in future aggregate consumption $C_{t+1}$, for a given interest rate $R_t$. By implication, future interest rates have greater effect than current interest rates on current consumption.

This latter result is driven by the fact that when $\psi < 1$ lower spending leads to higher employment risk. When consumption is expected to be low next period, households face high income risk. As a result, they have high precautionary savings motives. This depresses current consumption further. The assumption that labor income risk is countercyclical in this way is standard in the asset pricing literature seeking to explain high historical equity premia (Constantinides and Duffie, 1996; Alvarez and Jermann, 2001). The assumption of countercyclical idiosyncratic risk has also received empirical support (Storesletten et al., 2004; Guvenen et al., 2014).\(^{15}\)

A similar feedback loop between the aggregate spending and idiosyncratic risk is at the heart of the amplification mechanism in Ravn and Sterk (2012) (see also Chamley, 2012).

\(^{15}\)Earlier literature specified and found support for income processes with a countercyclical variance for the shocks e.g. Storesletten et al. (2004). Recent work with administrative data has instead found support for a different specification. In particular, Guvenen et al. (2014) find that it is the left-skewness of shocks that is strongly cyclical. For our purposes downside risk is likely to be what matters most and their evidence supports strong countercyclicality of this risk.
They build a full macroeconomic model with search unemployment, mismatch shocks, incomplete markets, nominal rigidities, taking into account the zero lower bound that is calibrated to the US economy and show that this mechanism is capable of inducing deep recessions. Notably, they also exploit the tractability afforded by assuming zero liquidity. However, their quantitative approach still requires solving their full model numerically and they do not express the aggregate Euler relation featuring only aggregate consumption and interest rates, as I have done here to show the extra sensitivity that results from these assumptions.

The present formulation assumed that the intensive margin effect is symmetric in the two employment states, so that household income had elasticity $\psi$ with respect to aggregate income. Appendix A treats a case of extreme asymmetry: the high state is unchanged, but income is fixed in the low state. The appendix shows that the same results are then guaranteed to hold as long as $\lambda$ or $\psi$ are not too large, since this guarantees that uncertainty rises when aggregate income falls.

4 Positive Liquidity

Incomplete markets are likely to have more bite when liquidity is scarce, since this makes it harder for households to smooth transitory income fluctuations or cushion permanent labor income shocks using their savings. Thus, the extreme case without liquidity studied in the previous section may isolate the strongest case for incomplete markets. Nevertheless, it is interesting to investigate situations with positive liquidity.

4.1 ‘As If’ Representative Agent Result: Acyclical Liquidity

Obtaining sharp results is more challenging with positive liquidity because the allocation no longer coincides with autarky. Liquidity allows agents to smooth their consumption and the resulting equilibrium allocation is nontrivial, as in the general equilibrium studies of income fluctuations problems (e.g. Huggett, 1993; Aiyagari, 1994). Despite these challenges, I now show that with logarithmic utility aggregate implications can be worked out. Indeed, I obtain a standard representative agent Euler equation condition.

Utility is given (14) with $\sigma = 1$ so that the utility function is logarithmic

$$U(c) = \log(c).$$

As before household labor income satisfies (15), so that it is proportional to aggregate
income. It then follows from identity (6) that

\[ D_t(Y_t) = d_t \cdot Y_t, \]

for some \( \{d_t\} \), so that dividends are also proportional to total income. Finally, we also assume that borrowing constraints are proportional to aggregate income

\[ B_i^t(s, Y) = \tilde{B}_i^t(s) Y, \quad (17) \]

for some function \( \tilde{B}_i^t(s) \).

When \( d_t = 0 \) and \( \tilde{B}_i^t(s) = 0 \) we are back to the case zero liquidity studied in the previous section. We say there is positive liquidity if the asset’s dividend is positive or if borrowing is allowed, if \( d_t > 0 \) or \( \tilde{B}_i^t(s) > 0 \) for some \( t \) and \( s \).\(^{16}\)

**Revaluation Effects.** As it turns out, when studying comparative statics of equilibria with given initial conditions, the initial portfolio households hold matters due to revaluation effects. Recall, at \( t = 0 \) the budget constraint features initial wealth \( q_0 a_0 + R - b_0 \); this is the only period where the asset price enters the budget constraint. The interest rate \( R \) is predetermined and fixed, but \( q_0 \) is endogenous. Different interest rate paths \( \{R_t\} \) imply different \( q_0 \) affecting the value of initial wealth. The redistribution channel is the focus of Auclert (2015), who gives a detailed analysis of the different possibilities. In the present paper, these revaluation effects are present, but will play out in the background. I will consider two cases depending on the initial asset and bond holdings. In the first case, the revaluation effect is proportional to output, leading to the standard Euler representation.

### 4.1.1 Zero Initial Bond Holdings

Recall that, along an equilibrium, bonds and assets are perfect substitutes and households are indifferent to borrowing and saving in one or the other. As a result, the equilibrium is indeterminate and for any equilibrium with \( b_i^t \neq 0 \) there is another equilibrium with \( b_i^t = 0 \) and identical interest rates and allocations. It is convenient to first consider the case where initial bond holdings are zero for all households,

\[ b_i^0 = 0. \]

\(^{16}\)Strictly speaking, when \( d_t = 0 \) and \( \tilde{B}_i^t(s) > 0 \) for some \( s \), then in some cases the equilibrium coincides with autarky. This is the case if borrowing is only allowed for the agent setting the interest rate.
Formally, the initial distribution $\Lambda^i_0$ has full mass over $b^i_0 = 0$. Note that this initial condition does not rule out or even constrain borrowing. It restricts initial bonds holdings to be zero, but as long as $\bar{B}^i_t(s) > 0$ future borrowing and saving in bonds, $b^i_t \neq 0$ for $t > 0$, is permitted. Indeed, even initial indebtedness is possible if it takes the form of negative positions in the asset, so that $a^i_t < 0$.

The next result shows that under these conditions we recover the standard Euler equation.

**Proposition 5.** Suppose utilities satisfy (14) with $\sigma = 1$, household income satisfies (15) and borrowing constraints satisfy (17). In addition, suppose initial bond holdings are zero $b^i_0 = 0$ for all households. Then $\{C_t, R_t\}$ is part of an equilibrium if and only if

$$U'(C_t) = \beta_t R_t U'(C_{t+1})$$

(18)

for some sequence of discount factors $\{\beta_t\}$, independent of both $\{R_t\}$ and $\{C_t\}$.

