The Race Between Man and Machine: Implications of Technology for Growth, Factor Shares and Employment*

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Abstract

The advent of automation and the simultaneous decline in the labor share and employment among advanced economies raises concerns that labor will be marginalized and made redundant by new technologies. This paper examines this proposition in a task-based framework wherein not only are tasks previously performed by labor automated, but also more complex versions of existing tasks, in which labor has a comparative advantage, can be created. We fully characterize the structure of equilibrium in this model, establishing how the allocation of factors to tasks and factor prices are determined by the available technology and the endogenous choices of firms between capital and labor. We then demonstrate that although automation tends to reduce employment and the share of labor in national income, the creation of more complex tasks has the opposite effect.

Our full model endogenizes the direction of research and development towards automation and the creation of new complex tasks. We show that, under reasonable conditions, there exists a stable balanced growth path in which the two types of innovations go hand-in-hand. Consequently, an increase in automation reduces the wage to rental rate ratio, and thus discourages further automation, encourages greater creation of more labor-intensive tasks, and restores the share of labor in national income and the employment to population ratio back towards their initial values. Though the economy is self-correcting, the equilibrium allocation of research effort is not optimal: to the extent that wages reflect quasi-rents for workers, firms will engage in too much automation. Finally, we extend the model to include workers of different skills. We find that inequality increases during transitions, but the self-correcting forces also serve to limit the increase in inequality over longer periods.

Still in Progress. Comments Welcome.

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1 Introduction

The accelerated automation of tasks performed by labor raises concerns that these new technologies will make labor redundant (e.g., Brynjolfsson and McAfee, 2012, Akst, 2014, Autor, 2015). The recent declines in the share of labor in national income and the employment to population ratio in the US economy, shown in Figure 1, are often interpreted to support the claims that as digital technologies, robotics and artificial intelligence penetrate the economy more deeply, workers will find it increasingly difficult to compete against machines, and their compensation will experience a relative or even absolute decline. Yet, a comprehensive framework incorporating such effects, as well as countervailing forces, remains to be developed. The need for such a framework stems not only from the importance of understanding how and when automation will have these transformative effects on the labor market, but also from the fact that similar claims have been made, but have not always come true, about previous waves of new technologies. Keynes (1930), for example, famously foresaw the steady increase in per capita income in the 20th century from the introduction of new technologies, but incorrectly predicted that this would create widespread technological unemployment as machines replaced men. Economic historian Robert Heilbroner confidently stated in 1965 that “as machines continue to invade society, duplicating greater and greater numbers of social tasks, it is human labor itself — at least, as we now think of ‘labor’ — that is gradually rendered redundant” (quoted in Akst, 2014), while another observer of mid-century automation, economist Ben Seligman, similarly predicted a future of work without men (Seligman, 1966). Wassily Leontief was equally pessimistic about the implications of new machines. He drew an analogy with the technologies of the early 20th century that made horses redundant and speculated “Labor will become less and less important… More and more workers will be replaced by machines. I do not see that new industries can employ everybody who wants a job” (Leontief, 1952).

This paper is a first step in developing a conceptual framework which both shows how machines replace human labor and why this may or may not lead to the disappearance of work and stagnant wages. Our main conceptual innovation is to introduce into a unified framework both automation replacing tasks previously performed by labor and the creation of new complex tasks where labor has a comparative advantage. The role of these new tasks is well illustrated by the technological and organizational changes during the Second Industrial Revolution, which not only involved the replacement of the stagecoach by the railroad, sailboats by steamboats, and of manual dock workers.
Figure 1: Trends in the employment to population ratio among men between 25-54 years, and the labor share in the nonfarm business sector in the United States.

by cranes, but also the creation of new labor-intensive tasks — including a new class of engineers, machinists, repairmen, and conductors as well as modern managers and financiers involved with the introduction and operation of these new technologies (e.g., Landes, 1969, Chandler, 1977, and Mokyr, 1990).

Today, while digital technologies and computer-controlled machines replace labor, we are simultaneously witnessing the emergence of new tasks ranging from engineering and programming functions, to new professional jobs including audio-visual specialists, executive assistants, data administrators and analysts, meeting planners or computer support specialists. In the U.S. labor market, the creation and expansion of these new tasks appears to have played a central role in generating employment. To document this fact, we use data on “task novelty” from Lin (2011), which measures the share of jobs and tasks in an occupation for which there were no previous job titles and which are considered by employers as different from existing ones. For instance, in 2000, about 70% of the tasks performed by computer software developers (an occupational group employing 1 million people at the time) did not appear in the 1990 Index of Occupations and are classified as new. Similarly, radiology technologies are considered new in 1990 and management analysts are new in 1980. Figure 2 shows that in each decade since 1980, employment growth has been faster in occupations with more novel jobs and tasks. The regression line depicts the empirical relationship,
which implies that occupational groups with 10 percentage point more novel jobs at the beginning of each decade grow 5.2% faster (standard error= 1.3%). From 1980 to 2007, employment grew by 17.1%, of which about half (9%) is explained by the additional growth in occupations with more novel tasks and jobs — relative to a benchmark category with no new tasks.³

![Figure 2: Scatter plot of employment growth and the share of novel jobs at the beginning of the decade across 330 occupational groups. Data from 1980 to 1990 (in dark blue), 1990 to 2000 (in blue) and 2000 to 2007 (in light blue, scaled). See the Appendix for sources and data construction details.](image)

This paper develops a tractable but rich framework to study how automation and the creation of new tasks performed by labor impact factor prices, factor shares in national income and employment. In contrast to the more commonly-used models featuring factor-augmenting technological change, in this task-based framework new technologies that facilitate automation not only reduce the share of labor in national income, but may also reduce wages and employment. Conversely, the creation of new labor-intensive tasks increases wages, employment and the share of labor, and may reduce the rate of return to capital. These comparative statics follow because factor prices are determined by the range of tasks performed by capital and labor (see also Acemoglu and Autor, 2011).

We then embed this framework in a dynamic economy in which capital accumulation is endoge-

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³The data for 1980, 1990 and 2000 are from the U.S. Census. The data for 2007 are from the American Community Survey. Additional information on data and samples are provided in the Appendix, where we also present regression evidence to further document the relationship depicted in Figure 2 and its robustness.
nous. We characterize restrictions under which the model delivers balanced growth — which we take to be a good approximation to economic growth in the United States and the United Kingdom over the last two centuries. The key restriction is that there is exponential productivity growth from the creation of new tasks and that the two types of technological changes — automation and creation of new labor-intensive tasks — advance at equal paces.

Our full model endogenizes the rate of improvement of these two types of technologies by marrying our task-based framework with a canonical directed technological setup. This full version of the model remains tractable, and under natural assumptions, generates asymptotically stable balanced growth: in the long run, there is equal advancement of the two types of technologies, and if one type of technology runs ahead of the other, market forces induce countervailing advances in the other type of technology. The economics of these self-correcting forces are instructive and highlight a crucial new force: a wave of automation innovations will push wages down relative to the rental rate of capital. But when technology is endogenous, this will encourage the creation of new tasks. Even though there is an indirect market size effect due to induced capital accumulation, the (factor) price effect dominates and makes it more profitable to use the now cheaper labor. Put differently, in our model where new technologies replace tasks, relative factor prices emerge as the key object regulating the future path of technological change, and thus generate a powerful force that tends to restore employment and the share of labor to their values before the wave of automation innovations.

The most important implication of the stability of the balanced growth path is that, in this model economy, periods in which automation runs ahead of the creation of new more complex tasks will tend to self-correct. Thus, contrary to the increasingly widespread concerns discussed above, our model raises the (theoretical) possibility that rapid automation may not signal the demise of labor, but may be a prelude to a phase of new technologies favoring labor. Our model also highlights other types of structural changes that may have different long-run consequences. For example, if the developments we observe are triggered by a change in the innovation possibilities frontier (the technology of creating technologies) that make it easier to invent automation technologies, then the economy may undergo an extended period of automation and ultimately settle in a new balanced growth path with a greater share of tasks performed by capital and a lower labor share in national income.

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4 Our analysis also reveals another (partially) self-correcting economic force, a productivity effect: automation substitutes the cheaper capital for labor, thus increasing productivity and the demand for all factors. This effect is present in our model throughout, and does not change the fact that automation reduces the share of labor in national income and may even reduce the wage rate. It becomes more powerful in the long run, however, when (and if) the interest rate is constant, e.g., due to capital accumulation, as we show below.

5 The role of technologies replacing tasks in this result can be seen by noting that with factor-augmenting technological changes, the impact on relative factor prices is ambiguous (depending on the elasticity of substitution between factors), and the incentives determining the direction of innovation may be dominated by a strong market size effect (e.g., Acemoglu, 2002).
The final major implication of our framework concerns the efficiency of equilibrium. In addition to the standard inefficiencies due to monopoly markups and appropriability problems in endogenous technological change models, our analysis identifies a new source of inefficiency in the direction of technological advance, pushing towards too much automation and too little creation of new tasks. This is because the market economy responds to factor prices, and thus when wages are high, automation becomes profitable as it enables firms to economize on wage payments; but when some of the wage payments accruing to workers are rents, these incentives generate too much automation relative to what the social planner would prefer.

We consider two extensions of our model. In our baseline framework, all workers have the same skill level. In our first extension, we introduce heterogeneity in skills, and assume that skilled labor has a comparative advantage in newer tasks, which we view as a natural assumption (in particular in view of the evidence presented in the next section). Automation then tends to increase inequality by taking jobs from unskilled labor. The creation of new complex tasks also increases inequality at first, since skilled workers have comparative advantage in such tasks, but reduces it over longer periods as new tasks are standardized and can employ unskilled labor more productively. This extension formalizes claims in the literature suggesting that both automation and new, more complex tasks, increase inequality, but also pointing out that short-run dynamics following such technological changes might be quite different — especially from their medium-term implications in the case of new labor-intensive tasks. Our second extension establishes that under different assumptions on patents and the resulting creative destruction effects, there are similar qualitative forces, but the model might generate multiple and/or unstable steady-state equilibria.

Our paper relates to several literatures. It can be viewed as a combination of task-based models of the labor market with directed technological change models. Task-based models have been developed both in the economic growth and labor literatures, dating back at least to Roy’s seminal work (1955). The first important recent contribution is Zeira (1998), which proposed a model of economic growth based on capital-labor substitution and constitutes a special case of our model when technology (both automation and the set of tasks) are held fixed. Acemoglu and Zilibotti (2000) developed a simple task-based model with endogenous technology and applied it to the study of productivity differences across countries, illustrating the potential mismatch between new technologies and the skills of developing economies (see also Zeira, 2006, Acemoglu, 2010). Autor, Levy and Murnane (2003) suggested that the increase in inequality in the U.S. labor market reflects

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the replacement of routine, labor-intensive tasks by technology. The static, exogenous-technology part of our model is most similar to Acemoglu and Autor (2011). Our full model extends this model not only because of the dynamic equilibrium incorporating directed technological change, but also because tasks are combined with a general elasticity of substitution (a feature that turns out to be important), and because the equilibrium allocation of tasks depends both on factor prices and the state of technology. Acemoglu and Autor’s model, like ours, is one in which a discrete number of labor types are allocated to a continuum of tasks. Costinot and Vogel (2010) develop a complementary model in which skills and tasks form continuum sets.

Three papers from the economic growth literature that are particularly related to our work are Acemoglu (2003a), Jones (2005), and Hemous and Olson (2015). The first two develop growth models in which the aggregate production function is endogenous and, in the long run, adapts to make balanced growth possible. In Jones (2005), this occurs because of endogenous choices about different combinations of activities/technologies being used. In Acemoglu (2003a), which is more closely related, this long-run behavior is a consequence of directed technological change. However, in contrast to the framework here, the two types of technologies that advance endogenously are both factor augmenting. The task-based framework developed here enables us to address questions related to automation and creation of new more complex tasks. In addition, it provides a more robust economic force ensuring the stability of the balanced growth path: in models with factor-augmenting technologies, such as Acemoglu (2003a), stability requires an elasticity of substitution less than 1 for the more abundant factor to command a lower share of national income. In a task-based framework, in contrast, further automation increases the relative price of capital to labor, directly exerting a stabilizing force. Finally, Hemous and Olson (2015) develop a model of automation and horizontal innovation with endogenous technology and use it to study the income inequality consequences of different types of technologies. In their model too, high wages (this time for low-skill workers) encourage automation. But unlike our model, the unbalanced dynamics that this generates are not countered by other types of innovations in the long run.

The rest of the paper is organized as follows. Section 2 presents our basic task-based framework in the context of a static economy. Section 3 introduces capital accumulation and clarifies the structure of task productivity that is necessary for balanced growth in this economy. Section 4 presents our full model with endogenous technology and establishes, under some weak conditions, the existence and stability of a balanced growth path with two types of technologies advancing simultaneously. Section 5 compares the equilibrium composition of new technologies to the social

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8 Acemoglu and Autor (2011), Autor and Dorn (2011), Jaimovich and Siu (2014), Foote and Ryan (2014), Burstein and Vogel (2012), and Burstein, Morales and Vogel (2014) provide various pieces of empirical evidence and quantitative evaluations on the importance of the endogenous allocation of tasks to factors in recent labor market dynamics.

9 See also the recent paper by Hawkins, Ryan and Oh (2015), which shows how a task-based model is more successful than standard models in matching the co-movement of investment and employment at the firm level.
planner’s allocation, establishing that the equilibrium will tend to have too much automation and too little creation of new labor-intensive tasks. Section 6 considers the two extensions mentioned above. Section 7 concludes. The Appendix contains the omitted proofs and the details of the empirical analysis described above.

2 Static Model

We start with a static environment with exogenous technology, which will enable us to introduce our main setup in the simplest fashion and characterize the impact of different types of technological change.

2.1 Environment

The economy contains a unique final good \( Y \), produced by combining a continuum of tasks \( y(i) \) with an elasticity of substitution \( \sigma \in (0, \infty) \). Namely,

\[
Y = \left( \int_{N-1}^{N} y(i) \frac{\sigma-1}{\sigma} \, di \right)^{\frac{\sigma}{\sigma-1}}. \tag{1}
\]

The final good and each task is produced competitively.

The new feature in the aggregate production function (1) is that the index of tasks runs from \( N - 1 \) to \( N \), guaranteeing that the total measure of tasks performed at any point in time is 1. As described in the Introduction, the economy will feature creation of new more complex tasks, represented here by an increase in \( N \). By assuming that the range of tasks is between \( N - 1 \) and \( N \) we are imposing that the creation of new tasks always corresponds to the destruction of the lowest-index task, capturing the replacement or upgrading of an existing task — a feature we model explicitly below.\(^{10}\)

Each task is produced combining labor or capital with a task-specific intermediate \( q(i) \), which embeds the technology used both for production and for the possible automation of tasks. In preparation for our full model in Section 4, we assume that property rights to each intermediate is held by a technology monopolist which can produce it at the marginal cost \( \mu \psi \) in terms of the final good, where \( \mu \in (0, 1) \) and \( \psi > 0 \). The technology for each intermediate can be copied by a fringe of competitive firms, which can produce each at a higher marginal cost of \( \psi \). We assume that \( \mu \) is such that the unconstrained monopoly price of an intermediate would be greater than \( \psi \), ensuring that the unique equilibrium price in the presence of the competitive fringe will be a limit price at \( \psi \) for all types of intermediates.

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\(^{10}\) This formulation imposes that once a new task is created at \( N \), it will automatically be utilized and as a consequence, also replace the lowest available task located at \( N - 1 \). In Section 3, we provide conditions under which firms will indeed prefer to do so (see also footnote 13.
All tasks can be produced by labor. We model the technological constraints on automation by assuming that there exists $I \in [N-1, N]$ such that tasks $i \leq I$ are *technologically automated* in the sense that it is technologically feasible for them to be produced by capital as well. Conversely, tasks $i > I$ are not technologically automated, so cannot be produced by capital. Though tasks $i < I$ are technologically automated, the equilibrium may not involve all of those being produced by capital depending on factor prices as we will next describe.

For tasks $i > I$, which are not technologically automated, the production function takes the form

$$y(i) = B \left[ \eta q(i) \frac{\zeta-1}{\zeta} + (1 - \eta) (\gamma(i) l(i)) \frac{\zeta-1}{\zeta} \right] \frac{\zeta}{\zeta-1}, \quad (2)$$

where $\gamma(i)$ denotes the productivity of labor in task $i$, $\zeta \in (0, \infty)$ is the elasticity of substitution between intermediates and labor, $\eta \in (0, 1)$ is the distribution parameter of this constant elasticity of substitution production function, and finally, $B$ is a normalizing constant, set equal to $B \equiv (1 - \eta)^{\zeta/(1 - \zeta)}$ to simplify the algebra.

In contrast, tasks $i \leq I$ can be produced using labor or capital, and their production function takes the form

$$y(i) = B \left[ \eta q(i) \frac{\zeta-1}{\zeta} + (1 - \eta) (k(i) + \gamma(i) l(i)) \frac{\zeta-1}{\zeta} \right] \frac{\zeta}{\zeta-1}. \quad (3)$$

All of the parameters are thus common between the production function of tasks above and below the threshold $I$, with the only difference that those below $I$ can be produced by capital as well as labor. This feature is embedded in (3) via the assumption that capital and labor are perfect substitutes — so that capital can fully replace labor at the task level.\textsuperscript{11}

Though all of our main results apply with the task production functions (2) and (3), we sometimes illustrate our results with one of two special cases, which lead to easier-to-interpret and particularly insightful expressions (without sacrificing any of the qualitative effects in the model): these are either $\eta \to 0$ (so that the share of revenues going to intermediates is very low) or $\zeta \to 1$ (so that the production functions for tasks become Cobb-Douglas between factors and intermediates).

The key assumption we make throughout is that $\gamma(i)$ is strictly increasing, so that labor has a *comparative advantage* in higher-indexed tasks. In the next section, we strengthen this assumption by imposing a parametric form for $\gamma(i)$, which will ensure that productivity gains from the creation of new tasks is consistent with balanced growth (see in particular, equation (12)), but this functional form assumption plays no role in the analysis in this section. The important implication of strict comparative advantage is that, in equilibrium, there will exist some threshold task $I^* \leq I$ such

\textsuperscript{11}The assumption implicit in writing this expression — that the same intermediate can be used regardless of whether this task is being produced by capital or labor — is for simplicity, and our results remain entirely unchanged if we have separate labor- and capital-specific intermediates at the task level.

Another simplifying feature of (3) is that capital has the same productivity in all tasks — while labor has different productivity. This is a very convenient simplifying assumption, and could be relaxed, though at the cost of additional complexity.
that all tasks \( i \leq I^* \) are produced using capital, while all tasks \( i > I^* \) use labor (see Acemoglu and Zilibotti, 2001, and Acemoglu and Autor, 2011). The argument for the existence of such a threshold in our model is provided in the next subsection.

Figure 1 diagrammatically represents the allocation of tasks to factors and also how, as already noted, the creation of new tasks replaces existing tasks from the bottom of the distribution.

In the static model, we take the capital stock to be fixed at \( K \) (which will be endogenized via household decisions in Section 3). In addition, since we wish to study the impact of new technologies not just on factor prices but also on employment, we assume that the employment level is given by a quasi-labor supply taken to be an increasing function of the wage rate \( W \) relative to capital payments \( rK \), i.e., \( L^s(\frac{W}{rK}) \). This quasi-labor supply curve thus implies that as the wage rate increases relative to payments to capital, the employment level increases as well. Though we impose this as a reduced-form in the text, it is straightforward to derive it from various micro foundations. In Appendix B, we show how an efficiency wage model generates this relationship, while Acemoglu and Restrepo (2016) derives this relationship from a search-matching model in a task-based framework. With this specification of the supply side, capital and labor market clearing

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\(^{12}\)We impose without loss of any generality that when indifferent, firms use capital. This explains our convention of writing that all tasks \( i \leq \tilde{I} \) (rather than \( i < \tilde{I} \)) are produced using capital.
can be written as
\[
\int_{N-1}^{N} k(i) di = K \\
\int_{N-1}^{N} l(i) di = L^s \left( \frac{W}{rK} \right).
\]
We assume that \(L^s(0) > 0\), so that labor never entirely disappears from the economy.

2.2 Equilibrium in the Static Model

We now characterize the equilibrium in this static economy. As noted above, all intermediates will be priced at \(\psi\), and strict comparative advantage ensures that there exists a threshold task \(I^*\) below which all tasks are produced using capital. Given these intermediate prices and the threshold structure, an equilibrium can be represented as a function of the wage rate, \(W\), the rental rate, \(r\), and the equilibrium threshold \(I^*\).

