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A constructive proof of the ordinal efficiency welfare theorem

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Abstract

We provide a non-geometric constructive short proof of the ordinal efficiency welfare theorem.

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1. Introduction

Bogomolnaia and Moulin [2] consider the problem of allocating n indivisible objects to n agents, with each agent entitled to receiving one object. [1] The allocation is based on the agents' ordinal preferences over objects, and fairness motivates a random allocation. Bogomolnaia and Moulin define a random allocation to be ordinally efficient if it is not first-order stochastically dominated for all agents by any other random allocation. In the context of allocation mechanisms based solely on ordinal preferences, ordinal efficiency is perhaps the most compelling efficiency notion. There is a growing literature [1,4,5,7–9] studying ordinal efficiency.

Bogomolnaia and Moulin note that any (expected) utilitarian welfare maximizing random allocation for some profile of von Neumann–Morgenstern utilities compatible with the ordinal preferences is ordinally efficient. McLennan [9] allows for indifferences in the preference profile, and proves that for any ordinally efficient random allocation there exists a profile of von Neumann–Morgenstern utilities that is consistent with the preferences and with respect to which the allocation

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[1] This problem, also known as the house allocation problem, has been introduced by Hylland and Zeckhauser [3], and is closely related to the Shapley and Scarf [11] housing market.

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is a utilitarian welfare maximizing outcome. The proof applies a new version of the separating hyperplane theorem in which the sets to be separated are polyhedra. We provide a non-geometric constructive short proof of McLennan's ordinal efficiency welfare theorem.

Our proof is based on the Bogomolnaia and Moulin characterization of ordinal efficiency, extended by Katta and Sethuraman [4] to the case of weak preferences, in terms of the acyclicity of the binary relation defined as follows. For a preference profile and a random allocation, one object dominates another object according to the binary relation if there exists one agent who prefers the first object to the second, and receives the second object with positive probability. [2] The intuition is that the respective agent would be willing to move probability weight away from the less preferred object to the more preferred one, and a cycle in the binary relation would enable all corresponding agents to trade probability weights so that all of them are made better off in the sense of first-order stochastic dominance. For any ordinally efficient random allocation, the acyclicity of the binary relation permits the construction of a weak utility representation, which can be extended for each agent to a utility function consistent with the ordinal preferences, such that the allocation is a utilitarian welfare maximizing outcome for the resulting utility profile.

2. Framework

We consider a framework with a set N of n **agents** and a set O of n **objects**, in which each agent is entitled to exactly one object. An **allocation** is a one-to-one function $\alpha : N \to O$. Each allocation α is associated with a **permutation matrix** (a matrix with entries in $\{0, 1\}$, with each row and each column containing exactly one 1), $\Pi^\alpha = (\pi^\alpha_{io})_{(i,o)\in N\times O}$, with $\pi^\alpha_{io} = 1$ if $\alpha(i) = o$, and $\pi^\alpha_{io} = 0$ otherwise. An **allocation lottery** w is a probability distribution over the set of allocations. We associate to each allocation lottery w a **random allocation** $\Pi = (\pi_{io})_{(i,o)\in N\times O}$, which is a **bistochastic** $n \times n$ **matrix** (a matrix with non-negative entries, with each row and column summing to 1), $\Pi = \sum_\alpha w(\alpha)\Pi^\alpha$; π_{io} is agent i's probability of receiving object o. By the Birkhoff–von Neumann theorem [10], any bistochastic matrix can be written (not necessarily uniquely) as a convex combination of permutation matrices, thus the set of random allocations is identical to the set of $n \times n$ bistochastic matrices (when each agent's preferences depend only on his own reduced lottery over objects).

