Multi-unit open ascending price efficient auction*

Sergei Izmalkov
MIT†

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Abstract

In this paper I present an open ascending price mechanism that allocates efficiently M units of the same good among N bidders with interdependent values. The mechanism consists of a number of sequential English auctions with reentry and has the following attributes. In each of the individual auctions all the bidders compete simultaneously in the open ascending price format. The most distinctive feature of the mechanism is that winners are determined first, and then additional auxillary auctions are conducted to determine prices. Total number of auctions depends only on the number of goods to be allocated and not on the number of bidders.

1 Introduction

Auctions are among the oldest economic institutions in place—they have been used since antiquity to sell a wide variety of goods. Their basic form has remained unchanged—an extraordinary feat stemming from simplicity in practice and remarkable economic properties. The most important of which are capacities to generate high revenues to the seller and to allocate goods efficiently.

Design and analysis of selling mechanisms that obtain efficiency—a requirement that the good is to be sold to the buyer who values it the most—was almost exclusively a theorist's concern, and this is rapidly changing. The most prominent source of interest is apparent successes of auctions as means of sale in large-scale privatization activities in many countries of the world, including United States, United Kingdom, Russia, and China.[1] In many of these auctions, the seller (a government typically)

*Suggestions and comments are very welcome.

†50 Memorial Drive, Department of Economics, MIT, E52-252C, Cambridge, MA 02142. E-mail: izmalkov@mit.edu.

[1] For example, the sale of air, the third-generation mobile-phone licence auction conducted by British government, raised an estimated 34 billion dollars for five licences or 2.5% of GNP (see Binmore & Klemperer (2002)).

explicitly declares efficiency as the main objective. The most important problem is to find or design an auction that is efficient under particular circumstances and is simple enough to use.

As known from theory, in *the private values setting*—when no buyer's private information affects how others value the assets—many auction forms are efficient: second-price sealed-bid or open ascending price (English) auctions when one unit of a good is for sale, the Vickrey or Ausubel auctions when many units of the same good are being sold, many other forms under specific circumstances (Vickrey (1961), Milgrom & Weber (1982), Ausubel (2002)). When sizeable in value assets are being offered for sale, *the interdependent values setting*—a buyer's private information can be essential in how others are evaluating the assets—seems to be a more plausible description of the environment. For instance, Porter (1995) reports that in U.S. offshore oil and gas lease auctions, "firms are permitted to gather seismic information prior to the sale.... On-site drilling is not permitted, but firms owning adjacent tracts can conduct off-site drilling, which may be informative." Clearly, other firms might be interested in the results of such private tests.

In the interdependent values setting, achieving efficiency is a highly non-trivial problem. All simple auction forms with a remarkable exception of the English auction, that are efficient in the private values setting, cease to be efficient if the values are interdependent even in the case of a single unit for sale. In practice, many units of the same good are often being auctioned simultaneously, spectrum licences is a typical example. Direct simultaneous or sequential application of single-unit efficient auctions for sale of multiple units does not produce an efficient outcome except under very special circumstances. Known mechanisms that are efficient under most general conditions are complex and almost prohibitively unsuitable for actual implementation.

In this paper, I consider a situation when M units of the same good are offered for sale. Each bidder's marginal valuation for an additional unit is non-increasing in the number of units in possession, so the units are substitutes for the bidders. Each bidder receives a one-dimensional private signal that might affect marginal valuations of the others.[2] To *allocate efficiently* in this setting is to assign units to the bidders with M highest *ex post* marginal values.

I present a mechanism that consists of a number of sequential single-unit English auctions. The distinctive feature of the mechanism is that the winners and the prices they pay are determined separately. The main result is that the proposed auction with the English auction with reentry introduced in Izmalkov (2003) as the primary mechanism has an *ex post* equilibrium that is efficient under multi-unit versions of the single-crossing and signal intensity conditions. Similarly to the single-unit case, the single-crossing condition is necessary for efficiency—without it the direct efficient

[2] As indicated in Maskin (1992) and as a direct corollary to the results of Jehiel & Moldovanu (2001), if the private information received by the bidders prior to the sale is multi-dimensional, no individually rational, incentive compatible, balancing the budget and at the same time efficient mechanism exists unless this information can be summarized by a one-dimensional signal.

mechanism or generalized Vickrey-Clark-Groves (*VCG*) mechanism that is individually rational and balances the budget does not exist, the signal intensity condition is relatively weak. The proposed auction is flexible enough, any single-unit efficient mechanism—an English auction with or without reentry, the Vickrey auction in the case of two bidders—can serve as a building block to obtain efficiency under corresponding multi-unit conditions.

The main difficulty in constructing an efficient mechanism in the interdependent values setting is that it must guarantee that the winners pay the Vickrey prices. Perry & Reny (1999) present a revenue equivalence theorem establishing the uniqueness of the marginal prices under ex-post incentive compatibility. Therefore the prices paid by the winners have to be the same as in *VCG* mechanism. The existence of the direct efficient auction under the single-crossing condition was indirectly shown in Crémer & McLean (1985), Ausubel (1999) contains complete description of the mechanism. A major drawback of the direct mechanism is the requirement that the auctioneer knows everything that bidders know about each other. Dasgupta & Maskin (2000) offer a detail-free "contingent bid" mechanism: a bidder submits prices she is willing to pay for different numbers of units conditional on the realized values of all the other bidders, the auctioneer then calculates a fixed point, declares winners and prices. Both the direct mechanism and the "contingent bid" mechanism allocate efficiently provided the multi-unit single-crossing is satisfied. Dasgupta & Maskin (2000) mechanism is a theoretically "perfect" mechanism, it works under most general conditions and not necessarily for identical objects, but it is not practical.

The Vickrey price a winner is obliged to pay for her kth unit is obtained by the following counterfactual exercise. True signal of the winner is being lowered until it reaches a level at which her kth marginal valuation becomes equal to the $(M-k+1)$th highest of the marginal valuations of the others. If the bidder were to have this particular signal, the auctioneer would have been indifferent between allocating a unit to the bidder (kth for her) or to somebody else. The price the bidder is obliged to pay for this unit is the kth marginal value, or, equivalently, the $(M-k+1)$th highest of the marginal valuations of the others, calculated at this "virtual" signal.

In the private values setting, in the process of finding the Vickrey prices, the ranking of the marginal values of the others is preserved when the signal of one of the bidders changes. This allows for the open ascending counterpart of the Vickrey auction, proposed in Ausubel (2002), to work. The Ausubel auction, which is a multi-unit extension of the English auction, relies on the clinching rule. A bidder *clinches* (wins) a unit at the price when some other bidder reduces her demand and the sum of total demands of all the others becomes lower than the number of units available. Immediately thereafter, the total number of units available and the demand of the winner are reduced by one. If no other bidder clinches at this price, the auction continues until all units are allocated.

In the interdependent values setting, since the ranking of the marginal values of the others can change when the signal of some bidder increases, the Ausubel auction generically cannot provide the Vickrey prices and so cannot be efficient.[3] Perry & Reny (2001) recognize that the Ausubel auction is efficient in the important special case of only two bidders in the auction. They propose an open ascending price auction, in which at a given price bidders submit directed demands against each of the other bidders. A bidder clinches a unit when the sum of demands of the others directed at her becomes less than the number of remaining units. When calculating demand against bidder j, bidder i uses information she inferred about private signals of all the others except j from the prices those bidders reduced their demands at. Bidders' values are supposed to be non-degenerate for all M units, so at the start the demand of every bidder against any other bidder is equal to M. As a result, at the moment a bidder clinches a unit, the signals of all the others are truthfully revealed in the proposed equilibrium. Since each pair of bidders literally engages in the two-bidder efficient auction with the signals of the others being fixed, winners clinch their units exactly at the Vickrey prices, and so Perry & Reny (2001) obtain an efficient auction.

Perry & Reny (2002) present another construction of an efficient mechanism in the interdependent values setting, which is a generalization of the Vickrey auction and is a sealed-bid relative of Perry & Reny (2001).[4] Recognizing that in the interdependent values setting the Vickrey auction is efficient with only two bidders present, they propose a two-stage auction. In the first stage players make public announcements in a way the others infer their true signals. For example, they may announce their signals directly. In the second stage, all possible pairs of bidders are formed, each pair plays the Vickrey auction. After all the bids are submitted, an allocation and payments are determined. First, the units are assigned one by one to the bidders. Eventually, bar ties, a bidder wins at least k units if the combined number of bids she does not defeat with her kth highest bid in each of the $N-1$ auctions she participates does not exceed $M-k$. After all units are assigned, prices are determined. A winner of K units pays the sum of Mth, $(M-1)$th, $\ldots$, $(M-K+1)$th highest among the bids submitted against her in all $N-1$ auctions she participated combined.

