English Auctions with Reentry

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Abstract

The English auction is usually modeled (since Milgrom & Weber (1982)) as an open continuously ascending price auction in which bidders choose when to drop out but, once they do so, are forbidden from reentering. It is known that this auction allocates a single object efficiently in many but not all circumstances. In particular, if there are three or more bidders with interdependent values, the English auction may not be efficient. This paper considers a modification of the standard model in which bidders can exit and reenter at will. The main result is that the English auction with reentry has an efficient equilibrium under weak conditions. These are much weaker than the conditions under which the standard English auction is efficient. Thus the modification is not only a more realistic model of the real-world auction but has superior theoretical properties. The failure of the English auction to allocate efficiently stems not from some defect in the institution *per se* but rather from the way it has been traditionally modeled as a game. *JEL Classification Numbers:* $C72$, $C73$, $D44$, $D82$. *Keywords:* efficient auction, English auction, interdependent values, ex post equilibrium, reentry.

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1 Introduction

The English, or open ascending price, auction is one of the oldest economic institutions in place—its use has been recorded since antiquity. In an English auction—say of the kind used to sell art—the auctioneer sets a relatively low initial price. This price is then increased and the exact manner in which this is done varies. In some auctions the bidders themselves call out prices and the auction stops when some bid is not raised any further. In other auctions, the auctioneer raises prices in small increments until only one bidder is willing to buy the object.

The English auction is especially attractive as a mechanism because of its strategic simplicity. The current price is commonly known and a bidder need only to decide whether he wants to buy the object at that price or not. Once the price reaches a level that would make it unprofitable for the bidder to purchase the object he drops out of the bidding. The practical simplicity of the mechanism, when combined with interest stemming from its widespread use—both historical and current—has made the English auction a frequent object of study.

Game theoretic models of the English auction abstract away from many details of the real-world institution. For instance, it is assumed that the price is raised in a continuous fashion by an idealized automated auctioneer—a price clock. The bidders need only indicate whether they are active, that is, willing to buy the object at the current price, or not. More importantly, the strategic problem facing bidders is simplified by requiring that once a bidder drops out of the auction while others are still active, his decision is irrevocable—there are no circumstances under which he can win the object. This model was introduced in Milgrom & Weber (1982), and in what follows I refer to it as a *standard* English auction.

In this paper I consider a more realistic model of the real-world English auction. Specifically, bidders are given the freedom to indicate their willingness to buy at any price and at any time. In particular, a bidder may drop out or exit at a low price and then "reenter" the auction at a higher price.

This model is used to study the problem of allocating a single object among a number of asymmetric, privately informed bidders with *interdependent* values—the *ex post* value of the object to a particular bidder depends on both his own signal and the signals of other bidders. The main question concerns circumstances under which the allocation will be *efficient*, that is, the object will end up in the hands of the bidder with the highest *ex post* value. Note that with private values—when the value depends only on a bidder's own signal—the standard English auction is efficient, that is, it has an equilibrium which always results in an efficient allocation.

It is well known that the possibility of achieving efficiency at all via any mechanism hinges on the so-called single-crossing condition, requiring that a bidder's signal has a greater influence on his own value than on some other bidder's value. It is also well known that the standard English auction is efficient when the number of buyers is two and the single-crossing condition is satisfied. When the number of bidders is more than two the standard English auction may not be efficient. An example below shows that this may happen under quite natural circumstances.

In this paper I argue that the failure of the English auction to achieve efficiency results not from some defect in the institution itself but rather from the way it has been traditionally modeled. In particular, the inefficiency stems from the rule that once a player exits, this decision is permanent. He is not allowed to reenter at a higher price. In the context of private values this restriction is of no consequence. With interdependent values, however, the price carries valuable information about other players' signals, which may be revealed after a player has exited, causing him to regret his exit decision. Thus the restriction of no reentry is substantive.

The main result of this paper is that once the rules of the standard model are amended to allow reentry, the English auction has an efficient equilibrium as long as the *single-crossing* condition and a new condition, called the *signal intensity* condition are satisfied.

Both conditions pertain to situations in which a group of players has the same value and this is the highest among all players. Both specify circumstances in which as a result of an increase in his own signal, a particular member of this group becomes the sole player with the highest value. The single-crossing condition requires that this must happen if the *signals* of the rest of the group are kept fixed whereas the signal intensity condition requires that this happen if their *values* are kept fixed. In this sense, the conditions are dual to one another.

The single-crossing condition is known to be necessary for efficiency and so cannot be avoided. In any case, it appears to be quite weak. I will argue that the signal intensity condition is also relatively weak. It is weaker than the conditions under which the standard English auction has an efficient equilibrium. Moreover, when there are only two players it is implied by the single-crossing condition itself.

A second result of this paper is that whenever the standard English auction has an efficient equilibrium it can be duplicated as an equilibrium in the English auction with reentry. In particular, if the efficient equilibrium of the standard English auction is "regular" as in Maskin (1992) or Krishna (2003), the equilibrium in the English auction with reentry identified in the first result is isomorphic—all actions and information processing are identical.

Both as a model of the real-world institution and on theoretical grounds, the English auction with reentry dominates the standard English auction.

1.1 An illustration

Consider a hypothetical "real-life" situation. There is a painting for sale, and there are three bidders: two experienced dealers and a novice art collector. The dealers are experts and have an accurate assessment of the worth of the object to them. The novice collector is unsure about the value and indeed cares about how the dealers value the painting.

Suppose that the painting is sold by means of an open ascending price auction and consider the following scenario. All three participants are active—they keep raising the price or indicating their interest in some other manner—until the price reaches a level such that the novice collector stops bidding. This may be because the price reached the maximum amount he was willing to spend for a nice, but of unknown origin, piece of art, and there was no indication from the behavior of the others that the painting is genuine. But now suppose that the two dealers continue to outbid each other for some time. This may cause the collector to reevaluate his decision—the aggressive bidding of the dealers may convince him that the painting is genuine. Thus the collector may wish to reenter the auction.

The following example is an abstract model of preferences of the collector (player 1) and the dealers (players 2 and 3) intended to capture the scenario outlined above. This example is a simplified version of the example from Perry & Reny (2001).

Example 1. *There is a single good for sale and there are three interested players who receive signals s_1, s_2 and s_3 from $[0, 1]$ and have the following value functions*

$$V_1(s_1, s_2, s_3) = s_1 + s_2 s_3$$
$$V_2(s_1, s_2, s_3) = s_2$$
$$V_3(s_1, s_2, s_3) = s_3$$

Consider what happens in the standard model of the English auction. Obviously, it is dominant for each of players 2 and 3 (the dealers) to stay in until price reaches their signals. Suppose $s_2 = s_3 = \frac{1}{2}$ then $V_1 = s_1 + \frac{1}{4}$, and as long as $s_1 < \frac{1}{4}$ both dealers have the highest value. If $s_2 = s_3 = \frac{4}{5}$ then player 1 (the collector) has the highest value if $s_1 > \frac{4}{5} - \frac{16}{25} = \frac{4}{25}$. Suppose the signal of the collector lies between $\frac{1}{4}$ and $\frac{4}{25}$, say $s_1 = \frac{2}{9}$. If the object is to be allocated efficiently when both dealers' signals are $\frac{1}{2}$ and also when they are $\frac{4}{5}$, player 1 has to drop out before $p = \frac{1}{2}$, but at the same time stay in at least until $p = \frac{4}{5}$. So, full efficiency is infeasible.[1]

We now argue that the standard English auction is unable to achieve efficiency because it forbids the collector from reentering the auction once he drops out. We showed above that while it is imperative for the collector to exit before the price reaches $\frac{1}{2}$ there are situations in which he would like to be active at $p = \frac{4}{5}$. Giving players the option to exit and enter at will allows player 1 to behave optimally in both situations.

In particular, suppose that player 1 adopts the following strategy in the modified English auction with reentry. If the other players are active, calculate $w_1(s_1, p) = V_1(s_1, p, p) = s_1 + p^2$—the minimal inferred value at price p. Stay active as long as $w_1(s_1, p) > p$, and be inactive if $w_1 < p$.

For $s_1 = \frac{2}{9}$ the collector has to solve $s_1 + p^2 = p$ and this results in two solutions: $p' = \frac{1}{3}$ and $p'' = \frac{2}{3}$. The proposed strategy implies that when both dealers are active, player 1 should exit at $p' = \frac{1}{3}$ and then reenter at $p'' = \frac{2}{3}$; when some dealer exits first then play as in the "regular" equilibrium of the two-bidder English auction (as in Maskin (1992)).

As we can see, if the current price is below p' or above p'' then the minimal inferred value of the good to the collector exceeds the price. If one of the dealers drops out at such a price, then the auction continues as a two-bidder English auction, and the collector may win the good while paying less than the value, so the expected profit is positive. Thus it is optimal for him to be active when $w_1 > p$.

If the first dealer to drop out, say player 2, does so at a price p_2 in between p' and p'', there is no opportunity for the collector to win the good and pay

[1] If the collector chooses to stay until some other player exits first when all players are still actively bidding, then an efficient allocation can be achieved almost always, that is there exists an almost efficient equilibrium which fails to allocate efficiently only at some signal profiles with $s_2 = s_3$. This fact, however, is specific to the particular forms of value functions of players 2 and 3. In the full example in Perry & Reny (2001) with $V_2 = s_2 + \frac{1}{2}s_1$ there are no efficient (or almost efficient) equilibria.

less. Indeed, $w_1(s_1, p_2) = V_1(s_1, p_2, p_2) = V_3(p_2)$, so $V_1(s) < V_3(s) = s_3$ by the single-crossing condition since $s_3 > p_2$. Therefore, if the collector is active at p_2 and would win the good later, he would have to pay s_3 which is more than the value of the good to him. So, it is optimal for the collector to stay inactive whenever $w_1 < p$.

It is easy to see that the resulting allocation is always efficient.

1.2 Related literature

In his pioneering work Vickrey (1961) already observed that the standard English auction allocates efficiently in the private values setting. Milgrom & Weber (1982) studied the English auction in a setting with symmetric interdependent values. Their characterization of the equilibrium strategies implies that in this setting efficiency obtains as long as the single-crossing property is satisfied.

Maskin (1992) shows that even with asymmetric bidders the standard English auction has an efficient equilibrium when the number of bidders is two, but may not have an efficient equilibrium when there are more than two players. Krishna (2003) specifies the circumstances under which the standard N-bidder English auction possesses the specific kind of efficient equilibrium which we refer to as the *regular* equilibrium.[2] This is the same kind of equilibria that are analyzed in Maskin (1992). The example from Perry & Reny (2001) shows that conditions suggested by Krishna (2003) are quite restrictive.

Crémer & McLean (1985), among other things, introduce the single-crossing condition and show that when it is satisfied there exists a direct mechanism that efficiently allocates one object among N buyers. Maskin (1992) shows the necessity of the single-crossing for the existence of the efficient equilibrium. He also shows that with multidimensional signals an efficient equilibrium may not exist even in the case of only two bidders. Dasgupta & Maskin (2000) and Perry & Reny (2002) propose alternative mechanisms that allocate efficiently under the single-crossing condition on value functions. These mechanisms, however, are quite complicated. The former presents a mechanism in which bids are contingent on the realized values of others, and the latter proposes a mechanism in which each bidder

[2] Regular equilibrium bidding strategies are such that if all active bidders were to drop out at the current price, their estimated values at the inferred signals would be the same and equal to the price.

submits a vector of bids each component of which is directed at a specific bidder.

Perry & Reny (2001) present an efficient ascending price auction applicable to both multiple and single goods. The key feature here is that even with a single good, at any given price players submit a set of directed demands against all other players. While the mechanism is of the ascending price variety, the strategic problem facing bidders is more complex than in an English auction—each player has to decide whether he is to be active or inactive against each of the other $n-1$ players separately. An important feature of this mechanism is that any player may trigger a stoppage of the price clock by reducing his demand against some player, and once the clock is stopped, any other player may increase some of his demands.[3]

For all basic facts and a brilliant coverage of classic results in auction theory the reader is referred to Krishna (2002). Maskin (2003) provides a survey of the major facts and problems related to efficiency analysis in auctions.

The rest of the paper is organized as follows. Section 2 specifies the environment in which the presented auction is studied. In Section 3 I introduce an English auction with reentry and consider a simplified version of the auction and of the main result. Section 4 contains the complete set of rules and the full analysis of the presented auction. The two main results are formulated in Section 2.2 and proved in Section 4.4. Section 5 concludes.