Proposition 2 applied with zero liquidity where the outcome was financial autarky. In contrast, Proposition 5 applies to situations with positive liquidity with nontrivial allocations that depart from financial autarky, with households attempting to smooth their consumption by saving and borrowing. Indeed, the equilibrium may be nonstationary and involve rich dynamics, as in the deleveraging episodes modeled by Eggertsson and Krugman (2012) and Guerrieri and Lorenzoni (2011) where borrowing constraints suddenly tighten and upset the original steady state, inducing a slow transition.

How is it that all these nontrivial household decisions under uncertainty can be subsumed in as simple an aggregate relation as (18)? Indeed, the underlying household allocation cannot generally be solved in closed form—it is well known that incomplete market models demand a numerical solution approach. The key innovation behind Proposition 5 is to change the question one seeks to answer, so as to avoid having to solve all equilibria. Instead, I turn to the more manageable problem of characterizing how a given reference equilibrium adjusts to changes in interest rates. Although the household allocations in the reference equilibrium are nontrivial, across equilibria with different interest rates, household consumption scales up and down proportionally with aggregate consumption. Obtaining the sequence of discount factors $\{\beta_t\}$ requires computing the reference equilibrium, one with constant output. In general this may be a demanding task—as demanding as any of the existing equilibrium analysis of an incomplete markets economy e.g. Guerrieri and Lorenzoni (2011).

To see this more clearly, it is useful to spell out the argument behind the result in some detail. The discount factors $\beta_t$ and the household allocation behind this result are
obtained as follows. Consider a normalized economy with constant income $\tilde{Y}_t = 1$ and its associated equilibrium, including the interest rate $\{\tilde{R}_t\}$, household consumption and wealth $\{\tilde{c}(s^t; a_0), \tilde{a}(s^t; a_0)\}$ and asset prices $\tilde{q}_t = \sum_{s=0}^{\infty} (\tilde{R}_t \tilde{R}_{t+1} \cdots \tilde{R}_{t+s})^{-1} d_{t+1+s}$. Define $\beta_t \equiv \frac{1}{\tilde{R}_t}$, then by construction, the aggregate Euler equation (18) holds for $C_t = Y_t = 1$.

Now, for any other sequence $\{C_t, R_t\}$ satisfying the aggregate Euler equation (18), construct the rest of the equilibrium objects as follows. Renormalize household consumption and wealth proportionally

$$c(s^t; a_0) = \tilde{c}(s^t; a_0)C_t \quad \text{and} \quad \tilde{a}(s^t; a_0) = \tilde{a}(s^t; a_0)C_t$$

and adjust the interest rate and asset prices by

$$R_t = \tilde{R}_t \frac{C_{t+1}}{C_t} \quad \text{and} \quad q_t = \tilde{q}_tC_t.$$ 

With this guess, one can verify all equilibrium conditions. The crucial observation is that the Euler equation, the budget constraints and borrowing constraints are all linear homogeneous in $Y_t = C_t$. Notably, the $t = 0$ budget constraint is

$$c^i(s_0; a_0) + \tilde{a}^i(s_0; a_0) = \tilde{\gamma}_0^i(s_0)C_0 + (\tilde{q}_0 + d_0)a_0^iC_0,$$

which is homogenous in $C_0$ thanks to the fact that $b_0^i = 0$.

**Does the Level of Liquidity Matter?** Yes and no. As Proposition 5 makes clear, the amount of liquidity has absolutely no effect on the response of consumption to current and future interest rates. Indeed, the response is identical to that of a representative agent with the same preferences. The response of current consumption $C_t$ to changes in the interest rate $R_t$ or changes in future consumption is identical to those implied by a representative agent.

Liquidity is still important at the microeconomic level, with greater liquidity allowing greater consumption smoothing. In the representative-agent Euler equation representation, this shows up by affecting discounting. In this way, the level of liquidity does have a macroeconomic effect in levels: it affects interest rates for a given consumption path or consumption for a given interest rate path. However, these level effects do not affect the response of aggregate consumption to changes in the interest rate path, which is independent of the amount of liquidity.
Discounting and Natural Interest Rates. When primitives are stationary (i.e. \( s_t \) is a Markov process) and the economy is initialized with an invariant distribution for \( (s_0, a_0) \) then the discount factor is constant \( \beta_t = \frac{1}{\bar{R}} \), where \( \bar{R} \) is the steady state interest rate. As shown by Huggett (1993) and Aiyagari (1994) steady states require \( \beta \bar{R} < 1 \), implying that a higher discount factor appears in the representation, compared to the true subjective discount factor.

In general, the discount factors \( \beta_t \) may not be constant, but this is a feature, not a bug. For example, periods of higher idiosyncratic uncertainty are likely to increase \( \beta_t \) temporarily. Likewise, deleveraging episodes—situation where some household start with high initial debt but must lower their debt over time—as modeled, for example, by Eggersson and Krugman (2012) and Guerrieri and Lorenzoni (2011), may also increase the discount factor temporarily. A higher discount factor \( \beta_t \) is associated with a lower natural interest rate, in the New Keynesian model. Indeed, the economy may be pushed up against the zero interest rate bound if \( \beta_t \) is high enough, as stressed by the liquidity trap literature.

4.1.2 Arbitrary Initial Bond Holdings

I now allow any initial distribution for initial bonds holdings, \( b_0^i \). The main result is a similar characterization to that obtained when initial bonds holdings are zero, except for an adjustment to the discount factors that depends on initial consumption.

Proposition 6. Suppose utilities satisfy (14), household income satisfies (15) and borrowing constraints satisfy (17). Then \( \{C_t, R_t\} \) is part of an equilibrium if and only if

\[
U'(C_t) = \beta_t R_t U'(C_{t+1})
\]

for some a sequence of discount factors \( \{\beta_t\} \) that depends on \( C_0 \), i.e. \( \beta_t = \hat{\beta}_t(C_0) \).

According to this result, a standard Euler equation continues to hold, except that now the relevant discount factors are endogenous. Conveniently, these new effects are entirely summarized by the initial aggregate consumption level, \( C_0 \). Thus, the aggregate dynamics for consumption remains tractable and tied to a standard Euler equation.