Perry & Reny (2002), Perry & Reny (2001), and Eső (1999) share a common feature—bidders compete in pairs. While this allows to achieve efficiency, the number of bids or demands of a particular bidder depends on N. The construction presented here has a significant advantage over these mechanisms, all bidders compete simultaneously in each of the single-unit auctions. Therefore, at any given moment any bidder has at most one demand for a single unit, and so complexity of the bidding is similar to the complexity of the English auction. The total number of auctions conducted and so the combined number of bids submitted by each bidder does not depend on the number of bidders. Note also that when $M = 1$ the auction presented here unlike the other mechanisms reduces to a single-unit English auction.

[3]The exact conditions for efficiency of the Ausubel auction are not yet known.

[4]Eső (1999) contains a mechanism that is similar to the one in Perry & Reny (2002). In it, each of the pairs engages in the English auction instead of the sealed-bid Vickrey auction.

The paper is organized as follows. Section 2 describes the considered setup in detail. Section 3 introduces the multi-init auction. Section 4 contains results. Section 5 concludes.

2 Preliminaries

2.1 The environment

The environment considered here is the multi-unit extension of the environment of Izmalkov (2003). There are M homogeneous units of some good offered for sale. There are N potential buyers, each of whom receives a private signal $s_i \in [0, 1]$. Let $\mathcal{N}$ denote the set of buyers. Given the signals $\mathbf{s} = (s_1, s_2, \ldots, s_N)$, the marginal value of the kth unit to bidder i is $V_i^k(\mathbf{s})$. If bidder i receives K units and pays p in total, her payoff is $\sum_{k=1}^{K} V_i^k(\mathbf{s}) - p$. The valuation functions $\{V_i^k\}_{i=1..N}^{k=1..M}$ are assumed to be commonly known among all the bidders. How signals are distributed and whether that distribution is commonly known among the bidders is not important for the results since the equilibrium presented here is an *ex post* equilibrium, see Appendix A.1 for the definition and discussion. It is important that the signals are one-dimensional, see Footnote 2; $[0, 1]$ is only a normalization, the support need not be bounded as long as the value functions are bounded.

For any player i value functions V_i^k have the following properties: for any k, V_i^k is twice-differentiable, $V_i^k(0, \ldots, 0) = 0$ and

$$\forall i \ \frac{\partial V_i^k}{\partial s_i} > 0, \qquad \forall j \neq i \ \frac{\partial V_i^k}{\partial s_j} \geq 0. \tag{1}$$

The marginal value functions are assumed to be non-increasing for each additional unit (no complementarities):

$$V_i^k \geq V_i^{k+1} \text{ for all } k < M. \tag{2}$$

Denote $\mathbf{V^k}$ to be the profile of value functions $V_i^{k_i}$ picked for every bidder $i \in \mathcal{N}$ according to vector $\mathbf{k} = (k_1, \ldots, k_N)$. If every bidder $i \in \mathcal{N}$ has been assigned $(k_i - 1)$ units, then profile $\mathbf{V^k}$ is the set of marginal valuations of all bidders for an additional unit.

We call $\mathbf{k}$ *admissible* if $k_i \geq 1$ for all i and $0 \leq \sum_{i=1}^{N}(k_i - 1) < M$, that is the total number of "allocated" units is less than M. Given $\mathbf{k}$, one can define winners circle $\mathcal{I}^{\mathbf{k}}(\mathbf{s})$ as the set of bidders with maximal values at $\mathbf{s}$ according to $\mathbf{V^k}(\mathbf{s})$.

We also assume that for any admissible $\mathbf{k}$, a player with the lowest possible own signal, $s_i = 0$, cannot be a *"free rider"*—have the highest value at $\mathbf{s}$ (unless $\mathbf{s} = \mathbf{0}$) according to $\mathbf{V^k}(\mathbf{s})$, and that the set of functions $\mathbf{V}_{\mathcal{J}}^{\mathbf{k}_{\mathcal{J}}}(\mathbf{s})$ is *regular* at $\mathbf{s}_{\mathcal{J}}$ for any

$\mathcal{J} \subset \mathcal{I}^{\mathbf{k}}(\mathbf{s})$, that is det $\frac{d\mathbf{V}_{\mathcal{J}}^{\mathbf{k}_{\mathcal{J}}}(\mathbf{s}_{\mathcal{J}},\mathbf{s}_{-\mathcal{J}})}{d\mathbf{s}_{\mathcal{J}}} \neq 0$, where $\mathbf{s}_{\mathcal{J}}$ and $\mathbf{s}_{-\mathcal{J}}$ denote signals of bidders from $\mathcal{J}$ and $\mathcal{N} \setminus \mathcal{J}$ correspondingly, $\mathbf{s} = (\mathbf{s}_{\mathcal{J}}, \mathbf{s}_{-\mathcal{J}})$.[5]

The main result relies on the following multi-unit versions of the single-crossing (*A1*) and signal intensity (*A2*) assumptions.[6]

A1M (*single-crossing*) For any admissible $\mathbf{k}$, for all $\mathbf{s}$ and any pair of players $\{i, j\} \subset \mathcal{I}^{\mathbf{k}}(\mathbf{s})$,

$$\frac{\partial V_i^{k_i}(\mathbf{s})}{\partial s_i} > \frac{\partial V_j^{k_j}(\mathbf{s})}{\partial s_i}. \tag{3}$$

A2M (*signal intensity*) For any admissible $\mathbf{k}$, for all $\mathbf{s}$ and $i \in \mathcal{I}^{\mathbf{k}}(\mathbf{s})$ there exists an $\varepsilon > 0$ such that for all $\mathbf{s}'$ satisfying (i) $s_i < s_i' < s_i + \varepsilon$; (ii) $\forall j \in \mathcal{I}(\mathbf{s}) \setminus \{i\}$, $V_j^{k_j}(\mathbf{s}') = V_j^{k_j}(\mathbf{s})$ and (iii) $\forall l \notin \mathcal{I}^{\mathbf{k}}(\mathbf{s})$, $s_l' = s_l$, it is the case that $\mathcal{I}^{\mathbf{k}}(\mathbf{s}') = \{i\}$.

The signal intensity condition requires that if we increase the signal s_i of some member i of the winners' circle $\mathcal{I}^{\mathbf{k}}(\mathbf{s})$ and change the signals s_j of other players $j \in \mathcal{I}^{\mathbf{k}}(\mathbf{s})$ in a way that their marginal values are unchanged, offsetting the effect of the change in s_i, then i's marginal value, $V_i^{k_i}$, goes up (the signals of all players $l \notin \mathcal{I}(\mathbf{s})$ are kept fixed). In other words, the combined effect on player i's marginal value, directly from the increase in her own signal and indirectly through the changes in signals of other members of the winners' circle, is positive: the direct effect outweighs the indirect effect.

It is useful to view the signal intensity condition as a *dual* to the single-crossing condition. Single-crossing prescribes what should happen if we fix the *signals* of everyone else in the winners' circle and increase the signal of one particular player—she should become the sole member of the winners' circle. Signal intensity prescribes what should happen if we fix the *values* of everyone else in the winners' circle and increase the signal of one particular player—again, she should become the sole member of the winners' circle.

Note that these conditions are required to be satisfied only for admissible $\mathbf{k}$—in situations when less than the total number of units has been "allocated." The single-crossing condition in this "weak" form is the same condition that is required in Dasgupta & Maskin (2000), and that guarantees existence of the direct efficient mechanism.

[5] Regularity guarantees that equilibrium strategies are well defined. "No free riders" assumption together with $V_i^{k_i}(\mathbf{0}) = 0$ are to ensure complete and analytically attractive consideration of the most general case. When these are not satisfied, a similar analysis can be provided (assumptions *A1M* and *A2M* below will need to be strengthened somewhat), it will give a rise to the existence of "waiting" efficient equilibria, where one of the players' strategy at some point of the auction is to "wait" (or stay forever) until some other bidder exits the auction.

[6] For detailed description, discussion, and comparison of these assumptions the reader is referred to Izmalkov (2003).