It is most convenient to proceed by characterizing the unit cost of producing tasks as a function of factor prices and the automation technology represented by \(I\). Since tasks are produced competitively, their prices will be equal to these unit costs. Thus

\[
p(i) = \begin{cases} 
    c^u \left( \min \left\{ r, \frac{W}{\gamma(i)} \right\} \right)^{\eta} \psi^{1-\eta} \psi^{1-\eta} & \text{if } i \leq I, \\
    c^u \left( \psi, \frac{W}{\gamma(i)} \right) \equiv \left[ \left( \frac{\eta}{1-\eta} \right)^{\eta} \psi^{1-\eta} + \min \left\{ r, \frac{W}{\gamma(i)} \right\} \right]^{1-\eta} & \text{if } i > I.
\end{cases}
\]

(4)

Here \(c^u\) is the constant unit cost of production of task \(i\), derived from the task production functions, (2) and (3). This unit cost also depends on the price of intermediates, \(\psi\), but we suppress this dependence to simplify notation. The reason why the unit cost for tasks \(i \leq I\) is written as a function of \(\min \left\{ r, \frac{W}{\gamma(i)} \right\}\) is simply that, given perfect substitution between capital and labor, firms will choose whichever factor has a lower effective cost — where the effective cost for labor is \(W/\gamma(i)\) in view of the fact that the productivity of labor in task \(i\) is \(\gamma(i)\). Notice also that this expression distinguishes between \(i \leq I\) and \(i > I\) (and not \(i \leq I^*\) and \(i > I^*\)), since it refers to what is technologically feasible, not to the equilibrium allocation of tasks to capital and labor.

We choose the final good as the numeraire, which from (1) gives the demand for task \(i\) as

\[
y(i) = Y p(i)^{-\sigma}.
\]

(5)

From equations (4) and (5), equilibrium levels of task production can be written as

\[
y(i) = \begin{cases} 
    Y c^u \left( \min \left\{ r, \frac{W}{\gamma(i)} \right\} \right)^{-\sigma} & \text{if } i \leq I, \\
    Y c^u \left( \frac{W}{\gamma(i)} \right)^{-\sigma} & \text{if } i > I.
\end{cases}
\]
These expressions immediately imply that, given strict comparative advantage, there will exist a threshold \( \tilde{I} \) such that tasks below \( I^* \equiv \min\{I, \tilde{I}\} \) will be produced using capital and the remaining more complex tasks will be produced using labor. Specifically, whenever \( \min\{r, \frac{W}{\gamma(i)}\} \) picks \( r \), the relevant task is produced using capital, and whenever it picks \( W/\gamma(i) \), it is produced using labor.\(^13\)

Since \( \gamma(i) \) is strictly increasing, this implies that there exists a threshold \( \tilde{I} \) at which, conditional on technological feasibility, firms are indifferent between using capital and labor. Namely, at task \( \tilde{I} \), we have that \( r = W/\gamma(\tilde{I}) \), or that \( \frac{W}{r} = \gamma(\tilde{I}) \).

Put differently, this condition determines the cost-minimizing allocation of tasks between capital and labor. However, if \( \tilde{I} > I \), firms will not be able to use capital all the way up to task \( \tilde{I} \) and achieve this cost-minimizing allocation because of the constraint imposed by the available automation technology. For this reason, the equilibrium threshold below which tasks are produced using capital is given by

\[
I^* = \min\{I, \tilde{I}\},
\]

meaning that \( I^* = \tilde{I} \) when this is technologically feasible, and \( I^* = I \) otherwise.

To fully characterize the static equilibrium, we next need to derive the quantities of task production as functions of the equilibrium threshold \( I^* \). Factor demands from each intermediate task can be derived from (2) and (3) as

\[
k(i) = \begin{cases} 
Ye^{u(r)c^u \frac{r}{\gamma(i)}} & \text{if } i \leq I^*, \\
0 & \text{if } i > I^*.
\end{cases}
\]

and

\[
l(i) = \begin{cases} 
0 & \text{if } i \leq I^*, \\
g(i)^{\zeta-1}Ye^{u\frac{W}{\gamma(i)}}c^u W^{\zeta} & \text{if } i > I^*.
\end{cases}
\]

Capital and labor market clearing conditions then yield the following equilibrium conditions,

\[
Ye^{u\left(\min\{I, \tilde{I}\} - N + 1\right)c^u \frac{r}{\gamma(i)}} = K, \tag{7}
\]

and

\[
Ye^{u\left(N\min\{I, \tilde{I}\} - N + 1\right)c^u \frac{r}{\gamma(i)}} = L^s\left(\frac{W}{rK}\right). \tag{8}
\]

The following proposition summarizes our characterization of the equilibrium.

\(^{13}\)This discussion reveals an asymmetry in our treatment of automation and new labor-intensive technologies; as already noted in footnote 10, we have assumed that the latter type of technology is always used when it is created (and hence we have not distinguished \( N, N^* \) and \( N^\dagger \)). This is because, as we show in Proposition 3, in the interesting part of the parameter space, where the interest rate is not too small (which in turn results from the discount rate in our full model, \( \rho \), being at least some \( \rho \)), all new labor-intensive technologies will be used immediately, whereas all new automation technologies may or may not be depending on the relative state of the two types of technologies.
Proposition 1 (Equilibrium in the static model) For any range of tasks \([N-1, N]\), automation technology \(I \in (N-1, N]\), and capital stock \(K\), there exists a unique equilibrium characterized by factor prices, \(W\) and \(r\), and threshold tasks, \(\overline{I}\) and \(I^\ast\), such that: (i) \(\overline{I}\) is determined by equation (6) and \(I^\ast = \min\{I, \overline{I}\}\); (ii) all tasks \(i \leq I^\ast\) are produced using capital and all tasks \(i > I^\ast\) are produced using labor; (iii) capital and labor market clearing conditions, equations (7) and (8), are satisfied; and (iii) factor prices satisfy:

\[
(I^\ast - N + 1) c^u(r)^{1-\sigma} + \int_{I^\ast}^{N} c^u \left( \frac{W}{\gamma(i)} \right)^{1-\sigma} di = 1. \tag{9}
\]

Proof. All of the expressions in this proposition follow from the preceding derivations. Existence and uniqueness are proved in Appendix B. ■

The equilibrium characterized in Proposition 1 is illustrated in Figure 4. The equilibrium is represented by the intersection of an upward and downward-sloping curve determining \(\omega \equiv \frac{W}{K}\).

The downward-sloping curve, \(\omega(I^\ast, N, K)\), corresponds to the relative demand for labor, which is obtained by combining the market clearing conditions for capital and labor, (7) and (8), together with the expression for the levels of factor prices, which is derived from the ideal price index, given in equation (9). The upward-sloping curve represents the cost-minimizing allocation of tasks to capital and labor, as represented by equation (6), with the constraint that the equilibrium level of automation can never exceed \(I\) (explaining the vertical portion).

![Figure 4: Static equilibrium. The left panel is for the case in which \(I^\ast = I\) so that the allocation of factors is constrained by technology, while the right panel is for the case in which \(I^\ast = \overline{I} < I\) so that equilibrium factor allocation is not constrained by technology.](image)

The figure distinguishes between the two cases already highlighted above. In the right panel, we have the case where \(I^\ast = I < \overline{I}\) and the allocation of factors is constrained by technology, while the left panel plots the case where \(I^\ast = \overline{I} < I\) and firms choose the cost-minimizing allocation given factor prices. An immediate implication of our characterization and of Figure 4 is that an increase in \(N\) (the creation of new, more complex tasks) always expands the set of tasks performed by labor and contracts those performed by capital, while an increase in \(I\) (greater technological automation) expands the set of tasks performed by capital and contracts those performed by labor provided that \(I < \overline{I}\).
The following proposition gives a complete characterization of comparative statics.

**Proposition 2 (Comparative statics)** Let \( \omega \equiv \frac{W}{rK} \) be the ratio of wages to capital payments. Then:\(^{14}\)

- If \( I^* = I < \bar{I} \) — so that the allocation of tasks to factors is constrained by technology — then:

\[
\frac{d \ln \omega}{dI} = \frac{d \ln (W/r)}{dI} = \frac{\partial \ln (W/r)}{\partial I^*} < 0, \quad \frac{d \ln \omega}{dN} = \frac{d \ln (W/r)}{dN} = \frac{\partial \ln (W/r)}{\partial N} > 0
\]

and

\[
\frac{d \ln (W/r)}{d \ln K} = \frac{d \ln \omega}{d \ln K} + 1 = \frac{1}{\sigma_{SR}} - 1 \frac{1}{1 - \sigma_{SR} \varepsilon_L} + 1 > 0,
\]

where \( \varepsilon_L = \frac{d \ln L(\omega)}{d \ln \omega} \) is the elasticity of the quasi-labor supply schedule, and \( \sigma_{SR} \in [0, \infty) \) is the short-run elasticity of substitution between labor and capital holding the allocation of factors to tasks fixed, which is given by a weighted average of \( \sigma \) and \( \zeta \).

Moreover, if \( \sigma_{SR} \) is sufficiently large, \( \frac{d \ln W}{dI} < 0 \), and \( \frac{d \ln W}{dI} > 0 \) otherwise.

- If \( I^* = \bar{I} < I \) — so that tasks are allocated to factors in the unconstrained cost minimizing fashion — then

\[
\frac{d \ln \omega}{dI} = \frac{d \ln (W/r)}{dI} = 0, \quad \frac{d \ln \omega}{dN} = \frac{d \ln (W/r)}{dN} = \frac{\partial \ln (W/r)}{\partial N} \bigg|_{I^*} > 0 \quad \text{and}
\]

\[
\frac{d \ln \omega}{d \ln K} + 1 = \frac{d \ln (W/r)}{d \ln K} = \frac{1}{1 - \frac{1}{\varepsilon_{\gamma}} \frac{\partial \ln (W/r)}{\partial I^*}} > 0,
\]

where \( \frac{\partial \ln (W/r)}{\partial \ln x} \bigg|_{I^*} \) denotes derivative with \( I^* \) held constant, \( \varepsilon_{\gamma} = \frac{d \ln \gamma(I)}{dI} > 0 \) is the semi-elasticity of the comparative advantage schedule, and \( \sigma_{MR} \) is the medium-run elasticity of substitution between capital and labor (with the allocation of tasks between factors adjusting), given by

\[
\sigma_{MR} = \sigma_{SR} \left( 1 - \frac{1}{\varepsilon_{\gamma}} \frac{\partial \ln (W/r)}{\partial I^*} \right) > \sigma_{SR}.
\]

Moreover, if the medium-run elasticity of substitution between labor and capital, \( \sigma_{MR} \), is sufficiently large, \( \frac{d \ln r}{dN} < 0 \), and \( \frac{d \ln W}{dN} > 0 \) otherwise.

- Finally, in both parts of the proposition, the labor share and employment move in the same direction as \( \omega \).

**Proof.** See Appendix B. \( \blacksquare \)

The most important implication of Proposition 2 is that the two types of technological changes — automation and creation of new, more complex tasks — have polar implications. Automation, corresponding to an increase in \( I \), tends to reduce \( W/r \), the labor share and employment (unless

\(^{14}\)We do not cover the case in which \( I^* = I = \bar{I} \) to save on additional notation, since in this case left and right derivatives would respect to \( I \) are different.
I^* = \tilde{I} < I \text{ and firms are not constrained by technology in their automation choice), while the}
creation of new tasks, corresponding to an increase in N, increases W/r, the labor share and
employment.

These comparative static results are illustrated in Figure 4: automation moves us along the
relative labor demand curve in the technology-constrained case shown in the top panel (and has no
impact in the bottom panel), while the creation of new tasks shifts out the relative labor demand
curve.

Another important implication of Proposition 2 is that when I^* = I, automation — an increase
in I — can reduce wages. This happens because automation expands the range of tasks performed
by capital and contracts the set of tasks performed by labor. This last feature, combined with
the diminishing returns to the quantity of a task, puts downward pressure on the wage, but is
counteracted by a positive effect coming from the fact that tasks are (q-)complements in the ag-
gregate production function, (1). This positive effect is weaker when σ is greater, explaining why
the overall impact of automation on the wage rate is negative when σ is large.15 Similarly, again
when σ is large, the creation of new tasks — that is, an increase in N — can reduce the rental
rate on capital. Moreover, automation is always capital-biased (that is, it reduces W/r), while the
creation of new tasks is always labor-biased (that is, it increases W/r). Both of these are major
consequences of the task-based framework developed here. With factor-augmenting technologies,
technological improvements always increase the price of both factors, but this is no longer the case
when technological change alters the range of tasks performed by the two factors (see also Ace-
moglu and Autor, 2011).16 Furthermore, as is well known, with factor-augmenting technologies,
whether different types of technological changes are biased towards one factor or the other depends
on the elasticity of substitution, but this too is different in our task-based framework — again
because different types of technological changes directly alter the range of tasks performed by the
two factors. This last feature will play a critical role in our full model in Section 4.

A final implication of Proposition 2 is the difference between the short-run and the “medium-
run” elasticities of substitution between capital and labor. The short-run elasticity, \( \sigma_{SR} \), is obtained
when the range of tasks allocated to capital and labor is held fixed (as in the case where I^* = I);
the medium-run elasticity, \( \sigma_{MR} \), applies when the range of tasks responds to changes in factor
prices (as in the case where I^* = \tilde{I}).17

---

15This negative impact does not require σ to be unrealistically large. For example, if σ = 1, automation reduces
the marginal product of labor if K/Y < 2.7182.
16For instance, an increase in capital-augmenting technology increases the marginal product of labor because
factors are q-complements in any production function with constant returns to scale and two factors. To see this, let
\( F(A_K K, A_L L) \) be such a production function. Then W = F_L, and \( \frac{dW}{dA_K} = K F_{L K} = -L F_{L L} > 0 \) because of constant
returns to scale.
17Another observation about the elasticity of substitution following from this proposition is that a long-run negative
association between capital accumulation and the labor share is not sufficient to conclude that σ — the elasticity
Though Proposition 2 provides a complete characterization of the responses of relative factor prices, factor shares and employment to automation and creation of new tasks, the results are qualitative and the explicit expressions are complicated; this is because imperfect substitution between factors and intermediates (the $q(i)$’s) implies that as technology changes, the profits of intermediate producers change as well. Two special cases simplify this impact on profits and illustrate the workings of our model and the comparative statics more transparently: $\eta \to 0$, where these profits go to zero, and $\zeta \to 1$, where they become a constant fraction of revenue. We next provide the explicit expressions in these two special cases. We also simplify this illustration by taking $L(\omega) = L$, so that the quasi-labor supply coincides with the inelastic labor supply in the economy.

In both of these special cases we obtain a particularly revealing expression for aggregate output:

$$Y = \left[(I^* - N + 1)^{\frac{1}{\sigma}} K^{\frac{\sigma - 1}{\sigma}} + \left(\int_{I^*}^{N} \gamma(i)^{\frac{1}{\sigma}} L^{\frac{\sigma - 1}{\sigma}} \right) \int_{I^*}^{N} \gamma(i)^{\frac{1}{\sigma}} \right]^{\frac{\sigma}{\sigma - 1}}, \quad (10)$$

where $\hat{\sigma} \equiv \eta + (1 - \eta)\sigma$ (which also implies that when $\eta \to 0$, we have the particularly simple case with $\hat{\sigma} = \sigma$).

This expression emphasizes that aggregate output is a constant elasticity of substitution aggregate of capital and labor (with the short-run elasticity of substitution between capital and labor, $\sigma_{SR}$, simply being equal to $\hat{\sigma}$), but crucially, the distribution parameters are endogenous and depend on the state of the two types of technologies in the economy. Indeed, automation increases the importance of capital and reduces the importance of labor in the (derived) aggregate production function, while the creation of new, more complex tasks does the opposite.

Relative factor demands are also straightforward to derive from simple differentiation of (10):

$$\ln \omega = \left(\frac{1}{\sigma} - 1\right) \ln K + \frac{1}{\sigma} \ln \left(\int_{I^*}^{N} \gamma(i)^{\frac{1}{\sigma}} \right), \quad (11)$$

The next corollary provides a more explicit characterization of the comparative statics derived in Proposition 2 in these two special cases.

**Corollary 1** Suppose $\eta \to 0$ or $\zeta \to 1$. Then:

- If $I < \bar{I}$:

  $$\hat{\sigma} d \ln \omega = (1 - \hat{\sigma}) d \ln K - \left[\frac{\gamma(I)^{\frac{1}{\sigma}}}{\int_{I^*}^{N} \gamma(i)^{\frac{1}{\sigma}} di} + \frac{1}{I^* - N + 1}\right] dI + \left[\frac{\gamma(N)^{\frac{1}{\sigma}}}{\int_{I^*}^{N} \gamma(i)^{\frac{1}{\sigma}} di} + \frac{1}{I^* - N + 1}\right] dN.$$
• If \( \tilde{I} < I \):

\[
(\hat{\sigma} + \Lambda/\varepsilon) d\ln \omega = (1 - \hat{\sigma} - \Lambda/\varepsilon) d\ln K + \left[ \frac{\gamma(N)^{\hat{\sigma} - 1}}{\int_{\tilde{I}}^N \gamma(i)^{\hat{\sigma} - 1} di} + \frac{1}{I - N + 1} \right] d\ln N,
\]

where

\[
\Lambda \equiv \frac{\gamma(\tilde{I})^{\hat{\sigma} - 1}}{\int_{\tilde{I}}^N \gamma(i)^{\hat{\sigma} - 1} di} + \frac{1}{I - N + 1} > 0,
\]

and \( \hat{\sigma} \equiv \eta + (1 - \eta)\sigma \).

The labor share and employment move in the same direction as \( \omega \).

In this corollary, the difference between the short-run and the medium-run elasticity of substitution can be seen quite clearly: \( \sigma_{SR} = \hat{\sigma} \), and \( \sigma_{MR} = \hat{\sigma} + \Lambda/\varepsilon \).

3 Dynamics, Balanced Growth and the Productivity Effect

In this section, we extend our model to a dynamic economy in which the evolution of the capital stock is determined by households’ saving decisions. We then investigate the conditions under which the economy admits a balanced growth path, where output, the capital stock and wages grow at a constant rate. We conclude this section by discussing the effect of automation on wages in the long run (when the interest rate is constant as in the balanced growth path), which highlights an important “productivity effect,” creating a force from automation towards higher wages.

3.1 Balanced Growth

The most important assumption in this section is the parameterization of the comparative advantage schedule to ensure balanced growth. Since, as usual, balanced growth will be driven by technology, and in this model technological change comes in part from the creation of new tasks, exponential growth will require productivity improvements from new tasks to be exponential. In other words, we require

\[
\gamma(i) = e^{Ai} \text{ with } A > 0,
\]

which we impose in the remainder of the paper.\(^\text{18}\)

Let \( \{K(t), N(t), I(t)\}_{t=0}^\infty \) denote the path of the state variables, technology and capital. Also, let \( \{r(t), W(t), Y(t)\}_{t=0}^\infty \) denote the path of factor prices and equilibrium output at each period.

\(^\text{18}\)As usual we could impose this functional form only asymptotically, but simplify the analysis and exposition by imposing it throughout its range.

Notice also that the productivity of all tasks that are automated continues to be constant in this dynamic economy. This does not, however, imply that any of the previously automated tasks can be used regardless of \( N \). As \( N \) increases, as emphasized by equation (1), the set of feasible tasks shifts to the right, and only tasks above \( N - 1 \) can be combined with those currently in use.
We start by assuming exogenous technological change, and define
\[ n(t) \equiv N(t) - I(t) \]
as a summary measure of the state of technology, whereby a higher \( n \) favors new tasks more than automation. Clearly, as automation increases, \( n \) declines, and conversely, as more new tasks created, \( n \) increases. We further simplify the discussion and notation by assuming that \( I^*(t) = I(t) \). As noted in the next section, with endogenous technology, this is the relevant region, since \( I^*(t) < I(t) \) would imply that there are resources spent on automating tasks that will not be immediately produced with capital. We discuss conditions that ensure \( I^*(t) = I(t) \) in Proposition 3 below.

The economy is assumed to admit a representative household. This representative household’s preferences over consumption paths, \( \{C(t)\}_{t=0}^\infty \), are given by
\[ \int_0^\infty e^{-\rho t} C(t)^{1-\theta} - 1 \frac{1}{1-\theta} dt, \]
and the resource constraint faced by the household takes the form
\[ \dot{K}(t) = Y(t) - C(t) - \delta K(t) - \psi\mu \int_{N-1}^N q(i,t) di, \]
where \( Y(t) \) continues to be given by (1), and \( \delta \) is the depreciation rate of capital. Recall also that \( \psi\mu \), with \( \mu \in [0,1] \), parametrizes the marginal cost of producing intermediates (and thus there is an equilibrium markup of \( 1 - \mu \geq 0 \), which plays no important role until next section).

We characterize the equilibrium by defining the normalized variables \( y(t) \equiv Y(t)/\gamma(I(t)) \), \( k(t) \equiv K(t)/\gamma(I(t)) \), \( c(t) \equiv C(t)/\gamma(I(t)) \), and \( w(t) \equiv W(t)/\gamma(I(t)) \).