Why are the discount factors dependent on initial consumption? When output expands in the first period, this diminishes the relative value of bonds. In this relative sense, the distribution of bond holdings contracts towards zero. In contrast, the asset’s price and dividends expand proportionally with output, and the wealth held in the form
of assets remains in proportion to output. This explains why Proposition 6 requires an adjustment relative to Proposition 5.

Indeed, by a simple extension of the argument provided above, the discount factors are now precisely \( \beta_t = \frac{1}{\tilde{R}_t} \), the reciprocal of the equilibrium interest rates \( \tilde{R}_t \) for a normalized economy featuring constant output, but with initial bond holdings rescaled to \( \tilde{b}_0^i = \frac{1}{C_0} b_0^i \). This works since then the budget constraint at \( t = 0 \) becomes

\[
c^i(s_0; a_0) + \tilde{a}^i(s_0; a_0) = \tilde{\gamma}_0(s_0)C_0 + (\tilde{q}_0 + d_0)a_0^iC_0 + R - \tilde{b}_0^iC_0,
\]

which is homogeneous in \( C_0 \).

**Implications.** How is the dependence of \( \beta_t \) on \( C_0 \) likely to play out? What are the implications of this adjustment? I offer a few tentative, but informed, speculations.

As I just argued, a higher \( C_0 \) diminishes the dispersion in bond holding levels relative to output, scaling it down towards zero. For given interest rates, it is reasonable to expect lower dispersion in initial bond holdings to increase consumption (since consumption functions are concave as a function of wealth), at least in earlier periods. This pushes the interest rates \( \tilde{R}_t \) upward to reestablish equilibrium with constant unitary output. This argument suggests that the discount factors \( \beta_t = \frac{1}{\tilde{R}_t} \) decrease with \( C_0 \). The effects on \( \beta_t \) are likely to die out in later time periods \( t \), since \( \tilde{R}_t \) is likely to converge to a common steady state \( \tilde{R} \), regardless of initial conditions.

If, as seems likely, discount factors are decreasing in consumption, then interest rate paths that increase consumption, e.g. lower interest rates, are likely to have an additional effect, especially in earlier periods. In other words, non-zero initial bond holdings are likely to amplify the effects of interest rate changes.

### 4.2 Departures from ‘As If’ Result: Procyclical Liquidity

Away from the baseline studied in the previous subsection no aggregation result is immediately available. This makes studying the effects of interest rate changes challenging. For each interest rate path, one must solve the household allocations that are part of an equilibrium and, unlike the previous section, there is no obvious relation across these equilibria. One possibility is to turn to a numerical analysis, but this may not uncover the forces at work transparently. Fortunately, one can work out a few simple cases analytically that shed light on the mechanism and help understand the expected direction of departure from the previous results.
A Three Period Economy. Consider a three period economy, with \( t = 0, 1, 2 \); one should think of period \( t = 2 \) as collapsing the entire future in the infinite horizon setting. Agents are ex ante identical. There is no uncertainty at \( t = 0 \) nor \( t = 2 \), so that \( y_0 = (1 - d_0)Y_0 \) and \( y_2 = (1 - d_2)Y_2 \). At \( t = 1 \) households experience an idiosyncratic shock \( s_1 \) with c.d.f. \( F \) and collect labor income \( y_1 = s_1(1 - d_1)Y_1 \). Agents cannot borrow: \( B_t(s) = 0 \). The asset pays \( D_t(Y_t) = d_t Y_t \) and for simplicity we set \( d_0 = d_1 = 0 \). All agents start with one unit of the asset.

Households are identical at \( t = 0 \), so in equilibrium they all carry the asset into \( t = 1 \). Households are then hit with a temporary income shock. Those with a negative enough shock sell some or all of the asset, while those with a positive enough shock buy the asset. Intuitively, this helps households smooth consumption between \( t = 1 \) and \( t = 2 \), reducing the dispersion of consumption at \( t = 1 \).

Before studying this model, it is useful to note that identical results obtain without outside assets if one models borrowing constraints as dependent on interest rates, in the form of a present value. This is a common specification in the literature. In particular, if one supposes that \( B_1 = \frac{1}{R_1}d_2 Y_2 \) and \( y_2 = Y_2 \), then \( d_2 \) can be interpreted as the pledgeable fraction of labor income. All the results below apply to this formulation.

I now consider the sensitivity of aggregate consumption to changes in \( R_0 \) and \( R_1 \). The experiment fixes \( Y_2 \) at a constant and solves for \( C_1 = Y_1 \) and \( C_0 = Y_0 \). \(^{17}\)

Slack Liquidity Constraint Case. I first consider the case where no agent sells off their entire asset holdings, so that no agent is liquidity constrained.

Proposition 7. Consider the three-period economy with uncertainty at \( t = 1 \). Suppose borrowing constraints are never binding, so that the Euler condition holds with equality for all agents. If \( \sigma > 1 \),

\[
\frac{d \log C_0}{d \log R_1} < \frac{d \log C_1}{d \log R_1} = \frac{d \log C_0}{d \log R_0} = -\frac{1}{\sigma};
\]

the inequality is reversed if \( \sigma < 1 \). Finally, we have \( \frac{d \log C_1}{d \log R_0} = 0 \).

The effect of the interest rate on current spending at \( t = 1 \) is standard, exactly as in the representative agent case: a one percent drop in the interest rate \( R_1 \) leads to a rise in consumption of \( \frac{1}{\sigma} \) percent. The effects on earlier periods, however, depends on the value of \( \sigma \). When \( \sigma > 1 \) this interest rate drop makes consumption at \( t = 0 \) rise

\(^{17}\)Consumption in all periods scales proportionally with \( Y_2 \), given interest rates \( R_0 \) and \( R_1 \). One interpretation for fixing \( Y_2 \) in a monetary economy is when prices are assumed flexible at \( t = 2 \) or when monetary policy is able to achieve the flexible price equilibrium; in this case, output is at the flexible price at \( t = 2 \), but we are interested in aggregate demand in the short and medium run, \( t = 0, 1 \).
by a greater percentage; when $\sigma < 1$ the effect is smaller. The borderline logarithmic case, $\sigma = 1$, was covered earlier and boils down to the standard representative agent case, with consumption rising by the same percentage in both periods, i.e. the standard consumption smoothing property.