2.2 Overture

The difficulty in allocating multiple units efficiently via simple mechanisms arises from the fact that the prices the winners have to pay are not easily determined. If the mechanism has an ex post equilibrium that is efficient, it has to ensure that every winner pays the Vickrey price—the same price she is obliged to pay in the generalized *VCG* mechanism.

To illustrate the associated problems consider a simple situation when there are only two units for sale and three bidders. Suppose that at a particular $\mathbf{s}$ bidders 1 and 2 have the first and the second highest marginal values among all the bidders—they have to be the winners if efficiency is the allocation principle. The Vickrey price, p_1, that bidder 1 has to pay for her unit, is equal to her marginal value for the first unit, V_1^1, calculated at $(s_1^1, \mathbf{s}_{-1})$, where s_1^1 is the signal at which $V_1^1(s_1^1, \mathbf{s}_{-1})$ is equal to the *second* highest of all marginal valuations of bidders 2 and 3. When values are interdependent, the *first* highest valuation of the other bidders at $(s_1^1, \mathbf{s}_{-1})$ need not be V_2^1. This scenario may arise in a situation when V_2^1 is more sensitive to the value of s_i than V_3^1 is.

Suppose that it is indeed the case that $V_3^1(s_1^1, \mathbf{s}_{-1}) > V_2^1(s_1^1, \mathbf{s}_{-1})$, while $V_3^1(s_1, \mathbf{s}_{-1}) < V_2^1(s_1, \mathbf{s}_{-1})$. Suppose also that the marginal valuations for the second unit are equal to 0 for each of the bidders at any $\mathbf{s}$.

Then, for any s_1' slightly above s_1^1, an efficient mechanism has to assign one unit to bidders 1 and 3 each as follows from continuity of the value functions and from (*A1M*). If $s_1'' < s_1$, (*A1M*) implies that no units are to be assigned to bidder 1. To differentiate among three types of bidder 1, s_1, s_1', and s_1'', in order to determine correctly the winners, and at the same time to charge the same price p_1 in the first two cases—is practically a hopeless task for a dynamic mechanism such as an open ascending price auction. This is exactly the reason why the Ausubel auction fails to be efficient when values are interdependent. Indeed, bidder 1 with a signal equal to s_1 or s_1' has to win a unit at $p = p_1$. This can happen only if one of the bidders 2 and 3 reduces her demand to 0 at p_1, but then that bidder would not be able to win when the signal of bidder 1 is such that it is efficient to do so.

An assumption that each bidder demands at most one unit is important for the above argument, and it does not comply with (1). The fact that the signals of the bidders may be revealed in the early stages of the bidding, prior to any unit be clinched, seems to provide an opportunity for the coordination among the bidders on who should win and who should exit at certain prices. This is not correct in general due to an additional fact that the Vickrey prices are generically different among the bidders. As a result, particularly in the case when marginal valuations for the first unit of all the bidders are the three highest marginal valuations at almost all $\mathbf{s}$, the Ausubel auction is not efficient. Indeed, even if all the signals are revealed before any of the units are clinched, at the price the first unit is assigned the demands of each of the bidders are equal to 1. One of the bidders reduces her demand to 0 at this price, and the winner is randomly determined among the other two bidders. The price the winner pays cannot be the Vickrey price for both bidders since those are different.

To summarize, in an efficient mechanism, winners are determined by comparison of marginal values at the actual $\mathbf{s}$, while the prices they have to pay are determined separately, under a counterfactual exercise—bidder i has to pay for the kth unit she won the price p_i^k that is equal to her marginal valuation $V_i^k(s_i^k, \mathbf{s}_{-i})$, where s_i^k is the signal with which she would marginally win k units: $V_i^k(s_i^k, \mathbf{s}_{-i})$ is equal to the $(M-k+1)$th highest of the marginal valuations of the others.

The proposed auction separates processes of determining winners and prices, and by doing so achieves efficiency.

3 An open ascending price efficient auction

3.1 Structure

Overview. The proposed mechanism consists of a series of sequential single-unit English auctions with reentry. These auctions are subdivided into two phases. Phase 1 is composed of M auctions, each deciding a winner for one of the units. During Phase 2 the prices the winners have to pay are determined. The number and particular details of the auctions in Phase 2 depend on the results of Phase 1.

Rules. Each individual auction is conducted according to the rules of the English auction with reentry. The full set of rules is presented in Appendix A.2.1. The following is the abridged version that captures all essential features. The full set differs only to the extent it accounts for all technical complications related to multiple simultaneous entries and exits.

1. The auctioneer sets a low initial price, say zero, and constantly raises it. It is convenient to think of an automatic price clock publicly showing the current price.

2. At any price, each bidder is either active (willing to buy the unit at this price) or not. All bidders are active at a price of zero. As the price increases any bidder can exit (become inactive) and reenter (become active) the auction at will. The activity statuses of all bidders are commonly observed and known.

3. The auction ends (the price clock stops) when at most one person is active. The winner is the only remaining person (or is randomly chosen among those who exited last) and pays the price at which the last exit took place.

The identity of the winner and the winning price are publicly announced after each individual auction. So, before the start of a particular auction all public information of every previous auction is commonly known.

Phase 1. This phase spans M auctions. The units are sold one-by-one. The winner of each of the auctions is assigned a unit.

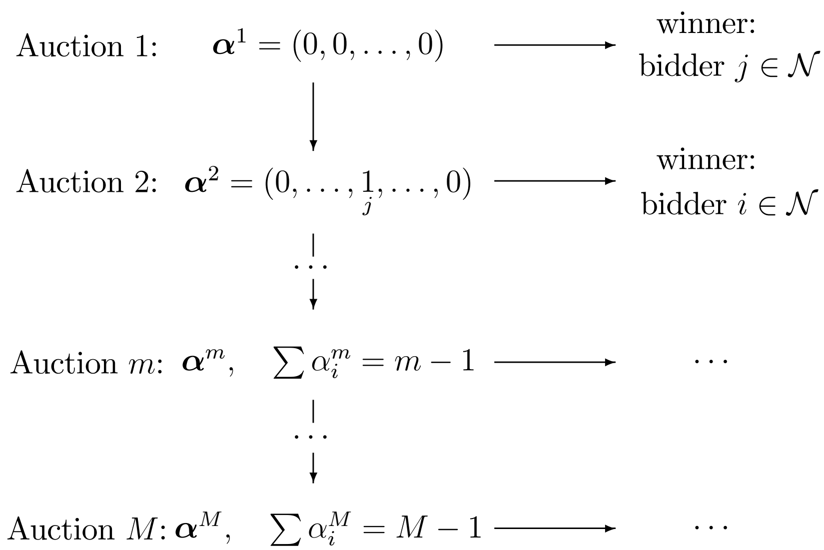

Figure 1: Phase 1

It is convenient to think that before the start of each of the auctions the auctioneer announces how many units have been already assigned and to whom. Let $\boldsymbol{\alpha}^m = (\alpha_1^m, \dots, \alpha_N^m)$ denote the announcement of the auctioneer at the start of auction m, where α_i^m is the number of units assigned to bidder i after $m-1$ auctions, $\sum_{i=1}^N \alpha_i^m = m - 1$, and $\boldsymbol{\alpha}^1 = (0, \dots, 0)$. The outline of Phase 1 is shown in Figure 1.

Phase 2. As a result of Phase 1 all M units are assigned to the bidders. All bidders who have won at least one unit are arbitrarily ordered. According to this order, for every winner a number of auxiliary auctions is conducted to determine the price for each unit won by this bidder. Suppose bidder $j \in \mathcal{N}$ has won K units. Then for each of her units, from 1st to Kth, the following procedure applies, outlined in Figure 2.

The price for the kth unit won by bidder j, p_j^k, is determined as follows. At the start of auxiliary auction 1, the auctioneer announces $\boldsymbol{\alpha}^1 = (0, \dots, \underset{j}{k-1}, \dots, 0)$—as if bidder j has won $k-1$ units, while any other bidder has won no units. A single-unit English auction with reentry is conducted with a restriction on the strategy of bidder j: she is prohibited from exiting, and so bidder j necessarily wins. Let $\mathcal{A}^1$ be the set of runners-up—those bidders who exited last, at price p^1. Most likely but not necessarily, set $\mathcal{A}^1$ is a singleton. Each runner-up is "assigned" a unit. If the total number of units "assigned" equals or exceeds M, $\sum_{i=1}^N \alpha_i^1 + \#\mathcal{A}^1 = k - 1 + \#\mathcal{A}^1 \geq M$, then p_j^k is determined, it is equal to p^1.