Each agent $i \in N$ has a **preference relation** $\succeq_i$ over O. We assume that for each agent there are no externalities from the object assignment for the other agents. Let $\succ_i$ and $\sim_i$ be the strict preference, and, respectively, the indifference relations, induced by $\succeq_i$. We assume that agents are expected utility maximizers. A von Neumann–Morgenstern utility function $u_i : O \to \mathbb{R}$ is **consistent** with (or **represents**) $\succeq_i$ if

$$u_i(o) \geqslant u_i(o') \quad \Leftrightarrow \quad o \succeq_i o', \quad \forall o, o' \in O.$$

Let $\succeq = (\succeq_i)_{i\in N}$ and $u = (u_i)_{i\in N}$ denote the preference and utility profiles; u is consistent with $\succeq$ if u_i is consistent with $\succeq_i$ for each agent i. A random allocation Π is **ex-ante utilitarian welfare maximizing** for u if it maximizes the **social welfare function**

$$\sum_{i\in N} \sum_{o\in O} \pi_{io} u_i(o).$$

[2] The definition, taking into account possible indifferences, is made precise in the next section.

A random allocation $\Pi = (\pi_{io})_{(i,o)\in N\times O}$ **ordinally dominates** another random allocation $\Pi' = (\pi'_{io})_{(i,o)\in N\times O}$ **at** $\succeq$ if for each agent i the lottery π_i first-order stochastically dominates the lottery π'_i,

$$\sum_{o':o'\succeq_i o} \pi_{io'} \geqslant \sum_{o':o'\succeq_i o} \pi'_{io'}, \quad \forall o \in O$$

with strict inequality for some i, o. The random allocation Π is **ordinally efficient at** $\succeq$ if it is not ordinally dominated at $\succeq$ by any other random allocation. If Π ordinally dominates Π' at $\succeq$ then every agent, irrespective of his von Neumann–Moregenstern utility index consistent with $\succeq$, prefers Π to Π'.

Extending Bogomolnaia and Moulin's acyclicity characterization of ordinal efficiency to the case of weak preferences, Katta and Sethuraman show that the **strict and weak domination via probability trade** binary relations $\rhd(\Pi,\succeq)$ and, respectively, $\bowtie(\Pi,\succeq)$ on O,

$$\begin{aligned} o\rhd(\Pi,\succeq)o' &\Leftrightarrow \exists i \in N, o \succ_i o' \,\&\, \pi_{io'} > 0,\\ o\bowtie(\Pi,\succeq)o' &\Leftrightarrow o\not\rhd(\Pi,\succeq)o' \text{ and } \exists i \in N, o \sim_i o' \,\&\, \pi_{io'} > 0 \end{aligned}$$

may be used to test Π's ordinal efficiency. The expression $o\rhd(\Pi,\succeq)o'$ ($o\bowtie(\Pi,\succeq)o'$) is read "object o strictly (weakly) $\succeq$-dominates object o' via some probability trade at Π." Intuitively, object o strictly $\succeq$-dominates object o' via some probability trade at Π if one of the agents strictly prefers (according to his corresponding preference in the profile $\succeq$) to trade probability weight away from o' to o in Π; a similar interpretation holds for weak domination via probability trade. The definitions of the domination via probability trade relations immediately imply the following remark.

Remark 1. Suppose that the random allocation Π is ordinally efficient at $\succeq$. For each $o, o' \in O$ if $o\rhd(\Pi,\succeq)o'$ or $o\bowtie(\Pi,\succeq)o'$, then $o'\not\rhd(\Pi,\succeq)o$.

We define the **comprehensive domination via probability trade** binary relation $\trianglerighteq(\Pi,\succeq)$ on O by the union[3] of $\rhd(\Pi,\succeq)$ and $\bowtie(\Pi,\succeq)$. When we write $\trianglerighteq(\Pi,\succeq)$, we refer to the $\rhd(\Pi,\succeq)$ $\cup \bowtie(\Pi,\succeq)$ decomposition. By definition, $(\rhd(\Pi,\succeq),\bowtie(\Pi,\succeq))$ is a partition of $\trianglerighteq(\Pi,\succeq)$, but it is not necessarily an asymmetric–symmetric decomposition of $\trianglerighteq(\Pi,\succeq)$. Note that when Π is ordinally efficient $\rhd(\Pi,\succeq)$ is asymmetric (Remark 1), but $\bowtie(\Pi,\succeq)$ is not necessarily symmetric, and $\trianglerighteq(\Pi,\succeq)$ is not necessarily complete.