2 Preliminaries

2.1 The setup

There is a single good for sale. There are N potential buyers, each of whom receives a signal $s_i \in [0,1]$. Given the signals $\mathbf{s} = (s_1, s_2, \ldots, s_N)$ the value of the object to the player i is $V_i(\mathbf{s})$. Since the value depends on all the signals, this is a situation of *interdependent* values. The valuation functions V_i are assumed to be commonly known to the players.

Let $\mathcal{N}$ denote the set of buyers. For any subset $\mathcal{I}$ of buyers denote $\mathbf{s}_{\mathcal{I}} = (s_i)_{i\in\mathcal{I}}$—the set of signals of players from $\mathcal{I}$. We also write $\mathbf{s} = \mathbf{s}_{\mathcal{N}}$, $\mathbf{s}_{-\mathcal{I}} = \mathbf{s}_{\mathcal{N}\setminus\mathcal{I}}$, and $\mathbf{s} = (\mathbf{s}_{\mathcal{I}}, \mathbf{s}_{-\mathcal{I}})$. Signals are distributed according to the joint

[3] In Section 4.5 the conditions for efficiency and the way information is processed in Perry & Reny (2001) are compared with those proposed in this paper.

probability density function $f(s_1, \ldots, s_N)$ which is assumed to be positive on all of $[0,1]^N$.

For any player i his value function V_i has the following properties: V_i is twice-differentiable, $V_i(0, \ldots, 0) = 0$ and

$$\forall i \ \frac{\partial V_i}{\partial s_i} > 0 \qquad \forall j \neq i \ \frac{\partial V_i}{\partial s_j} \geq 0 \tag{1}$$

Definition 1. Given signals $\mathbf{s}$ the *winners' circle* $\mathcal{I}(\mathbf{s})$ is the set of players with maximal values. Formally,

$$i \in \mathcal{I}(\mathbf{s}) \Longleftrightarrow V_i(\mathbf{s}) = \max_{j \in \mathcal{N}} V_j(\mathbf{s}) \tag{2}$$

If the allocation is to be efficient, then the object must go to a member of the winners' circle.

It is assumed that if $s_i = 0$ and there exists a j such that $s_j > 0$ then $i \notin \mathcal{I}(\mathbf{s})$. In words, a player with the lowest possible own signal cannot have the highest value.

It is also assumed that for any $\mathbf{s}$ and a subset of players $\mathcal{J}$ of the winners' circle $\mathcal{I}(\mathbf{s})$ the set of functions $\mathbf{V}_{\mathcal{J}}(\mathbf{s})$ is regular at $\mathbf{s}_{\mathcal{J}}$, that is $\det DV_{\mathcal{J}} \neq 0$, where $DV_{\mathcal{J}}$ is the matrix of partial derivatives (Jacobian), $DV_{\mathcal{J}} = \left(\frac{\partial V_i(\mathbf{s})}{\partial s_j}\right)_{i,j \in \mathcal{J}}$.

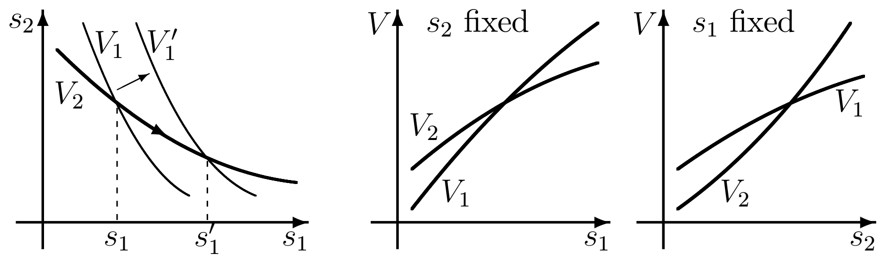
2.2 Main results

The main result relies on the following two assumptions.

A1 (*single-crossing*) For all $\mathbf{s}$ and any pair of players $\{i, j\} \subset \mathcal{I}(\mathbf{s})$,

$$\frac{\partial V_i(\mathbf{s})}{\partial s_i} > \frac{\partial V_j(\mathbf{s})}{\partial s_i} \tag{3}$$

The single-crossing condition says that the influence of a player's own signal on his value is greater than the influence of his signal on another player's value. It is required to hold only at signal profiles where both players' values are maximal. The single-crossing condition plays a key role in the analysis of auctions with interdependent values. In particular, if it is not satisfied then there may be no mechanism that allocates the object efficiently (Maskin (1992)).



a) Signal intensity b) Single-crossing

Figure 1: Signal intensity vs. single-crossing

A2 (*signal intensity*[4]) For all $\mathbf{s}$ and $i \in \mathcal{I}(\mathbf{s})$ there exists an $\varepsilon > 0$ such that for all $\mathbf{s}'$ satisfying (i) $s_i < s_i' < s_i + \varepsilon$; (ii) $\forall j \in \mathcal{I}(\mathbf{s}) \setminus \{i\}$, $V_j(\mathbf{s}') = V_j(\mathbf{s})$ and (iii) $\forall k \notin \mathcal{I}(\mathbf{s})$, $s_k' = s_k$, it is the case that $\mathcal{I}(\mathbf{s}') = \{i\}$.

The signal intensity condition requires that if we increase the signal s_i of some member i of the winners' circle $\mathcal{I}(\mathbf{s})$ and change the signals s_j of other players $j \in \mathcal{I}(\mathbf{s})$ in a way that their values are unchanged, offsetting the effect of the change in s_i, then i's value goes up (the signals of all players $k \notin \mathcal{I}(\mathbf{s})$ are kept fixed). In other words, the combined effect on player i's value, directly from the increase in his own signal and indirectly through the changes in signals of other members of the winners' circle, is positive: the direct effect outweighs the indirect effect.

Figure 1 illustrates the difference between the signal intensity and the single-crossing conditions when there are only two bidders in the winners' circle, say 1 and 2. Suppose $\mathbf{s}$ is such that $V_1(\mathbf{s}) = V_2(\mathbf{s}) = V$. If we increase s_1 along the isovalue curve $V_2 = V$ to s_1', the assumption requires that player 1's value V_1' be greater than V. Thus, as shown in Figure 1a, the isovalue curves of player 1 must be steeper than the isovalue curves of player 2, that is,

$$-\frac{\partial V_1}{\partial s_1} \Big/ \frac{\partial V_1}{\partial s_2} < -\frac{\partial V_2}{\partial s_1} \Big/ \frac{\partial V_2}{\partial s_2} \tag{4}$$

As shown in Figure 1b, the single-crossing requires that

$$\frac{\partial V_1}{\partial s_1} > \frac{\partial V_2}{\partial s_1} \text{ and } \frac{\partial V_2}{\partial s_2} > \frac{\partial V_1}{\partial s_2} \tag{5}$$

[4]This condition is so named because it is similar to the notion of factor intensity in the literature on international trade.

and clearly (5) implies, but is not implied by, (4). Thus if $\mathcal{I}(\mathbf{s})$ has only *two* players then the signal intensity condition is implied by the single-crossing condition. This reasoning also makes apparent why the single-crossing condition does not imply the signal intensity condition once there are *three or more* players. Now it is possible to move along a player's isovalue surface in a way that the pairwise comparisons of partial derivatives no longer provide enough information.

It is useful to view the signal intensity condition as a *dual* to the single-crossing condition. Single-crossing prescribes what should happen if we fix the *signals* of everyone else in the winners' circle and increase the signal of one particular player—he should become the sole member of the winners' circle. Signal intensity prescribes what should happen if we fix the *values* of everyone else in the winners' circle and increase the signal of one particular player—again, he should become the sole member of the winners' circle.

For a further discussion on the role and implications of the signal intensity assumption see also Section 4.5 and Appendix A.1.

Now, I can state the main result of this paper.

Theorem 1 (Main). *Under the single-crossing and the signal intensity conditions the English auction with reentry has an ex post equilibrium that is efficient.*[5]

The English auction with reentry is, from the perspective of efficiency, superior to the English auction without reentry. First, whenever the latter has an efficient equilibrium so does the former, and this is our second main result. Moreover, as the example in the introduction shows, there are circumstances in which the standard English auction fails to have an efficient equilibrium while the auction with reentry does.

The main result relies on two conditions. The single-crossing condition (A1) is known to be necessary for efficiency. The signal intensity condition (A2) is not necessary but in the dynamic framework of the auction proposed here, guarantees that no player suffers from any regret from exiting or entering. It is well recognized that the question of efficiency is closely linked to the

[5] A Bayesian-Nash equilibrium is called an *ex post* equilibrium if no player wants to change his actions once all the information about the actual realization of the signals becomes commonly known. This notion is closely related to the notion of the robust equilibrium (see Dasgupta & Maskin (2000)), which requires that the strategies remain optimal under any initial distribution of signals. In fact, the presented equilibrium is also robust.

absence of any *ex post* regret. Moreover, the signal intensity assumption is weaker than any other currently known condition under which the standard English auction without reentry possesses an efficient equilibrium.

In fact we can show

Theorem 2. *If the standard English auction without reentry has an efficient equilibrium then so does the English auction with reentry.*

3 English auction with reentry: a first look

Allowing the possibility that bidders can exit or enter the auction at will leads to some complications that need to be treated with care. In this section I attempt to convey the main ideas as simply and clearly as possible by, as a first step, neglecting some of these complications. The cost of doing this is not great—if players follow the suggested equilibrium the complications occur only with probability zero. For example, it may be that player 1 who is inactive up to that point, decides to enter at some price p, while player 2 who is active, decides to exit at the same price p. Such simultaneous exits or entries can be problematic to deal with. The decision of player 1 to enter may have been based on the fact that player 2 was active. If player 1 knew that player 2 was going to exit at p—an event viewed as bad news—he would have stayed out himself. Thus once player 1 enters he may want to exit at the lowest price $p' > p$ which, of course, cannot be defined.

Such complications can also arise as a result of deviations from equilibrium play and as a first pass at the problem, we do not check equilibrium conditions for such deviations. Later sections are devoted to taking care of all these details.

3.1 Rules

The English auction, as modeled in the literature, is conducted as follows.

1. The auctioneer sets a low initial price, say zero, and constantly raises it. It is convenient to think of an automatic price clock publicly showing the current price.

2. At any price, each player is either active or not. All players are active at a price of zero and a player can exit from the auction at any time.

The activity statuses of all players are commonly observed and known. *A player who exits cannot reenter the auction.*

3. The auction ends (the price clock stops) when at most one person is active. The winner is the only remaining person (or is randomly chosen among those who exited last) and pays the price at which the last exit took place.

This paper explores the implications of amending only one of the rules of the English auction. Rule 2 is modified so that the last clause reads:

2M. *A player can exit and reenter the auction at will.*

3.2 Information and strategies

It is assumed that at any price p, the history of exits and entries of all players that took place before p is common knowledge. Denote this public information as $H(p)$. Clearly, which players were active or inactive at a particular (time) price before p can be easily reconstructed.

A strategy β_i of player i determines the price level $p_i \equiv \beta_i(s_i, H(p)) > p$ at which player i is going to change his status, enter if he was out, or exit if he was in.

In order to state the suggested equilibrium strategies in the modified English auction, some definitions are needed. These incorporate the inferences that players make about the signals of other players from their exit and entry behavior.

Definition 2. Let $\mathcal{A}$ be the set of active players at some price p. The *estimated minimal signals at* p, $\mathbf{x}(p) = (x_1(p), x_2(p), \ldots, x_N(p))$ are defined as follows: (i) $\mathbf{x}(0) = \mathbf{0}$; (ii) if $j \notin \mathcal{A}$, $x_j(p) = x_j(p_j)$, where $p_j < p$ is the price at which player j last exited; (iii) for all active players, $\mathbf{x}_{\mathcal{A}}(p) = (x_i(p))_{i \in \mathcal{A}}$ is a solution to the system of equations

$$\mathbf{V}_{\mathcal{A}}(\mathbf{x}_{\mathcal{A}}(p), \mathbf{x}_{-\mathcal{A}}(p)) = p \tag{6}$$

Let $w_i(s_i, p) \equiv V_i(s_i, \mathbf{x}_{-i}(p))$ be the *estimated minimal value* of the good to player i when the current price is p.[6]

[6] Later we show that $\mathbf{x}(p)$ is well-defined and is unique for any finite history $H(p)$.