Otherwise, auxiliary auction 2 is conducted with the announcement $\boldsymbol{\alpha}^2$, where $\alpha_i^2 = \alpha_i^1 + 1$ for every runner-up $i \in \mathcal{A}$, $\alpha_i^2 = \alpha_i^1$ for $i \notin \mathcal{A}$, and the restriction that

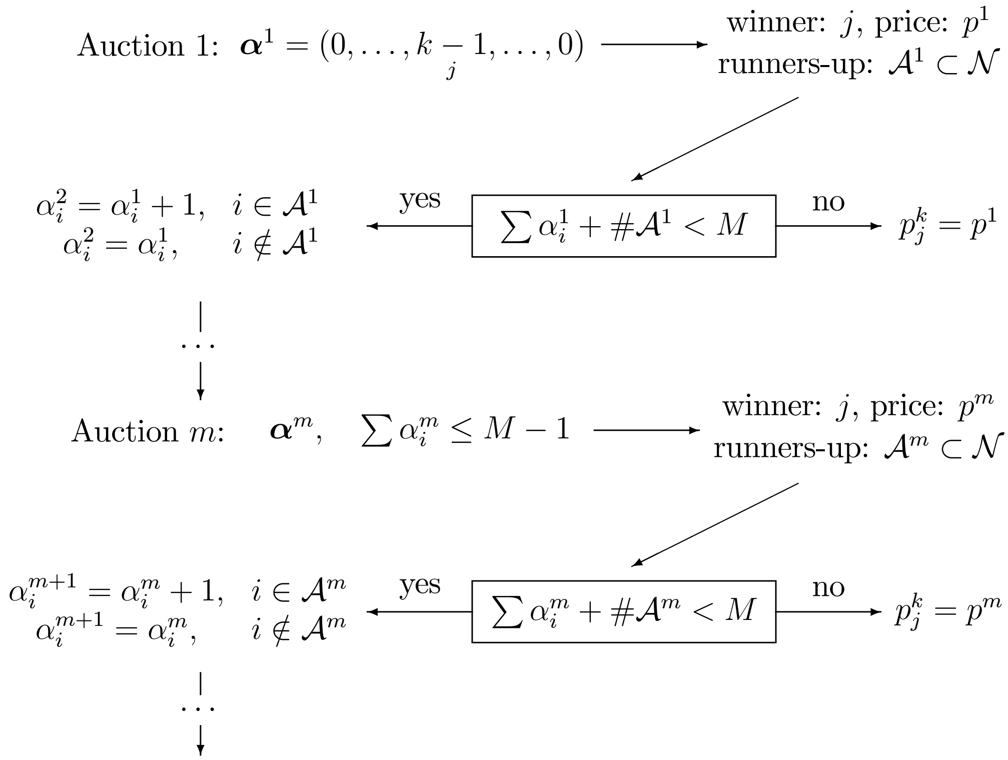

Figure 2: Phase 2

bidder j necessarily wins. Every runner-up is "assigned" a unit, ... This is repeated until as a result of auxiliary auction m, $\sum_{i=1}^{N} \alpha_i^m + \#\mathcal{A}^m$ equals or exceeds M, where $\mathcal{A}^m$ is the set of runners-up. Price p_j^k is set to be equal to p^m. Since there is always at least one runner-up, the total number of auxiliary auctions required to determine p_j^k does not exceed $M - k + 1$.

3.2 An illustration

Here we preview how the presented mechanism is supposed to work and provide an example. The complete proof is in Section 4.

In the beginning of every auction in Phase 1, bidders know how many units were previously allocated and to whom. At the start of auction m they obtain vector $\mathbf{k}^m$, where for every bidder i, $k_i^m = \alpha_i^m + 1$. In auction m, therefore, bidder i is bidding for her k_ith unit, so $\mathbf{V}^{\mathbf{k}^m}$ is an accurate representation of the value functions according to which bidders are going to bid. If, bidders follow the suggested strategies—the strategies that form an efficient equilibrium in each of the single-unit auctions according to $\mathbf{V}^{\mathbf{k}^m}$, the winner of auction 1 is the bidder with the highest marginal value among all the bidders, the winner of auction 2 is the bidder with the second highest value, and so on—the efficient allocation is obtained.

In Phase 2, during the procedure to determine p_j^k, before the start of auxiliary auction m, bidders can calculate $\mathbf{k}^m = \boldsymbol{\alpha}^m + \mathbf{1}$. If all the bidders, except j (she does not exit), follow the suggested strategies according to $\mathbf{V}^{\mathbf{k}^m}$, bidder j wins at price p^m, at which $p^m = V_j^k(s_j^m, \mathbf{s}_{-j}) = \max_{i \neq j} V_i^{k_i}(s_j^m, \mathbf{s}_{-j})$ for some signal s_j^m. After the first auction, if the number of runners-up is $n^1 = \#\mathcal{A}^1$, the value $V_j^k(s_j^1, \mathbf{s}_{-j})$ of bidder j is equal to the (n^1)th highest of the marginal values of the remaining bidders. If $(k-1) + n^1 \geq M$, or $n^1 \geq M - k + 1$, the price is calculated, $p_j^k = p^1$. Otherwise all runners-up are "assigned" one unit each, and another auction is conducted with the announcement $\boldsymbol{\alpha}^2$, and so on. As we will show, in auxiliary auction m, the final price, p^m, does not exceed any of the prices from the previous auctions, and so the marginal value of any runner-up in the previous auctions for the corresponding unit calculated at $(s_j^m, \mathbf{s}_{-j})$ is not lower than p^m. Therefore, $V_j^m(s_j^m, \mathbf{s}_{-j})$ is equal to the nth highest of the marginal values of the others, where n is the number of runners-up in the current and all previous auxiliary auctions, $n = \sum_{l=1}^m n^l$, $n^l = \#\mathcal{A}^l$. The procedure ends after auxiliary auction m such that $\sum_{i=1}^N \alpha_i^m < M$ and $\sum_{i=1}^N \alpha_i^m + n^m \geq M$. Since $\sum_{i=1}^N \alpha_i^m + n^m = (k-1) + \sum_{l=1}^m n^l = k - 1 + n$, we have that $n \geq M - k + 1$. Therefore, if p_j^k is set to equal $p^m = V_j^k(s_j^m, \mathbf{s}_{-j})$, it is also equal to the $(M-k+1)$th highest of the marginal values of the others, and so p_j^k is the Vickrey price.

The argument that the collections of the above strategies for each bidder form an ex post equilibrium is based on the following two main principles that are at the core of the presented construction. First, none of the winners in Phase 1 can affect the prices they have to pay for the units they won, since each winner is required to stay active in the corresponding auxiliary auctions and, thus, cannot affect the prices she has to pay. Therefore, only the number of units won in Phase 1 matters for the final payoff. Second, by virtue of the Vickrey prices, if the marginal valuation of the bidder for a given unit is among the M highest for all the bidders, then the Vickrey price for this unit is not higher than the value, otherwise it is not lower. Therefore, if a bidder wins more units that she is supposed to according to the efficient allocation, she pays more for additional units than receives in value. Similarly, if a bidder wins less units than she should, she forfeits some profits.

The following example illustrates how the proposed auction operates.

Example 1 *Suppose there are two units of the good offered for sale with three bidders interested. Their value functions are:*

$$\begin{aligned} V_1^1 &= s_1, & V_1^2 &= \tfrac{1}{2}s_1, \\ V_2^1 &= s_2 + \tfrac{1}{3}s_1, & V_2^2 &= \tfrac{1}{2}s_2, \\ V_3^1 &= s_3, & V_3^2 &= \tfrac{1}{2}s_3, \end{aligned}$$

Suppose that $s_i \in [0, 100]$ for each bidder $i = 1..3$.[7] *It is routine to verify that (*A1M*) and (*A2M*) hold.*

[7] This is only a renormalization intended to make the example more transparent.

Consider a particular profile of realized signals, $\mathbf{s} = (60, 20, 35)$. The two highest marginal values are $V_1^1 = 60$ and $V_2^1 = 40$, so both bidder 1 and bidder 2 have to win one unit each to obtain an efficient allocation.