At each point in time, technology and capital, \( n(t) \) and \( k(t) \), fully determine output, \( y(t) \), and factor prices \( w(t) \) and \( r(t) \) as in the static equilibrium (where, for consistency with our static analysis, \( r(t) \) is taken to be the rental rate of capital, so that the interest rate is \( r(t) - \delta \)). Specifically, the market clearing conditions for capital and labor, (7) and (8), and the ideal price index condition, (9), give the following equilibrium conditions in this case:

\[ k(t) = y(t)(1 - n(t))c^u(r(t))^{\zeta-\sigma} r(t)^{-\zeta}, \]
\[ L^e \left( \frac{w(t)}{r(t)k(t)} \right) = y(t) \int_0^{n(t)} \gamma(i)^{\zeta-1} c^u \left( \frac{w(t)}{\gamma(i)} \right)^{\zeta-\sigma} w^{-\zeta} di, \]
\[ 1 = (1 - n(t))c^u(r(t))^1-\sigma + \int_0^{n(t)} c^u \left( \frac{w(t)}{\gamma(i)} \right)^{1-\sigma} di. \]

The implied values for normalized output and factor prices can be written as \( y(t) = y^E(n(t),k(t)) \), \( w(t) = w^E(n(t),k(t)) \) and \( r^E(t)(n(t),k(t)) \), which are uniquely defined from Proposition 1. Importantly, we also have that \( w^E(n,k) \geq r \), because the endogenous allocation of tasks to factors implies \( W/\gamma(I^*) \geq r \) (or \( I \geq I^* \)). We also denote the output net of intermediate costs by \( f^E(n(t),k(t)) \).
Using this notation, we can describe the dynamic equilibrium of our model as a path for \( c(t) \) and \( k(t) \) satisfying the Euler equation,

\[
\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(r^E(n(t), k(t)) - \delta - \rho) - g,
\]

(13)
coupled with the household’s transversality condition,

\[
\lim_{t \to \infty} k(t)e^{-\int_0^t (\rho - (1-\theta)g)ds} = 0,
\]

(14)
and the resource constraint,

\[
\dot{k}(t) = f^E(n(t), k(t)) - c(t) - (\delta + g)k(t).
\]

(15)

Figure 5: Balanced growth path and dynamic equilibrium with exogenous technological change and \( n(t) \to n \).

Figure 5 presents the phase diagram for this system for \( n(t) \to n \). The structure of the above system is similar to the standard neoclassical growth model, with the slight exception that technology monopolists’ markups create a wedge between \( r^E \) and \( f^E_k \).

We define a balanced growth path as an allocation in which \( Y, C, K \) and \( w \) grow at a constant rate and \( r \) is constant. The next proposition characterizes the conditions under which the asymptotic behavior of this economy will converge to a balanced growth path and also establishes that this involves both types of technological change.

**Proposition 3 (Dynamic equilibrium with exogenous technological change)** Suppose that technology evolves exogenously. There exists a threshold \( \rho^* \) such that, for \( \rho > \rho^* \) we have:

1. There exists \( \pi^* \) such that for \( n(t) < \pi^* \), we have \( I^* < I \), while for \( n(t) \geq \pi^* \), \( I^* = I \).
2. Suppose that \( \lim_{t \to \infty} n(t) = n \in [\bar{\pi}, 1) \). Then a balanced growth path exists if and only if asymptotically \( \tilde{N} = \tilde{I} = \Delta \), and is moreover unique. In this balanced growth path, \( I^* = I, Y, C, K \) and \( w \) grow at a constant rate \( \Delta \), and \( r \) is constant. Suppose instead that \( \lim_{t \to \infty} n(t) < \bar{\pi} \). Then there exists a balanced growth path in which \( I^* < I \).

3. Moreover, given such a path of technological change (with \( \lim_{t \to \infty} n(t) = n \in [\bar{\pi}, 1) \), or \( n(t) \leq \bar{\pi} \) for all \( t \geq T \)), the dynamic equilibrium is unique starting from any initial level of capital stock and converges to the balanced growth path.

**Proof.** See Appendix B. ■

The most important implication of Proposition 3 is that balanced growth can emerge from the simultaneous process of automation and development of new tasks. But it also highlights that this process needs to be “balanced” itself: the race between machine and man cannot be dominated by either. This implies, in particular, that the two types of technologies need to advance at the same rate, and moreover that \( \lim_{t \to \infty} n(t) \geq \bar{\pi} \) (otherwise, not all available automation technologies will be used). We will see in the next section that the threshold \( \bar{\pi} \) also plays an important role when we endogenize technology.\(^{19}\)

The additional requirement in Proposition 3, \( \rho > \bar{\pi} \), ensures that the long-run equilibrium interest rate is not too close to 0. As we will see further in the next section, this is the interesting range of parameters for our focus (and this is the requirement that ensures that \( \bar{\pi} \) is well-defined). The Appendix also shows how the analysis needs to be modified in the case where \( \rho < \bar{\pi} \).

Combining this proposition together with Proposition 2, we also see that when automation runs ahead of the creation of new tasks (i.e., when \( \dot{I} > \dot{N} \), so that \( n(t) \) decreases), we will not only move away from balanced growth (presuming that we started at or near balanced growth), but also that this will reduce the share of labor in national income and employment. In light of this result, the patterns shown in Figure 1 in the Introduction can be interpreted as a consequence of automation outpacing the creation of new labor-intensive tasks over the last two decades.

Proposition 3 can be further illustrated and strengthened in the two special cases considered in the previous section, where \( \eta \to 0 \) or \( \zeta \to 1 \). Supposing also that \( \dot{N} = \dot{I} = \Delta \), the aggregate production function can be simplified to \( Y(t) = f(K(t), A(t)L) \) as given in equation (10). We further have that

\[
A(t) = \left( \int_{I(t)}^{N(t)} \gamma(i)^{\bar{\sigma} - 1} \, di \right)^{\frac{1}{\bar{\sigma} - 1}} = e^{A(t)} \left( \frac{e^{A(\bar{\sigma} - 1)n(t) - 1}}{A(\bar{\sigma} - 1)} \right)^{\frac{1}{\bar{\sigma} - 1}},
\]

\(^{19}\)We should also note that \( \rho > \bar{\rho} \) and \( \lim_{t \to \infty} n(t) \geq \bar{\pi} \) are not a very restrictive condition. For example, focusing on a standard annual parametrization of our model with \( \theta = 1, \delta = 0.06, g = 0.016, \sigma = 0.5, \zeta = 0.2 \) (so that the elasticity of substitution between capital and labor lies between 0.5 and 0.2), \( A = 2, \eta = 0.5 \) and \( \psi = 0.9 \), we obtain \( \bar{\pi} = 0.012 \), so that the standard value of the discount rate, \( \rho = 0.05 \), is comfortably above this threshold. These parameters also imply \( \bar{\pi} = 0.56 \).
so that $A(t)$ grows at a rate $A\Delta$. In this case, technology is purely labor augmenting on net because labor and capital perform a fixed share of tasks, while labor is used on tasks in which it is more productive over time. This provides a direct connection between our model and Uzawa’s Theorem, which implies that balanced growth requires purely labor-augmenting technological change (e.g., Acemoglu, 2009, or Grossman, Helpman and Oberfield, 2015). The combination of (12) and $\dot{N} = \dot{I}$ ensures that technology is purely labor-augmenting in our economy.

### 3.2 The Productivity Effect

We now study the dynamic implications of automation running ahead of the creation of new tasks. Though many of the insights from our static model apply in this case, the dynamic economy also highlights another economic force, which we will call the productivity effect: automation, by enabling the substitution of the cheaper capital for labor, increases productivity and thus the demand for labor. The productivity effect has been present in our analysis so far. But, as we show next, it becomes more powerful in the balanced growth path because the interest rate is constant (Proposition 3).

We continue to assume that $\rho > \bar{\rho}$ as in Proposition 3, and also focus on the case where $n(t) \to n \in (\bar{n}, 1)$, so that the balanced growth path involves simultaneous advancement and use of both types of technologies. In this balanced growth path, capital adjusts to make $r = \rho + \delta + \theta g$, and the long-run normalized wage as a function of the state of technology becomes

$$w^{LR}(n) = w^E(n, k(n)).$$

Given our normalization (where the wage, $W$, is divided by $\gamma(I) = \gamma(I^*)$), this is the wage per effective unit of labor paid in the least complex tasks performed by labor. The wage per effective unit of labor in the most complex task can then be written as $w^{LR}(n)/\gamma(n)$.

Proposition 3 implies that $I^*(t) = I(t)$ and that all new labor-intensive (new complex) tasks are utilized immediately. In terms of the notation we have just introduced, this is equivalent to:

$$w^{LR}(n)/\gamma(n) \leq r \leq w^{LR}(n).$$

(16)

The long-run productivity effect can now be seen from the ideal price condition, (9):

$$(I - N - 1) c^n(\rho + \delta + \theta g)^{1-\sigma} + \int_I^N c^n(\frac{W}{\gamma(t)})^{1-\sigma} = 1.$$  

(17)

With this notation, this ideal price condition can also be written as

$$(1 - n) c^n(\rho + \delta + \theta g) + \int_0^n c^n(\frac{w^{LR}(n)}{\gamma(t)})^{1-\sigma} = 1.$$  

(18)

20This is similar to the productivity or efficiency effect in models of offshoring such as Grossman and Rossi-Hansberg (2008), Rodriguez-Clare (2010) and Acemoglu, Gancia and Zilibotti (2015), which results from the substitution of cheaper foreign labor for domestic labor in certain tasks.
There are three important implications from the ideal price condition. First, automation cannot reduce wages in the long run. This claim follows from (17) when we use (16), which yields

\[
\frac{dW}{dI^*} \propto \frac{1}{\sigma - 1} \left[ c^u(r)^{1-\sigma} - c^u\left(w^{LR}(n)\right)^{1-\sigma} \right] \geq 0.
\]

Intuitively, because the interest rate is constant in the long run, automation increases the amount of capital used in production as well. Labor, which is the inelastic factor, earns the productivity gains in the form of higher wages.

Second, this time from (18), we also have that

\[
\frac{dw^{LR}(n)}{dn} \propto \frac{dW}{dN} \propto -\frac{1}{\sigma - 1} \left[ c^u(r)^{1-\sigma} - c^u\left(w^{LR}(n)\right)^{1-\sigma} \right] \geq 0,
\]

so that automation (corresponding to a decrease in \(n\)) reduces the wage per effective unit of labor in the least complex tasks (while the creation of new tasks increases it).

Finally, once again using (17), we further have

\[
\frac{dw^{LR}(n)/\gamma(n)}{dn} \propto \frac{1}{\sigma - 1} \left[ c^u\left(w^{LR}(n)\right)^{1-\sigma} - c^u(r)^{1-\sigma} \right] \leq 0,
\]

so that the creation of new tasks reduces wage per effective unit of labor in the most complex tasks; while automation increases it.

\[\text{Figure 6: The evolution of equilibrium wage following a permanent increase in automation.}\]

These observations thus establish that the majority of the results from the static model continue to apply, but because of the productivity effect, the potential negative impact of automation on the equilibrium wage level disappears in the long run. This is illustrated in Figure 6, which plots
the behavior of the wage level, $W$, following a permanent, one-time decline in $n$ due to additional automation (and continuation of the same technological paths thereafter): wages may fall in the short run following a surge in automation, but they necessarily increase in the long run because of the productivity effect.

4 Full Model: Tasks and Endogenous Technologies

The analysis in the previous section established the existence of a balanced growth path under the assumption that $\dot{N} = \dot{I}$. But why should these two types of technologies advance at the same rate? This is the question at the center of our paper, and to answer it, we now develop our full model, which endogenizes the pace at which automation and creation of new tasks proceeds.

4.1 Endogenous and Directed Technological Change

We endogenize technological change by allowing new intermediates, which either automate previously non-automated tasks or create new tasks, to be introduced by technology monopolists. We assume that successful innovations always achieve automation or the creation of new tasks in the order of the intermediate indices, $i \in [0, \infty)$, so that lower-indexed tasks will always be automated before higher-index tasks, and a new labor-intensive task will always correspond to the lowest-indexed task that has not been created yet (and the lower index of integration at $N - 1$ in the aggregate production function, (1), already imposes that new tasks replace the lowest-indexed task currently in use). As a consequence, the two types of endogenous technological changes will correspond to an increase in $I$ and to an increase in $N$, respectively. We continue to assume that all intermediates, including those that have just been invented, can be produced at the fixed marginal cost of $\mu \psi$, and that the fringe of competitive firms forces the technology monopolists to price at $\psi$, which of course implies a per-unit profit of $(1 - \mu)\psi$.

Though per-unit profits of technology monopolists are constant, their net present discounted value is a complex object for two reasons. First, the fact that $I$ and $N$ will grow at some fixed rate, for example in the balanced growth path as characterized in Proposition 3, implies that there will be a deterministic component to the length of time during which a monopolist will be able to enjoy profits from its technology. Despite this first complication, we will see that the dynamics of endogenous technology can be characterized, though this will involve somewhat different arguments than in the standard endogenous technological change models. Second, as in other models of quality improvements (e.g., Aghion and Howitt, 1992; Grossman and Helpman, 1991), new intermediates replace some existing ones. This generates the “creative destruction” of profits of existing producers by new firms, at least under the assumptions used in the literature, which is that new firms do not have to respect the intellectual property rights of the technology.
on which they are building. This assumption, however, creates more complex dynamics, especially coupled with the deterministic replacement of products in our model. For this reason, we adopt an alternative (and arguably equally plausible) structure of protection of intellectual property rights whereby building and replacing an existing technology is viewed as infringement of the patent of that technology. This implies that the inventor of a new technology will have to buy this existing patent (or license the technology). We assume that this takes place with the inventor making a take-it-or-leave-it offer to the holder of the patent of the technology on which it is building. Consequently, a firm automating a task previously performed by labor will have to license or buy the relevant patent from an existing firm supplying the intermediate to this task, and similarly, a firm creating a new task, which is effectively creating a more complex, labor-intensive version of an existing task, will have to obtain the patent from an existing firm for the intermediate used in this task (which, in any equilibrium with automation, will be an automated task, since it is the lowest-indexed task currently in use). This game-form ensures that each technology monopolist will receive the same flow of revenues regardless of whether its own product is replaced or not — either as profits when he is operating or as payments for its patent when it is replaced. We return to the analysis of how the results change when we allow for the creative destruction of profits in Section 6.

We are now in a position to describe the innovation possibilities frontier (the technology of creating new technologies). We assume that innovation requires scientists, and there is a fixed (inelastic) supply of $S$ scientists in this economy.\(^\text{21}\) At each point in time, $\dot{S}_I(t) \geq 0$ of these scientists are hired by monopolists at a competitive wage $W^S$ for automation, and $\dot{S}_N(t) \geq 0$ of them are hired at the same wage for creating new tasks. The market clearing condition for scientists is

$$S_I(t) + S_N(t) \leq S,$$

with the wage $W^S$ being equal to zero if this inequality is strict.

We assume that advances in automation and creation of new tasks are given by

$$\dot{I}(t) = \kappa_I S_I(t), \quad (19)$$

and

$$\dot{N}(t) = \kappa_N S_N(t), \quad (20)$$

where $\kappa_I$ and $\kappa_N$ are positive constants, representing the difficulty/ease of the corresponding type of technological change.

\(^{21}\)Focusing on an innovation possibilities frontier using just scientists, rather than variable factors such as in the lab-equipment specifications, is convenient because it enables us to focus on the direction of technological change — and not on the overall amount of technological change — especially when we turn to the welfare analysis in the next section (see, e.g., Acemoglu, 2002).
4.2 Equilibrium with Endogenous Technological Change

The key objects we need to compute to characterize the equilibrium with endogenous technological change are value functions determining the net present discounted value of new automation and labor-intensive innovations. We denote these by \( V_I(t) \) and \( V_N(t) \). More specifically, \( V_I(t) \) is the value of a new technology automating the task at \( i = I(t) \) (i.e., the highest-indexed task that has not yet been automated, or more formally \( i = I(t) + \varepsilon \) for \( \varepsilon \) arbitrarily small and positive). Likewise, \( V_N(t) \) is the value of a new technology creating a more complex task at \( i = N(t) \).

Given these value functions, an equilibrium with endogenous technology is given by paths \( \{K(t), N(t), I(t)\}_{t=0}^{\infty} \) for capital and technology (starting from an initial values \( K(0), N(0), I(0) \)), paths \( \{r(t), W(t), W^S(t)\}_{t=0}^{\infty} \) for factor prices, paths \( \{V_N(t), V_I(t)\}_{t=0}^{\infty} \) for the value functions of technology monopolists, and paths \( \{S_N(t), S_I(t)\}_{t=0}^{\infty} \) for the allocation of scientists such that all markets clear, all firms, including prospective technology monopolists, maximize profits, the representative household maximizes its utility, and \( N(t) \) and \( I(t) \) evolve endogenously according to equations (19) and (20).

We start by characterizing the value functions for technology monopolists. Suppose also that in this equilibrium, \( n > \bar{n} \) so that \( I^* = I \) and new automated tasks start being used immediately. We next compute the flow profits from automation:

\[
\pi_I(t, i) = (1 - \mu) \left( \frac{\eta}{1 - \eta} \right) ^\zeta \psi ^{1-\zeta} Y(t) e^{u(r(t))} \gamma^{-\sigma}.
\]

Intuitively, these profits come from the ability of firms to produce task \( i \) using capital (which is necessarily profitable given our assumption that \( I^*(t) = I(t) \)). Similarly, the flow profits of producing such a task using labor are

\[
\pi_N(t, i) = (1 - \mu) \left( \frac{\eta}{1 - \eta} \right) ^\zeta \psi ^{1-\zeta} Y(t) e^{u \left( \frac{W(t)}{\gamma(I)} \right)} \gamma^{-\sigma}.
\]

It is then straightforward to compute the offer that a monopolist with a new technology automating task \( I \) at time \( t \) needs to make to the firm currently holding the patent for the (labor-intensive) technology of that intermediate. This offer will be given by the net present discounted value of the profit streams, discounted using the path of future interest rates, that the existing patent-holder would obtain

\[
(1 - \mu) \left( \frac{\eta}{1 - \eta} \right) ^\zeta \psi ^{1-\zeta} \int_t^\infty e^{-\int_0^\tau (r(s) - \delta) ds} Y(\tau) e^{u \left( \frac{W(\tau)}{\gamma(I)} \right)} \gamma^{-\sigma} d\tau.
\]

Since this is a take-it-or-leave-it offer, the best response of the patent holder is to accept it.\(^{22}\)

\(^{22}\)This expression is written by assuming that the patent-holder will also turn down subsequent less generous offers in the future. Writing it in the value function form, using the one-step ahead deviation principle, leads to the same conclusion.
Similarly, the offer of the technology monopolist with a new technology for creating a new labor-intensive task while replacing task \( N-1 \) (which is necessarily automated in any equilibrium with automation) is given by

\[
(1 - \mu) \left( \frac{\eta}{1 - \eta} \right)^\zeta \psi^{1-\zeta} \int_t^\infty e^{-\int_t^\tau (r(s) - \delta) ds} Y(\tau) c^u (r(\tau)) \zeta^{-\sigma} d\tau.
\]

Both of these offers will be accepted by the patent-holders with the current technologies. Incorporating this, we can then compute the values of firms that innovate (respectively with automation and creation of new tasks):

\[
V_I(t) = (1 - \mu) \left( \frac{\eta}{1 - \eta} \right)^\zeta \psi^{1-\zeta} \int_t^\infty e^{-\int_t^\tau (r(s) - \delta) ds} Y(\tau) \left( c^u (r(\tau)) \zeta^{-\sigma} - c^u \left( \frac{W(\tau)}{\gamma(I(t))} \right) \zeta^{-\sigma} \right) d\tau, \tag{21}
\]

and

\[
V_N(t) = (1 - \mu) \left( \frac{\eta}{1 - \eta} \right)^\zeta \psi^{1-\zeta} \int_t^\infty e^{-\int_t^\tau (r(s) - \delta) ds} Y(\tau) \left( c^u \left( \frac{W(\tau)}{\gamma(N(t))} \right) \zeta^{-\sigma} - c^u (r(\tau)) \zeta^{-\sigma} \right) d\tau. \tag{22}
\]

Both of these expressions have a common form: they subtract the lower cost of producing a task with the factor for which the new technology is designed from the higher cost of producing the same task with the other factor (working with the older technology). Moreover, because of the same structure of offers that will be forthcoming in the future, the innovator will continue to earn a flow of revenues in the future, regardless of when its technology is replaced — and this is the reason why the time of replacement does not feature in these expressions. Observe also, for future reference, that these values are positive only when \( \sigma > \zeta \). Take, for example, (21). By virtue of the fact that a task performed by labor is being automated, \( c^u (r(\tau)) < c^u \left( \frac{w(\tau)}{\gamma(I(t))} \right) \). Thus if we had \( \zeta > \sigma \), the profit stream would be negative. The intuitive reason for this is that, in the case where \( \zeta > \sigma \), profits are lower when the firm is more productive, and thus when the holder of the new technology buys out the patent-holder of the less productive technology, it ends up with negative net profits.