Suppose that for all i, $\mathbf{x}_{-i}(p)$ represents the minimal signals that i estimates other players to have received. If $\mathbf{x}(p)$ is a consistent set of such estimates, then they have to satisfy (6). To see why notice that if for some active player i, $w_i(x_i(p), p) > p$, that is, if the estimated minimal value exceeds the current price, then for all signals $s_i < x_i(p)$ which are close to $x_i(p)$, we have $w_i(s_i, p) > p$. This means that it would be reasonable for player i to be active when his signal were $s_i < x_i(p)$, contradicting the fact that $x_i(p)$ is the minimal signal that i may have received. On the other hand, if $w_i(s_i, p) < p$ then if player i were to play conservatively—and the strategies specified below will call on a player to do so—he would be inactive at p. Thus the estimated minimal signals at p, $\mathbf{x}(p)$ have to satisfy (6).

Definition 3. The *potential minimal signals at* p, $\mathbf{y}(p) = (y_1(p), \ldots, y_N(p))$ are defined as follows: (i) if $i \in \mathcal{A}$, $y_i(p) = x_i(p)$; (ii) if $j \notin \mathcal{A}$, $y_j(p)$ is the solution to $w_j(y_j(p), p) = p$ or $y_j(p) = 1$ if such a solution does not exist.

Note that for any active player i, $w_i(x_i(p), p) = w_i(y_i(p), p) = p$. For an inactive player j, $w_j(x_j(p), p) < p$ and thus, $y_j(p) > x_j(p)$. In words, for an inactive player j, $y_j(p)$ is the signal at which j's estimated minimal value equals the price p. Thus $y_j(p)$ represents the hypothetical value of j's signal which would lead him to become active at p, all else being equal.

3.3 Equilibrium Strategies

The proposed strategies are based on the estimated minimal values $w_i(s_i, p)$ derived from the history of play up to price p.[7]

Player i should

1. if $w_i(s_i, p) > p$, then be active at p;
2. if $w_i(s_i, p) < p$, then be inactive at p;
3. if $w_i(s_i, p) = p$ and for ε small enough, $w_i(s_i, p - \varepsilon) \leq p - \varepsilon$ and $w_i(s_i, p + \varepsilon) > p + \varepsilon$, then become active at p;

[7] Note that $\mathbf{x}_{-i}(p)$ can be regarded as the set of beliefs player i has at price p. In particular, player i believes at p that $s_j = x_j(p)$ for any inactive player j and for any active player k, $s_k \in [x_k(p), 1]$. The joint distribution of $\mathbf{s}_{-i}$ at p is the ex ante distribution of $\mathbf{s}_{-i}$ conditional on $s_k \in [x_k(p), 1]$ for all active $k \neq i$, $s_j = x_j(p)$ for all inactive $j \neq i$, and the true s_i. The proposed strategies depend only on $\mathbf{x}_{-i}(p)$ and not on the distribution of signals.

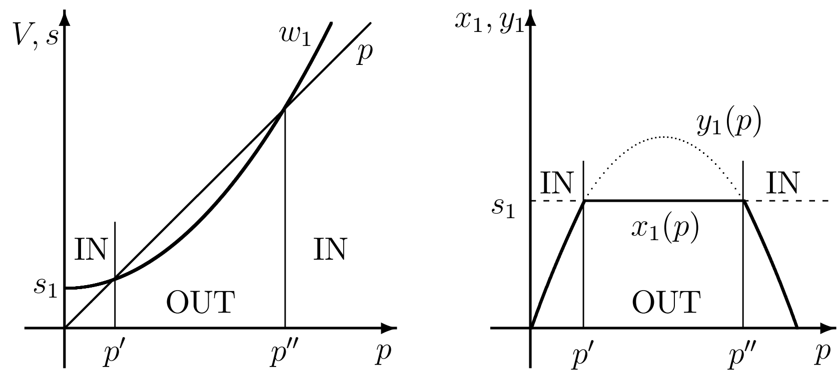


Figure 2: Player 1's decision

4. if $w_i(s_i, p) = p$ and for ε small enough, $w_i(s_i, p - \varepsilon) \geq p - \varepsilon$ and $w_i(s_i, p + \varepsilon) < p + \varepsilon$, then become inactive at p.

   The strategy can be summarized as: *Exit (Enter) whenever your estimated minimal value crosses the price from above (below).*

Equivalently, the strategies can be defined based on $\mathbf{x}(p)$ and $\mathbf{y}(p)$—Exit, whenever $x_i(p) = y_i(p)$ reaches s_i from below, Enter when $y_i(p)$ reaches s_i (and $x_i(p)$) from above.

Remark 1. The reader may wonder whether the strategies proposed above are well defined. The strategies prescribe how a player should change his status based on the estimated minimal values w_i. But these values depend in turn on the estimated minimal signals x_i which depend on the status of a player—active or inactive. The strategies do not explicitly say whether player i is considered to have maintained his status at p while $w_i(s_i, p)$ crosses p, or to have changed his status. As shown below (Lemma 4 in Appendix A.1), this slight ambiguity causes no difficulties. Assumption (A2) guarantees that if $w_i(s_i, p)$ crosses p when player i is considered active it also crosses p when player i is considered inactive.

Let us revisit example 1 for an illustration of the strategies. Figure 2 shows the decision rule for player 1 provided players 2 and 3 are still active. The proposed strategy for player 1 with a signal s_1 prescribes that he exit at p' and reenter at p''.

Remark 2. It is possible that for a particular s_i, $w_i(s_i, \cdot)$ would only touch and not cross the 45° line at p. For the moment, let us set aside all such signals

s_i in addition to all signal profiles $\mathbf{s}$ that result in simultaneous exits and (or) entries. The probability of such signal profiles is zero. This is because if we fix player i, no matter what are the signals of others, the probability that player i is involved in one of these two situations is zero; a small perturbation in the signal of player 1 necessarily pulls him out of any possible equalities.

3.4 Information processing

3.4.1 Equilibrium play

First, I will show that if everyone plays as suggested all $x_i(p)$, $y_i(p)$, $w_i(s_i, p)$ are uniquely defined and satisfy: for any active player i, $w_i(x_i(p), p) = p$; for any inactive player j who did not exit at p, $x_j(p) = s_j < y_j(p)$ and $w_j(s_j, p) < p$.

Suppose, some player k exited or entered at price p^0, and the estimated minimal signals at that price are $x_i^0(p^0)$. Suppose also that these estimates satisfy: (i) for any active player $i \in \mathcal{A}$, $V_i\left(\mathbf{x}_{\mathcal{A}}^0(p^0), \mathbf{x}_{-\mathcal{A}}^0(p^0)\right) = p^0$; (ii) for any inactive player $j \neq k$, $V_j\left(\mathbf{x}_{\mathcal{A}}^0(p^0), \mathbf{x}_{-\mathcal{A}}^0(p^0)\right) < p^0$; (iii) for player k, $V_k\left(\mathbf{x}_{\mathcal{A}}^0(p^0), \mathbf{x}_{-\mathcal{A}}^0(p^0)\right) = p^0$.

Starting from p^0 consider the system (6) for the current set of active players $\mathcal{A}$. By full differentiation of (6) we get

$$DV_{\mathcal{A}} \cdot \frac{d\mathbf{x}_{\mathcal{A}}}{dp} = \mathbf{1} \tag{7}$$

where $DV_{\mathcal{A}}$ is the matrix of partial derivatives $\left(\frac{\partial V_i(\mathbf{s})}{\partial s_j}\right)_{i,j\in\mathcal{A}}$ and $\mathbf{1}$ is the column of 1s of size $\#\mathcal{A}$.

System (7) together with $\frac{dx_j}{dp} = 0$ for any $j \notin \mathcal{A}$ gives a direction of change of the estimated minimal signals $\mathbf{x}(p)$ at the price p

$$\begin{cases} \frac{d\mathbf{x}_{\mathcal{A}}}{dp} = (DV_{\mathcal{A}})^{-1} \cdot \mathbf{1} \\ \frac{d\mathbf{x}_{-\mathcal{A}}}{dp} = \mathbf{0} \end{cases} \tag{8}$$

For a given initial condition $\mathbf{x}(p^0) = \mathbf{x}^0(p^0)$ system (8) has a unique continuous solution $\mathbf{x}(p)$. This solution is considered until some other player changes his status at some $p' > p^0$. After that, the terminal $\mathbf{x}(p')$ will be considered as the initial condition for the new system (8), corresponding to the new set $\mathcal{A}$ of active players.

Equilibrium play proceeds as follows. Initially, all players are active at $p = 0$, that is, $\mathcal{A} = \mathcal{N}$. The system (8) is solved for $\mathcal{A} = \mathcal{N}$ with the initial condition $\mathbf{x}(\mathbf{0}) = \mathbf{0}$ and all players are active until for some player k, $x_k(p)$ reaches s_k and $w_k(s_k, p)$ crosses p from above. (If $s_k = 0$ player k will exit right before the start of the auction). Player k exits at that point and a new system (8) is solved, and remains in effect until some other player exits or player k comes back and so on until the auction ends.

Lemma 5 in Appendix A.2 guarantees that $\mathbf{x}(p)$ is well-defined: it is positive for all $p > 0$. It is also the case that $w_i(x_i(p), p) = p$ for all (active) players $i \in \mathcal{A}$, and $w_j(x_j(p), p) \leq p$ for all inactive players $j \notin \mathcal{A}$, with the inequality being strict at non-decision points. This follows from the fact, that when player j exits, his $w_j(x_j, p)$ does cross p from above, that is he does not regret exiting. This property is guaranteed by the signal intensity condition and is proved in Lemma 4 in Appendix A.1. Similarly, there are no regrets about entries.

3.4.2 Off-equilibrium information processing

Suppose player j exits at p_j with estimated minimal signal $x_j(p_j)$. Suppose that at $p > p_j$ his potential estimated signal $y_j(p)$ touches $x_j(p) = x_j(p_j)$ from above. This means that other players expect player j to reenter at p. If player j deviates from the suggested strategy in a manner that is not anticipated, some special attention is needed. There are two possibilities.

Case 1a. Player j does not reenter at p, entering later or, possibly, not at all. Let us call such player j a *dormant* player and define $d_j \equiv x_j(p) = x_j(p_j)$. From the moment $y_j(p)$ reaches d_j from above and player j does not enter as the other players expect him to do, the following procedure applies. Player j is added to $\mathcal{A}$ as if he has entered (is considered by the others as if he is active), his $x_j(p)$ is estimated accordingly, so it is changing with p. This continues until one of the following three situations happens: player j enters; $x_j(p)$ reaches d_j or the auction ends. If player j enters then since $x_j(p)$ is appropriate, that is $V_j(\mathbf{x}(p)) = p$, no additional changes are necessary. If $x_j(p)$ reaches d_j, player j is removed from $\mathcal{A}$ as if he has exited.

Case 1b. Player j enters earlier than expected, at some p when $y_j(p) > x_j(p_j)$. In this case other players adjust player j's estimated minimal signal $x_j(p)$ slowly until a price p' such that $w_j(x_j(p'), p') = p'$. At the same time it is ensured that the auction does not end before the full adjustment takes place. This is achieved by adjusting $x_j(p)$ in a way that guarantees that for some player $i \in \mathcal{A}$, $x_i(p)$ is decreasing with p, so that $x_i(p)$ stays below s_i and player i does not exit. As an exception, full or partial instantaneous adjustment of $x_j(p)$ is possible if the value functions of the other players from $\mathcal{A}$ do not depend on s_j for some range of s_j. For a full description of the procedure see Appendix A.3.1.

The other possible information impasse happens when player j stays too long.

Case 2. Player j does not exit at p_j when $x_j(p_j) = 1$ and $w_j(1, p) < p$ for $p > p_j$. The solution is simple, player j is removed from $\mathcal{A}$ for the purposes of solving (8). Nothing happens if he exits in the process. If $w_j(1, p)$ crosses p from below, player j is added back to $\mathcal{A}$.

3.4.3 Summary

During the auction at each price level p for each player i, the other players have an estimate of the signal of player i that is consistent with the current status of player i. If player i is inactive that estimate is presumably the inferred true value of player i's signal. If player i is active, since it is plausible to think that he would have been active with a higher signal as well, the other players have an estimate of minimal possible signal of player i with which it is worth to be active. If all players think like that and behave accordingly, then one can obtain a set of such estimates for all the players at all price levels.