Phase 1 consists of two auctions. In the first auction, the auctioneer announces $\boldsymbol{\alpha}^1 = (0, 0, 0)$, so $\mathbf{k}^1 = (1, 1, 1)$, and each player bids according to V_i^1. Bidder 1 and bidder 3 remain active until price reaches their values, while bidder 2, when bidder 1 is active, infers that $V_1^1 = s_1 \geq p$ and $V_2^1 \geq 20 + \frac{1}{3}p$. The strategy of bidder 2 (if bidder 1 is active) calls on her to remain active as long as $p \leq 20 + \frac{1}{3}p$, or until price reaches 30. Therefore, bidder 2 exits first at $p_2 = 30$, then bidder 3 exits at $p_3 = 35 = p^1$. Bidder 1 wins, she is assigned a unit.

In the second auction, the auctioneer announces $\boldsymbol{\alpha}^2 = (1, 0, 0)$, so $\mathbf{k}^2 = (2, 1, 1)$, and players bid according to V_1^2, V_2^1, V_3^1. Similarly to the above bidders 1 and 3 stay active till the price reaches their values, and bidder 2 makes inferences about the s_1. Now, at p, $V_1^2 = \frac{1}{2}s_1 \geq p$ implies $V_2^1 \geq 20 + \frac{2}{3}p$, and so bidder 2 stays active until $p = 60$ if bidder 1 does not exit earlier. As a result, bidder 1 exits first at $p_1 = 30$. After inferring s_1, bidder 2 stays until price reaches her value. She wins after bidder 3 exits at $p_3 = 35 = p^2$ and is assigned a unit.

In Phase 2 the order at which p_1^1 and p_2^1 are determined is irrelevant. Suppose first p_1^1 is being calculated. In the first auxiliary auction (third overall), the auctioneer announces $\boldsymbol{\alpha}^1 = (0, 0, 0)$. This auction proceeds exactly as the first auction in Phase 1, so there is actually no need to conduct it. In any case, the runner-up is bidder 3. In the second auxiliary auction (forth overall), the auctioneer announces $\boldsymbol{\alpha}^1 = (0, 0, 1)$, so $\mathbf{k}^1 = (1, 1, 2)$, and players bid according to V_1^1, V_2^1, V_3^2. Now it is bidder 3 who exits first at $p_3 = V_3^2 = \frac{1}{2}s_3 = 17.5$. Then bidder 2 exits at $p_2 = 30 = p^2$, and this determines the price for the unit won by bidder 1, $p_1^1 = 30$. To verify that this is indeed the Vickrey price note that at $V_1^1 = s_1^1 = 30$, $V_2^1 = 30$, $V_3^1 = 35$ and the other marginal values are lower. So V_1^1 at $s_1^1 = 30$ equals the second highest of the marginal values of bidders 2 and 3.

To determine p_2^1, in the first auxiliary auction (fifth overall), the auctioneer announces $\boldsymbol{\alpha}^1 = (0, 0, 0)$. This auction, however, proceeds differently then before since bidder 2 has to remain active. As a result, bidder 3 exits first at $p_3 = 35$, then bidder 1 exits at $p_1 = 60 = p^1$. The second auxiliary auction (sixth overall) is conducted with $\boldsymbol{\alpha}^2 = (1, 0, 0)$, it will proceed exactly as the second auction of Phase 1. As a result, $p_2^2 = 35$. It is straightforward to verify that this is the Vickrey price.

This example highlights why it is important to ensure that winners pay the Vickrey prices. If, suppose, bidder 1 had to pay the price that obtains in the auction she wins the unit, $p = 35$, she can profitably deviate. Bidder 1 could exit at, say, $p = 31$ and let bidder 3 win the unit. Then, in the second auction bidder 3 would be bidding for her second unit, and so the auction would be similar to the second auxiliary auction (forth overall) conducted to determine p_1^1. If the others stick to their strategies, bidder 1 would win the unit at $p = 30$ and gain. Note that then bidders 1 and 3 would be the winners, which is not efficient.

4 Results

Theorem 1 *Under the multi-unit single-crossing and signal intensity conditions the proposed multi-unit open ascending price auction has an* ex post *equilibrium that is efficient.*

Proof. First, we define the equilibrium strategies. For every bidder i, the following strategy is proposed:

1. At the beginning of each individual auction based on the announcement, $\boldsymbol{\alpha}$, determine $\mathbf{k}$ and $\mathbf{V^k}$—profile of value functions according to which bidder i and the other bidders are supposed to bid in this auction: for each bidder $j \in \mathcal{N}$, $k_j = \alpha_j + 1$.

2. In each individual auction follow the efficient equilibrium strategy for the single-unit English auction with reentry corresponding to $\mathbf{V^k}$. The brief description of the strategies and of the efficient equilibrium construction is presented in Appendix A.2.2.

If bidders follow equilibrium strategies corresponding to selection $\mathbf{V^k}$, Theorem 1 in Izmalkov (2003) shows that the winner of each individual auction is bidder j with the highest value among $\mathbf{V^k}(\mathbf{s})$,

$$j \in \arg\max_{i \in \mathcal{N}} V_i^{k_i}(\mathbf{s}), \tag{4}$$

and the resulting price satisfies

$$p = V_j^{k_j}(s'_j, \mathbf{s}_{-j}) = \max_{i \neq j} V_i^{k_i}(s'_j, \mathbf{s}_{-j}), \tag{5}$$

where s'_j is a "virtual" signal of bidder j at which the (marginal) value of bidder j exactly equals the highest of the (marginal) values of the others. Signal s'_j is uniquely defined by (5).

Second, we show that at the end of Phase 1, the winners are the bidders with M highest marginal values. Indeed, by (4), the winner of the first unit is the bidder with the highest marginal value, the winner of the second unit is the bidder with the second highest marginal value, and so on.

Third, we show that during Phase 2, the Vickrey prices are determined. Suppose p_j^k, the price bidder j has to pay for the kth unit she won is to be determined. By (5), auxiliary auction m of the described procedure ends at

$$p^m = V_j^k(s_j^m, \mathbf{s}_{-j}) = \max_{i \neq j} V_i^{k_i^m}(s_j^m, \mathbf{s}_{-j}). \tag{6}$$

At p^m, the bidders in $\mathcal{I}^{\mathbf{k}^m}(s_j^m, \mathbf{s}_{-j})$, which includes bidder j and the group of runners-up $\mathcal{A}^m$, have equal and maximal values. For any runner-up, $i \in \mathcal{A}^m$, since $V_i^{k_i}(\cdot) \geq V_i^{k_i+1}(\cdot)$ for any $1 \leq k_i < k_i^m$, we have

$$V_i^{k_i}(s_j^m, \mathbf{s}_{-j}) \geq V_i^{k_i+1}(s_j^m, \mathbf{s}_{-j}) = p^m.$$

Consider any bidder $l \neq j$ who is not a runner-up, $l \notin \mathcal{I}^{\mathbf{k}^m}(s_j^m, \mathbf{s}_{-j})$. If $k_l^m > 1$ then, for each k_l such that $1 \leq k_l < k_l^m$, bidder l must have been a runner-up in one of the previous auctions, say auxiliary auction $r < m$. In that auction, $k_i^r \leq k_i^m$ for each $i \in \mathcal{A}^m$. The resulting price satisfies

$$p^r = V_j^k(s_j^r, \mathbf{s}_{-j}) = V_l^{k_l}(s_j^r, \mathbf{s}_{-j}) \geq V_i^{k_i^r}(s_j^r, \mathbf{s}_{-j}) \geq V_i^{k_i^m}(s_j^r, \mathbf{s}_{-j}). \tag{7}$$

Equation (6), single-crossing (*A1M*), and $V_j^k(s_j^r, \mathbf{s}_{-j}) \geq V_i^{k_i^m}(s_j^r, \mathbf{s}_{-j})$ from (7) imply that $s_j^r \geq s_j^m$, and so $p^r \geq p^m$. As a result, again by (*A1M*) and (7),

$$V_l^{k_l}(s_j^m, \mathbf{s}_{-j}) \geq V_j^k(s_j^m, \mathbf{s}_{-j}) = p^m.$$

How many bidders have their marginal values higher or equal to $V_j^k = p^m$ at $(s_j^m, \mathbf{s}_{-j})$? For each $i \neq j$ and $1 \leq k_i < k_i^m$, $V_i^{k_i} \geq p^m$; and for each $i \in \mathcal{A}^m$, $V_i^{k_i^m} = p^m$. Combined, we have that $\sum_{i \neq j}(k_i^m - 1) + \#\mathcal{A}^m = \sum_{i \neq j} \alpha_i^m + \#\mathcal{A}^m$ marginal values of the bidders other than j are higher or equal to V_j^k. The procedure stops once $\sum_{i \in \mathcal{N}} \alpha_i^m < M$, while $\sum_{i \in \mathcal{N}} \alpha_i^m + \#\mathcal{A}^m \geq M$. Since $\sum_{i \in \mathcal{N}} \alpha_i^m = (k-1) + \sum_{i \neq j} \alpha_i^m$, this is exactly the moment when $\sum_{i \neq j} \alpha_i^m + \#\mathcal{A}^m \geq M - k + 1$. Thus, $p^m = V_j^k(s_j^m, \mathbf{s}_{-j})$ is equal to the $(M-k+1)$th highest marginal value of the other bidders, so $p_j^l \equiv p^m$ is the Vickrey price.