Underlying these results and their dependence on $\sigma$ is the asset price relative to income, which affects the amount of liquidity, which in turn affects how much agents are able to smooth consumption in the intermediate period. To see this, note that the asset price at $t = 1$ rises in proportion to the fall in $R_1$, yet, output rises by $\frac{1}{\sigma} < 1$. Thus, if $\sigma > 1$, the asset value rises relative to output relative to output, increasing the supply of liquidity in this sense. This lowers the dependence of consumption on the current income shock. Consumption at $t = 0$ then rises for two reasons: the higher level of consumption at $t = 1$ and the lower uncertainty at $t = 1$. When $\sigma < 1$ this effect is reversed, since consumption at $t = 1$ expands more than the asset value.

This result underscores the importance of the response of asset prices to interest rate changes. It is worth clarifying that there are revaluation effects from the change in the asset value for any value of $\sigma$, regardless of whether $\sigma > 1$, $\sigma < 1$ or $\sigma = 1$. Instead, whether or not these revaluations effects increases or decreases the sensitivity of consumption to interest rate changes depends on $\sigma$ because this determines the relative strength of the standard substitution channel. In effect, what is relevant is whether the asset value rises relative to output. Finally, the revaluation effects I focus on here work through outside assets that are in net positive supply with their own stream of dividends. Thus, these effects are distinct from the redistribution effects between creditor and debtors focused on in Auclert (2015) or Sheedy (2014).

**Binding Liquidity Constraint Case.** Next I consider the case where some agents sell off all their assets at $t = 0$. These agents find themselves liquidity constrained and their Euler equation hold with strict inequality.

**Proposition 8.** Consider the three-period economy with uncertainty at $t = 1$. Suppose borrowing constraints bind at $t = 1$ for some households. equality for all agents. Then if $\sigma > 1$ we have that

$$
\frac{d \log C_0}{d \log R_1} < \frac{d \log C_1}{d \log R_1} < \frac{d \log C_0}{d \log R_0} = -\frac{1}{\sigma};
$$

the inequalities are reversed if $\sigma < 1$. Finally, we have $\frac{d \log C_1}{d \log R_0} = 0$.

When households find themselves liquidity constrained, the asset value relative to income determines the degree to which this constraint binds. When $\sigma > 1$, the asset value
rises relative to output. The increase in the asset value has a direct one-for-one impact on
the consumption at \( t = 1 \) for those households that are constrained; these households
have a marginal propensity to consume out of wealth of unity. This amplifies the effect of
\( R_1 \) on \( C_1 \).

5 Labor Markets, Nominal Rigidities and Supply Side

Up to this point I have worked with the incomplete market setting underlying the con-
sumption and savings problem faced by households. I have taken household labor in-
come as well as asset dividends as given functions of aggregate income \( Y_t \). This way of
 compartamentalizing proved fruitful, since it allowed us to focus on aggregate demand
and to judge the assumptions needed for various results more easily, without the details.
In this section I now briefly sketch how one can fill in the supply side of the model. I
consider a few variants of the New Keynesian model. This is a well known model, so I
will describe the main elements and omit the details.

I start with the common elements across the different variants. There is a single final
good that is produced by a continuum of varieties according to a Dixit-Stiglitz aggregator

\[
Y_t = A_t \left( \int B_t(i)Y_t(i)^{1-\frac{\alpha}{\varepsilon}} \right)^{\frac{1-\alpha}{1-\varepsilon}} K^\alpha_t
\]

where \( A_t \) is total factor productivity, \( B_t(i) \) are taste shifters across varieties, and \( \varepsilon > 1 \)
the elasticity and \( \alpha \) is the capital share. Capital is in fixed supply, representing the outside
asset held by households. The Cobb-Douglas specification implies that capital’s rents are
proportional to output, with \( d_t = \alpha_t \).

Each variety \( i \) is produced one-for-one from labor

\[
y_t(i) = N_t(i)
\]

Utility is additively separable between consumption and labor

\[
\sum_{t=0}^{\infty} \beta_i t \mathbb{E}_0[u_t(c(s^t), s_t) - v(n(s^t))].
\]

Households are subject to shocks to the labor productivity: if they work \( n(s^t) \) then they
supply \( N(s^t) = s_t n(s^t) \) units of quality adjusted labor.

The market is organized as follows. The final good is produced competitively, earning
zero profits. The varieties are produced by monopolists and will generally earn positive profits. Nominal rigidities may prevent the immediate adjustment of prices or wages. The exact nature of this rigidity will not be described here, as it is not crucial.

The different variants make different assumptions regarding the functioning of the goods and labor market.

**Yeomen Farmers: Collapsing the Goods and Labor Market.** In the first variant, intermediate goods are produced by households themselves. Each household owns a particular variety. In other words, there is no labor market; or alternatively, the labor market coincides with the goods market. Households set the price for their variety (or their wage in the alternative labor-market interpretation).

With zero liquidity households will consume their income. As a result, their optimization will lead to income that depends on their productivity shock as well as the taste shock for their variety. These are the idiosyncratic shocks.

**Sticky Prices and Flexible Labor Market.** The standard New Keynesian model instead assumes the existence of firms that set prices and hire labor in a competitive labor market, with a flexible wage. Assume profits of these firms are taxed 100% with the proceeds rebated by way of a labor subsidy to households. This assumption plays two roles. First, it is a standard way to obtain the efficient allocation, by undoing the monopolistic markup. Second, it implies that profits and dividends from the ownership of firms is zero. As a result, it has the convenient property that the only outside asset with positive income flow is capital.\(^{18}\)

**Sticky Wages and Rationing in the Labor Market.** Assume now that varieties are produced competitively, earnings zero profits. However, the nominal wage is rigid and may require rationing. Prices are flexible and set to marginal cost. Note that, since the wage is rigid, marginal costs are rigid and so prices inherit this rigidity. We assume wages are set above the market clearing wage, implying that labor equals labor demand, which in turn equals the demand for goods.