Example 1. Let $n = 3$, $N = \{1,2,3\}$, $O = \{o_1, o_2, o_3\}$, and preferences be given by $o_1 \succ_1 o_2 \succ_1 o_3$, $o_1 \succ_2 o_2 \succ_2 o_3$, and $o_3 \sim_3 o_1 \succ_3 o_2$. For Π specified by $\pi_{io_i} = 1$ for all $i \in N$, we obtain $\rhd(\Pi,\succeq) = \{(o_1,o_2)\}$ and $\bowtie(\Pi,\succeq) = \{(o_1,o_1),(o_2,o_2),(o_3,o_3),(o_1,o_3)\}$.

We say that $\trianglerighteq(\Pi,\succeq)$ is **strictly acyclic** if there exists no sequence $o_1, o_2, \ldots, o_k$ in O s.t. $o_1\trianglerighteq(\Pi,\succeq)o_2\trianglerighteq(\Pi,\succeq)\cdots\trianglerighteq(\Pi,\succeq)o_k\rhd(\Pi,\succeq)o_1$. In Example 1, Π is ordinally efficient and $\trianglerighteq(\Pi,\succeq)$ is strictly acyclic. The following generalization of Remark 1 holds.

Proposition 1 (*Katta and Sethuraman [4, Lemma 2]*). *The random allocation Π is ordinally efficient at $\succeq$ if and only if the relation $\trianglerighteq(\Pi,\succeq)$ is strictly acyclic.*

[3] Binary relations on O can be regarded as subsets of $O \times O$ [6].

One key step in our proof uses the following choice theory result. Let $\rhd$ and $\bowtie$ be two disjoint binary relations on O, and write $\unrhd = \rhd \cup \bowtie$; as above, $\unrhd$ is **strictly acyclic** if there exists no sequence $o_1, o_2, \ldots, o_k$ in O s.t. $o_1 \unrhd o_2 \unrhd \cdots \unrhd o_k \rhd o_1$. A function $v : O \to \mathbb{R}$ is a **weak representation** of $\unrhd$ if

$$o \rhd o' \Rightarrow v(o) \geqslant v(o') + 1, \quad \forall o, o' \in O, \tag{2.1}$$

$$o \bowtie o' \Rightarrow v(o) \geqslant v(o'), \quad \forall o, o' \in O. \tag{2.2}$$

Proposition 2. *Any strictly acyclic $\unrhd = \rhd \cup \bowtie$ admits a weak representation.*

Proof. Define the binary relation $\triangleq$ on O by

$$o \triangleq o' \quad \Leftrightarrow \quad \exists o_1 \unrhd o_2 \unrhd \cdots \unrhd o_k \unrhd o_1 \quad \text{with } o, o' \in \{o_l | l = \overline{1,k}\}$$

(the sequence $o_1, o_2, \ldots, o_k$ may have repeated terms). Obviously, $\triangleq$ is an equivalence relation on O. Denote by $O/\triangleq$ the set of equivalence classes of $\triangleq$, and by $[o]$ the $\triangleq$ equivalence class of o. Define $\gg$ on $O/\triangleq$ by

$$[o_1] \gg [o_2] \quad \Leftrightarrow \quad [o_1] \neq [o_2] \quad \text{and} \quad \exists o_1' \in [o_1], \ o_2' \in [o_2], \ o_1' \unrhd o_2'.$$

The relation $\gg$ has no cycles. Indeed, assume $[o_1] \gg [o_2] \gg \ldots \gg [o_k] \gg [o_1]$. Then there exist $o_l' \in [o_l]$ for $l = \overline{1,k}$ s.t. $o_1' \unrhd o_2' \unrhd \cdots \unrhd o_k' \unrhd o_1'$. Therefore, all $(o_l')_{l=\overline{1,k}}$ are in the same equivalence class of $\triangleq$, or $[o_1] = [o_2] = \cdots = [o_k]$.