At last, we argue that the proposed strategies form an equilibrium. By construction of Phase 2, no bidder can affect her payoff at that moment. Thus, only the number of units won in Phase 1 affects final payment. Since the resulting prices are the Vickrey prices, bidders receive non-negative payoff for any additional unit as long as the marginal valuation of that unit is one of the M highest. The payoff from any additional unit with marginal valuation not among the M highest is non-positive. Indeed, $p_j^k = V_i^k(s_i^k, \mathbf{s}_{-i})$ for the kth unit won by bidder j, and it is also equal to the $(M-k+1)$th highest of the marginal valuations of the others calculated at $(s_j^k, \mathbf{s}_{-j})$. Therefore, at $(s_j^k, \mathbf{s}_{-j})$, $V_j^k(s_j^k, \mathbf{s}_{-j})$ is exactly the Mth highest of the marginal valuations of all the bidders. By the single-crossing assumption, (*A1M*), if at the true signals, $V_j^k(s_j, \mathbf{s}_{-j})$ is among the M highest marginal values, then $s_j \geq s_j^k$ and $V_j^k(s_j, \mathbf{s}_{-j}) \geq p_j^k$, otherwise $s_j \leq s_j^k$ and $V_j^k(s_j, \mathbf{s}_{-j}) \leq p_j^k$. Thus, any deviation that results in a different number of units won cannot be profitable.

Presented equilibrium is *ex post*, even if $\mathbf{s}$ are commonly known prior to the auction, in each of the single-unit auctions the proposed strategies form an *ex post* equilibrium, and, as follows from the above arguments, no profitable deviations are possible. ■

An important feature of the proposed mechanism is that the total number of auctions is bounded from above by a number that depends only on M and not on the number of participating bidders as in Perry & Reny (2002), where the total number of two-bidder auctions in the second phase is $N(N-1)/2$. Indeed, Phase 1 involves M auctions, Phase 2 involves at most $M \times M$ auctions—if every winner is a different bidder the maximal value of auxiliary procedures is M per bidder. Some auctions need not be run. For instance, there is no need to repeat the first auction in Phase 1. The last auction in Phase 1 immediately determines the Vickrey price that the winner of the last auction has to pay, so for this bidder and her last unit no auxiliary auctions in Phase 2 need to be conducted. Therefore, we have established

Corollary 1 *The total number of the single-unit English auctions needed to achieve efficiency does not exceed* M^2*.*

Remark 1 *The proposed mechanism is quite flexible; it can be built upon any single-unit efficient auction with the appropriate multilateral extensions of assumptions. For example, if the English auction without reentry, the model introduced in Milgrom & Weber (1982) and extensively analyzed thereafter, is used as the primary auction, then the presented mechanism has an* ex post *efficient equilibrium if the generalized single-crossing condition is satisfied for* $\mathbf{V^k}$ *for any admissible* $\mathbf{k}$*.*[8]

5 Concluding remarks

5.1 Variations

In the presented efficient equilibrium, once the winner of the first auction of Phase 1 drops for the first time in the subsequent auctions, the signals of all the bidders are revealed. Since in each individual auction players bid as if they learn their signals anew, there is a lot of redundancy in the information exchange in the proposed mechanism. This suggests that the mechanism can be modified significantly depending on additional desirable properties without sacrificing efficiency. I would like to mention two directions for potential improvement: simplifying the structure and relaxing the assumption.

Here are some changes that can be made to the proposed auction. Phase 1 can be modified as follows. Conduct the first auction as before, suppose bidder j is the winner. Then conduct the auction with the announcement (not an actual assignment)

[8] Birulin & Izmalkov (2003) introduce the generalized single-crossing condition and show that it is both necessary and sufficient condition for efficiency of the English auction without reentry. In the single-unit case it requires that: if starting from a signal profile where the values of a group of bidders are equal and maximal the signals of a subset of the group are increased slightly, then no bidder outside of the subset can attain the value higher than the maximal value attained among the bidders from the subset. The generalized single-crossing condition both implies the (pairwise) single-crossing condition and reduces to it in the case of two bidders.

that $M-1$ units has been allocated to bidder j until she exits for the first time. If this happens, stop. As a result, the true signals of all the bidders are supposedly inferred. Otherwise, bidder j wins all M units, in which case the minimal signal she may have to win all M units is inferred. Whether the winner has this inferred signal or her true signal does not matter when determining the efficient allocation and the Vickrey prices as long as she obtains all the units.

As a result of this "abridged" Phase 1 all necessary information (for the bidders, not the auctioneer) to determine all the winners and the Vickrey prices is revealed. Phase 2, then, can be conducted in any manner that ensures that the winners of all the units pay the Vickrey prices. This, in turn, guarantees that the bidders do reveal their information in Phase 1 and that the efficient allocation is obtained. For example, Stage 2 of the sealed-bid auction proposed in Perry & Reny (2002) or a simpler version of the ascending auction proposed in Perry & Reny (2001) can be used. In the latter, since the signals are supposedly revealed, there is no need to make inferences, the bidders can simply reduce their demands in any manner that guarantees that every winner j clinches her kth unit at the Vickrey price p_j^k. Alternatively, the auctioneer can ask each bidder to propose the allocation and the payments. If the reports of at least $N-1$ bidders coincide, they define the outcome, otherwise conduct one of the above alternatives to Phase 2. It is straightforward to show that the strategies to bid as proposed in each of the auctions of Phase 1 and then to report the correct allocation and the prices based on the revealed signals form an *ex post* efficient equilibrium.

The proposed mechanism relies on the two main assumptions, the single-crossing (*A1M*) and signal intensity (*A2M*) conditions. And while the single-crossing is indispensable, the signal intensity is only necessary for the information processing to work smoothly in each of the single-unit English auctions. Once the structure is modified, (*A2M*) can potentially be relaxed. In particular, for the modified as above Phase 1 to work, it suffices to require (*A2M*) to be satisfied only for a subset of admissible $\mathbf{k}$: only for $\mathbf{k}=(1,\ldots,1)$ and $\mathbf{k}=(1,\ldots,\underset{j}{M},\ldots,1)$ for any $j\in\mathcal{N}$, since only auctions corresponding to $\mathbf{V^k}$ for these $\mathbf{k}$ may be conducted in Phase 1. For Phase 2, (*A2M*) is not required for the modifications proposed above.

While any of such modifications are potentially simplify the structure of the game and enlarge the space of applicability, whether it will be easier for the bidders to calculate and follow equilibrium strategies is unclear. At the same time,

Remark 2 *The structure of the proposed mechanism while potentially redundant is uniform. Each of the single-unit auctions has the same structure, and the requirements imposed on the bidders are the same and simple. The bidders, in each of the auctions, need only to decide whether they are willing to pay for an additional unit of the good the price that shows on the clock or not given their inferences (beliefs) about realized signals of the others at that time.*

5.2 On incentives to follow equilibrium strategies

The proposed mechanism has the following weakness: after all the signals are revealed, which happens quite early in the game, the most of the bidders are indifferent between following the proposed equilibrium strategies and not participating. And, in particular, during Phase 2, no bidder has strong incentives to participate. This is a typical problem, to somewhat lesser degree the same applies to Stage 2 of Perry & Reny (2002) and to the ascending price auction of Perry & Reny (2001) after all the bidders reduced one of their demands at least once.

The fact that the proposed auction has a uniform construction can actually help to reduce this problem significantly. Suppose that instead of bidding in person in each of the individual auctions each player is required to submit a program to bid in her place. Such programs would be required to be capable to bid in a single-unit English auction with reentry. The only input to the programs at the start of each individual auction is the announcement $\boldsymbol{\alpha}$ by the auctioneer.