Once these value functions are derived, the allocation of scientists to the two different types of technological progress follows immediately by noting that the market wage of scientists will be equal to their value in the activity where their productivity is greater. Therefore,

\[
\begin{align*}
S_N(t) &= S, & S_I(t) &= 0 & \text{if } \kappa_N V_N(t) > \kappa_I V_I(t) \\
S_N(t) &= 0, & S_I(t) &= S & \text{if } \kappa_N V_N(t) < \kappa_I V_I(t) \\
S_N(t) &\in [0, S], & S_I(t) &= S - S_N(t) & \text{if } \kappa_N V_N(t) = \kappa_I V_I(t).
\end{align*}
\]

Intuitively, whenever one of the two types of technologies (automation versus creation of new tasks) is more profitable, all scientists will be allocated to this activity, and their wage, \( W^S \), will
be equal to their value in this activity. But this also implies that this wage will exceed their value in the other technological activity, unless we are in the case where \( \kappa_N V_N(t) = \kappa_I V_I(t) \).

These observations enable us to represent, using the same normalizations as in the previous section, the equilibrium path with endogenous technology by the time path of the tuple \( \{n(t), k(t), c(t), S_I(t)\}_{t=0}^{\infty} \) such that:

- The evolution of the state variables is given by
  \[
  \dot{k}(t) = f^E(k(t), n(t)) - c(t) - (\delta + A\kappa_I S_I(t))k(t)
  \]
  \[
  \dot{n}(t) = \kappa_N (S - S_I(t)) - \kappa_I S_I(t).
  \]

- Consumption satisfies the Euler equation (13) coupled with the transversality condition in equation (14).

- The allocation of scientists satisfies:
  \[
  S_I(t) = \begin{cases} 
  0 & \text{if } \kappa_I V_I(t) < \kappa_N V_N(t) \\
  [0, S] & \text{if } \kappa_I V_I(t) = \kappa_N V_N(t) \\
  S & \text{if } \kappa_I V_I(t) > \kappa_N V_N(t)
  \end{cases}
  \]

with \( V_N(t) \) and \( V_I(t) \) given by equations (21) and (22).

We next characterize the dynamic equilibrium with endogenous technology. A balanced growth path is defined as in Proposition 3, as an allocation in which normalized capital \( k(t) \) and the interest rate \( r(t) \) are constant, except that now \( n \) will be determined endogenously. The next proposition gives another one of the main results of the paper. It establishes conditions for the existence of a unique balanced growth path in which there are both types of technological changes and also shows that, under the same set of conditions, it is (saddle-path) stable.

**Proposition 4 (Equilibrium with endogenous technological change)** Suppose that \( \sigma > \zeta \). There exist \( \overline{p} \) and \( \overline{S} \) such that for \( \rho > \overline{p} \) and \( S < \overline{S} \) (where \( \overline{p} \) is as defined in Proposition 3), the following are true:

1. There exists \( \overline{r} \) such that for \( \frac{\kappa_I}{\kappa_N} > \overline{r} \), there is a balanced growth path, where \( \ddot{N} = \dot{I} = \frac{\kappa I S_N}{\kappa_I + \kappa_N} S \), and \( Y, C, K \) and \( W \) grow at the constant rate \( g = A \frac{\kappa I S_N}{\kappa_I + \kappa_N} S \), and the interest rate, \( r \), the labor share and employment are constant. Along this path, we have \( N(t) - I(t) = n^D \), with \( n^D \) determined endogenously from the condition \( \kappa_N V_N = \kappa_I V_I \), and satisfying \( n^D \in (\bar{n}, 1) \), where \( \bar{n} \) is as defined in Proposition 3. In addition, there exists \( \overline{\rho} \geq \rho \) such that if \( \rho < \overline{\rho} \), the balance growth path is unique.

2. Suppose that \( \rho < \overline{\rho} \) so that the balance growth path is unique. Then, when \( \theta = 0 \), the dynamic equilibrium is globally asymptotically (saddle-path) stable. Moreover, there exists \( \overline{S} \leq \overline{S} \) such that provided that \( S < \overline{S} \), the dynamic equilibrium is unique in the neighborhood of the balanced growth path and is locally asymptotically (saddle-path) stable.

26
Proof. See Appendix B.

The first important result contained in this proposition is the existence of the balanced growth path and its uniqueness (when $\rho < \bar{\rho}$). The second critical result, established in the second part, is that this balanced growth path is locally asymptotically stable and also globally asymptotically stable when $\theta = 0$ (so that preferences have an infinite elasticity of intertemporal substitution). This result implies that there are powerful market forces pushing the economy towards the balanced growth path.

These results are established under several conditions. First, we have imposed that $\sigma > \zeta$. This condition ensures that innovations are directed towards technologies using the cheaper factors.23 Recall from Section 2 that more tasks are allocated to the factor that is cheaper. This creates a natural force that tends to push innovations to be directed towards the cheaper factor. One way of understanding this effect is that as a factor becomes cheaper, the range of activities in which it is used expands. Holding constant the proportions at which the factor in question is combined with intermediates in the task production functions, (2) and (3), this expansion increases the quantity of the corresponding intermediate, $q(i)$, raising the profitability of technologies working with this factor and encouraging innovation beneficial for this factor. The extent of this positive force is regulated by the elasticity of substitution $\sigma$: the greater is $\sigma$, the more powerful is this effect directing innovation towards the cheaper factor. There is a countervailing effect as well, however: as a factor becomes cheaper, it is substituted for the intermediate it is combined with, so that the quantity of the corresponding intermediate declines, holding the level of task production fixed. This countervailing effect creates a negative force, discouraging innovations directed towards the cheaper factor. Task production functions, (2) and (3), clarify that the extent of the substitution effect will depend on the elasticity of substitution between the factor in question and the intermediates, $\zeta$. The condition $\sigma > \zeta$ guarantees the positive effect dominates so that innovations are directed towards the cheaper factor.24

We should note that this condition is quite plausible: as already emphasized, when $\sigma < \zeta$, there will be no research at all because, somewhat pathologically, profits are higher when the producer is less productive, and thus the net present discounted values from innovation will be negative (recall equations (21) and (22)). Put differently, in this case, there is such a strong substitution effect allowing the substitution of the cheaper factor for intermediates that the present discounted value from innovation is negative and thus there is no incentive to innovate. This condition $\sigma > \zeta$ is therefore imposed even for the existence result in part 1 of the proposition in order to ensure

23By the term “innovation directed towards the cheaper factor”, we mean a comparative static statement: as the relative price of a factor declines, innovation is directed more towards this factor.

24Our assumption ruling out the creative destruction of profits of existing producers is also playing an important role in the stability result here. In Section 6, we show that with creative destruction of profits, the balanced growth path equilibrium remains very similar, but stability is no longer guaranteed.
positive incentives for innovation. Moreover, this condition is also empirically plausible. We expect the elasticity of substitution between factors and intermediates, $\zeta$, to be very low — in the limit zero as in the Leontief case since new technologies, for example enabling automation, are embedded in these intermediates.\(^{25}\)

Second, the assumption $S < \overline{S}$ is used for guaranteeing that the growth rate is not too high (and also ensures that the net present discounted value of the representative household is finite). If the growth rate is above the threshold implied by $\overline{S}$, the creation of new tasks is discouraged (even if current wages are low) because firms anticipate that wages will grow very rapidly, reducing the future profitability of these labor-intensive tasks. This requirement is strengthened to $S < \overline{S}$ in the second part of the proposition when we consider local stability based on Taylor approximations.

Third, as in Proposition 3, $\rho > \overline{\rho}$. As discussed in that context, this assumption ensures that the interest rate is not too close to 0, which in turn guarantees that there exists a well-defined threshold $\bar{n}$; when $n > \bar{n}$, all technologically automated tasks will be immediately produced with capital, and as also noted above, $\rho > \overline{\rho}$ guarantees that all new labor-intensive tasks will immediately start being used with labor. (The case in which $\rho < \overline{\rho}$ is studied in Appendix B). The proposition further shows that when $\rho > \overline{\rho}$, and the other conditions in the proposition are satisfied, the long-run equilibrium will indeed involve $n > \bar{n}$, so this latter condition does not need to be imposed, but follows as an implication. The condition that $\rho < \overline{\rho}$ is strengthened to $\rho < \overline{\rho}$ in order to obtain uniqueness (which requires the value functions to satisfy a form of single-crossing property).

Figure 7 draws the net present discounted value (normalized by output) of allocating scientists to creating new labor-intensive tasks or to automation as a function of $n$ for $\rho > \overline{\rho}$ and $S < \overline{S}$, denoted respectively by $v_N(n) \equiv V_I/Y$ and $v_I(n) \equiv V_N/Y$. In the region where $n > \bar{n}$, the value of automation decreases as the economy automates more tasks ($n$ decreases). This is the key economic force that generates stability in our model: greater automation increases wages per unit of production relative to the interest rate, and thus the relative value of creating new labor-intensive tasks rises with automation. Though there is also the productivity effect acting towards increasing the labor cost, the assumption that $\rho > \overline{\rho}$ ensures that this effect is dominated by the direct substitution effect, and thus guarantees that the balanced growth path is asymptotically stable.

However, these forces are not sufficient to guarantee that the curves for $\kappa_I v_I(n)$ and $\kappa_N v(n)$ intersect. Though the former always starts below the latter as shown in Figure 7, it could always remain below the latter. The condition in Proposition 4 that $\kappa_I/\kappa_N$ is sufficiently large ensures that such an intersection takes place and thus there exists a unique “interior” balanced growth path.

\(^{25}\)Observe also that this condition does not impose any restrictions on the short-run elasticity of substitution between capital and labor, which can be less than one as in many of the studies reviewed in Acemoglu and Robinson (2015).
This discussion also clarifies that when this condition does not hold, the long-run equilibrium will be one in which only new tasks are developed, and there is no automation.

In summary, the critical economic force highlighted by Proposition 4 is that, differently from models with factor-augmenting technologies, it is factor prices, not primarily the market sizes, that guide the direction of technological change.\textsuperscript{26} Consequently, there are stronger incentives to undertake the type of innovation that will work with the factor that has a relatively cheaper user cost.\textsuperscript{27}

The emphasis of our main result in this section, Proposition 4, has been to show that shocks to technology, for example in the form of a series of new automation technologies, will set in motion self-correcting forces, so that in the long run the economy returns back to its pre-shock balanced growth path with the same employment level and labor share in national income. This does not, however, imply that all changes will leave the long-run prospects of labor unchanged. The next corollary shows that if there is a change in the innovation possibilities frontier, making automation easier than before, then there will be a new balanced growth path with lower employment and lower share of labor in national income.

\textsuperscript{26}We should also note that this does not overturn the “weak bias” results in Acemoglu (2007), since these were derived in a setting that is general enough to nest the current environment.

\textsuperscript{27}We can also observe that the long-run elasticity of substitution between capital and labor, $\sigma_{LR}$, which allows for both the endogeneity of technology and capital accumulation, is equal to 1 because following a shock to technology or capital stock, the economy returns to its balanced growth path, where the share of labor in national income is constant. This implies that, interestingly, the long-run elasticity of substitution need not be larger than the medium-run and short-run elasticities $\sigma_{MR}$ and $\sigma_{SR}$ defined above. This is because it is not only technology but also the capital stock of the economy that adjusts in the long run (as emphasized by the productivity effect in the previous section).
Corollary 2 Suppose that there is a one-time permanent increase in $\kappa I / \kappa N$. Then the economy converges to a new balanced growth path with lower $n^D$, lower employment and lower share of labor in national income.

This corollary follows immediately because an increase in $\kappa I / \kappa N$ shifts the intersection in Figure 7 to the left, leading to a lower value of $n^D$ in the balanced growth path.

Therefore, this corollary and Proposition 4 clearly delineate the types of changes in technology that will set in motion self-correcting dynamics: those driven by faster than usual arrival of automation technologies. In contrast, those which alter the ability of the society to create new automation technologies will not generate such self-correcting dynamics and will result in lower prospects for labor in the future.\footnote{In principle, another type of shock might send the economy to a corner solution with only automation and no creation of new tasks. However, as we show in the Appendix, this is not possible when $\rho > \bar{\rho}$. Such a corner solution becomes possible only when $\rho < \bar{\rho}$.}

5 Welfare

In this section we turn to an analysis of the efficiency of the equilibrium described in Proposition 4. Our main finding is that the presence of rents for workers, as captured by our quasi-labor supply, distorts the composition of equilibrium technology towards too much automation and too little creation of new, more complex (labor-intensive) technologies — and this is in addition to other distortions that exist in this class of models. We present two complementary results shedding light on this inefficiency. First, we characterize the constrained efficient allocation of a social planner who is subject to the same quasi-labor supply schedule and the constraint that wages are equal to the marginal product of labor, and technologies evolve according to the same innovation possibilities frontier. We then show how this constrained efficient allocation can be decentralized by taxes and subsidies. This exercise shows that, in addition to the usual wedges (taxes/subsidies) between the social planner’s allocation and the decentralized equilibrium, workers’ rents create an additional reason to subsidize the creation of new tasks relative to automation. Second, focusing on one of the special cases already utilized in Section 2, which helps us isolate this novel inefficiency, we show the decentralized equilibrium Can be improved by altering the composition of R&D in the direction of the creation of new tasks.

We start by characterizing the constrained efficient allocation, which we will use in deriving both results. Let us denote by $F^P(N, I, K, L)$ the net aggregate output (net of the costs of producing intermediates) in the constrained efficient allocation when the level of employment is $L$, the capital stock is $K$, the state of technologies is represented by $N$ and $I$, and intermediates are priced at their marginal cost (which is the relevant net aggregate output expression for the social planner, since she would always price all intermediates at marginal cost). Also, let $W^P(N, I, K)$ and $r^P(N, I, K)$
denote the resulting marginal products of labor and capital (corresponding to the wage and interest rates in the decentralized allocation) with the level of employment given by the quasi-labor supply schedule, \( L = L^s(\omega) \). Finally, let \( \omega^P(N,I,K) \) denote the equilibrium value for \( W/rK \) in this case. It is then straightforward to prove that these variables satisfy the same comparative statics described in Proposition 2.

The constrained efficient allocation solves the problem

\[
\max_{\{C(t),L(t),S_N(t),S_I(t)\}} \int_0^\infty e^{-\rho t} \frac{C(t)^{1-\theta} - 1}{1-\theta} dt,
\]

subject to the endogenous evolution of the state variables:

\[
\begin{align*}
\dot{K}(t) &= F^P(N(t), I(t), K(t), L(t)) - C(t) - \delta K(t), \\
\dot{N}(t) &= \kappa_N S_N(t), \\
\dot{I}(t) &= \kappa_I S_I(t).
\end{align*}
\]

In using the net aggregate production function, \( F^P \), we have already incorporated that the planner will price all intermediates at marginal cost, \( \mu \psi \). Furthermore, we have written the objective function of the social planner as just maximizing the net present discounted value of consumption streams, thus imposing that there is no disutility or opportunity cost of labor supply and all wages received by workers are “quasi rents”. This formulation is justified by the microfoundation provided for the quasi-labor supply schedule in Appendix D, and we also note that the results are entirely analogous if the opportunity cost of labor is positive, provided that it is lower than the market wage; the reason why we have assumed that it is equal to zero is simply for notational convenience. An important implication of this structure is that, all else equal, the social planner would like to maximize employment as this increases net output and wage payments without any disutility cost.\(^29\)

Because the planner faces the same quasi-labor supply schedule and labor demand relations, we also have:

\[ L(t) \leq L^s(\omega^P(N,I,K)). \]

The relationship imposes that the planner will take into account the impact of technology and capital accumulation on employment.

Let \( \mu_N \) and \( \mu_I \) denote the shadow values of the two types of technology, respectively, and \( \mu_L \) and \( \mu_K \) the shadow values of labor and capital. The maximum principle implies that these satisfy

\(^29\)This program also imposes that the threshold the social planner will set for automation, \( I^*(t) \) in the terminology used in the previous section, is equal to \( I(t) \). In other words, all available automation technologies will be used. As in the previous section, this will be the case when \( \rho > \bar{\rho} \), since in this case the planner will always choose to operate in the region in which \( n > \bar{n} \).
the necessary conditions:

\[ \begin{align*}
\rho \mu_N - \dot{\mu}_N &= \mu_K F_N^P + \mu_L L_s^\omega \omega_N^P, \\
\rho \mu_K - \dot{\mu}_K &= \mu_K F_I^P + \mu_L L_s^\omega \omega_K^P, \\
\mu_L &= \mu_K W^P.
\end{align*} \]

All the functions in the above equations are evaluated at their corresponding arguments at time \( t \), and subscripts denote partial derivatives. Moreover, we show in Appendix C that the current value Hamiltonian associated with the planner’s problem is concave, so these conditions (plus the Euler equation for consumption and the transversality condition) are sufficient for characterizing the constrained efficient allocation.

Let \( \Psi_N(t) \equiv \mu_N(t)/\mu_K(t) \) and \( \Psi_I \equiv \mu_I(t)/\mu_K(t) \) be the shadow discounted net present values of new technologies (in terms of additional net output they create). The optimal allocation of scientists to the two different types of research then satisfies

\[ \begin{align*}
S_N(t) &= S, & S_I(t) &= 0 & \text{if } \kappa_N \Psi_N(t) > \kappa_I \Psi_I(t) \\
S_N(t) &= 0, & S_I(t) &= S & \text{if } \kappa_N \Psi_N(t) < \kappa_I \Psi_I(t) \\
S_N(t) &\in [0, S] , & S_I(t) &= S - S_N(t) & \text{if } \kappa_N \Psi_N(t) = \kappa_I \Psi_I(t).
\end{align*} \]

Intuitively, \( \Psi_N \) and \( \Psi_I \) play an analogous role to \( V_N \) and \( V_I \) in the decentralized allocation, and can be also written as integrals of future net benefits:

\[ \begin{align*}
\Psi_N &= \int_t^\infty e^{-\int_t^\tau (r^P - \delta + W^P L_s^\omega \omega_K^P) d\tau} \left( F_N^P + W^P L_s^\omega \omega_N^P \right) d\tau, \\
\Psi_I &= \int_t^\infty e^{-\int_t^\tau (r^P - \delta + W^P L_s^\omega \omega_K^P) d\tau} \left( F_I^P + W^P L_s^\omega \omega_I^P \right) d\tau.
\end{align*} \]

These equations are clearly analogous to the expressions for \( V_N \) and \( V_I \) in the decentralized equilibrium given by equations (21) and (22).

To complete our characterization, let \( f^P(n, k) = F^P(N, I, K, L(\omega^P(N, I, K))) / \gamma(I) \) denote the normalized net output; \( w^P(n, k) = W^P(N, I, K) / \gamma(I) \) the normalized wages; and \( r^P(n, k) \) the normalized interest rate obtained when intermediates are priced at their marginal cost. These are defined in the same way as in the decentralized equilibrium in Section 4. Summarizing, the constrained efficient allocation can be represented as \( \{n(t), k(t), c(t), S_I(t)\}_{t=0}^\infty \) such that:

- The evolution of the state variables is given by

\[ \begin{align*}
\dot{k}(t) &= f^P(n(t), k(t)) - c(t) - (\delta + A\kappa_I S_I(t))k(t) \\
\dot{n}(t) &= \kappa_N(S - S_I(t)) - \kappa IS_I(t).
\end{align*} \] (25)

- Normalized consumption satisfies the Euler equation

\[ \dot{c}(t) = c(t) \left( \frac{1}{\varrho} (r^P(n(t), k(t)) \left( 1 - \omega^P L_s^s \frac{\partial \ln L_s^s}{\partial \ln \omega} \frac{\partial \ln \omega^P}{\partial \ln K} \right) - \delta - \rho) - \kappa_I S_I(t) \right). \] (26)
The allocation of scientists satisfies:
\[
S_I(t) = \begin{cases} 
0 & \text{if } \kappa_I \Psi_I(t) < \kappa_N \Psi_N(t) \\
\in [0, S] & \text{if } \kappa_I \Psi_I(t) = \kappa_N \Psi_N(t) \\
S & \text{if } \kappa_I \Psi_I(t) > \kappa_N \Psi_N(t)
\end{cases}
\]
with \( S_N(t) = S - S_I(t) \).

- The transversality condition holds, i.e.,
\[ \lim \mu_k e^{-\rho t} = 0. \]

This characterization implies that the constrained efficient allocation has a similar structure to the equilibrium described in Proposition 4. The next proposition summarizes this result and shows that it also has the same asymptotic and stability properties. Most importantly, it also determines the set of taxes and subsidies that can decentralize this constrained efficient allocation.

**Proposition 5 (Constrained efficient allocation and decentralization)** Suppose that \( \sigma > \zeta \) and \( \rho > \bar{\rho} \). Then, the constrained efficient allocation is uniquely defined by the solution to (25)-(28). Moreover, under the same conditions derived in Proposition 4, this allocation locally converges to the unique constrained efficient balanced growth path, and if \( \theta \to 0 \), it globally converges to this constrained efficient balanced growth path.