When aggregate demand is low, labor demand is low and employment is rationed. One may make various assumptions about this rationing. But it may also include an

\(^{18}\)Relaxing this assumption lead to profits from monopolistic producers that endogenously vary with output, but this does not necessarily affect the results in any one direction. First one must decide whether firm ownership is transferrable, whether firms are modeled as public or private firms; in the former case the stream of profits is effectively part of the outside asset in our notation; in the latter case, profits are just a component of labor income \(\gamma_t\) in our notation. Profits in the standard New Keynesian may be countercyclical (due to the procyclical wage) or procyclical (due to the scale effects) depending on the calibration.
extensive and intensive margin, lending an interpretation to the specification in Section 4.2.

6 Exact Aggregation in a Real Business Cycle Model

Up to this point, I have considered economies with a fixed or mechanical supply of liquidity.\textsuperscript{19} In addition, consumption has been assumed to be the only component of demand, since the model lacked investment. Finally, although the results apply more generally and make no explicit assumptions regarding nominal rigidities, their natural application is the study of monetary policy in the presence of nominal rigidities.

The purpose of this section is to show that the results can be generalized and applied in a very different setting. To this end, I now consider a Real Business Cycle model where capital is accumulated by investment and as the outside asset. This real economy is assumed to operate under flexible prices and perfect competition.

My main result assumes utility is logarithmic and supposes full depreciation of capital, a well-known case referred to as Brock-Mirman. Under these conditions, aggregate dynamics behave exactly as those obtained in representative agent or complete market version of the model. This occurs despite potentially arbitrarily large departures at the microeconomic household level in the allocation.

Krusell and Smith (1998) studied a similar real business cycle model numerically, but without the restriction to full depreciation. Their main conclusion was approximate aggregation. My result complements theirs, providing conditions for exact aggregation.

6.1 Economic Environment

The economic environment is a standard real business cycle model, augmented to include idiosyncratic uncertainty and incomplete markets. Because most of it is well known, I will keep the description brief.

Preferences. All households have utility

\[
\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left( \log c_t - v(n_t) \right)
\]

\textsuperscript{19} Even with a fixed supply of assets, liquidity is arguably not exogenous, since dividends are a function on aggregate income, an endogenous variable, and the asset price is determined endogenously.
where 
\[ v(n) = \sigma \frac{n^{1+\gamma}}{1+\gamma} \]
with \( \sigma > 0 \) and \( \gamma \geq 0 \).

**Uncertainty.** Agents are subject to idiosyncratic productivity shocks so that if they work \( n_i^t \) they supply \( z_i^t \cdot n_i^t \) to the market. Denote the history of idiosyncratic shocks by \( z^t \), and the history of aggregate shocks by \( s^t \). Let \( \Lambda_t \) denote the cross sectional distribution of history of shocks for households of type \( i \).

**Technology.** Output is given by a constant returns to scale Cobb-Douglas production function,
\[ Y(s^t) = A(s_t)F(K(s_t-1), N(s_t)) = A(s_t)K(s_t^{-1})^\alpha N(s_t)^{1-\alpha}. \]
The resource constraint is
\[ C(s^t) + K_t(s^t) \leq A(s_t)F(K_t(s_t-1), N_t(s_t)) + (1-\delta)K(s^t). \]
We shall focus on the case with \( \delta = 1 \).

**Budget Constraints.** Households are subject to the budget constraints
\[ c^t(z^t, s^t) + k^t(z^t, s^t) \leq W_t(s^t)n^t(z^t, s^t) + k^t(z_t^{t-1}, s_t^{t-1})R_t(s^t) \]
To simplify, I have assumed no borrowing. The results are robust to the introduction of borrowing, as long as the borrowing constraints are proportional to output.

**Equilibrium Conditions.** The equilibrium conditions are standard and we relegate the details to the appendix. Households maximize utility subject to the budget constraints. Aggregate variables must be consistent with individual household choices. Firms maximize. Finally, the market for goods (consumption and investment), the market for labor and the market for capital must clear.

6.2 ‘As if’ Representative Agent Result
The next proposition states the main result of this section.
Proposition 9. Consider the Real Business Cycle model with $\delta = 1$. Then the aggregate dynamics of capital and labor are equivalent to their counterparts in the complete market, or representative agent economy. Namely,

$$C(s^t) = (1 - \omega_t) Y(s^t)$$
$$K(s^t) = \omega_t Y(s^t)$$
$$N(s^t) = \bar{N}_t,$$

for some deterministic sequence of saving rates $\{\omega_t\}$ and labor $\{\bar{N}_t\}$. Saving rates $\omega_t$ are constant if the initial distribution of wealth is at an invariant steady state.

Individual household consumption and labor are given by

$$c^i(z^t, s^t) = c^i(z^t) C(s^t),$$
$$n^i(z^t, s^t) = n^i(z^t) N(s^t),$$

where $c^i(z^t)$ and $n^i(s^t)$ is computed from an incomplete market equilibrium for a normalized economy with $C(s^t) = 1$ and $N(s^t) = 1$.

7 Conclusion

This paper studies the effect of financial market imperfections on aggregate consumption. Idiosyncratic uncertainty, incomplete markets and borrowing constraints may affect the level of aggregate demand as well as its sensitivity to the path of current and future interest rate. In terms of the level, greater uncertainty or more stringent borrowing constraints typically depress consumption, as one would expect. In terms of sensitivity, I provided a benchmark cases where the response of aggregate consumption to interest rates is exactly identical to that of a standard representative-agent model. Outside these benchmark cases there are many other possibilities, but consumption becomes more sensitive to current and future interest rate changes in plausible cases.

An interesting feature of the present model is that interest rates are not sufficient statistics. Liquidity conditions, determined by the value of assets and borrowing conditions, also matter. With incomplete markets, these liquidity conditions may not be fully reflected in interest rates, since they affect the underlying capacity of individual households to smooth consumption. If consumers anticipate liquidity shortages, they act prudently and lower consumption. Interestingly, current liquidity affects consumption directly, while future liquidity matters both directly and indirectly by lowering uncertainty.
References


A Asymmetric Intensive Margin

Assume now that income equals $\bar{y}Y^\psi$ with probability $1 - \lambda$ and, otherwise, income is fixed at $y$. Define $\lambda(Y)$ from the identity

$$Y \equiv (1 - \lambda(Y))\bar{y}Y^\psi + \lambda(Y)y,$$

and consider the range of $Y$ is such that $\lambda(Y) \in (0, 1)$ and $\bar{y}Y^\psi > y$. Note that since $\psi < 1$ then $\lambda(Y)$ is decreasing.