Normally, once a player, say j, exits, the estimate of his presumably true signal s_j is obtained. Given that the others can project future behavior of player j. In particular, they can forecast if and when player j is going to enter again and all other changes of i's status. In fact, the others may *simulate* the behavior of player j after the moment of first exit. It is exactly what they are supposed to do in the presented equilibrium if player j does not enter as projected.

At the same time, players *"forget"* the previously obtained estimate s_j once player j who exited earlier becomes active once again. All estimates of the lowest possible true signals of currently active players are based only on their current status and *not* on the history of previous actions. This is an important feature since the estimate $x_j(p)$ of the lowest possible signal of player j does affect in general the price player j has to pay for the good if he wins. If a higher estimate is taken into account than is necessary to win at a particular realization of signals, player j would have an incentive to shade his actions and exit for the first and all other times earlier, which, in turn, may render the allocation to be inefficient.

The manner in which information is processed is quite intuitive: players update the information they have about the other players only when they are active or become active, that is when players express their willingness to buy the good at the current price.[8]

3.5 Proof of the Main theorem: Simplified case

3.5.1 An efficient direct mechanism

It is well-known that there exists a direct mechanism which allocates efficiently based on the work of Crémer & McLean (1985). The single-crossing assumption (A1) is sufficient for the existence of an efficient direct mechanism. This mechanism is of interest because equilibrium play in the model of the English auction with reentry will be outcome equivalent to the direct mechanism.

For a given signal profile $\mathbf{s}$, for any player i define a signal s_i^* to be a unique solution to the equation

$$V_i(s_i^*, \mathbf{s}_{-i}) = \max_{j \neq i} V_j(s_i^*, \mathbf{s}_{-i}) \tag{9}$$

or set $s_i^* = 1$ if even at $s_i^* = 1$ player i does not have the highest value.

For every player i define p_i^* to be

$$p_i^* = \max_{j \neq i} V_j(s_i^*, \mathbf{s}_{-i}) \tag{10}$$

The uniqueness of the solution to (9) is guaranteed by the single-crossing condition and by the fact that $V_i(0, \mathbf{s}_{-i}) < \max_{j \neq i} V_j(0, \mathbf{s}_{-i})$ for $\mathbf{s}_{-i} \neq \mathbf{0}$. The solution is interior, $s_i^* < 1$, when $V_i(1, \mathbf{s}_{-i}) \geq \max_{j \neq i} V_j(1, \mathbf{s}_{-i})$; otherwise $s_i^* = 1$.

The prices in (10) were identified in Maskin (1992).

Lemma 1. *Suppose $i \in \mathcal{I}(\mathbf{s})$ for a signal profile $\mathbf{s}$. Then $p_i^* = \min_j p_j^*$, $V_i(\mathbf{s}) \geq p_i^*$ and the inequality is strict when $\mathcal{I}(\mathbf{s}) = \{i\}$. For any $j \notin \mathcal{I}(\mathbf{s})$, $V_j(\mathbf{s}) < p_j^*$.*

[8] Note that in special Case $1a$, when some player j does not enter when expected, his $x_j(p)$ is lowered temporarily. However, the others remember what it was at the moment of last exit and will keep changing $x_j(p)$ until it reaches that level or player j becomes active. Therefore, no actual updating of information takes place.

Proof. By the single-crossing, $s_i \geq s_i^*$ and $s_j < s_j^*$, where $j \notin \mathcal{I}(\mathbf{s})$. This implies $V_i(\mathbf{s}) \geq p_i^*$ and $V_j(\mathbf{s}) < V_i(\mathbf{s}) \leq V_i(s_j^*, \mathbf{s}_{-j}) \leq p_j^*$. □

Lemma 2. *There exists a direct mechanism that ex post implements an efficient outcome under (A1). Players submit their signals, the winner is player i (or randomly chosen among those) who has the lowest p_i^* and his payment is p_i^*. Others pay* 0*.*

Proof. It is straightforward from lemma 1. □

3.5.2 Proof

Now we can prove the Main theorem with specified restrictions on initial signal profiles and under a weaker equilibrium concept in which equilibrium conditions need not satisfy for deviations that lead to simultaneous exits and (or) entries. In addition, the strategies that result in an infinite number of entries and exits are also not considered for a moment. An example of such strategy is: exit at $p - \varepsilon$, enter at $p - \varepsilon^2$, exit at $p - \varepsilon^3$, and so on, for some small $\varepsilon > 0$. Therefore, p is a limiting point of switches of status for this particular player, and whether the player is active or inactive right before p cannot be determined. The rules of the auction have to be amended to deal with such cases, which is done in the next section. Meanwhile, it should be noted that in the equilibrium play no matter what the initial $\mathbf{s}$ is, only a finite number of exits and entries by any player is possible, which follows from the boundedness of the first derivative of $w(s_i, p)$ with respect to p, which in turn is guaranteed by the properties of value functions.

Proof of the Main theorem with restrictions. Suppose all bidders except possibly player i follow the strategies proposed in Section 3.3. If player i wins at a price p, he has to be active at p and have $w_i(x_i(p), p) = p$. There are only two situations considered previously, Case 1b and Case 2, when he can be active but have $V_j(\mathbf{x}(p)) < p$. If it is Case 2 and he wins, player i obviously pays more than the value of the good to him since $V_j(1, \mathbf{s}_{-j}) < p$. If it is Case 1$b$, by construction of the special procedure, his $x_j(p)$ will be adjusted to have $w_j(x_j(p), p)) = p$ before the auction ends.

If player i wins then the signals of the others have to be truthfully revealed: $x_j(p) = s_j$ for all $j \neq i$. In addition, for any player $j \neq i$, $V_j(x_i(p), \mathbf{s}_{-i}) = w_j(s_j, p) \leq p$ with $w_k(s_k, p) = p$ for at least one player k

(who exited at p). This means that $x_i(p) = s_i^*$ and $p = p_i^*$ as defined by (9) and (10) and player i cannot in any way affect the price he is obliged to pay.

If $s_i > s_i^*$, then by Lemma 1 it is player i who has the highest value. He will win the auction and get $V_i(\mathbf{s}) - p_i^* > 0$ if he follows the suggested strategy. If he deviates, his payoff is the same if he still wins the object. Obviously, he is worse off if he does not win. If $s_i < s_i^*$, player i is not supposed to win. If he does, by Lemma 1, $V_i(\mathbf{s}) < p_i^*$, so he has to pay more than the value of the object to him.

Obviously, if player i knows what are the true signals of the others, he cannot do better. There is no way he can manipulate the auction, say, to make some players exit without revealing their signals—the procedures of Cases 1a and 1b take care of that. And, if he wins, the signals of the others will be revealed anyway. □

By the revelation principle there exists a direct mechanism that is outcome equivalent to the equilibrium of the English auction with reentry constructed above. This is the mechanism, outlined in the lemma 2.

Remark 3. The proposed strategy of player j is the *unique* best response if every other player follows the proposed equilibrium strategies.

This is quite obvious. Suppose player j decides to switch status at p: exit earlier (enter later) or exit later (enter earlier) than p_j, specified by the proposed strategy. One can find an appropriate initial signal profile $\mathbf{s}$ such that all other players exit simultaneously in between p and p_j. As a result, player j either wins the auction but has to pay more than the value of the good to him or does not win when it is efficient and thus profitable for him to do so.

Remark 4. If everyone plays according to suggested strategies the set of functions $x_i(p)$ for all active players $i \in \mathcal{A}$ satisfies the same equations as the set of inverse bidding functions $\sigma_i(p)$ for $\mathbf{s}_{-\mathcal{A}} = \mathbf{x}_{-\mathcal{A}}(p)$ as defined in Krishna (2003). Thus parts of the strategies that deal with exits in the proposed equilibrium closely resemble those specified in the regular equilibrium (Krishna (2003) or Maskin (1992)).

In fact, whenever the standard English auction possesses an efficient equilibrium the English auction with reentry has an efficient equilibrium in which the behavior of the players is the same. In particular, in that case no exiting player exercises the option to reenter.

4 English auction with reentry. General case

In this section we are considering the most general case, that is, we define equilibrium strategies and information processing for all possible strategy profiles. All kinds of deviations are examined as well.

In order to accommodate all possible deviations, some of the rules governing play in the English auction with reentry need to be amended. Basically, the amendments are the following. The first amendment (to Rules 1 and 2) is that we now require that in order to change his or his status, a player must *stop* the price clock and a player has the option of doing so at any time. Stopping the clock is a simple and convenient solution to the problems of simultaneous exits and/or entries. For instance, player 1 may want to stay in the auction if some other player 2 is active at p, but wants to be out if 2 exits at p. Without stopping the clock at p player 1 will want to exit the auction as soon as possible if player 2 exits at p. Of course, there is no "as soon as possible" price level. Stopping the clock allows player 1 to exit at the same price as 2.

The second amendment (to Rules 2 and 3) is that the auctioneer can temporarily *suspend* players if their behavior becomes disruptive, in the sense, that they cannot determine their statuses. As an example, suppose that players 1 and 2 exit at some price p, while player 3 enters at the same price. Decisions of players 1 and 2 to exit were based on the assumption that the other would remain active while player 3 would remain inactive. Once they learn that player 3 has entered, there is the possibility that player 1 may want to be active only when player 2 is active but player 2 may want to be active if player 1 is not. With the clock stopped at p, neither player 1 nor 2 can make up their minds, causing a stalemate. In this case, we give the auctioneer the authority to suspend both players 1 and 2 so that the auction may continue.

Finally, Rule 5 below disallows an infinite number of exits and entries by the same player. The auctioneer sets an upper bound to the number of switches of status.

4.1 Complete rules for English auction with reentry

1. The auctioneer sets a low initial price, say zero, on a price clock and this is raised.

2. While the clock is ticking, each player is either active, inactive or *suspended*. All players are active at a price of zero. The activity statuses of all players are commonly observed and known. The status of the player can change only when clock is stopped. *A player who is not suspended can stop the clock at any time (say by raising his or her hand).* A suspended player cannot stop the clock.

3. Once the clock stops:

   (a) All suspensions are lifted.

   (b) All players are asked to indicate their intention to be active or inactive once the clock restarts. Players may also indicate that they are undecided. These intentions are communicated to the auctioneer simultaneously and observed by all.

   (c) Players who indicated their intention to be inactive or who were undecided are asked if they wish to change their intentions in light of the information revealed in (b). Undecided players are only allowed to indicate either that they now wish to be active or that they are still undecided.

   (d) If some player changes his intention, then (c) is repeated.

   (e) If no player changes his intention, then these are considered the current statuses. If only one player is undecided, then he must reconsider and must choose to be either active or inactive. Others are not allowed to change their statuses.

   (f) The auctioneer suspends all undecided players if there are at least two active players.

4. The clock restarts as long as there are more than two active players once all players have chosen as in Rule 3. Otherwise, the auction ends and the good is sold at the price showing on the clock. It is awarded to:

   (a) The only remaining active person, if there is such a player.

   (b) A randomly chosen player among those who exited last, if no active players remain.

5. If the number of exits and entries of a player after his last suspension (if he has been suspended before) or after the start of the auction (if he has never been suspended before) exceeds a commonly known number, pre-announced by the auctioneer, the player is automatically suspended at that price.[9]

To see how these rules work consider an auction with 5 players. Players 1 and 2 are active, player 3 is inactive and players 4 and 5 are suspended at the current price, the price clock is ticking. Later, at a price p, player 1 stops the clock. According to Rule 3 the suspensions of players 4 and 5 are lifted, all players are asked to indicate their intentions as in Rule 3b. Suppose players' intentions are as follows: player 2 wants to be active, players 1 and 5—inactive, players 3 and 4 remain undecided. These intentions are observed by all players. Rule 3c applies.

If no players are willing to change their intentions then player 2 as the only active player is declared the winner, the auction ends. Suppose instead, player 3, observing that player 2 will be active, decides to be active as well, and he is the only player to change his intentions in the second stage. As a result the intentions of the players are: players 2 and 3 want to be active, players 1 and 5 want to be inactive, player 4 is undecided.

If in the next stage no player changes his intention, by Rule 3e player 4 is asked to decide on his status. Once he decides, all players' statuses are fixed and, by Rule 4, the clock restarts. Lastly, suppose in the third stage, player 1 becomes undecided, while player 5 changes his intentions to be active, and no other player wants to change his intentions after this. Then both undecided players 1 and 4 get suspended by Rule 3f, the auction restarts with players 2, 3, and 5 being active. Note that Rule 3c insures that after a finite number of stages no player will change his intention.