Clearly, a collection of programs that at each individual auction calculate $\mathbf{k} = \boldsymbol{\alpha} + \mathbf{1}$, and then follow the efficient equilibrium strategies of the single-unit auction corresponding to the selection $\mathbf{V}^{\mathbf{k}}$, forms an *ex post* efficient equilibrium—no bidder is willing to change her program.

The major difference is that the programs can be copied and so a new copy for each of the bidders can be used in each of the auctions. As a result, the programs would not have a knowledge of the information revealed in any of the previous auctions and whether the currently conducted auction is the part of Phase 1 or Phase 2. The only information from the previous auctions the program can infer contains in the announcement $\boldsymbol{\alpha}$. A bidder cannot benefit from designing a program that infers such information if others submit the programs as proposed. In addition, the auctioneer, if necessary, can conduct "bogus" auctions with arbitrary $\boldsymbol{\alpha}$ to make any inferences out of $\boldsymbol{\alpha}$ sufficiently noisy. Therefore, each bidder will have strong incentives to submit a program that in each individual auction follows the equilibrium strategy.

5.3 The private values setting

Continuing the argument of Remark 1, in the private values setting the second-price sealed-bid auction can be chosen as the primary auction. Then, the presented mechanism provides an alternative to the Vickrey auction. While the alternative requires more bids to be submitted by each bidder, it also has an advantage—the bidders disclose less information about their value functions than in the Vickrey auction, where they reveal everything. Rothkopf, Teisberg & Kahn (1990) argue that because of this the Vickrey auction is rarely used in practice, the bidders, fearing that the seller or third parties can take an advantage of them, have an incentive to shade their bids.

In the proposed auction the above concern while not eliminated is reduced. Compared to the combined number of bids, $N \times M$, in the Vickrey auction, the total number of different bids submitted by all the bidders in the sequential mechanism does not exceed $N + 2M - 2$. This number corresponds to the worst-case scenario when all the units are won by one bidder, and the next M highest marginal values belong to some other bidder. If all units are won by different bidders then the combined number of bids does not exceed $N + M$, only the winners of the first $M - 1$ units and possibly the winner of the Mth unit reveal their top two marginal values. It is interesting to note that in the Ausubel auction the bidders reveal the opposite of what they do here—all of the bidders marginal valuations are disclosed except the winning ones.

In the private values setting, since the ranking of the marginal values of the others does not change when the signal and marginal valuations of a particular bidder change, the need for auxiliary auctions is significantly reduced. In Phase 1 there are only $M + 1$ different bids submitted by all the bidders, they coincide with the $M + 1$ highest values among all the bidders. Auxiliary auctions can be constructed in such a way that each additional auction determines the next highest value among all the bidders. In the worst case scenario, when all the units are won by the same bidder, since the price for her first unit needs to be equal to the Mth highest among marginal values of the others, $2M$ of the highest values of all the bidders need to be disclosed, so only $M - 1$ additional auxiliary auctions need to be conducted. In the best case scenario, when all $M + 1$ different bids in Phase 1 came from $M + 1$ different bidders, no auxiliary auctions need to be run, the price each winner pays is the price from the last (Mth) auction.

5.4 Searching for the ultimate mechanism

What are the practical requirements for a selling mechanism? First, it has to achieve its designed goal. Second, the weaker is the set of conditions under which it operates the better. Third, it has to be relatively transparent and simple to use. How does the presented construction performs according to these criteria and how it compares to the existing constructions?

It is efficient, so it works. Assumptions (*A1M*) and (*A2M*) are weak as argued in Izmalkov (2003). The single-crossing condition, exactly in (*A1M*) specification, is necessary for efficiency.[9] The Perry & Reny (2002) mechanism and Perry & Reny (2001) ascending auction use the "strong" form of the single-crossing, required to be satisfied for any pair of bidders, not only for the members of the winners circle, and not only for admissible $\mathbf{k}$. It is not comparable with the presented pair of conditions. The signal intensity condition, (*A2M*), imposes restrictions at exactly the same signal profiles at which the "weak" single-crossing, (*A1M*), does. This set of profiles has measure zero among the set of profiles on which the "strong" single-crossing is imposed. In this sense, the conditions presented here are weaker. Moreover, they can

[9] The single-crossing is also sufficient for efficiency of the Dasgupta & Maskin (2000) and generalized *VCG* mechanisms.

be further relaxed if variations of the presented mechanism are employed.

The proposed auction, since it is based on the English auction, is relatively transparent and simple. It consists of at most M^2 auctions, a number that does not depend on N. The amount of information that is required to be carried from one auction to another is minimal—only the identities of the winners or the runners-up depending on the phase need to be known. Compared to the existing simple constructions it has its advantages and disadvantages. Individual auctions in Perry & Reny (2002), having only two bidders each, are much simpler, but the number of those is of the order N^2. An ascending auction in Perry & Reny (2001) allocates all units simultaneously, and that makes it attractive from the strategic point of view—every bidder has strong incentives to participate at least for a part of the auction, but it also requires more from the bidders. Every buyer makes directed demands against all the others, and the process of information updating is static—inferences occur only when somebody adjusts his or her demands. This, in turn, necessitates imposing the "strong" form of the single-crossing condition to achieve efficiency. Although simple and effective, this type of information processing does not seem to be "natural"—the fact that no bidder changes his or her demands while the price have increased has to convey some information.

Note also, that the presented construction reduces to the single-unit English auction with reentry when $M = 1$. This is unlike Perry & Reny (2002) and Perry & Reny (2001), that are even with $M = 1$ still have pairwise auctions or pairwise demands.

As underscored in Maskin (2003), to find an open auction counterpart to the Vickrey-Clark-Groves mechanism in the case of multiple goods for sale is a very important issue, and a great deal of work remains to be done. I believe that the proposed construction is an important step on the way.

A Appendix

A.1 *Ex post* equilibrium

Definition 1 *A Bayesian-Nash equilibrium* $\boldsymbol{\beta}$ *is called an* ex post *equilibrium if for any realization of signals (types)* $\mathbf{s}$*, even if* $\mathbf{s}$ *were commonly known strategies* $\boldsymbol{\beta}$ *would form a Nash equilibrium.*

The fact that an equilibrium is *ex post* implies that it remains an equilibrium for any possible common prior distribution of the signals. Moreover, it is not necessary that the bidders even share a common prior over the simple type space (the domain of the signals) or any richer type space, an *ex post* equilibrium is a Bayesian-Nash equilibrium on the universal type space (see Bergemann & Morris (2003)).

Therefore, by presenting an efficient equilibrium that is *ex post*, we simultaneously show that the equilibrium is robust to variations in each bidder's assessment of a distribution of the signals of the others, and relax the assumed information burden on the bidder—it is no longer necessary to know or calculate what is the distribution of the signals of the others to play according to the proposed equilibrium.

A.2 English auction with reentry

A.2.1 Rules

1. The auctioneer sets a low initial price, say zero, on a price clock and this is raised.

2. While the clock is ticking, each player is either active, inactive or *suspended*. All players are active at a price of zero. The activity statuses of all players are commonly observed and known. The status of the player can change only when clock is stopped. *A player who is not suspended can stop the clock at any time (say by raising his or her hand).* A suspended player cannot stop the clock.

3. Once the clock stops:

   (a) All suspensions are lifted.

   (b) All players are asked to indicate their intention to be active or inactive once the clock restarts. Players may also indicate that they are undecided. These intentions are communicated to the auctioneer simultaneously and observed by all.

   (c) Players who indicated their intention to be inactive or who were undecided are asked if they wish to change their intentions in light of the information revealed in (b). Undecided players are only allowed to indicate either that they now wish to be active or that they are still undecided.

   (d) If some player changes her intention, then (c) is repeated.

   (e) If no player changes her intention, then these are considered the current statuses. If only one player is undecided, then she must reconsider and must choose to be either active or inactive. Others are not allowed to change their statuses.

   (f) The auctioneer suspends all undecided players if there are at least two active players.

4. The clock restarts as long as there are more than two active players once all players have chosen as in Rule 3. Otherwise, the auction ends and the good is sold at the price showing on the clock. It is awarded to:

   (a) The only remaining active person, if there is such a player.

   (b) A randomly chosen player among those who exited last, if no active players remain.