The Euler equation becomes

$$U'(\bar{y}Y^\psi) = \beta R \left( (1 - \lambda(Y'))U'(\bar{y}Y'^\psi) + \lambda(Y')U'(y) \right),$$

where $U'(c) = c^{-\sigma}$. Equivalently

$$U'(Y^\psi) = \hat{\beta}(Y')RU'(Y'^\psi)$$

where

$$\hat{\beta}(Y) \equiv \beta \left( (1 - \lambda(Y) + \lambda(Y) \frac{1}{U'(\bar{y}Y^\psi)}) \right).$$

Differentiating and rearranging,

$$\hat{\beta}'(Y) = \beta \frac{\lambda(Y)}{YU'(\rho)} \sigma \psi \left( \frac{-1 Y\lambda'(Y)}{\psi \sigma \lambda(Y)} \left( U'(\rho) - 1 \right) + 1 \right)$$

(20)
where $\rho \equiv \frac{y}{Y^\psi}$ is relative income in the two states.

Differentiating the identity defining $\lambda(Y)$ one obtains

$$\frac{-1}{\sigma \psi} \frac{Y \lambda'(Y)}{\lambda(Y)} = \frac{1}{\psi} \left( (1 - \psi) \frac{1 - \lambda(Y)}{\lambda(Y)} \rho + 1 \right) \frac{1}{\sigma} \frac{1}{\psi} \frac{1}{y^{\psi} Y^\psi - 1}.$$  

Substituting into (20) and rearranging gives,

$$\hat{\beta}'(Y) = \beta \frac{\lambda}{YU'(\rho)} \psi \sigma \left( \frac{1}{\psi} \left( (1 - \psi) \frac{1 - \lambda}{\lambda} \rho + 1 \right) \frac{1}{\sigma} \frac{U'(\rho) - 1}{\rho - 1} + 1 \right)$$

It follows that $\hat{\beta}'(Y) \leq 0$ if and only if

$$1 \leq \frac{1}{\psi} \left( (1 - \psi) \frac{1 - \lambda}{\lambda} \rho + 1 \right) \left( -\frac{1}{\psi} \frac{U'(\rho) - 1}{\rho - 1} \right). \quad (21)$$

This condition is relaxed for lower $\psi$ and for lower $\lambda$. Indeed, it is always satisfied if either $\psi$ or $\lambda$ close to 0. Thus, for any $\lambda$ there is an interval of $\psi$ containing zero for which the condition is satisfied. Moreover, this interval expands for lower $\lambda$, and converges to the entire support $(0, 1)$ as $\lambda \to 0$.

A simpler sufficient condition is possible. Since $U'(c) = c^{-\sigma}$ is convex then

$$-\sigma \rho^{-\sigma - 1} = U''(\rho) \geq \frac{U'(\rho) - 1}{\rho - 1}.$$  

Thus, a sufficient condition for (21) is

$$\rho^{\sigma + 1} \leq \frac{1}{\psi} \left( (1 - \psi) \frac{1 - \lambda}{\lambda} \rho + 1 \right).$$

**B  Proof of Proposition 7**

To recap, the economy is described by the following primitives

$$\gamma_0(Y_0) = Y_0 \quad d_0 = 0$$

$$\gamma_1(Y_1) = sY_1 \quad d_1 = 0$$

$$\gamma_2(Y) = (1 - d_2)Y \quad d_2 > 0$$
where $s$ is a random variable with c.d.f. $F$ and $E[s] = 1$. Finally, no borrowing is allowed: $B_t(s) = 0$ for all $s$ and $t$. Fix $Y_2$ at a constant.

In equilibrium the asset price is

$$q_1 = \frac{1}{R_1} d_2 Y_2.$$  \hfill (22)

The present value budget constraint at $t = 1$ is

$$c_1(s) + \frac{1}{R_1} (c_2(s) - (1 - d_2)Y_2) = sY_1 + q_1.$$  \hfill (23)

Assuming borrowing constraints do not bind, the Euler equation between $t = 1$ and $t = 2$ gives

$$c_1(s) = (\beta R_1)^{-\frac{1}{\sigma}} c_2(s).$$  \hfill (24)

Substituting (22) and (24) into (23) gives

$$c_1(s) + \frac{1}{R_1} \beta^\frac{1}{\sigma} R_1^{\frac{1}{\sigma} - 1} c_1(s) = sY_1 + \frac{1}{R_1} Y_2.$$  

Solving gives

$$c_1(s) = \frac{sY_1 + \frac{1}{R_1} Y_2}{1 + \beta^\frac{1}{\sigma} R_1^{\frac{1}{\sigma} - 1}}.$$  

Setting $C_1 = Y_1$ and integrating gives the relation

$$C_1 = \int c_1(s) dF(s) = \frac{C_1 + \frac{1}{R_1} Y_2}{1 + \beta^\frac{1}{\sigma} R_1^{\frac{1}{\sigma} - 1}},$$  

and solving for $C_1$ gives

$$C_1 = R_1^{-\frac{1}{\sigma}} \beta^{-\frac{1}{\sigma}} Y_2,$$

so that $\frac{d\log C_1}{d\log R_1} = -\frac{1}{\sigma}$.

Returning to individual household consumption, one observes that

$$c_1(s) = \hat{c}(s; R) C_1$$

where

$$\hat{c}(s; R_1) = \omega(R_1)s + 1 - \omega(R_1),$$

41
\[ \omega(R_1) = \frac{\beta^{-\frac{1}{\sigma}}R_1^{1-\frac{1}{\sigma}}}{\beta^{-\frac{1}{\sigma}}R_1^{1-\frac{1}{\sigma}} + 1} \in (0, 1). \]

Thus, \( c(s) \) is a proportion \( \hat{c}(s; R_1) \) of aggregate income \( C_1 \); for \( \sigma = 1 \) this proportion varies with \( s \) but is independent of the interest rate \( R_1 \). However, \( \hat{c}(s; R_1) \) varies with \( R_1 \) when \( \sigma \neq 1 \). When \( \sigma > 1 \) we have \( \omega \) increasing and so an increase in \( R_1 \) increases \( \hat{c} \) for \( s > 1 \) and decreases \( \hat{c} \) for \( s < 1 \), i.e. if \( R'_1 > R_1 \) then \( \hat{c}(.; R'_1) \) single crosses \( \hat{c}(.; R_1) \) from below at \( s = 1 \); this implies a mean-preserving spread in the distribution for \( \hat{c} \). For \( \sigma < 1 \), we have \( \omega \) decreasing and so the reverse is true: a decrease in \( R_1 \) implies a mean preserving spread in \( \hat{c} \).