To complete the description of the auction as a game we specify payoffs when two or more players decide to remain active forever, so an auction continues indefinitely. We can spell out what are utility implications from "infinite bidding" or add a rule that governs or eliminates it ensuring that the auction remains "detail-free" and continues long enough to be efficient. This is irrelevant for the presented analysis as long as the auction continues

[9]Note that the mechanism defined by these rules is "detail-free"—the auctioneer is not required to have or acquire any information that players know—value functions, distribution of the signals, or the actual realization of the signals—to make the auction work.

long enough and the payoff to any involved player is non-positive. Purely for convenience we assume that any such bidder obtains $-\infty$.

4.2 Equilibrium strategies

The equilibrium strategies proposed in Section 3.3 need to be amended only slightly to take account of the fact that some additional decisions have to be made when the price clock stops.

We also have to specify how the auctioneer determines the bound on the number of exits and entries. Actually, any large number will serve its purpose. If in addition, the auctioneer knows the value functions but not the actual signals, the auctioneer can compute the number of exits and entries by all the players on the assumption that they follow the proposed strategies. This number depends on the particular realization of the signals $\mathbf{s}$ but it suffices to choose the maximum over all possible $\mathbf{s}$.[10]

Player i should behave as follows:

1. if $w_i(s_i, p) > p$, then be active at p or indicate intention to be active if the clock is stopped at p;
2. if $w_i(s_i, p) < p$, then be inactive at p or indicate intention to be inactive if the clock is stopped at p;
3. if $w_i(s_i, p) = p$, then stop the clock at p, if for ε small enough, $w_i(s_i, p-\varepsilon) \leq p-\varepsilon$ and $w_i(s_i, p+\varepsilon) > p+\varepsilon$, or $w_i(s_i, p-\varepsilon) \geq p-\varepsilon$ and $w_i(s_i, p+\varepsilon) < p+\varepsilon$;
4. if $w_i(s_i, p) = p$, and the clock is already stopped,
    (a) then indicate that he is undecided;
    (b) and if forced to make a decision (when i is the only undecided player):
        i. then choose to be inactive, if for ε small enough, $w_i(s_i, p+\varepsilon) \leq p+\varepsilon$;

[10] In fact, the outcome is unaffected no matter what number is chosen by the auctioneer. When this is small Rule 5 will be applied often and there will be many suspensions. The equilibrium construction below ensures that a suspension of a player does not bar him from winning and does not affect the price he pays if wins. We choose the bound in the way specified so that in equilibrium Rule 5 does not come into play.

ii. then choose to be active, if for ε small enough, $w_i(s_i, p+\varepsilon) > p+\varepsilon$.

4.3 Proposed equilibrium information processing

For the general process of calculation of $\mathbf{x}(p)$, $\mathbf{y}(p)$ and $\mathbf{w}(\cdot, p)$ see Section 3.4. The only situations which need additional attention are those that deal with the estimation of $\mathbf{x}(p)$ when there are some suspended players and those that involve a particular type of the off-equilibrium behavior, Case $1b$, which was considered in Section 3.4.2.

Similar to the analysis of Section 3.4, suppose first that all players behave as suggested. If there are no suspended players, $\mathbf{x}(p)$ is calculated as the solution to (8) together with an appropriate initial condition, where $\mathcal{A}$ is the set of active players.

Suppose that some players get suspended at p'. Generally, these players are treated as if they are inactive. It is possible that for a suspended player i, if $x_i(p)$ is fixed at $x_i(p')$, for some $p > p'$ during the period of suspension, then $w_i(x_i(p'), p)$ becomes higher than p. This situation is quite similar to the Case $1a$ of the late reentry, which is considered in Section 3.4.2. Any such player is treated as *dormant* (see Section 3.4.2), he is added to $\mathcal{A}$ and his $x_i(p)$ is adjusted correspondingly until the next stop of the clock or until $x_i(p)$ reaches $d_i = x_i(p')$. The only difference is that there can be many such dormant players at the same time. While there are no difficulties in dealing with many dormant players simultaneously there is a potential problem with starting this procedure. Namely,

Case 3. Suppose at some p (possibly the price at which the clock was stopped), there is a subset $\mathcal{K}$ of suspended players, such that $w_j(x_j(p), p) = p$ for any $j \in \mathcal{K}$. Additionally suppose that any such player j would have been considered dormant starting at p, given no other player had changed his current extended status: active, inactive, suspended or dormant. If, however, all such players $\mathcal{K}$ are treated as dormant, for some of them $w_i(x_i(p), p)$ may stay lower than p, so $x_i(p)$ should not change. Lemma 7 in Appendix A.3.2 proves that except for some situations of mere theoretical possibility there exists a partition of suspended players into those who have to be considered dormant and those who do not require any special attention. All dormant players will be added to $\mathcal{A}$ for solution purposes. In those situations where such a partition does not exist a special procedure for determining $\mathbf{x}(p)$, which is described in detail in Appendix A.3.2 will be used. It should be noted that for such situations to arise, some very specific value functions are needed as well as a specific initial signal profile $\mathbf{s}$.

If some player j chooses to deviate, his actions will receive special attention in exactly the same situations and in exactly the same way as considered in Section 3.4.2 as Cases 1a, 1b, 2. The analysis of Cases 1a and 2 remains the same as before.

Case 1b (reconsidered). The previously considered Case 1b deals with a situation where a change of status by a particular player j at some price p is incompatible with the estimate of the signal of player j at p. In the simplified case of Section 3 there is only one kind of such situation, when player j enters earlier than expected. In the most general case considered in this section two such situations can arise. Namely, once the clock is stopped at a particular price p, other players may estimate that for some previously inactive (possibly suspended) player j, $w_j(x_j(p), p) < p$. Therefore, if player j chooses to be *undecided* or *active* at this clock stop, the decision is incompatible with the current estimate of $x_j(p)$. If player j chose at some stage to be active his estimated minimal signal will be adjusted in exactly the same way as before. If player j chose to be undecided, in an exceptional case when values of players from $\mathcal{A}$ do not depend on s_j, for some range of s_j that includes $x_j(p)$, an instantaneous, possibly partial, adjustment will take place. There will be no further adjustment if player j is not active once the clock is restarted. For a full description of the adjustment procedure see Appendix A.3.1. Note only that it insures that player j cannot win the auction before the adjustment is completed.

If some of the proposed procedures of Cases 1b and 2 are applied and there are some suspended players, the procedure of Case 3 can be simultaneously applied as well if needed.

4.4 Results

We are ready to formulate and prove our main theorem. The set of rules in Section 4.1 describes the proposed English auction with reentry.

Theorem 1 (Main). *Under the single-crossing and the signal intensity conditions the English auction with reentry has an ex post equilibrium that is efficient.*

Proof. The arguments presented in Section 3.5.2 in the proof of the simplified version of the result are all valid in the general case. In particular, the main facts that drive the result still stand: If some player i wins the good while the others follow the proposed strategies, then the signals of all the other players must have been revealed truthfully. And, if player i wins the good, the price he has to pay does not depend on his own actions and is equal to p_i^* from (10), the same price as in efficient direct mechanism.

What is needed to be shown is that under the new amended set of rules and new available or not considered previously options for deviations no player j can profitably deviate or create any other kind of problems. Many potential deviations have been already covered in the proof in Section 3.5.2. Deviations that cause simultaneous exits and entries, related to the suspension process and a possibility of creating a converging sequence of stops of the clock are the only additional cases to be considered.

First, we show that the converging sequence of exits and entries is not possible. The fact that the number of exits and entries by all players if finite when everyone follows the proposed strategies follows from the properties of the value functions which imply that the derivatives of $\mathbf{x}(p)$ and, thus, of $\mathbf{w}(\cdot, p)$ are all bounded from above. Using calculus one can show then that $w_j(s_j, p)$ can cross p only finitely many times on a given segment. As a result, some player, say j, has to deviate to cause the converging sequence of switches of status(es). Moreover, by the same reasons as above, he himself has to switch the status in a manner that creates a limiting point of his own switches of status. Rule 5, however, effectively prohibits that, by suspending a player if the number of exits and entries exceeds the bound. In particular, player j will be suspended occasionally, and if the limiting point of switches of statuses exists, so does the limiting point of suspensions of player j. Player j, however, cannot stop the clock while being suspended. Again, it can be shown that, since the others follow the proposed strategies, there can be only a finite number of entries and exits of other players before the supposedly problematic price level p. Since some other player has to stop the clock to end the suspension of player j it is impossible to have a limiting point of suspensions of player j and so, the limiting point of entries and exits of all players.

Second, whenever player j causes a clock stop at which some players, possibly including j, are suspended, the proposed information processing ensures that there is no sudden jump in any of $x_i(p)$ and $w_i(s_i, p)$ for $i \neq j$. Since any suspension of player i can happen only at the clock stop and only when $w_i(s_i, p) = p$, whenever some player i exits or gets suspended, his signal is properly revealed. If several players are suspended at the same time, procedures developed in Section 4.3, especially in Case 3, guarantee that $\mathbf{x}(p)$ and $\mathcal{A}$ are at all times such that $\mathcal{A} \subset \mathcal{I}(\mathbf{x}(p))$, thus the system (8) can always be resolved.

For the rest of the proof see Section 3.5.2. □

Remark 5. The equilibrium constructed here satisfies conditions $B(i) - B(iv)$ and P, amended to account for the continuous nature of the game, that define a *perfect Bayesian equilibrium* in Fudenberg & Tirole (1991, p. 333). In particular, player k reveals by his own actions no information about player j's signal; beliefs $\mathbf{x}_i(p)$ about player i are the same for all other players; all types of player i have the same beliefs $\mathbf{x}_{-i}(p)$ about the others; and, of course, Bayes' rule is used to determine beliefs whenever possible. Whenever the status of player j is incompatible with the current estimate $x_j(p)$, the others believe that player j is making a mistake and that he will change the status immediately. Remarkably, continuous nature of the game allows simultaneously not to make any instantaneous adjustments at any given p and to adjust $x_j(p)$ continuously if the status of player j remains incompatible for a range of prices.

Theorem 2. *If the standard English auction without reentry has an efficient equilibrium then so does the English auction with reentry.*

Proof. In the case the standard English auction possesses the "regular" equilibrium $\boldsymbol{\beta}$ described in Krishna (2003), calculated $\mathbf{x}(p)$ are exactly the same as the inverse bidding strategies $\boldsymbol{\sigma}(p)$ described in Krishna (2003). The same price levels are chosen by the players to stop the clock and become inactive in the presented auction as those prices chosen to exit by the players in the "regular" equilibrium $\boldsymbol{\beta}$. The only difference is that in the presented auction the price clock stops at any exit. Additionally, whenever two or more players exit simultaneously, they first become suspended until the next player exits, but they never reenter. Thus the sequence of prices at which bidders become *not* active is exactly the same in both the proposed equilibrium in the English auction with reentry and in the "regular" equilibrium in the standard English auction. Clearly, such an equilibrium is also ex post.

Suppose the standard English auction has an efficient equilibrium of any kind in some setting, not necessarily the one of this paper. To construct an isomorphic Bayesian-Nash equilibrium in the English auction with reentry we only need to specify how players will react to an unexpected entry, what some players may find reasonable to do.

We suggest the following beliefs and the strategy: Whenever (re)entry happens by some player i who had previously exited, other players will believe that the actual signals of all other players are the highest possible, that is they are equal to 1. Any player j is supposed to stay in the auction until $p_j = V_j(s_j, \mathbf{1})$. Clearly, if player i still wins the good, the price he has to pay will be higher than he would have to pay otherwise. Thus, no entries will happen on the equilibrium path and, therefore, such beliefs are consistent. □

It may seem unreasonable that players change their beliefs not only about the reentered player, but about the others as well. In particular, the presented equilibrium may not survive some of the refinements of the equilibrium concept—it does not satisfy $B(iii)$ condition of a perfect Bayesian equilibrium. This set of beliefs is chosen only for the purposes of simplicity, it is possible to construct more complicated equilibrium without involving so harsh changes in the structure of beliefs. In particular, if the payment of the winner in an efficient equilibrium of the standard English auction is never higher than in the efficient direct mechanism, then, once a reentry occurs, the auction may proceed according to the presented strategies and information processing. In such a case there will be some adjustment done first (as in Case 2), sequentially to all players that are in need of adjustment, to receive a set of estimates $\mathbf{x}(p)$.