Moreover, this constrained efficient allocation can be decentralized by using the following sets of taxes and subsidies:

1. a proportional subsidy at the rate \( 1 - \mu \) on intermediate prices to remove the monopoly markups;
2. a proportional tax/subsidy of \( \tau_k = -\omega P L_s \alpha_{\ln \omega} \alpha_{\ln K} \) on savings to correct for the impact of capital on employment (this expression is positive, i.e., a tax, when \( \sigma_{SR} > 1 \), zero when \( \sigma_{SR} = 1 \), and a subsidy, i.e., negative, when \( \sigma_{SR} < 1 \));
3. additive taxes/subsidies for successful innovators who entered the market at time \( t_0 \), which correct for the technological externality generated by the two different types of innovation;
4. an additive subsidy \( W P L_s \alpha_{\ln \omega} \alpha_{\ln K} \geq 0 \) for successful innovators of new more complex tasks, and an additive tax \( W P L_s \alpha_{\ln \omega} \alpha_{\ln I} \leq 0 \) on successful innovators of new automation technologies; this tax and subsidy correct for the fact that technology monopolists do not take into account the effect of technologies on the level of equilibrium employment.

**Proof.** See Appendix C. □

Note first that in contrast to neoclassical models of capital taxation (e.g., Chamley, 1986 and Judd, 1985, but also see Straub and Werning, 2014), the decentralization of the constrained efficient allocation requires taxing or subsidizing capital accumulation. This is because the capital stock affects wages and thus the level of employment through the quasi-labor supply schedule. For instance, if \( \sigma_{SR} < 1 \), capital increases employment in the short run (see Proposition 2) which is, as
noted above, beneficial. Thus in this case, the social planner would set \( \tau_K < 0 \), further encouraging capital accumulation, while when \( \sigma_{SR} > 1 \), the opposite applies.

Second, the quality ladder structure in the creation of new labor-intensive complex tasks introduces a technological externality. By undertaking this type of innovation and thus increasing \( N \), a technology monopolist also allows new entrants to create more productive new tasks (because \( \gamma(N) \) is increasing). The externality created by automation is somewhat more subtle. Because capital has the same productivity in all automated tasks, there is no direct technological externality. But automation today forces future innovators to automate higher-indexed tasks, which are the ones where labor has a comparative advantage (because \( \gamma(I) \) is increasing), and this reduces the incremental profits of future innovators.

Finally and most importantly, the quasi-labor supply schedule creates an additional, and novel, distortion in the equilibrium relative to the constrained efficient allocation. Because firms do not internalize the quasi-rents received by workers, they automate tasks taking into account the wage rate. In contrast, the social planner internalizes these quasi-rents, and thus at the margin prefers to create more employment (or equivalently, at the margin she uses the opportunity cost of labor rather than the market wage in the automation decision). The resulting greater incentives of firms to automate tasks with given technology then translate into too much R&D directed towards automation and too little R&D directed towards the creation of new, more complex tasks. For this reason, the social planner would like to encourage the creation of more labor-intensive new tasks and less automation, and she achieves this by using taxes on automation innovations and subsidies to innovations, creating new labor-intensive tasks as outlined in part 4 of the proposition.

Proposition 5 focused on how the constrained efficient allocation can be decentralized. A key result, as we have just emphasized, is that conditional on the other taxes and subsidies necessary for dealing with markups and technological externalities, there needs to be an additional set of taxes and subsidies to encourage less automation and more effort towards the creation of new, more complex tasks. The complementary question is whether, starting from a decentralized allocation, and without this full set of subsidies, the social planner would still like to discourage automation.

The next proposition answers this question (in the affirmative), focusing on the configuration where \( \zeta \to 1 \) which, as we have already emphasized, is a particularly tractable special case of our model (we also continue to assume that the proportional subsidy at the rate \( 1 - \mu \) removing the main effect of monopoly markups is present, or equivalently that \( \mu \to 1 \)).

**Proposition 6 (Excessive automation)** Suppose that \( \rho > \overline{\rho} \) and \( S < \overline{S} \) as in Proposition 4, and that \( \sigma > \zeta \to 1 \). Moreover, suppose that intermediate goods are subsidized and can be purchased at their marginal cost (or equivalently \( \mu \to 1 \)). Consider the decentralized equilibrium path starting from some initial level of capital, \( K(0) \), and technologies, \( N(0) \) and \( I(0) \), converging to the balanced growth path described in Proposition 4 (i.e., \( n^D(t) = N(t) - I(t) \) converging to \( n^D \)). Then there
exists a feasible allocation satisfying \( n^P(t) \geq n^D(t) \) with \( \lim_{t \to \infty} n^P(t) > n^D \) that achieves strictly greater welfare than the decentralized equilibrium.

**Proof.** See Appendix C. □

This proposition therefore establishes that even without the full set of other taxes and subsidies, departing from the equilibrium in the direction of discouraging automation and further encouraging the creation of new, more complex tasks will be welfare improving. The assumption that \( \zeta \to 1 \) plays an important in this result, because, in this special case, the production function for intermediates becomes Cobb-Douglas, and monopoly profits are proportional to revenues. This, coupled with the assumption that monopoly markups are removed, ensures that incentives to undertake different types of innovations, as summarized by the value functions \( V_N \) and \( V_I \), are proportional to social values except for the distortion working through the quasi-labor supply schedule. This then enables us to focus on this novel source of distortion in the composition of R&D and direction of technological change.

6 Extensions

In this section, we discuss two extensions. First we introduce heterogeneous skills, enabling us to analyze the impact of the two types of technological changes we have studied in this paper on inequality between different skill types. Second, we reintroduce the creative destruction of profits, and show how similar balanced growth path results continue to apply in this case, though there may also exist other balanced growth path or steady states.

6.1 Automation, New Tasks and Inequality

In this subsection, we introduce heterogeneous skills and study how automation and the creation of new tasks impact inequality.

This extension is motivated by the observation that, since new tasks are more complex, their creation favors high-skill workers who may have a comparative advantage in new and complex tasks. This natural assumption receives support from the data. As the left panel of Figure 8 shows, in each decade since 1980, employment growth has been faster in occupations with more skill requirements — as measured by the average years of education among employees at the start of each decade.

Though the left panel of Figure 8 confirms that it is the skilled occupations that grow faster, the right panel shows a pattern of “mean reversion” in skill requirements, whereby average education

\[ \text{30The same result can also be established without these assumptions if the quasi-labor supply curve is sufficiently elastic, so that the benefits from small increases in wages (in terms of expanding employment) outweigh costs that may come from other nonlinear effects that are not removed by taxes and subsidies in this case.} \]
Figure 8: Scatter plots of employment growth within each occupational group (left panel) or the change in average skills among employees (right panel), and its skill requirements at the beginning of the decade. Employment growth or the change in average skills over the next 10 years are plotted in dark blue, over the next 20 years in blue, and over the next 30 years in light blue. Employment counts and skill requirements for 1980, 1990 and 2000 are from the U.S. Census; while data for 2007 is from the American Community survey.

declines over subsequent years in these high skill requirement occupations as they become more open to lower-skill workers.

We incorporate these features into our model by assuming there are two types of workers: skilled and unskilled. The pattern of comparative advantage is slightly more complicated and reflects our interpretation of the patterns in the data. We assume that high-skilled labor has productivity analogous to what we have assumed so far for labor overall:

\[ \gamma_H(i) = e^{A_H i}. \]

For low-skilled workers, we assume

\[ \gamma_L(i,t) = e^{A_L i + (A_H - A_L) \Delta(t - t_0(i))}, \]

where \( A_L < A_H \) and \( t \) is calendar time and \( t_0(i) \) the date at which task \( i \) was first introduced. This structure implies that the productivity of low-skill labor increases as time passes from the initial date at which a task was first invented/introduced. This assumption captures the feature that new technologies and tasks are standardized over time (e.g., Acemoglu, Gancia and Zilibotti, 2010) or that low-skill workers may not be good at adapting to a changing environment or new technologies (e.g., Schultz, 1965, Nelson and Phelps, 1966, Greenwood and Yorukoglu, 1997, Caselli, 1999, Galor and Moav, 2000, and Beaudry, Green and Sand, 2013).

The implication of this assumption for our setup is that capital has a comparative advantage in low-indexed tasks that have been automated, high-skill labor has a comparative advantage in high-
indexed tasks that have recently been introduced, and low-skill labor will perform intermediate-indexed tasks. In particular, it follows straightforwardly that there exists a threshold task $M$ such that high-skill labor performs tasks in $(M, N]$, low-skill labor performs tasks in $(I, M]$, and tasks in $[N - 1, M]$ are performed by capital.

In addition, we assume there is a quasi-labor supply of low-skill labor given by $L^s(w_L r K)$, and a quasi-labor supply of high-skill labor given by $H^s(w_H r K)$. The respective wages of these two types of labor are denoted by $w_L$ and $w_H$. For simplicity, we focus on the dynamic economy with exogenous technology.

The main implications of this model with heterogeneous labor are summarized in the next proposition.

**Proposition 7 (Automation, new tasks and inequality)** Suppose technology evolves exogenously. Then:

1. Suppose that $\hat{N} = \hat{I} = \Delta$ (and $A_H (1 - \theta) \Delta < \rho$ so that net present discounted value of household income is finite). Then there exists a unique balanced growth path. In this balanced growth path $Y, C, K, w_H$ and $w_L$ grow at a constant rate $A_H \Delta$ and $r$ is constant. Moreover, the wage ratio between high-skilled and low-skilled workers ($w_H / w_L$) is constant but depends on $n = N - I$

2. Given such a path of technological change, the dynamic equilibrium is unique starting from any initial condition and converges to the balanced growth path.

3. The immediate effect of increases in both $I$ and $N$ is to raise $w_H / w_L$. But the medium-run impact of an increase in $N$ is to reduce inequality.

**Proof.** The proof is essentially identical to that of Proposition 3 and is omitted.

A number of features are worth noting. First, this extended model generates not only an endogenous distribution of income between capital and labor, but also inequality between high-skill and low-skill workers. This inequality reflects comparative advantage — now the comparative advantage of high-skill workers relative to their low-skill brethren. This comparative advantage structure also implies that automation, by squeezing out tasks previously performed by low-skill labor, increases inequality between the two types of skills. Interestingly, however, the creation of new tasks also tends to increase inequality at first because it is high-skilled labor that has a comparative advantage in the higher-index, new tasks. However, given our standardization assumption (that tasks become standardized over time), the productivity of low-skill workers increases over time, and the medium-term implications of automation and creation of new tasks are very different. The former increases inequality both in the short and the medium run. In contrast, the creation of new tasks increases inequality in the short run, but not in the medium run. In fact, low-skill workers gain relative to capital in the medium run from the creation of new tasks.
Interestingly, inequality may be particularly high following a period of adjustment in which the labor share first declines—due to increases in automation—and then recovers—due to the introduction of new complex tasks. Inequality may remain large for a while, until learning by low-skilled workers pushes their wages up.

6.2 Creative Destruction of Profits

In this subsection, we modify the assumption we have made on the structure of intellectual property rights, reverting to the classical setup in the literature where new technologies destroy the rents/profits of existing technologies. We will show that this has little effect on the balanced growth path in our model, but makes dynamics and stability more complicated. Formally, we follow the standard models of quality improvements such as Aghion and Howitt (1992) and Grossman and Helpman (1991), and assume that a new innovation building on the previous technology directly replaces the previous technology without making any licensing nor patent payments.

Let us first define $V_N(t,i)$ and $V_I(t,i)$ as the values at time $t$ of having introduced different technologies for the production of task $i$ (respectively, new labor-intensive tasks and automation). As before, flow profits from introducing new technologies are given by $\pi_I(t,i)$ and $\pi_N(t,i)$, respectively for automation and creation of new tasks. Value functions then satisfy the following Bellman equations:

$$r(t)V_N(t,i) - \dot{V}_N(t,i) = \pi_N(t,i)$$
$$r(t)V_I(t,i) - \dot{V}_I(t,i) = \pi_I(t,i).$$

For a firm creating a labor-intensive technology for task $i$, let $T^N(i)$ denote the time at which it will be replaced by a technology allowing the automation of this task. Likewise, for a firm automating task $i$ at time $t$, let $T^I(i)$ denote the time at which it will be replaced by a more complex technology using labor. Since firms anticipate these deterministic replacement dates, their value functions also satisfy the boundary conditions $V_N(T^N(i),i) = 0$ and $V_I(T^I(i),i) = 0$.

Using the Bellman equations together with the boundary conditions derived above, we find the following formula for these value functions:

$$V_N(t) = V_N(N(t),t) = \int_t^{T^N(N(t))} e^{-\int_t^s r(\tau)d\tau}(1-\mu)Y(\tau)Bc^{\mu} \left( \frac{W(\tau)}{\gamma(N(t))} \right)^{\zeta-\sigma} d\tau,$$
$$V_I(t) = V_I(I(t),t) = \int_t^{T^I(I(t))} e^{-\int_t^s r(\tau)d\tau}(1-\mu)\kappa Y(\tau)Bc^{\mu} \left( \min \left\{ r(\tau), \frac{w(\tau)}{\gamma(I(t))} \right\} \right)^{\zeta-\sigma} d\tau.$$

In addition, for reasons that will become readily clear, we modify the evolution of the technology frontier and assume that advances in automation take the form

$$\dot{I}(t) = \kappa_I \phi(n(t))S_I(t). \quad \text{(29)}$$
Here, the function $\phi(n(t))$ is included and assumed to be nondecreasing to capture the possibility that automating tasks closer to the frontier (defined as the highest available task) may be more difficult. If $n(t)$ is close to 0, then it will be the recently invented tasks that are being automated, which may be more difficult than the case in which $n(t)$ is close to 1. It is straightforward to verify that Proposition 4 remains unaffected if we replace (19) with (29). However, the dynamic properties of the equilibrium now changed as outlined in next proposition.

**Proposition 8 (Equilibrium with creative destruction)** Suppose that $\sigma > \zeta$, $\rho > \bar{\rho}$ and $S < \underline{S}$ (where $\overline{\rho}$ and $\underline{S}$ are defined as in Proposition 4). If there is creative destruction of profits of existing technologies. Then:

1. There exist $\phi < \overline{\phi}$ such that if $\phi(0) < \phi$ and $\phi(1) > \overline{\phi}$, there exists at least one stable balanced growth path with both automation and creation of new tasks. In this balanced growth path, we have $N(t) - I(t) = n^{DR}$, $\kappa N V_N(t) = \kappa I \phi(n^{DR}) V_I(t)$ and $\dot{N} = \dot{I} = \kappa \phi(n^{DR}) S$. Also, $Y, C, K$ and $w$ grow at the constant rate $g = A \kappa \phi(n^{DR}) S$, $r$ is constant, and the labor share and employment are constant.

2. There may also exist other steady states or balanced growth paths.

**Proof.** See Appendix C. ■

The first part of the proposition follows using analogous lines of argument to the proof of Proposition 4, with the only difference being that, because of the presence of the $\phi(n)$ in (29) in this case, the key condition determining a balanced growth path becomes $\kappa I \phi(n^{DR}) V_I(n) = \kappa N V_N(n)$. In addition, in a balanced growth path, we have $T^N(N(t)) - t = n^{DR}/\Delta$, and $T^I(I(t)) - t = 1 - n^{DR}/\Delta$. Thus, both types of innovations are replaced at a fixed length of time, ensuring that the creative destruction of rents does not change the balance of the incentives for innovation.

The major difference with our previous analysis is that, in the presence of the creative destruction of rents, $V_N > 0$ is increasing in $n$ and $V_I > 0$ is constant. Thus, if $\phi(n)$ were constant, the intersection between these $\kappa N V_N(n)$ and $\kappa I V_I(n)$ schedules would give an unstable balanced growth path. Economically, this is a consequence of the productivity effect combined with the fact that the creative destruction of rents implies that, in contrast to the socially planned economy and to our baseline model, firms’ innovations incentives depend on the total revenues that a technology generates — not on the incremental value created (which is the difference between these revenues and the revenues that the alternative technology would have generated). For example, the net present discounted value of introducing new labor-intensive tasks is increasing in $n$ in this case, because wages always increase following both types of innovations due to the productivity effect (see subsection 3.2). Hence, in this case, it is the level of wages, not the the ratio wages to the interest rate, that guides the direction of innovation, and this creates a force towards instability. Stability is guaranteed in this setting by the presence of the $\phi(n)$ function, and in particular, the
condition that $\phi(0) < \bar{\phi}$ and $\phi(1) > \bar{\phi}$, which ensures that the intersection of these two curves takes place where $\kappa_I\phi(n)V_I(n)$ is steeper than $\kappa_NV_N(n)$. Proposition 8 then shows that most of the qualitative results concerning the nature of the balanced growth path still hold, but additional need to be imposed to guarantee stability.

7 Conclusion

As more and more tasks performed by labor have been automated, concerns that these new technologies will make labor increasingly redundant have also intensified. This paper has attempted to develop a comprehensive framework in which these forces can be analyzed and contrasted with countervailing effects. At the center of our model is a task-based framework in which tasks are allocated between capital and labor. Automation is modeled as (endogenous) expansion of the set of tasks that can be performed by capital, thus replacing labor in tasks that it previously produced. The main new feature of our framework is that, in addition to automation, there is another type of technological change enabling the creation of new, more complex versions of existing tasks, and it is labor that tends to have a comparative advantage in these new tasks. We fully characterize the structure of equilibrium in such a model, showing how, given factor prices, the allocation of tasks between capital and labor is determined both by available technology and the endogenous choices of firms between capital and labor. One attractive feature of task-based models is the link they highlight between factor prices and the range of tasks allocated to factors. More generally, as the equilibrium range of tasks allocated to capital increases (for example as a result of automation), the wage relative to the rental rate of capital and the share of labor in national income decline, and the equilibrium wage rate may also fall. Conversely, as the equilibrium range of tasks allocated to labor increases, the opposite result obtains. In our model, because the supply of labor is elastic, automation tends to reduce employment, while the creation of new tasks increases employment. These results highlight that, while both types of technological changes underpin economic growth, they have very different implications for the factor distribution of income and also for employment.

Our full model endogenizes the direction of research and development towards automation and the creation of new, more complex tasks, showing how this framework naturally leads to a (unique) balanced growth path in which both types of innovations go hand-in-hand. Moreover, the dynamic equilibrium is also unique and locally converges to balanced growth path. Underpinning this stability result is the impact of relative factor prices on the direction of technological change. The task-based framework — differently from the standard models of directed technological change which are based on factor-augmenting technologies — implies that as a factor becomes cheaper, this not only expands the range of tasks allocated to it, but also generates stronger incentives for the type of technological change working with this factor. These economic incentives then imply
that automation, by reducing wages relative to the rental rate of capital, encourages the creation of new labor-intensive tasks and generates a powerful self-correcting force towards stability.

Though market forces ensure the stability of the balanced growth path, they do not necessarily generate the efficient composition of technology. If the elastic labor supply relationship results from rents (so that there is a wedge between wage and the opportunity cost of labor), then there is an important and new distortion the type of technologies created. Firms tend to have an excessive bias for automation, because they derive profits by replacing labor with cheaper capital. The social planner, on the other hand, recognizes that part of the wage is rent captured by workers, and has weaker incentives to replace labor with capital.

In addition to claims about automation leading to the demise of labor, several commentators are concerned about the inequality implications of automation and related new technologies. In one of our extensions, we study this question by introducing a distinction between low-skilled and high-skilled labor, where the latter has a comparative advantage in producing with newer technologies. This structure implies that both automation, which squeezes out tasks previously performed by low-skill labor, and the creation of new tasks, which directly benefits high-skill labor, will increase inequality between the two labor types. Nevertheless, the medium-term implications of creation of new tasks could be very different, because new tasks are later standardized and used by low-skill labor. As a result of this effect, we show that there exists a unique balanced growth path in which not only the factor distribution of income (between capital and labor) but also inequality between the two skill types is constant.

We consider our paper to be a first step towards a systematic investigation of different types of technological changes that impact capital and labor differentially. Several areas of research appear fruitful based on this first step. First, we introduced labor market distortions in this model in the form of a reduced-form quasi-labor supply curve. Going beyond this, an important set of issues center around how the process of automation and replacement of workers by capital into plays with the costly and potentially slow reallocation of workers across tasks and firms. We take some steps in this direction in our companion paper, Acemoglu and Restrepo (2016). Second, there may be major differences in the ability of technology to automate and also to create new tasks across industries (e.g., Polanyi, 1966, Autor, Levy and Murnane, 2003). An interesting step is to construct realistic models in which the sectoral composition tasks performed by capital and labor as well as technology evolve endogenously and are subject to industry-level technological constraints (e.g., on the feasibility or speed of automation). Third and perhaps most importantly, our model highlights the need for additional empirical evidence on how automation takes place and incentives for automation and creation of new tasks respond to policies and changes in the environment. One interesting direction would be to construct measures of automation and creation of new tasks,
potentially at the industry level, and then exploit the impact on technology choices and innovation of industry-level variation in wages and institutional restrictions on capital-labor substitution.