Turning to \( t = 0 \) we have

\[ C_0 = (\beta R_0)^{-\frac{1}{\sigma}} \left( \int \hat{c}(s; R_1)^{-\sigma} dF(s) \right)^{-\frac{1}{\sigma}} C_1, \]

so that \( \frac{d \log C_0}{d \log R_0} = -\frac{1}{\sigma} \). Since the marginal utility function \( \hat{c}^{-\sigma} \) is convex for all \( \sigma > 0 \) then the expression

\[ \left( \int \hat{c}(s; R_1)^{-\sigma} dF(s) \right)^{-\frac{1}{\sigma}} \]

is decreasing in \( R_1 \) when \( \sigma > 1 \). This follows by Jensen’s inequality since \( R_1 \) implies a mean preserving spread in \( \hat{c} \). This implies that \( C_0 \) changes more than proportionally with \( C_1 \), due to changes in \( R_1 \); thus, this establishes that \( \frac{d \log C_0}{d \log R_1} < -\frac{1}{\sigma} \) when \( \sigma > 1 \). The reverse inequality holds when \( \sigma < 1 \), since then a decrease in \( R_1 \) implies a mean-preserving spreads in \( \hat{c} \).

C Proof of Proposition 8

The argument is similar to proof of Proposition 7, so I only sketch a proof of the arguments that are different.

We now impose the borrowing constraint

\[ c_1(s; Y_1) \leq sY_1 + q_1 \]

and suppose it is binding for some households, but not everyone. It follows that

\[ c_1(s) = \min \left\{ \frac{sY_1 + \frac{1}{R_1}Y_2}{1 + \beta \frac{1}{\sigma} R_1^{1-\frac{1}{\sigma}}}, sY_1 + \frac{1}{R_1} d_2 Y_2 \right\} \quad (25) \]
or equivalently

\[
c_1(s; Y_1, \hat{s}) = \begin{cases} 
  sY_1 + \frac{1}{R_1}d_2Y_2 & s \leq \hat{s} \\
  sY_1 + \frac{1}{R_1}Y_2 + \frac{\beta}{1 + \beta \frac{1}{\sigma} R_1^{-1}} & s > \hat{s}
\end{cases}
\]

where the cutoff \( \hat{s} \) is defined by the equality

\[
\frac{\hat{s}Y_1 + \frac{1}{R_1}Y_2}{1 + \beta \frac{1}{\sigma} R_1^{-1}} = \hat{s}Y_1 + \frac{1}{R_1}d_2Y_2.
\]

This provides a strictly decreasing relation between \( Y_1 \) and \( \hat{s} \), given \( R_1 \).

Aggregating,

\[
C_1 = \min_{\hat{s}} \int c_1(s; C_1; \hat{s})dF(s)
\]

\[
= \int_{\hat{s}} \left( sC_1 + \frac{1}{R_1}Y_2 \right) dF(s) + \int_{\hat{s}} \left( sC_1 + \frac{1}{R_1}d_2Y_2 \right) dF(s)
\]

\[
= \frac{C_1 \int_{\hat{s}} sdF(s) + \frac{1}{R_1}Y_2 (1 - F(\hat{s}))}{1 + \beta \frac{1}{\sigma} R_1^{-1}} + C_1 \int_{\hat{s}} sdF(s) + \frac{1}{R_1}d_2Y_2 F(\hat{s})
\]

The first equality follows from (25). It implies, by an Envelope condition argument, that we can compute the equilibrium derivative of \( C_1 \) and \( R_1 \) without considering the change in \( \hat{s} \). Rearranging, for given \( \hat{s} \), we obtain

\[
C_1 = h(R_1, \hat{s}) \beta^{-\frac{1}{\sigma}} R_1^{-\frac{1}{\sigma}} Y_2 = m(R_1, \hat{s}) \beta^{-\frac{1}{\sigma}} R_1^{-1} Y_2
\]

where

\[
h(R_1, \hat{s}) \equiv \frac{1 - F(\hat{s}) + d_2F(\hat{s}) + \beta \frac{1}{\sigma} R_1^{-1} d_2F(\hat{s})}{\int_{\hat{s}} sdF(s)}
\]

\[
m(R_1, \hat{s}) \equiv \frac{R_1^{-\frac{1}{\sigma}} (1 - F(\hat{s}) + d_2F(\hat{s})) + \beta \frac{1}{\sigma} d_2F(\hat{s})}{\int_{\hat{s}} sdF(s)}
\]

When \( F(\hat{s}) > 0 \) then \( h \) is decreasing if \( \sigma > 1 \) and decreasing if \( \sigma < 1 \). This in turn implies that when \( \sigma > 1 \) then \( \frac{d \log C_1}{d \log R_1} < -\frac{1}{\sigma} \) and the inequality is reversed when \( \sigma < 1 \).

When \( \sigma > 1 \) then \( m \) is increasing; if \( \sigma < 1 \) then \( m \) is decreasing. This implies that we can write

\[
Y_2 = M(R_1) R_1 C_1
\]

for some function \( M \) that is decreasing if \( \sigma > 1 \) and increasing if \( \sigma < 1 \).
To see that a change in \( R_1 \) changes \( C_0 \) in the same direction but in a greater proportion than the change resulting in \( C_1 \), note that in equilibrium

\[
c_1(s) = \hat{c}(s, R_1) C_1,
\]

where

\[
\hat{c}(s, R_1) = \min \left\{ \frac{s}{1 + \beta \frac{1}{R_1^{\frac{1}{2}}}} + \frac{M(R_1)}{1 + \beta \frac{1}{R_1^{\frac{1}{2}}}}, s + M(R_1) d_2 \right\}.
\]

Write \( \hat{c} \) in the form

\[
\hat{c}(s, R_1) = \min \{ \delta(R_1) s + \phi_2(R_1), s + \phi_2(R_1) \}.
\]

When \( \sigma > 1 \) then the slope \( \delta(R_1) \) is increasing in \( R_1 \) and the intercept \( \phi_2(R_1) \) is decreasing. An increase in \( R_1 \) shifts \( \hat{c} \) creating a single crossing at some interior point from below, i.e. if \( R_1' > R_1 \) then there exists a \( S \in (0, 1) \) such that \( \hat{c}(s, R_1') \leq \hat{c}(s, R_1) \) for \( s \leq S \) and \( \hat{c}(s, R_1') \geq \hat{c}(s, R_1) \) for \( s \geq S \). We also have by construction \( \int \hat{c}(s, R_1) dF(s) = 1 \) for all \( R_1 \).