Remark 6. Suppose the standard English auction has an efficient equilibrium. Then, if a player i exits at p, he should not have any immediate ex post regret. If he does then it is not optimal for him to exit. If the information is processed in the manner similar to that in the "regular" equilibrium of the standard English auction, then the signal intensity condition has to hold. This makes the signal intensity condition weaker than any known condition under which the standard English auction has an efficient equilibrium, and weaker than any condition under which a "regular" equilibrium exists. Note also that the signal intensity condition is vacuously satisfied whenever there is a player who has the highest value for all signal vectors.

4.5 Is (A2) necessary?

It is well-known that the direct mechanism (see Section 3.5.1) allocates the object efficiently as long as the pairwise single-crossing assumption (A1) alone is satisfied. Since the signal intensity condition (A2) is not needed in the direct mechanism, it is natural to ask what role this assumption plays in ascending price auctions.

The signal intensity condition (A2) allows players to process information in a simple and natural manner—they estimate minimal possible signals $\mathbf{x}(p)$ for other players and use these estimates to make decisions. In particular, once a player, say j, exits, the other players treat his signal as fixed while he is inactive. The signal intensity condition ensures that the imputed value of the object to a newly inactive player j does not rise faster than the imputed values of active players, implying that player j does not suffer from regret when becomes inactive. If the signal intensity condition did not hold player j's imputed value would rise faster than p, which would make player j regret exiting. In addition, since player j would become the only member of the winners' circle $\mathcal{I}(\mathbf{x}(p))$ and the single-crossing assumption (A1) is required to hold only for players in the winners' circle, other players' signals might not be accurately inferred thereafter.

So, what one can do in the absence of the signal intensity assumption? It seems that if players intentionally reduce $x_j(p)$ of any player j, for whom (A2) is violated, at the appropriate time and then use the procedure as in Case $1b$ (see Appendix A.3.1) to return $x_j(p)$ to the previous level, then an efficient equilibrium may possibly be constructed. This way of processing information, however, is unnatural and may need to be repeated a large if not infinite number of times. Additional complications arise when the number of such players exceeds one. Not only one will need to run simultaneous procedures for several players, but also the issues of how to effectively partition the players into those that are considered active and inactive arise.

An ascending price mechanism proposed in Perry & Reny (2001) obtains efficiency without imposing (A2). Apart of relying on directed demands it also requires two additional features: a stronger version of the single-crossing condition—required to be satisfied at all $\mathbf{s}$ for any pair of bidders, not only for those whose values are equal and maximal—and a "static" way of information processing—any inferences about a player's signal are drawn only when the player reduces one of his directed demands. If applied to the auction presented here, such information processing would at least require that $x_i(p) = 0$ for any player i if no player exited before p. It is exactly the combination of these two features that allows Perry & Reny (2001) mechanism not to require (A2) for efficiency: dynamic inconsistency in updating is not an issue if there is no dynamic updating, the stronger form of the single-crossing condition allows to make inferences about signals at any point.

We conjecture that in the absence of the signal intensity assumption (A2) it may still be possible to construct an *ex post* efficient equilibrium, but to do so information has to be processed in a highly unintuitive and complicated manner, and a stronger version of the single-crossing condition or some other condition has to be imposed.

5 Conclusion

In an English auction a player faces a simple choice: *To be in or not to be in.* This apparent strategic simplicity is what makes an English auction so attractive both from the perspective of the analyst and from the point of view of actual bidders. This is why it is important to know the strengths and weaknesses of the institution.

From the perspective of an economist an English auction is of special interest since it is known to allocate efficiently and, in some circumstances, simultaneously yield a high revenue to the seller. This paper emphasizes the efficiency aspect. As previous studies have shown, the extensively analyzed variant of an English auction—the standard English auction where exits are irrevocable, may fail to allocate efficiently under quite natural circumstances when the number of buyers is more than two.

This paper shows that the failure to allocate efficiently is not due to an institutional defect of the auction itself. I present a modified model of an English auction in which a buyer who had exited previously can return at will. I have shown that the modified mechanism has a number of remarkable properties. First, it is efficient under the single-crossing and the signal intensity conditions. The former is weak and known to be necessary. The latter ensures that in the dynamic nature of the auction no player regrets his decisions to exit or enter the auction and is also relatively weak—it is implied by any other condition under which the standard English auction is known to possess an efficient equilibrium. In the case of only two active bidders it is guaranteed by the single-crossing condition itself.

Second, the strategic complexity of the English auction with reentry necessary for efficiency is the same as that of the standard English auction. In fact, the presented equilibrium strategies and the information processing are exactly the same as in the "regular" equilibria of the standard English auction analyzed in Maskin (1992) and Krishna (2003). Moreover, whenever the standard English auction has an efficient equilibrium, regular or not, it can be duplicated—the behavior of the players will be exactly the same—as an efficient equilibrium of the modified auction. Therefore, the modified English auction with reentry augments and subsumes the standard English auction.

Third, since it allows reentry the presented model is obviously a better approximation of the "real-life" auction. The possibility of reentry means that now one can generate and analyze certain kinds of strategic behavior that occur in practice but are not possible to capture in the standard model. For instance, a player may exit "early" or "late" in order to mislead other bidders about his signal. This paper shows that the presented auction obtains efficiency despite all of these "new" possibilities. In particular, a bidder cannot gain by abstaining from bidding for a while.

Finally, the rules of the modified auction presented here are quite flexible. They may be amended so that, for example, reentry may be allowed only at the points when otherwise the good would have been awarded to some player. With this amendment, an efficient equilibrium with the same information processing still exists. Likewise, the rules determining what happens when the price clock stops or even when to stop the clock can also be amended. Thus the mechanism is flexible. This may be a desirable feature for practical purposes.

It remains to be seen how the model presented here can be extended to the case of allocating many goods via an open ascending price auction.

A Appendix

A.1 Implications of the signal intensity condition (A2)

First, note that since $\mathbf{V}$ is twice differentiable, the signal intensity condition (A2) is equivalent to the requirement that the directional derivative of $V_i(\mathbf{s})$ with respect to s_i is positive along the path with V_j being fixed for all $j \neq i$, $j \in \mathcal{I}(\mathbf{s})$, and s_k being fixed for all $k \in \mathcal{N} \setminus \mathcal{I}(\mathbf{s})$. Denote $\mathcal{A} = \mathcal{I}(\mathbf{s})$, since $\det DV_{\mathcal{A}} \neq 0$ (value functions are regular) we can write at $\mathbf{x}(p) = \mathbf{s}$

$$\frac{d\mathbf{x}_{\mathcal{A}}}{dp} = B \cdot \frac{d\mathbf{V}_{\mathcal{A}}}{dp} \tag{11}$$

where $B = (b_{ij}) = (DV_{\mathcal{A}})^{-1}$, $\frac{d\mathbf{x}_{\mathcal{A}}}{dp}$ and $\frac{d\mathbf{V}_{\mathcal{A}}}{dp}$ are the columns of $\frac{dx_i}{dp}$ and $\frac{dV_i}{dp}$ for all $i \in \mathcal{A}$, and the signals of players not from $\mathcal{A}$ are fixed, $\frac{dx_k}{dp} = 0$ for $\forall k \notin \mathcal{A}$. If the values of all players from $\mathcal{A} \setminus \{i\}$ are fixed, $\frac{dV_j}{dp} = 0$ for all

$j \in \mathcal{A}$, $j \neq i$, we have $\frac{dx_i}{dp} = b_{ii}\frac{dV_i}{dp}$. So, the value of the directional derivative in question is b_{ii}, and the signal intensity condition requires $b_{ii} > 0$.

Lemma 3. *Given* $\mathbf{s} \neq \mathbf{0}$*, Assumption (A2) implies that for any proper subset of at least two players* $\mathcal{J} \subset \mathcal{I}(\mathbf{s})$ *and any player* $j \in \mathcal{J}$*,* $(DV_{\mathcal{J}})^{-1}_{jj} > 0$*.*

Proof. The regularity of value functions, $\det DV_{\mathcal{J}} \neq 0$ and $\det DV_{\mathcal{L}} \neq 0$, where $\mathcal{L} = \mathcal{J} \setminus \{j\}$, implies $b_{\mathcal{J},jj} = (DV_{\mathcal{J}})^{-1}_{jj} = \det DV_{\mathcal{L}} / \det DV_{\mathcal{J}} \neq 0$. To show that $b_{\mathcal{J},jj} > 0$ we disturb $\mathbf{s}$ to obtain $\mathbf{s}(\boldsymbol{\varepsilon})$, at which $\mathcal{J} = \mathcal{I}(\mathbf{s}(\boldsymbol{\varepsilon}))$. Then, by Assumption (A2) at $\mathbf{s}(\varepsilon)$, $b_{\mathcal{J},jj}(\mathbf{s}(\boldsymbol{\varepsilon})) > 0$. Thus, $\lim_{\boldsymbol{\varepsilon} \to \mathbf{0}} b_{\mathcal{J},jj}(\mathbf{s}(\boldsymbol{\varepsilon})) = b_{\mathcal{J},jj}(\mathbf{s}) > 0$.

The signals are disturbed as follows. Suppose that players 1 to K are from $\mathcal{K} = \mathcal{I}(\mathbf{s}) \setminus \mathcal{J}$. Note that $s_i > 0$ for all $i \in \mathcal{I}(\mathbf{s})$ since $\mathbf{s} \neq \mathbf{0}$, and no player can have the maximal value with the lowest possible signal. Consider $\mathbf{s}^{(1)} = (\mathbf{s}^{(1)}_{\mathcal{I}(\mathbf{s})}, \mathbf{s}_{-\mathcal{I}(\mathbf{s})})$, where $\mathbf{s}^{(1)}_{\mathcal{I}(\mathbf{s})}$ is constructed as: $s^{(1)}_1 = s_1 - \varepsilon_1$ for some small $\varepsilon_1 > 0$, $\left(s^{(1)}_i\right)_{i \in \mathcal{I}(\mathbf{s}) \setminus \{1\}}$ are such that $V_i(\mathbf{s}^{(1)}) = V_i(\mathbf{s})$ for all $i \in \mathcal{I}(\mathbf{s}) \setminus \{1\}$. If ε_1 is small enough, then by Assumption (A2), V_1 decreases and $\mathcal{I}(\mathbf{s}^{(1)}) = \mathcal{I}(\mathbf{s}) \setminus \{1\}$. Given ε_1, one can in a similar manner construct $\mathbf{s}^{(2)}$, such that $\mathcal{I}(\mathbf{s}^{(2)}) = \mathcal{I}(\mathbf{s}) \setminus \{1, 2\}$ for small enough $\varepsilon_2(\varepsilon_1)$, where the values of players from $\mathcal{I}(\mathbf{s}) \setminus \{1, 2\}$ are fixed, the signals of player 1 and players $\mathcal{N} \setminus \mathcal{I}(\mathbf{s})$ are fixed, and $s^{(2)}_2 = s^{(1)}_2 - \varepsilon_2$. Proceeding further, one can obtain $\mathbf{s}^{(K)} = \mathbf{s}(\boldsymbol{\varepsilon})$, for $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_K)$, with $\mathcal{I}(\mathbf{s}(\boldsymbol{\varepsilon})) = \mathcal{J}$. □

The following lemma shows that the signal intensity condition implies that no player would immediately regret his action after he exits or (re)enters the auction.