References


A1 Details of the Empirical Analysis

In this part of the Appendix, we provide information about the data and samples used in constructing Figure 2 and provide a regression analysis documenting the robustness of the pattern illustrated in that figure.

**Data:** We use demographic and employment data from the U.S. Censuses for 1980, 1990 and 2000 and the American Community Survey for 2007, and aggregate all data to the 330 consistently defined occupational groups proposed by David Dorn (see [http://www.ddorn.net/data.htm](http://www.ddorn.net/data.htm)).

The measure of new tasks and jobs is from Lin (2011), who uses new occupational titles added to new waves of the Dictionary of Occupational Titles to create measures of new jobs in each census occupational group for 1980 and 1990. He also compares the 1990 Census Index of Occupations with its 2000 counterpart, as well as technical documentation provided by the Census to determine the share of new job classes in each occupational category of the 2000 Census. The data are available from his website [https://sites.google.com/site/jeffrlin/newwork](https://sites.google.com/site/jeffrlin/newwork).

**Analysis:** To document the role of novel tasks and jobs, we estimate the regression

\[
\ln E_{it+10} - \ln E_{it} = \beta N_{it} + \delta_t + \Gamma_t X_{it} + \varepsilon_{it}. \tag{A1}
\]

Here, the dependent variable is the percent change in employment from year \( t \) to \( t + 10 \), in each occupational category \( i \). We stack this model using data for \( t = 1980, 1990, 2000 \). For \( t = 2000 \), we use the change from 2000 to 2007 as the dependent variable and scale it to match a 10-year change. In all regressions we include a full set of decadal effects \( \delta_t \), and in some models we also control for differential decadal trends by occupational characteristics, \( X_{it} \). These characteristics include the share of employees in 5-year age brackets, from different races (Black, Hispanic) and origins (foreigner), and that are male. These flexibly control for demographic changes that may affect labor supply and for potential differential sectors of demand. \( \varepsilon_{it} \) is an error term that may be correlated over time within occupational groups.

The coefficient of interest is \( \beta \), which represents the additional employment growth in occupations with more novel tasks and jobs, \( N_{it} \). Throughout, all standard errors are robust against arbitrary heteroscedasticity and serial correlation.

Panel A in Table A1 presents estimates of equation (A1). Column 1 contains no additional covariates (the number of observations in this column is 328, including all occupational groups for which we have data from Lin). Our estimates indicate that occupational groups with 10 percentage point more novel jobs at the beginning of each decade grow 5.2% faster (standard error= 1.3%). If occupational groups with more novel jobs did not grow any faster than the benchmark category with no novel jobs, employment growth for each decade from 1980 to 2007 would have been, on
average, 2.7% instead of the actual 5.7%, implying that approximately 3% of the 5.7% growth is accounted for by novel jobs and tasks as reported at the bottom.

Table A1: Differential employment growth in occupational groups with more new jobs and tasks

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>Weighted by size</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: Stacked differences over decades.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Share of novel tasks and jobs at start of decade</td>
<td>0.522***</td>
<td>0.584***</td>
<td>0.495***</td>
<td>0.381***</td>
<td>0.199</td>
</tr>
<tr>
<td></td>
<td>(0.131)</td>
<td>(0.130)</td>
<td>(0.140)</td>
<td>(0.144)</td>
<td>(0.176)</td>
</tr>
<tr>
<td>log of employment at start of decade</td>
<td>-0.035***</td>
<td>-0.048***</td>
<td>-0.044***</td>
<td>-0.000***</td>
<td>0.004</td>
</tr>
<tr>
<td></td>
<td>(0.012)</td>
<td>(0.014)</td>
<td>(0.013)</td>
<td>(0.011)</td>
<td>(0.011)</td>
</tr>
<tr>
<td>Average years of education at start of decade</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>10.574***</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(1.841)</td>
</tr>
<tr>
<td>R-squared</td>
<td>0.03</td>
<td>0.04</td>
<td>0.11</td>
<td>0.15</td>
<td>0.13</td>
</tr>
<tr>
<td>Observations</td>
<td>986</td>
<td>986</td>
<td>986</td>
<td>986</td>
<td>986</td>
</tr>
<tr>
<td>Employment growth by decade in p.p.</td>
<td>5.7</td>
<td>5.7</td>
<td>5.7</td>
<td>5.7</td>
<td>5.7</td>
</tr>
<tr>
<td>Contribution of novel tasks and jobs</td>
<td>3.0</td>
<td>3.3</td>
<td>2.8</td>
<td>2.2</td>
<td>1.1</td>
</tr>
</tbody>
</table>

|                          |          |          |          |          |          |          |
| Share of novel tasks and jobs in 1980             | 1.241*** | 1.401*** | 1.452*** | 1.153*** | 0.056  | 0.516    |
|                          | (0.391)  | (0.343)  | (0.347)  | (0.322)  | (0.492) | (0.407)  |
| log of employment in 1980                          | -0.155***| -0.193***| -0.164***| -0.048   | -0.038 |          |
|                          | (0.031)  | (0.035)  | (0.034)  | (0.031)  | (0.031) | (0.030)  |
| Average years of education in 1980                 |          |          |          |          | 23.943***| 16.357***|
|                          |          |          |          |          | (4.128) | (4.390)  |
| R-squared                 | 0.02     | 0.08     | 0.17     | 0.26     | 0.17   | 0.17     |
| Observations              | 328      | 328      | 328      | 328      | 328    | 325      |
| Employment growth from 1980-2007 in p.p.           | 15.8     | 15.8     | 15.8     | 15.8     | 15.8   | 15.8     |
| Contribution of novel tasks and jobs               | 7.0      | 7.9      | 8.2      | 6.5      | 0.3    | 2.9      |

*Covariates:*

Decade fixed effects ✓ ✓ ✓ ✓ ✓ ✓
Demographics × decade effects ✓ ✓ ✓ ✓ ✓

*Notes:* The table presents 10-years stacked-differences estimates (Panel A) and long-differences estimates (Panel B) of the share of novel tasks and jobs in an occupational group on subsequent employment growth. The bottom row in each panel reports the observed growth and the share explained by growth in occupations with more novel tasks and jobs. Additional covariates that are not reported are indicated in the bottom of the table. In Column 5 we re-weight the data using the share of employment in each occupation, and in Column 6 we exclude three large employment categories that are outliers in the model of column 5. These include office supervisors, office clerks, and production supervisors. Standard errors robust to heteroskedasticity and serial correlation within occupations are presented in parentheses.

In column 2 we control for the log of employment at the beginning of the decade (year t). The coefficient of interest increases slightly to 0.584 and continues to be precisely estimated. The log of employment at year t appears with a negative coefficient, which indicates that smaller occupations tend to grow more over time and suggests that employment growth is not driven by already well-established occupations employing a large share of the population. In any case, the quantitative
The contribution of new tasks and jobs remains very similar to column 1, increasing slightly to 3.3%.

In column 3 we control for the demographic covariates described above, with little effect on the qualitative or quantitative pattern of results. In column 4, we also control for the average education of employees at the beginning the decade. Consistent with the patterns documented in Figure 8, occupations with more skilled/educated workers tend to grow more rapidly. This control also reduces the quantitative magnitudes of the share of novel tasks and jobs, which nevertheless remains highly significant. The contribution of these novel tasks and jobs is now estimated at 2.2% out of the 5.7% average decadal growth between 1980 and 2007.

Column 5 repeats the specification of column 4, but this time using the share of employment in each occupation as weights. This weakens the relationship of interest, and the share of novel tasks and jobs is no longer statistically significant. However, this lack of significance is driven by a few large occupations that are outliers in the estimated relationship. (In contrast, there are no major outliers in the unweighted regressions reported in columns 1-4). These outliers include office supervisors, office clerks, and production supervisors, three occupational groups with combined employment of about 4 million workers in 1980 and have been on the decline since then. Though these occupational groups introduced a significant number of novel jobs and tasks in 1980, they shed a large amount of workers from 1980 to 1990. In column 6, we exclude these occupational groups from our analysis, and obtain a similar pattern to column 4.

Finally, in Panel B, we present regressions that focus on long differences between 1980 and 2007. The overall patterns are very similar, and now the contribution of novel tasks and jobs to the 15.8% growth in employment between 1980 and 2007 is between 6.5 and 8.2%.

A2 Main Proofs

A2.1 Proofs from Section 2

Proof of Proposition 1: We proceed in three steps. First, given $I^*, N$ and $K$, equilibrium values of $r, W$ and $Y$ are uniquely determined, thus allowing us to define the function $\omega(I^*, N, K)$ representing the relative demand for labor, which was introduced in the text. Second, we show that $\omega(I^*, N, K)$ is nonincreasing in $I^*$. Third, we show that $\min\{I, \bar{I}\}$ is nondecreasing in $\omega$ and conclude that there is a unique pair $\{\omega^*, I^*\}$ such that $I^* = \min\{I, \bar{I}\}$ and $\omega^* = \omega(I^*, N, K)$. This pair uniquely determines the equilibrium relative factor prices and automation decision.

Step 1: Consider $I^*, N$ and $K$ such that $I^* \in (N - 1, N)$. Then, $r, W$ and $Y$ satisfy the system of equations given by capital and labor market clearing, equations (7) and (8), and the ideal price index, equation (9).
Taking the ratio of (7) and (8), we obtain

\[
\frac{\int_{I^*}^{N} \gamma(i)^{\zeta-1} e^{u \left( \frac{W}{\gamma(i)} \right)^{\zeta-\sigma}} \frac{W}{(I^* - N + 1)} \zeta - \sigma \, di}{L^s \left( \frac{W}{rK} \right)} = \frac{1}{K}
\]  

(A2)

The left-hand side is decreasing in \( W \) and increasing in \( r \) in view of the fact that \( L^s \) is nondecreasing and the function \( c^u(x)^{\zeta-\sigma} x^{-\zeta} \) is decreasing everywhere in \( x \) (as it can be verified directly by differentiation). Therefore, (A2) defines an everywhere upward-sloping relationship between \( W \) and \( r \). Inspection of equation (9), in turn, readily shows that the ideal price index equation gives a downward-sloping locus between \( r \) and \( W \) as shown in Figure A1. The intersection point defines equilibrium factor prices given \( I^* \), \( N \) and \( K \). Since one curve is upward sloping and the other one downward sloping, there can at most be one intersection.

To prove that there always exists an intersection, we verify “boundary conditions”. We distinguish two cases. First, suppose \( \zeta < 1 \) and \( \sigma < 1 \) or \( \zeta > 1 \) and \( \sigma > 1 \). In this case, it can be verified that there exists a maximum level of the wage rate, \( \tilde{W} \), reached when \( r = 0 \), and a maximum level of the interest rate, \( \tilde{r} \), reached when \( W = 0 \), consistent with the ideal price condition, (9). (This follows because, under these parameter configurations, if we let \( W \to \infty \) or \( r \to \infty \), (9) cannot be satisfied). This implies that the downward-sloping locus in Figure A1 hits both axes, and thus must necessarily have an intersection with the relative demand locus. Second, suppose \( \zeta > 1 \) and \( \sigma < 1 \) or \( \zeta < 1 \) and \( \sigma > 1 \). In this case, it can be verified straightforwardly that \( \lim_{x \to 0} c^u(x)^{\zeta-\sigma} x^{-\zeta} = \infty \), and that \( \lim_{x \to \infty} c^u(x)^{\zeta-\sigma} x^{-\zeta} = 0 \). This implies that as \( W \to 0 \), the numerator of (A2) limits to infinity, and hence, so must the denominator, proving that the relative demand curve locus starts from the origin. Similarly, as \( W \to \infty \), the numerator of (A2) limits to zero, and so must the denominator, which then implies that the relative demand curve locus goes to infinity as \( r \to \infty \). This implies that the downward-sloping relative demand curve locus necessarily starts below and ends up above the ideal price condition, ensuring that there always exists an intersection in this case as well. The unique intersection uniquely defines the function \( \omega(I^*, N, K) = \frac{W}{rK} \). The level of aggregate output, \( Y^* \), can then be computed from (7), or (8), using the fact that in these expressions \( \min \{ I, \tilde{I} \} = I^* \).

\textbf{Step 2:} As we have just seen, the equilibrium values of \( W \) and \( r \) are given by the intersection of the index condition, (9), and equation (A2). Now totally differentiating these two equations with respect to \( I^* \), we obtain

\[
d \ln \left( \frac{W}{r} \right) \propto -dI^* \left( \frac{\gamma(I^*)^{\zeta-1} e^{u \left( \frac{W}{\gamma(I^*)} \right)^{\zeta-\sigma}}}{\int_{I^*}^{N} \gamma(i)^{\zeta-1} e^{u \left( \frac{W}{\gamma(i)} \right)^{\zeta-\sigma}} \, di} + \frac{1}{I^* - N + 1} + \frac{\sigma - \zeta}{1 - \sigma} \left[ e^u \left( \frac{W}{\gamma(I^*)} \right)^{1-\sigma} - e^u(r)^{1-\sigma} \right] \right).
\]

Recall that \( \tilde{I} \) is defined such that \( \gamma(\tilde{I}) = W/r \). Since \( I^* = \min \{ I, \tilde{I} \} \), \( \gamma(I^*) \leq W/r \), and thus \( c^u \left( \frac{W}{\gamma(I^*)} \right) \leq c^u(r) \). Then the term in square brackets is positive when \( \sigma < 1 \) and is multiplied by a positive constant, and is negative when \( \sigma > 1 \) and is multiplied by a negative constant, ensuring
that determine brackets on the right-hand side is always positive, and thus $W/r$, and hence $\omega$, is decreasing in $I$.

**Step 3:** As a last step, we will establish that $I^* = \min\{I, \tilde{I}\}$ is uniquely defined. Since $\gamma(\tilde{I}) = \omega K$, $\tilde{I}$ is increasing in $\omega$, and thus $I^* = \min\{I, \tilde{I}\}$ is nondecreasing in $\omega$. Now considering the pair of equations $\omega = \omega(I^*, N, K)$ and $I^* = \min\{I, \tilde{I}\}$ plotted in Figure 4, we can conclude that because $\omega = \omega(I^*, N, K)$ is decreasing in $I^*$ and $I^* = \min\{I, \tilde{I}\}$ is increasing in $\omega$, there exists at most a single pair ($\omega, I^*$) satisfying these two equations (or a single intersection in the figure).

To prove existence, we again verify the appropriate boundary conditions. Suppose that $I^* \to N - 1$. Then from (7), $r \to 0$, while $W > 0$, and thus $\omega \to \infty$. This ensures that the relative demand relationship $\omega(I^*, N, K)$ starts above $I^* = \min\{I, \tilde{I}\}$ in Figure 4. Since $I^* = \min\{I, \tilde{I}\}$ has a maximum at $I$, this implies that it cannot be below $\omega = \omega(I^*, N, K)$ at $I^* = I$, ensuring that there must exist a unique intersection between the two curves over the interval $I^* \in (N - 1, I]$, completing the proof of Proposition 1.

**Proof of Proposition 2:** We first establish the comparative statics of $\omega$ and $I^*$ with respect to $I$, $N$ and $K$ when both $I^* = I < \tilde{I}$ and $I^* = \tilde{I} < I$, and then turn to their effects on wage levels.

**Comparative statics with respect to $I$:** The relative demand locus $\omega = \omega(I^*, N, K)$ does not directly depend on $I$, so that the comparative statics in this case are entirely determined by the effect of changes in $I$ on the $I^* = \min\{I, \tilde{I}\}$ schedule in Figure 4. When $I^* < I$, small changes in $I$ have no effect as claimed in the proposition. Suppose next that $I^* = I < \tilde{I}$. In this case, an increase in $I$ shifts the curve $I^* = \min\{I, \tilde{I}\}$ to the right in Figure 4, increasing $I^*$ and reducing $\omega$ as stated in the proposition.

**Comparative statics for $N$:** Changes in $N$ only shifts the relative demand curve up in Figure
This follows from the fact that from (A2), it shifts up the relative demand locus in Figure A1, and because $c^u \left( \frac{W}{\gamma(N)} \right) \leq c^u(r)$ in the range we are concerned with, it shifts up the downward-sloping locus corresponding to the ideal price condition. Then, in the case where $I^* = I < \bar{I}$, we have

$$\frac{d \ln \omega}{dN} = \frac{d \ln(W/r)}{dN} = \left. \frac{\partial \ln W/r}{\partial N} \right|_{I^*} > 0$$

(where $\left. \frac{\partial \ln W/r}{\partial N} \right|_{I^*}$ denotes the partial derivative with $I^*$ held constant).

Turning next to the case where $I^* < I$, $\gamma(\bar{I}) = \omega/K$, note that $dI^* = \frac{1}{\varepsilon_\gamma} d\ln \omega$ (where recall that $\varepsilon_\gamma$ is the semi-elasticity of the $\gamma$ function as defined in the proposition). Therefore,

$$\frac{d \ln \omega}{dN} = \frac{d \ln(W/r)}{dN} = \left. \frac{\partial \ln \omega}{\partial N} \right|_{I^*} + \frac{1}{\varepsilon_\gamma} \frac{\partial \ln \omega}{\partial I^*} \frac{d \ln \omega}{dN}$$

which gives

$$\frac{d \ln \omega}{dN} = \frac{d \ln(W/r)}{dN} = \frac{\left. \frac{\partial \ln \omega}{\partial N} \right|_{I^*}}{1 - \frac{1}{\varepsilon_\gamma} \frac{d \ln \omega}{dI^*}}$$

as claimed in proposition.

**Comparative statics for $K$:** The curve $I^* = \min\{I, \bar{I}\}$ does not depend on $K$, so the comparative statics are entirely determined by the effect of these variables on $\omega(I^*,N,K)$. Using an analogous argument to that in step 2 of the proof of Proposition 1, we have that an increase in $K$ shifts up the relative demand locus in Figure A1, increasing $W$ and reducing $r$. The impact on $\omega = W/rK$ depends on whether this impact has elasticity greater than one (since $K$ is in the denominator). To determine this impact and express these effects in terms of the elasticity of substitution between capital and labor, defined as $\sigma_{SR} = \left. \frac{\partial \ln(K/L)}{\partial \ln(W/r)} \right|_{K}$, let us start with the case where $I^* = I < \bar{I}$ and write

$$\frac{d \ln \omega}{d \ln K} = \frac{d \ln(W/r)}{d \ln K} - 1$$

$$= \left. \frac{\partial \ln(W/r)}{\partial \ln K} \right|_{L} - \left. \frac{\partial \ln(W/r)}{\partial \ln L} \right|_{K} \frac{d \ln L}{d \ln K} - 1$$

$$= \frac{1}{\sigma_{SR}} - \frac{1}{\sigma_{SR}} \varepsilon_L \frac{d \ln \omega}{d \ln K} - 1.$$  

$$= \frac{1}{\frac{1}{\sigma_{SR}} - \frac{1}{\sigma_{MR}} \varepsilon_L}$$  

(A3)

where we have used the fact that $\left. \frac{\partial \ln(W/r)}{\partial \ln K} \right|_{L} = \left. \frac{\partial \ln(W/r)}{\partial \ln L} \right|_{K} = \frac{1}{\sigma_{SR}}$ and the definition in the proposition that $\frac{d \ln L}{d \ln \omega} = \varepsilon_L$. This establishes the claims about the comparative statics with respect to $K$ when $I^* = I < \bar{I}$. The results for the case where $I^* = \bar{I} < I$ can be derived analogously by substituting the medium-run elasticity of substitution, $\sigma_{MR}$, for $\sigma_{SR}$.

**Effects on factor price levels:** Consider an increase in $I$ in the case where $I^* = I < \bar{I}$. In Figure A1, this shifts down the relative demand locus, but shifts up the downward-sloping locus
corresponding to the ideal price condition. Thus in general the effects on wage levels are ambiguous. When the shift down of the relative demand locus is large, the impact will be negative, and when it is small, it will be positive. From our comparative statics with respect to \( K \), this shift down is given by (A3), which is monotone increasing in \( \sigma_{SR} \), and hence the conclusion follows. The proofs of the results on the effects of \( N \) on the interest rate are analogous. ■

**A2.2 Proofs from Section 3**

Let us define \( w^N(n) = W/\gamma(N) \) and \( w^I(n) = W/\gamma(I^*) \) as effective wages for tasks, respectively, at the margin of introduction and automation. In the balanced growth path, because \( r = \rho + \delta + \theta g \), these effective wages will be just functions of \( n = \lim_{t \to \infty} N(t) - I(t) \). The next lemma characterizes their behavior in the balanced growth path.

**Lemma A1 (The behavior of effective wages \( w^N(n) \) and \( w^I(n) \))** There exists \( \overline{\rho} > 0 \) such that:

1. For \( \rho > \overline{\rho} \), there exists \( \overline{n} \in (0, 1) \) such that:
   - for \( n \geq \overline{n} \), we have \( I^* = I \), and for \( n < \overline{n} \), we have \( I^* < I \);
   - for \( n \geq \overline{n} \), \( w^N(n) \) is increasing and \( w^I(n) \) decreasing in \( n \). Both wages are constant for \( n < \overline{n} \);

2. For \( \rho \leq \overline{\rho} \), there exists a different threshold \( \tilde{n} \in [0, 1) \) such that:
   - for \( n \geq \tilde{n} \), both technologies are used, while for \( n < \tilde{n} \), firms do not create or use new tasks (because labor is not productive or cheap enough compared to capital);
   - for \( n \geq \tilde{n} \), \( w^N(n) \) is increasing and \( w^I(n) \) decreasing in \( n \). Both wages are decreasing in \( n \) for \( n < \tilde{n} \).