For object o, we define $v(o)$ as the length k of the longest chain of $\gg$ starting at $[o]$, $[o] = [o_1] \gg [o_2] \gg \cdots \gg [o_k]$. Since $\gg$ has no cycles, any such chain is finite ($k \leqslant n$), so v is well defined. Obviously, v satisfies 2.1 and 2.2. If $o \rhd o'$, then $\unrhd$'s **strict acyclicity** and $\triangleq$'s definition imply that $[o] \neq [o']$ and therefore $[o] \gg [o']$; hence, any chain of $\gg$ starting at $[o']$ may be made into a longer chain starting at $[o]$ by appending $[o]$, so $v(o) \geqslant v(o') + 1$. If $o \bowtie o'$ and $[o] = [o']$ then $v(o) = v(o')$. If $o \bowtie o'$ and $[o] \neq [o']$ then, again $[o] \gg [o']$, and any chain of $\gg$ starting at $[o']$ may be made into a longer chain starting at $[o]$ by appending $[o]$, so $v(o) \geqslant v(o') + 1$. $\square$

3. The constructive proof

We provide a constructive proof of McLennan's ordinal efficiency welfare theorem.

Theorem 1 (*McLennan [9, Theorem 1]*). *Any random allocation that is ordinally efficient at $\succeq$ is ex-ante utilitarian welfare maximizing for some utility profile u consistent with $\succeq$.*

Let $\Pi = (\pi_{io})_{(i,o) \in N \times O}$ be an ordinally efficient random allocation at $\succeq$. Without risk of confusion, we write $\rhd$, $\bowtie$ and $\unrhd$ for $\rhd(\Pi, \succeq)$, $\bowtie(\Pi, \succeq)$ and $\unrhd(\Pi, \succeq)$, respectively, hereafter. By Proposition 1, $\unrhd$ is strictly acyclic, and hence by Proposition 2, $\unrhd$ admits a weak representation $v : O \to \mathbb{R}$. The main idea of the proof is that, for each agent i, v represents $\succeq_i$'s restriction to the set of objects o with $\pi_{io} > 0$, and if we set $u_i(o) = v(o)$ for $\pi_{io} > 0$, the definitions of $\unrhd$ and v imply that u_i can be extended to represent $\succeq_i$ over all O in such a way that $u_i(o) \leqslant v(o)$, $\forall o \in O$; when $\pi_{io} = 0$ we set $u_i(o)$ close to $u_i(o'')$ where o'' is one of i's most preferred objects that are not preferred to o satisfying $\pi_{io''} > 0$. Since for each object o, $u_i(o) \leqslant v(o)$ for all agents i, the overall contribution of o to the social welfare for any random allocation Π' cannot be larger than $v(o)$, and equals $v(o)$ for Π (as $u_i(o) = v(o)$ when $\pi_{io} > 0$); hence Π is utilitarian welfare maximizing for u.

Proof. Let Π, $\rhd$, $\bowtie$, $\unrhd$, and v be as specified in the preceding paragraph. The proof proceeds in 3 steps.

Step 1: For all agents i we define

$$u_i(o) = v(o) \quad \text{if } \pi_{io} > 0. \tag{3.1}$$

Then u_i represents $\succeq_i$'s restriction to the set of objects o with $\pi_{io} > 0$. Indeed, if $o \succ_i o'$ and $\pi_{io} > 0$, $\pi_{io'} > 0$, then $o \rhd o'$, so $u_i(o) = v(o) > v(o') = u_i(o')$. If $o \sim_i o'$ and $\pi_{io} > 0$, $\pi_{io'} > 0$, then $o \unrhd o'$ and $o' \unrhd o$, hence $v(o) = v(o')$ and $u_i(o) = v(o) = v(o') = u_i(o')$.