Thus, if \( \sigma > 1 \) we have a mean preserving spread in \( \hat{c} \) when \( R_1 \) increases. The reverse is true when \( \sigma < 1 \). The result then follows as before.

### D Proof of Proposition 9

The required equilibrium conditions are

\[
c^i(z^t, s^t) + k^i(z^t, s^t) = W_i(s^t)n^i(z^t, s^t) + k^i(z^{t-1}, s^{t-1})R_i(s^t)
\]

\[
\frac{v^i(n^i(z^t, s^t))}{u^i(c^i(z^t, s^t))} = z_t W_t
\]

\[
u^i(c^i(z^t, s^t)) \geq \beta \mathbb{E}_t \left[ R_{t+1} u^i(c^i(z^{t+1}, s^{t+1})) | z^t, s^t \right]
\]

with equality whenever \( k^i(z^t, s^t) > 0 \). The aggregate conditions are

\[
C(s^t) + K_{t+1}(s^t) = A(s_t) F(K_i(s^{t-1}), N_i(s^t)) + (1 - \delta) K(s^t)
\]

\[
R(s^t) = A(s_t) F_r(K(s^{t-1}), N(s^t)) + 1 - \delta
\]

\[
W(s^t) = A(s_t) F_N(K(s^{t-1}), N(s^t))
\]
with aggregates consistent with household choices

\[ C(s^t) = \sum \mu^i \int c^i(z^t, s^t) d\Lambda(z^t), \]
\[ N(s^t) = \sum \mu^i \int n^i(z^t, s^t) d\Lambda(z^t), \]
\[ K(s^t) = \sum \mu^i \int k^i(z^t, s^t) d\Lambda(z^t). \]

Guess and verify that the equilibrium satisfies

\[ c^i(z^t, s^t) = c^i(z^t)C(s^t) \]
\[ n^i(z^t, s^t) = n^i(z^t)N(s^t) \]

For each period \( t \) and aggregate history \( s^t \) such a decomposition is without loss of generality; define \( c^i(z^t) \) and \( n^i(z^t) \) to be the household decomposition of the allocation associated with some aggregate history \( s^t \) for each period. I now verify that this decomposition works for all other histories.

Substituting we obtain

\[ c^i(z^t)C(s^t) + k^i(z^t)K(s^t) = W(s^t)N(s^t)n^i(z^t) + k^i(z^{t-1})K(s^{t-1})R(s^t) \]
\[ \frac{c^i(z^t)}{\hat{v}t} \nu^t\left(n^i(z^t)\right) \nu^t(N(s^t))N(s^t)C(z^t) = W(s^t)N(s^t) \]
\[ u'(C(s^t)) \geq \left( \frac{\beta E[u'(c^i(z^{t+1})) | z^t]}{u'(c^i(z^{t+1}))} \right) \cdot E[R(s^{t+1})u'(C(s^{t+1})) | s^t] \]

By definition these equations hold for one history \( s^t \); to check whether these conditions hold for other histories, rewrite them as

\[ c^i(z^t)C(s^t) + k^i(z^t)K(s^t) = \frac{W(s^t)N(s^t)}{Y(s^t)}n^i(z^t) + \frac{K(s^{t-1})R(s^t)}{Y(s^t)} \cdot k^i(z^{t-1}) \]
\[ \hat{v}t\nu'(N(s^t))N(s^t)C(s^t) = \frac{W(s^t)N(s^t)}{Y(s^t)} \]
\[ u'(C(s^t)) = \hat{\beta}_t \cdot E[R(s^{t+1})u'(C(s^t)) | s^t] \]

where

\[ \hat{\beta}_t \equiv \int \frac{c^i(z^t)}{\hat{v}z^t} \nu^t\left(n^i(z^t)\right) \]
\[ \hat{\beta}_t \equiv \max_{z^t} \beta \mathbb{E} \left[ u'(c^t(z^t)) \mid z^t \right] \]

The first equation will hold as long as the terms

\[
\frac{C(s^t)}{Y(s^t)} \quad \frac{K(s^t)}{Y(s^t)} \quad \frac{W(s^t)N(s^t)}{Y(s^t)} \quad \frac{K(s^t-1)R(s^t)}{Y(s^t)}
\]

are independent of history \( s^t \); the last two ratios are guaranteed to be constant by the Cobb-Douglas assumption. The second ratio is implied by the first. Thus, we require

\[ C(s^t) = (1 - \omega_t) Y(s^t) \]

for some saving rate \( \omega_t \) that does not depend on the history \( s^t \). Turning to the second equation, we determine \( N(s^t) \),

\[ \hat{\nu}'(N(s^t))N(s^t) = \frac{1 - \alpha}{1 - \omega_t} \]

Finally

\[
\frac{1}{C(s^t)} = \hat{\beta}_t \mathbb{E} \left[ \frac{\alpha Y(s^t+1)}{k(s^t)} \frac{1}{C(s^t+1)} \mid s^t \right] \\
\frac{1}{(1 - \omega_t)Y(s^t)} = \hat{\beta}_t \frac{1}{k(s^t)} \frac{\alpha}{1 - \omega_t} \\
\frac{\omega_t}{1 - \omega_t} = \hat{\beta}_t \frac{\alpha}{1 - \omega_t}
\]

Note that at with a steady state invariant distribution we have \( \hat{\beta}_t \) constant, so we obtain \( \omega_t = \alpha \hat{\beta} \) as in the standard Brock-Mirman solution.