**Proof.** Let us first suppose that there exists \( x \in [0, 1] \) for which

\[
  w^I(x) > \rho + \delta + \theta g > w^N(x). \tag{A4}
\]

(We discuss the cases in which this inequality does not hold below).

We proceed in several steps.

**Step 1:** \( w^I(n) \) is increasing and \( w^N(n) \) decreasing in \( n \) for all \( n \geq x \).

To prove this claim, rewrite the ideal price index condition, (9), as in subsection 3.2, substituting for the balanced growth path value of the interest rate, \( r = \rho + \delta + \theta g \), which yields (18). Differentiating the expressions for effective wages using this balanced growth path value of the interest rate yields

\[
  \frac{\partial w^N(n)}{\partial n} = \frac{c^u(\rho + \delta + \theta g)^{1-\sigma} - c^u(w^I(n))^{1-\sigma}}{(1-\sigma) \int_0^{\gamma(n)} c^u(w^I/\gamma(i))^{-\sigma} c^{ui}(w^I/\gamma(i)) di},
\]

\[
  \frac{\partial w^I(n)}{\partial n} = \frac{c^u(\rho + \delta + \theta g)^{1-\sigma} - c^u(w^N(n))^{1-\sigma}}{(1-\sigma) \int_0^{\gamma(i)} c^u(w^I/\gamma(i))^{-\sigma} c^{ui}(w^I/\gamma(i)) di}. \tag{A5}
\]
Suppose first that $\sigma < 1$. Then, for $n \geq x$, the numerator of the first expression is positive, and the numerator of the second expression is negative, and their denominators are positive, and thus $\frac{dw^I(n)}{dn} > 0 > \frac{dw^N(n)}{dn}$. Suppose next that $\sigma > 1$. In this case, the signs of the numerators are flipped, but the denominators are negative, so we reach the same conclusion. From the Fundamental Theorem of Calculus, we can then also conclude that

$$w^I(n) > \rho + \delta + \theta g > w^N(n) \quad (A6)$$

for all $n \geq x$.

**Step 2:** Note that for $n < x$, provided that (A6) holds, $w^I(n)$ continues to be increasing and $w^N(n)$ decreasing in $n$ with the same argument as in Step 1. We next show that equation (A6) cannot hold for all $n \in [0, x]$. To obtain a contradiction, suppose that it did. Then,

$$w^I(0) > \rho + \delta + \theta g > w^N(0),$$

but this is impossible, since at $n = 0$, $w^I(0) = \frac{W(I_n)}{\gamma(0)} = w^N(0)$, thus yielding a contradiction.

**Step 3:** We next show that there exist $\overline{\pi}$ and $\bar{n}$ as in the lemma, but only one of these thresholds will be relevant for each parameter configuration. Since (A6) holds at $n = x$, but not at $n = 0$, and both $w^I(n)$ and $w^N(n)$ are continuous, there exists either $\overline{\pi}$ such that $w^I(\overline{\pi}) = \rho + \delta + \theta g$ and $\bar{n}$ such that $w^N(\bar{n}) = \rho + \delta + \theta g$.

Now, we show that only one of these cases may occur, and that $\rho$ determines which case it is.

First, suppose that as we move from $n = x$ to the left, we reach $\overline{\pi}$ first. At this point, there is no incentive to further automate tasks, and $I^* < I$. Further increases in $I$ — or reductions in $n$ — do not change the equilibrium allocation. Thus, for $n < \overline{\pi}$, $w^I(\overline{\pi})$ and $w^N(n)$ are constant, as shown in the left panel of Figure A2.

Suppose next that as we move from $x$ to the left, we reach $\bar{n}$ first. Now with an analogous argument, we have that for $n < \bar{n}$, and labor will not be used in any of the tasks, and $w^I(n)$ and $w^N(n)$ are constant to the left of $\bar{n}$ as shown in the right panel of Figure A2.

But the configurations in Figure A2 imply that one of $\overline{\pi}$ and $\bar{n}$ must be reached before the other, and then the other threshold becomes irrelevant. Which one of these two thresholds is reached first is determined by the discount rate, $\rho$. In particular, straightforward differentiation again yields

$$\frac{\partial w^N(n)}{\partial \rho} = -\frac{1}{\gamma(n)} \int_0^1 \frac{1}{\gamma(i)} c^u(\rho + \delta + \theta g)^{-\sigma} c^u(w^I / \gamma(i))^{-\sigma} c^l(w^I / \gamma(i))d\gamma(i) < 0,$$

$$\frac{\partial w^I(n)}{\partial n} = -\frac{1}{\gamma(n)} \int_0^1 \frac{1}{\gamma(i)} c^u(\rho + \delta + \theta g)^{-\sigma} c^u(w^I / \gamma(i))^{-\sigma} c^l(w^I / \gamma(i))d\gamma(i) < 0.$$

Therefore, as $\rho$ increases, both $w^I(n)$ and $w^N(n)$, and thus there exists a unique value $\overline{\pi}$ such that

$$w^I(0) = \overline{\pi} + \delta + \theta g,$$

A.8
and for all $\rho > \bar{\rho}$, the relevant threshold is $\bar{n}$, and for all $\rho < \bar{\rho}$, the relevant threshold is $\tilde{n}$.

**Step 4:** Suppose next that (A4) does not hold because of the right inequality. But in this case, all tasks will be produced using capital, leading to a wage of zero (and recall that $L^*(0) > 0$). But in that case, $w^N(n) = 0$ for all $n$, contradicting the supposition that the right inequality does not hold for any $n \in [0, 1]$. Suppose next that the left inequality does not hold. Then we can simply define $\bar{n} = 1$, and the lemma applies as is. This conclusion proof of the lemma.

Before proceeding further, we will now state a result claimed in the text as a corollary Lemma A2:

**Corollary A1** Suppose $\rho > \bar{\rho}$. Then in the balanced growth path, all new tasks will be utilized immediately.

This corollary follows as we have just established that, in this case $\rho + \delta + \theta g > w^N(n)$ for all $n$. But our analysis also implies that for $\rho < \bar{\rho}$, this cannot true (because in this case, for $n \leq \tilde{n}$, we have $w^N(n) = \rho + \delta + \theta g$). This corollary justifies our focus in the text that when $\rho > \bar{\rho}$, all new, more complex tasks will be utilized immediately (and thus the fact that we did not introduce a separate notation for the margin of adoption of these tasks).

We now return to the rest of the proof of Proposition 3.

**Proof of Proposition 3:** We prove each part of the proposition separately.

**Part 1:** This part follows as a corollary of Lemma A2 since we have imposed $\rho > \bar{\rho}$.
**Part 2:** Suppose that \( \lim_{t \to \infty} n(t) = n \geq \overline{n} \). Then given the functional form of \( \gamma \) in (12), all normalized variables grow at the same rate \( \dot{N} = \dot{I} = A\Delta \), and the interest rate, \( r \), is constant, proving the “if” part of the proposition for this case. For the “only if” part, first note that in any balanced growth path, \( Y, C, K \) and \( w \) must grow at some constant rate \( g \), and thus \( y, c, k \) and \( w \) must also grow at some constant rate \( \tilde{g} \), and \( r \) must remain constant at \( \rho + \delta + \theta g \). First note that \( \tilde{g} = 0 \) if and only if \( \dot{N} = \dot{I} = A\Delta \). We thus need to show that \( \tilde{g} = 0 \). Suppose to obtain a contradiction that \( \tilde{g} < 0 \). Then, for \( t \) large enough, we will have \( w(t) < r(t) \), contradicting the result that \( w(t) \geq r = \rho + \delta + \theta g \) derived in Lemma A2. Next, suppose once again to obtain a contradiction that \( \tilde{g} > 0 \). This implies that for \( t \) large enough, we will have \( \frac{w(t)}{\gamma(n(t))} > r(t) \), contradicting once again the result from Lemma A2 that \( \frac{w(t)}{\gamma(n(t))} < r = \rho + \delta + \theta g \). This establishes the “only if” direction of the proof in this case.

Next suppose that \( \lim n(t) < \overline{n} \). In this case, from Lemma A2, in the balanced growth path \( I^* < I \), and thus there will be no automation. The model is then equivalent to an endogenous growth model with just labor-augmenting technological change, and endogenous growth requires that such growth takes place at the constant rate, which is equivalent to \( \dot{N} = \dot{I} = A\Delta \), completing the proof of this part of the proposition.

**Part 3:** Starting with any initial value of \( k(0) \) and \( n(0) \), the equilibrium behavior is given by equations (13), (14) and (15). This is identical to the equations characterizing dynamics in the canonical neoclassical growth model (see, for example, Proposition 8.5 and 8.6 in Acemoglu (2009)), and the condition that \( \rho > \overline{\rho} \) guarantees that \( \rho > A(1 - \theta)\Delta \), which ensures that the transversality condition holds, establishing part 3. ■

A2.3 Proofs from Section 4

Once again, we start with an important lemma, this time concerning the behavior of normalized value functions, \( v_N(n) = V_N(n)/Y \) and \( v_I(n) = V_I(n)/Y \), which are, respectively, the values for creating new tasks and automating technologies for existing tasks, normalized by aggregate output. These two value functions depend on the interest rate and time in general, but in the balanced growth path, they will once again only depend on \( n \).

**Lemma A2 (Asymptotic behavior of the normalized value functions)** Suppose \( \sigma > \zeta \), and that the conditions required in Lemma A1 hold. Let \( v_N \) and \( v_I \) denote the normalized value functions evaluated in the balanced growth path, and let \( \overline{\rho} \) be as defined in Proposition 3. Then there exist thresholds \( \overline{S} \) and \( \overline{\rho} \) such that:

1. For \( \rho > \overline{\rho} \) and \( S < \overline{S} \):
   - if \( n < \overline{n} \), we have \( \kappa_N v_N > \kappa_I v_I > 0 \);
   - if \( n \geq \overline{n} \), \( v_N \) and \( v_I \) are strictly increasing in \( n \);
• if in addition $\rho > \overline{\rho}$, then for $n \geq \overline{n}$, $v_I/v_N$ is increasing in $n$.

2. For $\rho \leq \overline{\rho}$:

• for $n < \overline{n}$, we have $v_I > 0$ and $v_N \leq 0$;
• for $n \geq \overline{n}$, both $v_N$ and $v_I$ are positive and strictly increasing.

Proof. Let $g = \frac{\kappa I N}{\kappa I + \kappa N}S$ be the growth rate of the economy in the balanced growth path. Suppose $\rho > \overline{\rho}$. Then for $n \geq \overline{n}$, we can write the value functions in the balanced growth path as:

$$v_N(n) = \frac{V_N(n)}{Y(n)} = (1 - \mu) \left( \frac{\eta}{1 - \eta} \right) \psi^{1-\zeta} \int_0^\infty e^{-(\rho - (1-\theta)g)\tau} \left[ c^u (w^N(n)e^{\sigma})^{\zeta-\sigma} - c^u (\rho + \delta + \theta g)^{\zeta-\sigma} \right] d\tau,$$

$$v_I(n) = \frac{V_I(n)}{Y(n)} = (1 - \mu) \left( \frac{\eta}{1 - \eta} \right) \psi^{1-\zeta} \int_0^\infty e^{-(\rho - (1-\theta)g)\tau} \left[ c^u (\rho + \delta + \theta g)^{\zeta-\sigma} - c^u (w^I(n)e^{\sigma})^{\zeta-\sigma} \right] d\tau.$$ 

Thus, the value functions only depend on the unit cost of labor $w^N(n)$ and $w^I(n)$, and on the interest rate, which is equal to $\rho + \delta + \theta g$ in the balanced growth path.

Now consider Taylor expansions of both of these expressions (which are continuously differentiable) around $S = 0$, which implies $g = 0$:

$$v_N(n) = \frac{(1 - \mu)\psi^{1-\zeta}}{\rho} \left( \frac{\eta}{1 - \eta} \right) \zeta \left[ c^u (w^N(n))^{\zeta-\sigma} - c^u (\rho + \delta)^{\zeta-\sigma} \right] + o(S), \quad (A7)$$

$$v_I(n) = \frac{(1 - \mu)\psi^{1-\zeta}}{\rho} \left( \frac{\eta}{1 - \eta} \right) \zeta \left[ c^u (\rho + \delta)^{\zeta-\sigma} - c^u (w^I(n))^{\zeta-\sigma} \right] + o(S),$$

where $o(S)$ denotes terms that are second-order or less in $S$, and we have used the fact that when $g = 0$, wages are constant and thus the integrals can be solved out. Moreover, when $n < \overline{n}$, $v_I(n) = O(S)$, while the expression for $v_N(n)$ still applies and remains bounded away from zero. Thus, there exists $\overline{S} > 0$ such that for $S < \overline{S}$, $\kappa N v_N(n) > \kappa I v_I(n) > 0$ as claimed in the lemma.

Differentiating the value functions in (A7) when $n \geq \overline{n}$ immediately establishes that they are both strictly increasing (since $w^I(n) > \rho + \delta > w^N(n)$ from Lemma A1). The implied behavior of the value functions is depicted in Figure A3.

We now prove the existence of a threshold $\overline{\rho}$, which guarantees that the curves $\kappa N v_N(n)$ and $\kappa I v_I(n)$ cross at most once. To prove this result, note that:

$$\frac{\partial v_I(n)}{\partial n} v_N - \frac{\partial v_N(n)}{\partial n} v_I \propto (\sigma - \zeta) l^I c^u (\rho + \delta)^{1-\sigma} - c^u (w^N(n))^{1-\sigma},$$

$$\frac{\partial v_N(n)}{\partial n} v_I = \frac{\partial v_I(n)}{\partial n} v_N \propto (\sigma - \zeta) l^N c^u (w^I(n))^{1-\sigma} - c^u (\rho + \delta)^{1-\sigma},$$

where $l^I$ is labor employed in producing task $I$ and $l^N$ is share of labor employed in producing task $N$.

A.11
Figure A3: Behavior of value functions in steady state with respect to changes in $n = N - I$.

For $\rho$ large enough $\frac{\partial v_I(n)}{\partial n} v_I \to 0$. Thus, there exist threshold a threshold $\bar{\rho}$ such that for $\rho > \bar{\rho}$ and $S < \bar{S}$, the term $\frac{\partial v_I(n)}{\partial n} v_N - \frac{\partial v_N(n)}{\partial n} v_I > 0$ for all $n \geq \bar{n}$, which ensures that $v_I/v_N$ is increasing in $n$. (If $\bar{\rho}$ that ensures this property is strictly less than $\rho_{\text{opt}}$, then simply set $\bar{\rho} = \rho_{\text{opt}}$).

The second part of the lemma has an analogous proof, which we omit. ■

Proof of Proposition 4: We will now prove two propositions, one on the existence of a balanced growth path with endogenous technology and the other one on the properties of the dynamic equilibrium. Proposition 4 then follows as a corollary of these two propositions.

Proposition A1 There exist thresholds $\bar{\rho}, \bar{\rho}, \bar{S}$, and $\bar{\kappa}$ such that for $S < \bar{S}$, the following are true:

1. For $\rho > \bar{\rho}$:
   - If $\frac{\kappa_I}{\kappa_N} > \bar{\kappa}$, there is a balanced growth path, where both technologies are immediately used (so that $I^* = I$) and $\kappa_N V_N = \kappa_I V_I > 0$. Moreover, if $\rho > \bar{\rho}$, the balance growth path is unique.
   - If $\frac{\kappa_I}{\kappa_N} < \bar{\kappa}$, we have that $\kappa_N V_N > \kappa_I V_I$ and all scientists are allocated to create new labor-intensive tasks, and thus $n \to 1$ and the economy converges asymptotically to an $A(t)L$ economy which only uses labor to produce the final good and labor productivity increases at a constant rate over time.

2. If $\bar{\rho} > \rho$, and $\rho \in (\bar{\rho}, \bar{\rho})$, there might be multiple steady states.

3. For $\rho < \bar{\rho}$, starting from a small $n$, the economy admits a unique equilibrium path in which $\kappa_N V_N < \kappa_I V_I$ and it converges to an $AK$ economy, where the economy grows at a constant rate by accumulating more capital.

A.12
**Proof. Part 1:** The existence of a balanced growth path follows by noting that, as stated in Proposition 3, it emerges if and only if ˙\(I = \dot{N}\), and  \(n(t) = n^D\). Since all scientists are allocated to developing one of the two available technologies, we must have:

\[
S_I = \frac{\kappa_N}{\kappa_I + \kappa_N} S \quad \quad \quad \quad S_N = \frac{\kappa_I}{\kappa_I + \kappa_N} S,
\]

and the growth rate of the economy is therefore  \(g = \frac{\kappa_N \kappa_I}{\kappa_I + \kappa_N} S\).

In the balanced growth path, the Euler equation, (13), implies the interest rate equals  \(r = \rho + \delta + \theta g\), and wages are then given by  \(w^N(n)\) and  \(w^I(n)\). Moreover, when  \(\rho > A \frac{\kappa_I \kappa_N}{\kappa_I + \kappa_N} S(1 - \theta)\), the transversality condition will be automatically satisfied.

In any equilibrium with  \(S_I > 0\) and  \(S_N > 0\), the following equilibrium condition is to hold:

\[
\kappa_I v_I(n) = \kappa_N v_N(n).
\]

Then, a balanced growth path exists if and only if there exists a solution  \(n^D\) to this equation.

Suppose first that  \(\rho > \overline{\rho}\) and  \(\overline{S} > S\), then from Lemma A2

\[
\kappa_N v_N(\overline{\rho}) > 0 = \kappa_I v_I(\overline{\rho}),
\]

On the other hand, for  \(n > \overline{\rho}\), we have  \(v_I(n), v_N(n) > 0\). Then as the ratio  \(\frac{\kappa_I}{\kappa_N}\) increases — starting from zero — the curves for  \(\kappa_I v_I(n)\) and  \(\kappa_N v_N(n)\) eventually become equal to each other, proving that for  \(\kappa > \overline{\rho}\), there exists a balanced growth path represented by the first intersection at  \(n^D \in (\overline{\rho}, 1)\) between these two curves.

For  \(\kappa < \overline{\rho}\), we have that  \(\kappa_I v_I < \kappa_N v_N\) throughout, so  \(S_I = 0\), and the balanced growth path is identical to that of an endogenous growth model with purely labor-augmenting technological change.

Next, if  \(\rho > \overline{\rho}\), from Lemma A2  \(v_I/v_N\) is increasing in  \(n\), so  \(n^D\) defined by  \(\kappa_I v_I(n^D) = \kappa_N v_N(n^D)\) is unique.

**Part 2:** It also follows from the preceding argument that if  \(\overline{\rho} > \rho\), and  \(\rho \in (\overline{\rho}, \overline{\rho})\), there exists a balanced growth path, but it may not be unique.

**Part 3:** Finally, if  \(\rho \leq \overline{\rho}\), Lemma A2 implies that at  \(n = \overline{n}\), we always have

\[
\kappa_I v_I(\overline{n}) > \kappa_N v_N(\overline{n}),
\]

so that the balanced growth path involves only automation (and further creation of automated tasks).

The last step in our analysis is to establish the stability of a balanced growth path in some of the cases covered above. The precise statement we prove is summarized in the following Proposition:

**Proposition A2** Suppose that the conditions such that there is a unique steady state in Proposition A1 hold. Then,
1. When $\theta = 0$, the dynamic equilibrium is unique and the balanced growth path is globally asymptotically saddle-path stable.

2. For $\theta > 0$, there exists a threshold $\bar{S} \leq S$ such for $S < \bar{S}$, the dynamic equilibrium is unique around the balanced growth path and the balanced growth path is locally asymptotically saddle-path stable.

Proof. Part 1: Suppose $\theta = 0$. In this case, capital adjusts immediately and its equilibrium stock only depends on $n$, which becomes the unique state variable of the model. The rental rate of capital is fixed at $r = \rho + \delta$ at each point in time, and wages are given by $w^I(n)$ and $w^N(n)$.

Define $v(t) = \kappa_I v_I(t) - \kappa_N v_N(t)$, with $v_I(t) = V_I(t)$ and $v_N(t) = V(t)/Y(t)$ the value functions that determine the incentives to introduce different types of innovations.