Consider a signal profile $\mathbf{s}$ and a subset $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$ of at least two players, denote $p^0 = V_i(\mathbf{s})$, $i \in \mathcal{I}$. Pick any player $i \in \mathcal{A}$. Let $\mathbf{x}(p)$ be a solution to the system (8) for $\mathcal{A}$ with the initial condition $\mathbf{x}(p^0) = \mathbf{s}$. Let $\mathbf{x}^{\mathcal{B}}(p)$ be a solution to the system (8) for $\mathcal{B} = \mathcal{A} \setminus \{i\}$ with the same initial condition. The difference between $\mathbf{x}(p)$ and $\mathbf{x}^{\mathcal{B}}(p)$ is that in the first case player i is active, while in the second case he is treated as inactive. If player i is considered inactive his value $V_i(\mathbf{x}(p))$ changes as

$$\frac{dV_i^{\mathcal{B}}}{dp} = \sum_{j \in \mathcal{B}} \frac{\partial V_i}{\partial s_j} \frac{dx_j^{\mathcal{B}}}{dp} \tag{12}$$

Lemma 4. *With the notation specified above, Assumption (A2) guarantees that player i will not regret his decision to enter or exit. Formally, if $\frac{dx_i}{dp} > 0$ then $\frac{dV_i^{\mathcal{B}}}{dp} < 1$. If $\frac{dy_i}{dp} = \left(1 - \frac{dV_i^{\mathcal{B}}}{dp}\right) / \frac{\partial V_i}{\partial x_i} < 0$, meaning $\frac{dV_i^{\mathcal{B}}}{dp} > 1$, then $\frac{dx_i}{dp} < 0$.*

Proof. Suppose in (8), $\frac{dx_i}{dp} > 0$. Substituting $\frac{d\mathbf{V}_{\mathcal{A}}}{dp} = \mathbf{1}$ to (11) we obtain

$$0 < \frac{dx_i}{dp} = \sum_{j \in \mathcal{B}} b_{ij} + b_{ii} \tag{13}$$

Now, if we consider system (8) for $\mathcal{B}$, we have

$$0 = \frac{dx_i^{\mathcal{B}}}{dp} = \sum_{j \in \mathcal{B}} b_{ij} + b_{ii} \frac{dV_i^{\mathcal{B}}}{dp} \tag{14}$$

Since $b_{ii} > 0$ by Lemma 3 we must have $\frac{DV_i^{\mathcal{B}}}{dp} < 1$. The other part of the statement follows similarly. □

A.2 Properties of $\mathbf{x}(p)$ and $\mathbf{w}(\cdot, p)$

Lemma 5. *For all p and every player $i \in \mathcal{I}(\mathbf{x}(p))$, $x_i(p) > 0$ for $p > 0$.*

Proof. To make the proof as straightforward as possible, let us linearly extend all value functions into the domain of negative signals. For each $s_i < 0$ and any player j, let $V_j(s_i, \mathbf{s}_{-i}) = V_j(0, \mathbf{s}_{-i}) + s_i \frac{\partial V_j(0, \mathbf{s}_{-i})}{\partial s_i}$. In case of two or more negative signals the above extension can be done sequentially.

First, we show the result in the neighborhood of $p = 0$ by induction. For $\mathcal{J} = \{i, j\}$ from (8) by the single-crossing (where $V_{ij} = \frac{\partial V_i}{\partial s_j}$)

$$\frac{dx_i(\mathbf{0})}{dp} = \frac{V_{jj} - V_{ij}}{V_{ii}V_{jj} - V_{ij}V_{ji}} > 0, \qquad \frac{dx_j(\mathbf{0})}{dp} = \frac{V_{ii} - V_{ji}}{V_{ii}V_{jj} - V_{ij}V_{ji}} > 0 \tag{15}$$

Now, suppose for any subset $\mathcal{K} \subset \mathcal{N}$ of K players, $x_i(p) > 0$ for any $i \in \mathcal{K}$, $x_j(p) = 0$ for all $j \notin \mathcal{K}$ for all $p < \varepsilon$ for some $\varepsilon > 0$. We will show that for any arbitrary subset $\mathcal{J} \subset \mathcal{N}$ of $K + 1$ players there exists such an open neighborhood of 0, such that $x_i(p) > 0$ for all $i \in \mathcal{J}$ in the solution to (8) with $\mathcal{A} = \mathcal{J}$ and $\mathbf{x}(0) = \mathbf{0}$.

Suppose not, that is there exist $\varepsilon > 0$ and a player j, such that in the solution to (8) for $\mathcal{A} = \mathcal{J}$, $x_j(p) \leq 0$ for $p < \varepsilon$. Then, there exist a (possibly smaller) ε_1 such that $\frac{dx_j}{dp} \leq 0$ for $p < \varepsilon_1$. Lemma 4 implies that $\frac{dV_j^{\mathcal{B}}}{dp} \geq 1$, hence $V_j(\mathbf{x}^{\mathcal{B}}(p)) \geq \max_{i\in\mathcal{B}} V_i(\mathbf{x}^{\mathcal{B}}(p))$ for all $p < \varepsilon_1$, where $\mathcal{B} = \mathcal{J} \setminus \{j\}$. Note that $x_j^{\mathcal{B}}(p) = 0$ and, by the step of induction, $x_i^{\mathcal{B}}(p) > 0$ for all $i \in \mathcal{B}$. Thus, it is either $j \in \mathcal{I}(\mathbf{x}^{\mathcal{B}}(p))$ or $\mathcal{J} \cap \mathcal{I}(\mathbf{x}^{\mathcal{B}}(p)) = \emptyset$. By construction, if $k \notin \mathcal{J}$ then $x_k^{\mathcal{B}}(p) = 0$. So, there exists a player in the winners' circle at $\mathbf{x}^{\mathcal{B}}(p) \neq \mathbf{0}$ with the signal equal to 0, which is a contradiction. Induction is complete.

The above analysis shows that there are no problems with starting the auction. By construction, $\mathcal{A} \subset \mathcal{I}(\mathbf{x}(p))$ for all p. So, if $p > 0$, we must have $x_i(p) > 0$ for all $i \in \mathcal{I}(\mathbf{x}(p))$. Otherwise, let $p^0 > 0$ be the lowest price that $\exists i \in \mathcal{I}(\mathbf{x}(p^0))$ with $x_i(p) = 0$. Such p^0 is well defined since the result holds in the neighborhood of 0, so we have a contradiction to the assumption that a player cannot have the maximal value with the lowest possible signal. $\square$

Lemma 6. *For any given initial signal profile* $\mathbf{s}$, $\mathbf{x}(p)$ *and* $\mathbf{y}(p)$ *satisfy*

1. *At any price level* p, $V_i(\mathbf{x}(p)) = p$ *for all* $i \in \mathcal{A}$ *and* $V_j(\mathbf{x}(p)) \leq p$ *for all* $j \notin \mathcal{A}$, *where* $\mathcal{A}$ *is the set of players used to solve for* $\mathbf{x}(p)$.

2. $\mathbf{x}(p)$, $\mathbf{y}(p)$, *and* $w_i(\cdot, p)$ *are continuous in* p.[11]

Proof. The whole price line (as a time line) can be divided into a sequence of points $p_0 = 0, p_1, \ldots, p_n, \ldots$ such that for $p \in [p_{n-1}, p_n]$ a system like (7) is solved for some $\mathcal{A}_n$. Since the initial condition for $\mathbf{x}(p)$ at p_{n-1} is the same as the terminal value of $\hat{\mathbf{x}}(p_{n-1})$ of the system solved in the previous segment for $\mathcal{A}_{n-1}$, $\mathbf{x}(p)$ is continuous. Obviously then, $\mathbf{y}(p)$ and $w_i(\cdot, p)$ are continuous as well.

At $p = 0$, $\mathbf{x}(0) = \mathbf{0}$ and $V_i(\mathbf{x}(0)) = 0$ for any i. This means that the first statement of the lemma is satisfied at $p = 0$ for $\mathcal{A}_0 = \mathcal{N}$. Suppose now, $x(p_n)$ is such that $V_i(x(p_n)) = p_n$ for any $i \in \mathcal{A}_n$ and $V_j(x(p_n)) \leq p_n$ for any $j \notin \mathcal{A}_n$. By construction of the auction, $\mathcal{A}_{n+1}$ is defined in such a way, that in some right neighborhood of p_n, the solution $\mathbf{x}(p)$ to (8) with $\mathcal{A}_{n+1}$ satisfies the required properties as well. Again by construction, p_{n+1} is defined as the next price at which someone exits or enters, or the next price at which $\mathcal{A}_{n+1}$ has to be amended to keep $V_i(\mathbf{x}(p)) = p$ for all $i \in \mathcal{A}$ and

[11] There is one exception. If player j unexpectedly enters earlier than expected (Case 1b), if the signal of player j does not affect values of the other players, then there will be an instantaneous adjustment of $x_j(p)$ at p (see Appendix A.3.1 below). This adjustment concerns player j only, none of the $\mathbf{y}(p)$, $\mathbf{x}_{-j}(p)$, and $w_i(\cdot, \mathbf{x}_{-i})$ for all $i \neq j$ are affected.

$V_j(\mathbf{x}(p)) \leq p$ for all $j \notin \mathcal{A}$. In both cases it is either the suggested strategies or the proposed off-equilibrium information processing that guarantees that the required properties are satisfied on the whole $[p_n, p_{n+1}]$. $\square$

A.3 Special procedures

A.3.1 Case 1b. Early entry

Suppose at some p^0 (possibly when the price clock is stopped) some player $j \notin \mathcal{I}(\mathbf{x}(p^0))$ becomes active, intends to be active, or is undecided. Suppose an instantaneous adjustment of $x_j(p^0)$ to $y_j(p^0)$ is made. Typically, set $\mathcal{K} = \mathcal{I}(\mathbf{x}(p^0)) \setminus \mathcal{I}(y_j(p^0), \mathbf{x}_{-j}(p^0))$ of the players, who had the highest value at $\mathbf{x}(p^0)$ but are no longer members of the winners' circle after the adjustment, is not empty. Since (A1)-(A2) are imposed only on the members of the winners' circles, the true signals of players from $\mathcal{K}$ may not be inferred properly thereafter.

The procedure below outlines how $x_j(p)$ can be adjusted gradually. It guarantees that the auction does not end before the full adjustment takes place. This is done by ensuring that $x_i(p)$ decreases for at least some player i during the adjustment.

The whole procedure is based on the following two subprocedures.

Step 1. This is an exceptional case when a full or partial adjustment can be made instantaneously. Suppose for any player $i \in \mathcal{I}(\mathbf{x}(p^0))$, $\frac{\partial V_i(s_j, \mathbf{x}_{-j}(p^0))}{\partial s_j} = 0$ for all $s_j \in [x_j(p^0), t_{ji}]$, that is any increase of j's signal up to t_{ji} has no effect on player i's value. For all $k \notin \mathcal{I}(\mathbf{x}(p^0))$ (including player j) define t_{jk} to be the lowest j's signal at which $V_k(t_{jk}, \mathbf{x}_{-j}(p^0)) = p^0$ or set $t_{jk} = 1$ if $V_k(s_j, \mathbf{x}_{-j}(p^0)) < p^0$ for any s_j. Define $t_j^* \equiv \min_{l \in \mathcal{N}} t_{jl}$. Instantaneously adjust $x_j(p^0)$ to t_j^*. If $t_j^* = t_{jj}$ no additional adjustment is needed, otherwise further steps will follow. If $t_j^* = t_{jk}$ for $k \notin \mathcal{I}(\mathbf{x}(p^0))$, then k becomes a member of the new winners' circle $\mathcal{I}(t_j^*, \mathbf{x}_{-j}(p))$.

Step 2. This is the main subprocedure. It applies whenever there exists $i \in \mathcal{I}(\mathbf{x}(p))$ with $\frac{\partial V_i(s_j, \mathbf{x}_{-j}(p^0))}{\partial s_j} > 0$ for $s_j \in (x_j(p^0), x_j(p^0) + \varepsilon)$ for some $\varepsilon > 0$. It is proposed that players find $\mathbf{x}(p)$ as the solution to the following system with an initial condition $\mathbf{x}(p^0)$

$$\begin{cases} \frac{dx_j}{dp} = g(p) \geq 0 \\ \frac{d\mathbf{x}_{\mathcal{A}}}{dp} = (DV_{\mathcal{A}})^{-1} \cdot \left(\mathbf{1} - g(p)\frac{\partial \mathbf{V}_{\mathcal{A}}}{\partial s_j}\right) \\ \frac{d\mathbf{x}_{-\mathcal{A}\setminus\{j\}}}{dp} = \mathbf{0} \\ \min_{i\in\mathcal{A}} \frac{dx_i}{dp} \leq G < 0 \end{cases} \tag{16}$$

where $\mathcal{A} \subseteq \mathcal{I}(\mathbf{x}(p^0))$ is a set of players, who will be considered active, $j \notin \mathcal{A}$. Normally, $\mathcal{A}$ is the set of currently active players. Section A.3.2 below describes what to do in exceptional situations to find such $\mathcal{A}$.

This system is a modified version of (8), here $x_j(p)$ has growth rate $g(p)$. Any negative number can serve as G, the above system has a unique solution for $g(p)$ if $\min_{i\in\mathcal{A}} \frac{dx_i}{dp}$ is required to be equal to G. For any $i \in \mathcal{A}$ we have

$$\frac{dV_i}{dp} = 1 = (DV_{\mathcal{A}})_i \cdot \frac{d\mathbf{x}_{\mathcal{A}}}{dp} + g(p)\frac{\partial V_i}{\partial s_j}$$

so $g(p)\frac{\partial V_i}{\partial s_j}$ represents the effect of p on V_i through a change in $x_j(p)$. If $\min_{i\in\mathcal{A}} \frac{dx_i}{dp} > G$ then $DV_{\mathcal{A}} \cdot \frac{d\mathbf{x}_{\mathcal{A}}}{dp}$ is limited from below for all the players.