5. If the number of exits and entries of a player after her last suspension (if she has been suspended before) or after the start of the auction (if she has never been suspended before) exceeds a commonly known number, pre-announced by the auctioneer, the player is automatically suspended at that price.[10]

To see how these rules work consider an auction with 5 players. Players 1 and 2 are active, player 3 is inactive and players 4 and 5 are suspended at the current price, the price clock is ticking. Later, at a price p, player 1 stops the clock. According to Rule 3 the suspensions of players 4 and 5 are lifted, all players are asked to indicate their intentions as in Rule 3b. Suppose players' intentions are as follows: player 2 wants to be active, players 1 and 5—inactive, players 3 and 4 remain undecided. These intentions are observed by all players. Rule 3c applies.

If no players are willing to change their intentions then player 2 as the only active player is declared the winner, the auction ends. Suppose instead, player 3, observing that player 2 will be active, decides to be active as well, and she is the only player to change her intentions in the second phase. As a result the intentions of the players are: players 2 and 3 want to be active, players 1 and 5 want to be inactive, player 4 is undecided.

If in the next stage no player changes her intention, by Rule 3e player 4 is asked to decide on her status. Once she decides, all players' statuses are fixed and, by Rule 4, the clock restarts. Lastly, suppose in the third stage, player 1 becomes undecided, while player 5 changes her intentions to be active, and no other player wants to change her intentions after this. Then both undecided players 1 and 4 get suspended by Rule 3f, the auction restarts with players 2, 3, and 5 being active. Note that Rule 3c ensures that after a finite number of stages no player will change her intention.

A.2.2 Equilibrium construction

This is a brief summary of the equilibrium strategies and information processing in the efficient equilibrium of the single-unit English auction with reentry. For the complete construction, including detailed description of the off-equilibrium information processing, examples, the proof that the strategies are well-defined, and, of course, the proof that these strategies form an efficient *ex post* equilibrium see Izmalkov (2003).

Histories: At any point of the auction, whether the price clock runs or is stopped, the history of all actions that bidders or the auctioneer took prior to this point—stops of the clock, intended choices, exits, entries, suspensions—is public and is common knowledge among the bidders.[11] The strategy of a bidder specifies an action the

[10] Note that the mechanism defined by these rules is "detail-free"—the auctioneer is not required to have or acquire any information about the value functions, distribution of the signals or the actual realization of the signals to make the auction work.

[11] In the sequential construction proposed here, the bidders would actually know all the histories of the previous auctions but this is irrelevant to the equilibrium construction in each of the individual auctions.

bidder takes or will take given current public history: the price at which the bidder will stop the price clock if the clock is running provided no other bidder stops it before, the intended choice—be active, be inactive, or be undecided—if the clock is stopped and given the current set of intended choices of all the bidders.

Running ahead, in the equilibrium, for any $\mathbf{s}$ but the set of measure zero, each bidder when stopping the clock intends to change her status: exit—become inactive if were active, or enter—become active if were inactive. Only when intensions of some bidder to be active or inactive after a certain price depend on whether some other bidder exits or enters at this price, the equilibrium strategies are richer. This provides the reason to express the strategies in a slightly different format without loss of generality. The strategy of a bidder specifies: at any price p with the clock running, the price at which the bidder will change her status—exit (stop the clock and intend to be inactive) if is active, enter if is inactive; at any price p with the clock stopped by some other bidder, to maintain the status. An exception: the situation described above, in which case a player should stop the clock and act depending on the observed intended choices of the others.

In what follows we only describe the main part of the strategies in this simplified form.

Information processing: In the equilibrium bidders are constantly making inferences about signals of the other bidders based on their bidding behavior.

Let $\mathcal{A}$ be the set of active bidders at some price p. The *estimated minimal signals at* p, $\mathbf{x}(p) = (x_1(p), x_2(p), \ldots, x_N(p))$ are defined as follows: (i) $\mathbf{x}(0) = \mathbf{0}$; (ii) if $j \notin \mathcal{A}$, $x_j(p) = x_j(p_j)$, where $p_j < p$ is the price at which player j last exited; (iii) for all active players, $\mathbf{x}_{\mathcal{A}}(p) = (x_i(p))_{i \in \mathcal{A}}$ is a solution to the system of equations

$$\mathbf{V}_{\mathcal{A}}\left(\mathbf{x}_{\mathcal{A}}(p), \mathbf{x}_{-\mathcal{A}}(p)\right) = p \tag{8}$$

Let $w_i(s_i, p) \equiv V_i(s_i, \mathbf{x}_{-i}(p))$ be the *estimated minimal value* of the good to player i when the current price is p.

The set of estimated minimal signals, $\mathbf{x}(p)$, carries all the necessary inferred information to determine the strategies of the bidders. The solution to the system (8) can be calculated as a solution to the system of differential equations,

$$\begin{cases} \frac{d\mathbf{x}_{\mathcal{A}}}{dp} = \left(DV_{\mathcal{A}}\right)^{-1} \cdot \mathbf{1}, \\ \frac{d\mathbf{x}_{-\mathcal{A}}}{dp} = \mathbf{0}, \end{cases} \tag{9}$$

which is unique given an initial condition. During the course of the auction, $\mathbf{x}(p)$ are determined as follows, at $p = 0$, the set of active bidders is $\mathcal{A} = \mathcal{N}$. The solution to (9), $\mathbf{x}(p)$, is calculated with the initial condition $\mathbf{x}(0) = \mathbf{0}$. This continues until some bidder exits, say at p'. Then $\mathcal{A}$ is updated and $\mathbf{x}(p)$ is further calculated with the initial condition equal to the terminal condition of the previous system, and so on. At any price level at which some bidder enters or exits, the set of active bidders, $\mathcal{A}$, is updated, new system (9) being solved with the initial condition equal to the terminal condition of the previous system.

Equilibrium strategies: The proposed strategies are based on the estimated minimal values $w_i(s_i, p)$ derived from the history of play up to price p.

Bidder i should

1. if $w_i(s_i, p) > p$, then be active at p;
2. if $w_i(s_i, p) < p$, then be inactive at p;
3. if $w_i(s_i, p) = p$ and for ε small enough, $w_i(s_i, p - \varepsilon) \leq p - \varepsilon$ and $w_i(s_i, p + \varepsilon) > p + \varepsilon$, then become active at p;
4. if $w_i(s_i, p) = p$ and for ε small enough, $w_i(s_i, p - \varepsilon) \geq p - \varepsilon$ and $w_i(s_i, p + \varepsilon) < p + \varepsilon$, then become inactive at p.

   The strategy can be summarized as: *exit (enter) whenever your estimated minimal value crosses the price from above (below).*

If bidders do follow these strategies, then for each bidder $i \in \mathcal{N}$, $x_i(p)$ indeed represents the minimal signal bidder i could have to be consistent with her status at p, and $w_i(s_i, p)$ is the estimated minimal value of the good to player i at p.

Off-equilibrium information processing: If one of the bidders, say j, deviates from the proposed strategies, this may affect information processing and, potentially, the actions of the others only when the observed action of bidder j is unexpected. There are two main scenarios that can happen: bidder j does not (re)enter when expected, bidder j enters when not expected. By construction, (unexpected) exits do not cause any problems.

In the first case, bidder j is treated as being dormant, for the purposes of information processing all the other bidders treat her as an active bidder starting from the price she was supposed to enter till the price she is supposed to exit based on the previous information. It is important, that $x_j(p)$ of bidder j is being updated in the process.

In the second case, bidder j is treated as if she has entered by mistake. All the other bidders expect bidder j to exit immediately, they do not adjust $x_j(p)$ instantaneously. Instead, $x_j(p)$ is being adjusted as long as bidder j stays active in the manner that ensures that at least some other bidder stays active, and so the auction does not end before the full adjustment takes place.

The proof: The argument that these strategies form an equilibrium and it is efficient is based on the fact that, by construction, no bidder can affect the price she pays if she wins and of the others follow the proposed strategies. Moreover, this price is the Vickrey price, and so only for the bidder with the highest value at a given $\mathbf{s}$, this price does not exceed the value. Therefore, no profitable deviation is possible for any bidder: only deviations that change whether a bidder wins the good or not affect the bidder's payoff. The bidder forfeits positive gains if she does not win when it is efficient for her to win; she pays more than the value of the good to her if she wins when it is efficient to allocate to good to somebody else.

The argument that the equilibrium is *ex post* is trivial: at no point the strategies of the bidders depend on what they know or believe the distribution of the signals of the others is. The proposed strategies form a Nash equilibrium even if $\mathbf{s}$ is commonly known prior to the start of the auction.

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