Starting from any $n(0)$ the equilibrium path with endogenous technology is given by a tuple $\{n(t), S_I(t)\}_{t=0}^\infty$ such that:

- The evolution of the state variable is given by
  $$\dot{n}(t) = \kappa_N S - (\kappa_N + \kappa_I) S_I(t).$$

- The allocation of scientists satisfies:
  $$S_I(t) = \begin{cases} 
 0 & \text{if } v < 0 \\
 0 if v = 0 \\
 0 if v > 0 
\end{cases}$$

with $v$ satisfying the forward looking differential equation:

$$\rho v - \dot{v} = (1 - \mu) \left( \frac{\eta}{1 - \eta} \right)^\zeta \psi^{1-\zeta} \kappa_I \left( e^\mu (\rho + \delta)^{\zeta-\sigma} - e^\mu (w^I(n))^{\zeta-\sigma} \right) - (1 - \mu) \left( \frac{\eta}{1 - \eta} \right)^\zeta \psi^{1-\zeta} \kappa_N \left( e^\mu (w^N(n))^{\zeta-\sigma} - e^\mu (\rho + \delta)^{\zeta-\sigma} \right) + O(S).$$

Let $n^D$ denote the steady state value for $n(t)$ in the unique BGP. Since $\kappa_I v_I(n) - \kappa_N v_N(n)$ crosses zero only at $n^D$, this is the unique BGP and the equilibrium can be represented graphically as we do in Figure 7. We now prove it is globally stable. Figure A4 presents the phase diagram of the system in $(v, n)$. Importantly, the locus for $\dot{v} = 0$ crosses $v = 0$ at $n^D$ from below only once. This follows from the fact that $\kappa_I v_{I}^{D} > \kappa_N v_{N}^{D}$ at this interception (see Figure A3).

Thus, $n^D$ is saddle path stable, and for each $n(0)$ there is a unique $v(0)$ in the stable arm of the system.

In order to show all equilibria must be along the stable arm, we need to rule out other paths. From the figure it is clear that either the equilibrium settles at $n^D$, or it reaches the region with $\dot{v} > 0$ and $\dot{n} < 0$, or the region with $\dot{v} < 0$ and $\dot{n} > 0$. In the first case, $v$ is strictly increasing and...
n is strictly decreasing, and hence there are no interior limit points. Moreover, n cannot cross π because in this region we have ˙n > 0 (there are no incentives for automation). This implies v → ∞ along any such path, which violates the transversality condition for households entitled to profits from automation. In the second case, v → −∞ and n → 1; which again violates the transversality condition for households entitled to profits from the creation of new tasks.

**Part 2:** Let us next turn to the case in which θ > 0.

In this case, an equilibrium is given by a solution to the following differential inclusion:

Starting from any n(0), k(0) the equilibrium path with endogenous technology is given by a tuple \( \{n(t), k(t), c(t), v(t), S_I(t)\}_{t=0}^{\infty} \) such that:

- The evolution of the state variables is given by
  \[
  \dot{n}(t) = \kappa_N S - (\kappa_N + \kappa_I) S_I(t),
  \]

- The paths for c(t) and k(t) satisfy the Euler equation,
  \[
  \frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(r^E(n(t), k(t)) - \delta - \rho) - O(S) \tag{A8}
  \]
  coupled with the household’s transversality condition,
  \[
  \lim_{t \to \infty} k(t)e^{-\int_0^t (\rho - (1-\theta)\delta) ds} = 0, \tag{A9}
  \]
  and the resource constraint,
  \[
  \dot{k}(t) = f^E(n(t), k(t)) - c(t) - \delta k(t) + O(S). \tag{A10}
  \]
The allocation of scientists satisfies:

\[
S_I(t) = \begin{cases} 
0 & \text{if } v < 0 \\
\in [0, S] & \text{if } v = 0 \\
S & \text{if } v > 0
\end{cases}
\]

with \( v \) satisfying the forward looking differential equation:

\[
\rho v - \dot{v} = \kappa_I \pi_I(n, k) - \kappa_N \pi_N(n, k) + O(S),
\]

with

\[
\pi_N(n, k) = (1 - \mu) \left( \frac{\eta}{1 - \eta} \right) \zeta \psi^{1 - \zeta} \left( e^u \left( \frac{w(n, k)}{\gamma(n)} \right)^\zeta - e^u \left( r(n, k) \right)^\zeta \right)
\]

\[
\pi_I(n, k) = (1 - \mu) \left( \frac{\eta}{1 - \eta} \right) \zeta \psi^{1 - \zeta} \left( e^u \left( w(n, k) \right)^\zeta - e^u \left( r(n, k) \right)^\zeta \right)
\]

denoting the flow profits for innovators. Also, define the partial derivatives

\[
Q_k = \kappa_I \frac{\partial \pi_I}{\partial k} - \kappa_N \frac{\partial \pi_N}{\partial k},
\]

\[
Q_n = \kappa_I \frac{\partial \pi_I}{\partial n} - \kappa_N \frac{\partial \pi_N}{\partial n},
\]

evaluated at their balanced growth path values.

By continuity, there exists a threshold \( S \leq S_e \) such that, for \( S < S \), the local behavior of the above system matches that of the system in which we take the limit \( S \to 0 \). Moreover, because the value of \( \dot{n} \) jumps at \( v = 0 \), the resulting system corresponds to a differential inclusion. Corollary 1 of Lemma 1 in Section 7 of Filippov (1988) shows that, locally, the behavior of a solution to this differential inclusion can be approximated by a solution to the continuous system in which we replace the equation for \( \dot{n} \) by the continuous perturbation:

\[
\dot{n}(t) = \begin{cases} 
\kappa_N & \text{if } v < -\epsilon \\
\kappa_N - (v - \epsilon)^{\frac{\kappa_N + \kappa_I}{2\epsilon}} & \text{if } v \in (-\epsilon, \epsilon) \\
-\kappa_I & \text{if } v > 0
\end{cases}
\]

and consider its behavior when \( \epsilon \to 0 \).

Denote the unique steady-state values by \( n^D, v^D, k^D, c^D \). The local behavior of the perturbed system can be determined by its linearization around the unique steady state, which is given by:

\[
\dot{n} = -Q_v v
\]

\[
\dot{v} = \rho v - Q_k [k(t) - k^D] - Q_n [n(t) - n^D],
\]

\[
\dot{c} = \frac{c^D}{\theta} r_{n}[n(t) - n^D] + \frac{c^D}{\theta} r_{k}[k(t) - k^D]
\]

\[
\dot{k} = f_{n}(n(t) - n^D) + (f_{k} - \delta)[k(t) - k^D] - c.
\]
Here, \( Q_v = \frac{k_N + \kappa_I}{2\epsilon_v} > 0 \). As \( Q_v \to \infty \), the above system approximates the local behavior of the dynamic economy near the steady state.

The characteristic polynomial of this system of differential equations (with all derivatives still evaluated at their balanced growth path values) can be written as

\[
P(\lambda) = \begin{vmatrix}
-\lambda & -Q_v & 0 & 0 \\
-Q_n & \rho - \lambda & 0 & -Q_k \\
\frac{c}{\theta} r_k^E & 0 & -\lambda & \frac{c}{\theta} r_k^E \\
f_n^E & 0 & -1 & f_k^E - \delta - \lambda
\end{vmatrix},
\]

or expanding it:

\[
P(\lambda) = \lambda^4 - \lambda^3(f_k^E - \delta + \rho) + \lambda^2(-Q_v Q_n + \frac{c}{\theta} r_k^E + r(f_k^E - \delta)) - \lambda(Q_v (f_n^E Q_k - f_k^E Q_n) + r \frac{c}{\theta} r_k^E) + Q_v (r_n^E Q_k - r_k^E Q_n).
\]

We now show that this polynomial has exactly two positive and two negative eigenvalues. First, note that \( r_n^E Q_k - r_k^E Q_n > 0 \) is equivalent to the fact that, in steady state, the curve \( \kappa_I v_I \) cuts \( \kappa_N v_N \) from below (recall we assume that the conditions for a unique steady state are met). To see this, notice that the term \( r_n^E Q_k - r_k^E Q_n > 0 \) corresponds to the change in flow profits, \( \kappa_I v_I - \kappa_N v_N \) that results from an increase in \( n \) when capital adjusts to keep the interest rate constant.

Let \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) be the eigenvalues of the above system. Then \( \lambda_1 \lambda_2 \lambda_3 \lambda_4 = Q_v (r_n^E Q_k - r_k^E Q_n) > 0 \). This implies that either all eigenvalues are negative, or all are positive, or two are negative and two are positive.

All eigenvalues cannot be negative either, since their sum is \( f_k^E - \delta + \rho > 0 \) (this is the trace of the system matrix). The last inequality follows by noting that \( f_k^E > \delta \).

We are left with showing that we cannot have all positive eigenvalues. Notice that:

\[
\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_1 \lambda_4 + \lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4 = -Q_v Q_n + \frac{c}{\theta} r_k^E + \rho(f_k^E - \delta),
\]

The comparative statics for the static case imply that \( Q_n > 0 \). That is, as \( n \) increases—holding capital constant—the incentives to do automation increase. Thus, for \( \epsilon \to 0 \), we have that \( Q_v \to \infty \), and the sum \( \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_1 \lambda_4 + \lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4 \) is negative. This rules out the possibility that all eigenvalues are positive.

The conclusion is that the perturbed and linearized system has two positive and two negative eigenvalues, and since there are two state variables (\( k \) and \( n \)), it is locally saddle path stable.\[\blacksquare\]
A2.4 Proofs from Section 5

Proof of Proposition 5: We start by providing formulas for $F_N^P$ and $F_I^P$:

$$F_N^P = \frac{1}{\sigma - 1} Y c^u \left( \frac{w^P}{\gamma(n)} \right)^{\zeta - \sigma} \left( \sigma \left( \frac{w^P}{\gamma(n)} \right)^{1 - \zeta} + (\mu \psi)^{1 - \zeta} \left( \frac{\eta}{1 - \eta} \right)^{\zeta} \right)$$

$$- \frac{1}{\sigma - 1} Y c^u (r_P)^{\zeta - \sigma} \left( \sigma r_P^{1 - \zeta} + (\mu \psi)^{1 - \zeta} \left( \frac{\eta}{1 - \eta} \right)^{\zeta} \right)$$

$$F_I^P = \frac{1}{\sigma - 1} Y c^u (r_P)^{\zeta - \sigma} \left( \sigma r_P^{1 - \zeta} + (\mu \psi)^{1 - \zeta} \left( \frac{\eta}{1 - \eta} \right)^{\zeta} \right)$$

$$- \frac{1}{\sigma - 1} Y c^u (w^P)^{\zeta - \sigma} \left( \sigma (w^P)^{1 - \zeta} + (\mu \psi)^{1 - \zeta} \left( \frac{\eta}{1 - \eta} \right)^{\zeta} \right).$$

Using these formulas, we establish the decentralization result by construction.

First, assume the planner subsidizes a fraction $1 - \mu$ to the price of intermediate goods, and sets a tax/subsidy to capital savings of $\tau_k = \omega^PL^s \frac{\partial \ln L^s}{\partial \ln \omega^P} \frac{\partial \ln \omega^P}{\partial \ln K}$. This guarantees households discount future income at a rate $r^P - \delta - r^P \omega^P L^s \omega^P K$, which coincides with the planner’s discount rate.

Without the subsidies/taxes for successful innovators, the value of automating jobs or creating new tasks are given by a small modification of equations (21) and (22), which take into account that firms sell intermediates at a price $\psi$, but buyers perceive a price $\mu \psi$ because of the subsidy. These values also discount future profits as the same rate the planner does, because of the taxes/subsidies to capital accumulation. Thus:

$$V_i(t) = (1 - \mu) \psi \left( \frac{\eta}{1 - \eta} \right)^{\zeta} \int_t^\infty e^{-\int_t^\tau r_P - \delta + W^P L^s \omega^P K dt} y(\tau) \left( c^u \left( r_P(\tau) \right)^{\zeta - \sigma} - c^u \left( w^P(\tau) \frac{\gamma(I(\tau))}{\gamma(t)} \right)^{\zeta - \sigma} \right) d\tau,$$

$$V_N(t) = (1 - \mu) \psi \left( \frac{\eta}{1 - \eta} \right)^{\zeta} \int_t^\infty e^{-\int_t^\tau r_P - \delta + W^P L^s \omega^P K dt} y(\tau) \left( c^u \left( w^P(\tau) \frac{\gamma(I(\tau))}{\gamma(t)} \right)^{\zeta - \sigma} \right) d\tau.$$

Now, we can define the flow subsidies/taxes for successful innovators as follows. First, we have a component to adjust for the appropriability problems, $\tau_N^A(t), \tau_I^A(t)$. These are given by:

$$\tau_N^A(t) = \frac{1}{\sigma - 1} Y c^u \left( \frac{w^P}{\gamma(n)} \right)^{\zeta - \sigma} \left( \sigma \left( \frac{w^P}{\gamma(n)} \right)^{1 - \zeta} + (\mu \psi)^{1 - \zeta} \left( \frac{\eta}{1 - \eta} \right)^{\zeta} \left( 1 - (\sigma - 1) \frac{\mu}{1 - \mu} \right) \right)$$

$$- \frac{1}{\sigma - 1} Y c^u (r_P)^{\zeta - \sigma} \left( \sigma r_P^{1 - \zeta} + (\mu \psi)^{1 - \zeta} \left( \frac{\eta}{1 - \eta} \right)^{\zeta} \left( 1 - (\sigma - 1) \frac{\mu}{1 - \mu} \right) \right)$$

$$\tau_I^A = \frac{1}{\sigma - 1} Y c^u (w^P)^{\zeta - \sigma} \left( \sigma (w^P)^{1 - \zeta} + (\mu \psi)^{1 - \zeta} \left( \frac{\eta}{1 - \eta} \right)^{\zeta} \left( 1 - (\sigma - 1) \frac{\mu}{1 - \mu} \right) \right)$$

$$- \frac{1}{\sigma - 1} Y c^u (w^P)^{\zeta - \sigma} \left( \sigma (w^P)^{1 - \zeta} + (\mu \psi)^{1 - \zeta} \left( \frac{\eta}{1 - \eta} \right)^{\zeta} \left( 1 - (\sigma - 1) \frac{\mu}{1 - \mu} \right) \right).$$

A.18
The wedge between the private and social values of innovation captured by \( \tau_N^T(t), \tau_I^T(t) \) is well known, and arises because monopolists cannot extract the full value of introducing new tasks in models of expanding varieties. Here, this is the case for monopolists automating jobs or creating new tasks, so the taxes/subsidies \( \tau_N^A(t), \tau_I^A(t) \) have ambiguous signs and orderings.

Second, \( \tau_N^T(t_0,t), \tau_I^T(t_0,t) \), for \( t \geq t_0 \), are given by:

\[
\tau_N^T(t_0,t) = (1 - \mu) \psi \left( \frac{\eta}{1 - \eta} \right) \xi \left( \mu \psi \right) - \xi \left[ e^u \left( \frac{w^S(t)}{\gamma(n(t))} \right)^{\zeta - \sigma} \right] \geq 0
\]

\[
\tau_I^T(t_0,t) = (1 - \mu) \psi \left( \frac{\eta}{1 - \eta} \right) \xi \left( \mu \psi \right) - \xi \left[ e^u \left( \frac{w^S(t) \gamma(I(t))}{\gamma(n(t)) \gamma(I(t_0))} \right)^{\zeta - \sigma} - e^u \left( \frac{w^S(t)}{\gamma(I(t))} \right)^{\zeta - \sigma} \right] \leq 0.
\]

\( \tau_N^T(t_0,t) \geq 0 \) and \( \tau_I^T(t_0,t) \leq 0 \) correct for a technological externality: by inventing new tasks and increasing \( N \), monopolists improve the quality of intermediates that future entrants will develop. The opposite occurs for automation: by automating task \( I \), new entrants will be forced to automate more complex tasks, receiving fewer profits. These taxes/subsidies, depend on the time at which a task was introduced \( t_0 \)— since they are a compensation (or charge) for all technologies built on top of them.

Finally, \( \tau_N^W \) and \( \tau_I^W \) correct for the fact that technology monopolists do not take into account the effect of technologies on the quasi-supply of labor.

It is straightforward to verify that once we add these flow subsidies/taxes to the private profits from developing new technologies, we obtain these become \( \Psi_N \) and \( \Psi_I \), establishing the decentralization result.

Notice that the scientist allocation can be decentralized in many ways. In particular, since there is a fixed supply of scientists, we only need to get the relative expected profits from each type of innovation right. The particular decentralization outlined here guarantees the level of innovators’ profits also matches the social value of innovation. Even if both types of technology end up being subsidized in equilibrium, this does not matter because the money can be recovered by taxing scientists.

**Proof of Proposition 6**: Let \( S^P_I(t) \) and \( S^D_N(t) \) denote the allocation of scientists, and consider the allocation obtained by a small deviation \( S^P_N(t) = \min \{ S^D_N(t) + \epsilon, 0 \} \) and \( S^D_I(t) = \max \{ S^D_I(t) - \epsilon, 0 \} \) if \( S^D_I < 1 \), and \( S^P_N(t) = S^D_N(t), S^P_I(t) = S^D_I(t) \) otherwise. We prove in the appendix that for a small \( \epsilon > 0 \), such deviation increases welfare and reduces the extent of automation.

Clearly, the new allocation satisfies \( n^P(t) \geq n^D(t) \) as wanted. For \( \epsilon \) small enough, we have that the above allocation changes welfare by \( \epsilon (\kappa_N \mu_N - \kappa_I \mu_I) \), whenever \( S^D_I(t), S^D_N(t) \in (0, 1) \). Moreover, in these cases \( \kappa_N V_N(t) = \kappa_I V_I(t) \).

Thus, to prove our deviation increases welfare, it is enough to verify \( \kappa_N \mu_N - \kappa_I \mu_I > 0 \) whenever \( \kappa_N V_N(t) = \kappa_I V_I(t) \). In fact, we prove the stronger statement, that at all points in time \( \Psi_N(t) > \Psi_I(t) \).
Notice that, as $\epsilon \to 0$, we are along the market allocation. Thus, we can compute $\Psi_N$ and $\Psi_I$ as:

$$
\Psi_N(t) = \int_t^\infty e^{-\int_0^t (r^P(nD(s),kD(s)) - \delta)ds} \left( \frac{\sigma + \frac{\mu}{\sigma - 1}}{\sigma - 1} Y(\tau) \left[ c^u \left( \frac{w^P(\tau)}{\gamma(n(\tau))] \right) \gamma - c^u \left( r^P(\tau) \right) \zeta - \sigma \right] + W^P L^S\omega_N^P \right) d\tau,
$$

$$
\Psi_I(t) = \int_t^\infty e^{-\int_0^t (r^P(nD(s),kD(s)) - \delta)ds} \left( \frac{\sigma + \frac{\mu}{\sigma - 1}}{\sigma - 1} Y(\tau) \left[ c^u \left( r^P(\tau) \right) \zeta - \sigma - c^u \left( w^P(\tau) \right) \zeta - \sigma \right] + W^P L^S\omega_I^P \right) d\tau.
$$

However, this implies the inequalities:

$$
\frac{\Psi_N(t)}{\Psi_I(t)} = \int_t^\infty e^{-\int_0^t (r^P(nD(s),kD(s)) - \delta)ds} \left( \frac{\sigma + \frac{\mu}{\sigma - 1}}{\sigma - 1} Y(\tau) \left[ c^u \left( \frac{w^P(\tau)}{\gamma(n(\tau])] \right) \gamma - c^u \left( r^P(\tau) \right) \zeta - \sigma \right] + W^P L^S\omega_N^P \right) d\tau

- \int_t^\infty e^{-\int_0^t (r^P(nD(s),kD(s)) - \delta)ds} \left( \frac{\sigma + \frac{\mu}{\sigma - 1}}{\sigma - 1} Y(\tau) \left[ c^u \left( r^P(\tau) \right) \zeta - \sigma - c^u \left( w^P(\tau) \gamma(I(\tau - t)) \right) \zeta - \sigma \right] + W^P L^S\omega_I^P \right) d\tau

\geq \int_t^\infty e^{-\int_0^t (r^P(nD(s),kD(s)) - \delta)ds} \left( \frac{\sigma + \frac{\mu}{\sigma - 1}}{\sigma - 1} Y(\tau) \left[ c^u \left( \frac{w^P(\tau)}{\gamma(n(\tau)] \gamma(I(\tau - t)) \right) \zeta - \sigma \right] + W^P L^S\omega_N^P \right) d\tau

- \int_t^\infty e^{-\int_0^t (r^P(nD(s),kD(s)) - \delta)ds} \left( \frac{\sigma + \frac{\mu}{\sigma - 1}}{\sigma - 1} Y(\tau) \left[ c^u \left( r^P(\tau) \right) \zeta - \sigma - c^u \left( w^P(\tau) \gamma(I(\tau - t)) \right) \zeta - \sigma \right] + W^P L^S\omega_I^P \right) d\tau

= \frac{V_N(t)}{V_I(t)},
$$

as wanted.

The first inequality follows from the technological externality; which as explained above pushes towards the underprovision of new tasks. The second inequality results from the novel inefficiency underscored in this paper: the fact that labor gets rents in equilibrium pushes towards the underprovision of new tasks and excessive automation. This inequality is strict whenever $L^S_0 > 0$ — that is, labor gets rents.