We extend u_i to the set of all objects, so that u_i is consistent with $\succeq_i$, and

$$u_i(o) < \min_{o' \in O} v(o') \quad \text{if } \pi_{io} = 0 \text{ and } \{o' | o \succeq_i o' \& \pi_{io'} > 0\} = \emptyset, \tag{3.2}$$

$$u_i(o) < \max_{\{o' | o \succeq_i o' \& \pi_{io'} > 0\}} v(o') + 1 \quad \text{if } \pi_{io} = 0 \text{ and } \{o' | o \succeq_i o' \& \pi_{io'} > 0\} \neq \emptyset. \tag{3.3}$$

Such a representation of $\succeq_i$, satisfying (3.1)–(3.3), obviously exists. We argue that Π is an ex-ante utilitarian welfare optimum for the vector of utility indices $(u_i)_{i \in N}$.

Step 2: We show that

$$u_i(o) \leqslant v(o), \quad \forall (i, o) \in N \times O. \tag{3.4}$$

For $\pi_{io} > 0$, or $\pi_{io} = 0$ and $\{o' | o \succeq_i o' \& \pi_{io'} > 0\} = \emptyset$, (3.4) follows immediately.

For $\pi_{io} = 0$ and $\{o' | o \succeq_i o' \& \pi_{io'} > 0\} \neq \emptyset$, if o'' is one maximizer of v over $\{o' | o \succeq_i o' \& \pi_{io'} > 0\}$, we have that $u_i(o) < v(o'') + 1$, and $o \succeq_i o''$, $\pi_{io''} > 0$, so $o \unrhd o''$ and $u_i(o'') = v(o'') \leqslant v(o)$ (since v is a weak representation of $\unrhd$). There are two possible cases. If $o \succ_i o''$ then $o \rhd o''$, so $u_i(o) < v(o'') + 1 \leqslant v(o)$ (since v is a weak representation of $\unrhd$). If $o \sim_i o''$ then $u_i(o) = u_i(o'') = v(o'') \leqslant v(o)$.

Step 3: For any random allocation $\Pi' = (\pi'_{io})_{(i,o) \in N \times O}$, using (3.4),

$$\sum_{i \in N} \sum_{o \in O} \pi'_{io} u_i(o) \leqslant \sum_{i \in N} \sum_{o \in O} \pi'_{io} v(o) = \sum_{o \in O} v(o) \sum_{i \in N} \pi'_{io} = \sum_{o \in O} v(o)$$

with equality if and only if $\pi'_{io}(v(o) - u_i(o)) = 0$ for all i, o. In particular, $\pi_{io}(v(o) - u_i(o)) = 0$ by (3.1), hence Π is an ex-ante utilitarian welfare maximizing random allocation for $(u_i)_{i \in N}$. □

Remark 2. Π is not necessarily the unique ex-ante utilitarian welfare optimum of the utility profile u constructed in the proof. There exist ordinally efficient random allocations Π such that no utility profile u consistent with $\succeq$ yields Π as the unique welfare maximizer. Consider an example with $n = 2$, $o_1 \succ_1 o_2$, $o_1 \succ_2 o_2$. All the random allocations

$$\Pi(x) = \begin{pmatrix} x & 1 - x \\ 1 - x & x \end{pmatrix}$$

($x \in [0, 1]$) are ordinally efficient. Any utility profile that is consistent with $\succeq$ and yields $\Pi(1/2)$ welfare optimal needs to satisfy $u_1(o_1) - u_1(o_2) = u_2(o_1) - u_2(o_2)$, and in that case any other $\Pi(x)$, $x \in [0, 1]$, yields the same welfare, and is therefore utilitarian welfare maximizing.

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