If $\exists i \in \mathcal{I}(\mathbf{x}(p^0))$ with $\frac{\partial V_i(\mathbf{x}(p^0))}{\partial s_j} > 0$, then, in particular, $g(p)\frac{\partial V_i}{\partial s_j}$ can be made arbitrarily large, thus at least for some $k \in \mathcal{A}$ (not necessarily i), $\frac{dx_k}{dp}$ has to be lower than G.

If $\frac{\partial V_i(\mathbf{x}(p^0))}{\partial s_j} = 0$ for all $i \in \mathcal{I}(\mathbf{x}(p^0))$, there exist i, such that $\frac{\partial V_i(s_j,\mathbf{x}_{-j}(p^0))}{\partial s_j} > 0$ at $s_j \in (x_j(p), x_j(p)+\varepsilon)$ for some $\varepsilon > 0$. In this case the solution to (16) still exists, only $g(p) \to \infty$ when $p \searrow p^0$. We have that $\frac{\partial V_i(\mathbf{z}(p))}{\partial s_j} > 0$ for $p > p^0$ for almost all continuous and piecewise smooth paths $\mathbf{z}(p)$ with $\mathbf{z}(p^0) = \mathbf{x}(p^0)$, $z_j'(p) = g(p) \to \infty$ when $p \searrow p^0$ and $z_i'(p) = \frac{dx_i}{dp}$ as in the system (16). So, the requirement $\min_{i\in\mathcal{A}} \frac{dx_i}{dp} \leq G < 0$ can be met by choosing a path $\mathbf{z}(p)$ such that $g(p)\frac{\partial V_i(\mathbf{z}(p))}{\partial s_j} \to g^o > 0$ when $p \searrow p^0$ for at least one player i and large enough g^0.

The resulting $\mathbf{x}(p)$ may not be extendable beyond some $\hat{p}$, with the full adjustment not yet attained at $\hat{p}$. This means that at $\mathbf{x}(\hat{p})$, $\frac{\partial V_i(\mathbf{x}(\hat{p}))}{\partial s_j} = 0$ for all $i \in \mathcal{I}(\mathbf{x}(\hat{p}))$, so another subprocedure will be started at $\hat{p}$, possibly the one of Step 1.

Full procedure can be summarized as follows. Apply an appropriate subprocedure. Repeat if necessary. Stop if player j exits, or if p' is reached such that $w_j(x_j(p'), p') = p'$—the full adjustment takes place. In case some player exits or enters and the clock is not stopped (as in the simplified analysis of Section 3), or a player needs to be added or removed from $\mathcal{A}$ (as in Cases 1a, 2 or 3) at some interim p, $g(p)$ is fixed and the corresponding solution is considered until later p, at which the minimal negative growth of any player from previously considered $\mathcal{A}$ equals $G/2$. If some player (not j) exits or enters at that p, different objective can be further set, such as $G/3$. Once this is finished, the procedure continues for a new $\mathcal{A}$.

Note that the above procedure can be easily extended to adjust the minimal estimated signals for more than one player. These adjustments can be done sequentially or simultaneously.

A.3.2 Case 3. Looking for a partition

In this section we establish that the partition of players on active and inactive, in which no player regrets his status, does exist at any given p^0 and appropriate $\mathbf{x}(p^0)$.

Suppose $\mathbf{x}(p^0)$ is such that for all $i \in \mathcal{I}(\mathbf{x}(p^0))$, $w_i(x_i(p^0), p^0) = p^0$. Denote $A = DV_{\mathcal{I}(\mathbf{x}(p^0))}(\mathbf{x}(p^0))$, $B = A^{-1}$. Since players $j \in \mathcal{N} \setminus \mathcal{I}(\mathbf{x}(p^0))$ are of no particular interest—their $x_j(p)$ are fixed and they will stay inactive at least for some time, we omit them from the consideration to save the notation, as if $\mathcal{N} = \mathcal{I}(\mathbf{x}(p^0))$.

Instead of actual value functions consider the linearized at $\mathbf{x}(p^0)$ system of value functions with $DV_{\mathcal{I}(\mathbf{x}(p^0))}(\mathbf{s}) = A$ for all $\mathbf{s}$.

Lemma 7. *Fix any subset of players $\mathcal{K} \subset \mathcal{I}(\mathbf{x}(p^0))$. For the linearized at $\mathbf{x}(p^0)$ system of value functions there exist a subset $\mathcal{A} \subset \mathcal{I}(\mathbf{x}(p^0))$ of players such that for $p > p^0$ the solution $\mathbf{x}(p)$ to the system (8) for $\mathcal{A}$ has the following properties: $\frac{dx_k}{dp} \leq 0, \forall k \in \mathcal{K} \cap \mathcal{A}$; $\frac{dV_k(\mathbf{x}(p))}{dp} \leq 1, \forall k \in \mathcal{K} \setminus \mathcal{A}$.*

Proof. Let $n = \#\mathcal{I}(\mathbf{x}(p^0))$ and $K = \#\mathcal{K}$. Denote $z_i \equiv \frac{dx_i(p)}{dp}$ and $v_i \equiv \frac{dV_i(p)}{dp}$ for all players. We have that $A_i\mathbf{z} = v_i$ and $z_i = B_i\mathbf{v}$, where A_i and B_i are ith rows of matrices A and B correspondingly. Equations $z_i = 0$ and $v_i = 1$ define hyperplanes in the n-dimensional Euclidean space E^n of vectors $\mathbf{v}$.

Lemma states, equivalently, that there exists a point $T \in E^n$, which is an intersection of exactly n hyperplanes $\mathbf{v}_{\mathcal{A}} = 1$ and $\mathbf{z}_{-\mathcal{A}} = 0$, and which lies below hyperplanes $v_k = 1$ for all $k \in \mathcal{K} \setminus \mathcal{A}$, and $z_k = 0$ for $k \in \mathcal{K} \cap \mathcal{A}$. We will prove that such a point exists.

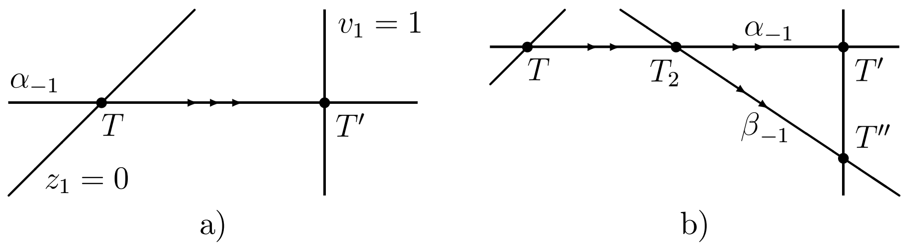


Figure 3: v_1 is decreasing

We restrict our attention to subspace $E_{\mathcal{K}}$ of dimensionality K, which is an intersection of E^n and $n - K$ hyperplanes $v_i = 1$ for all $i \in \mathcal{A} \setminus \mathcal{K}$ (these players are considered active and do not regret it for sure). It is enough to find an appropriate point in this subspace. Suppose that all $2K$ hyperplanes are in general position, that is any point which is an intersection of K of them does not belong to any other hyperplane. Consider an arbitrary T, which is an intersection of K hyperplanes, where exactly one of $z_i = 0$ and $v_i = 1$ is fixed for each $i \in \mathcal{K}$. Call any instance of $z_i(T) > 0$ or $v_i(T) > 1$ a *violation*. Among all possible 2^K points pick the one with the least number of violations. If this number is zero, we have found an appropriate partition.

Suppose the minimal number of violations is higher than zero and is reached at T. Without loss of generality assume that there is a violation for player 1. Consider the case $z_1(T) = 0$ with $v_1(T) > 1$. The other situation can be considered similarly. Fix the line α_{-1} which is formed by the intersection of the fixed at T hyperplanes of the others (see Figure 3a). Let T' be the intersection of α_{-1} with $v_1 = 1$. Starting at T, by moving along α_{-1} in the direction of decreasing v_1, T' is necessarily reached . By Lemma 3, z_1 has to decrease,[12] so $z_1(T') < 0$. Thus, if no additional violations were made in the process, we have managed to reduce the number of violations by one, which contradicts the minimality of this number.

By moving from T to T', the other (not fixed) hyperplane of some other player, say 2, may have been crossed and a new violation occurred. Suppose that at T, $v_2(T) = 0$, that is $v_2 = 1$ was fixed at T for player 2. Again, the case of $z_2(T) = 0$ is completely similar. Let β_{-1} be the line in which $z_2 = 0$

[12] To see that, let $\mathcal{J}$ to include player 1 and all players i, for whom $v_i = 1$ is fixed. Lemma 3 states that once the values of players from $\mathcal{J} \setminus \{1\}$ are fixed and the signals of players not from $\mathcal{J}$ are fixed, the signal and the value of player 1 has to move in the same direction.

is intersected with the fixed hyperplanes for all players from $\mathcal{K} \setminus \{1, 2\}$. Let T_2 be an intersection of β_{-1} and α_{-1}. Start moving from T along α_{-1} by decreasing v_1. If β_{-1} is reached it has to happen at T_2. Starting from T_2 and decreasing v_1 further, by moving along α_{-1} T' is reached, by moving along β_{-1} T'', an intersection of β_{-1} and $v_1 = 1$, is reached (see Figure 3b). As above, at both T' and T'' there is no violation for player 1 and at one of them there is no violation for player 2 as follows from Lemma 3. Indeed, if there is a violation for player 2, say at T', meaning $z_2(T') > 0$, by moving along the line connecting T' and T'' toward decreasing z_2, T'' is reached when $z_2 = 0$, but then v_2 has to decrease as well, so $v_2(T'') < 1$.

The above analysis shows that if, when moving along an appropriate line by decreasing v_1, the other (not fixed) hyperplane is reached for some other player j, there is a direction to continue along which no additional violation for player j is created. In fact, if there were a violation for player j it would be corrected. By continuing is this manner until v_1 reaches 1, subsequently changing directions if necessary for any player j whose second hyperplane is reached, a point T^* with less violations than at T is obtained.

If the position of hyperplanes is not general, that is there are points of intersection of more than K hyperplanes in $E_{\mathcal{K}}$, one can disturb them a little, such that (A2) is still satisfied (to keep positive a finite number of coefficients of the finite number of matrices), find the solution and take the limit. □

If at given p, the position of all relevant hyperplanes is general, then one can start (or continue) the information processing at p with the found $\mathcal{A}$, and the required inequalities for players from $\mathcal{K}$ will also be satisfied in the neighborhood of p by continuity of $DV_{\mathcal{A}}$.

The only problem with the above partition that can arise is that the partition may not be extendable beyond current price. This may happen if at the point in $E_{\mathcal{K}}$ space, found in Lemma 7, both hyperplanes corresponding to the same player intersect for some of the players. Thus, each of such players can be treated either active or inactive. It is possible that no choice of $\mathcal{A}$ at p is dynamically extendable. That is for any $\mathcal{A}$ there will be some player from $\mathcal{K}$, such that either his imputed value increases faster than p if the player is considered inactive or $x_i(p)$ increases if he is considered active. It should be noted that this situation is extremely unlikely, not only several players need to have some specific signals (the probability of this alone is zero), but some special value functions that have these problems at these specific signals are needed as well (some specific conditions on second order derivatives have to be satisfied). Therefore, it is the case of mere theoretical possibility.

If such a case arises the proposed solution is to take all the problematic players and instead of looking at the hyperplanes $z_i = 0$ consider $z_i = \varepsilon_i < 0$ for all such i. That is, starting at p, $x_i(p)$ will be lowered intentionally. The choice of ε_i has to insure that in the new problem the position of hyperplanes is general. A subset $\mathcal{A}$ found in Lemma 7 will be used to calculate $\mathbf{x}(p)$ until price reaches $p+\varepsilon$, for some small $\varepsilon > 0$. Starting at $p+\varepsilon$, or at the next stop of the price clock if it happens before, revert to the normal play and by using a procedure of the Case 1b, return an estimate of the minimal possible signal of any affected (who was considered inactive) player i to the level $x_i(p)$.

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