Business Cycle Anatomy

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NBER Summer Institute: July 10, 2019
Motivation and Contribution

“One is led by the facts to conclude that, with respect to the qualitative behavior of co-movements among series, business cycles are all alike. To theoretically inclined economists, this conclusion should be attractive and challenging, for it suggests the possibility of a unified explanation of business cycles.” (Lucas 1977)

- **A theorist's ambition:** account for bulk of the business cycle with a single-shock model i.e., multiple triggers but a dominant propagation mechanism
“One is led by the facts to conclude that, with respect to the qualitative behavior of co-movements among series, business cycles are all alike. To theoretically inclined economists, this conclusion should be attractive and challenging, for it suggests the possibility of a unified explanation of business cycles.” (Lucas 1977)

- **A theorist's ambition:** account for bulk of the business cycle with a single-shock model i.e., multiple triggers but a dominant propagation mechanism

- **This paper’s contribution:** provide an empirical template of it
What We Do

- Estimate a VAR (or VECM) on a few key variables
- Recover shock that has max contribution to volatility of $U$ over BC frequencies
- Repeat exercise by targeting other variables (e.g., TFP) or other frequencies (e.g., LR)
What We Do

• Estimate a VAR (or VECM) on a few key variables

• Recover shock that has max contribution to volatility of $U$ over BC frequencies

• Repeat exercise by targeting other variables (e.g., TFP) or other frequencies (e.g., LR)

⇒ "Business Cycle Anatomy" = large collection of one-dimensional cuts of the data

= rich set of restrictions on models of any size and type
Main Findings and their Use

- Establish existence of a "main business cycle (MBC) shock"
  - shocks that target $u$, $Y$, $h$, $I$, and $C$ over BC frequencies produce similar IRFs
  - supports hypothesis of common propagation mechanism

- Document its properties
  - transitory
  - disconnected from TFP at all horizons
  - orthogonal to shock that targets inflation
  - ...

- Use its properties and overall anatomy to guide theory
  - parsimonious, semi-structural perspective
  - fully structural DSGE models
Lessons for Theory

• Good news for parsimonious theories with a dominant shock/propagation mechanism

• Bad news for the following candidates
  
  • technology shocks
    RBC model
  
  • financial, uncertainty, or other shocks that map to TFP fluctuations
    Benhabib and Farmer (1992), Bloom et al (2016)
  
  • news about future TFP
  
  • inflationary demand shocks of the textbook type
  
  • propagation mechanisms in state-of-the-art DSGE models
    Smets & Wouters, Justiniano, Primiceri & Tambalott, Christiano, Motto & Rostagno
Lessons for Theory

- What fits the MBC template best?

- Non-inflationary, non-specialized, demand shocks

- Perhaps they exist (even) outside realm of sticky prices and Philips curves?
  
  example: our earlier Ecma paper (ACD 2018)
  
Outline

• Empirical Analysis

• Main Findings and Lessons

• Application to Three DSGE Models
Empirical Analysis
Baseline VAR

- Quarterly U.S data: 1955Q1-2017Q4
  - **Macro Quantities**: Unemployment, GDP, Hours, Invest. (inclusive of durables), Cons.
  - **Productivity**: util-adjust TFP, NFB labor productivity;
  - **Nominal**: Inflation (GDP Delator), Federal Fund Rate, Labor Share

- Bayesian VAR, 2 Lags (robust to 4 or 6 lags and VECM)
Baseline VAR

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- Bayesian VAR, 2 Lags (robust to 4 or 6 lags and VECM)

- **What next? Construct the “shock to variable X”**
  Linear combination of the VAR residuals that has the maximal contribution to the volatility of a variable X at the business-cycle frequencies, 6-32 quarters.

› Technicalities
Main Business Cycle Shock: Targeting Unemployment

Impulse Response Functions

Variance Contributions, Business-Cycle Frequencies

<table>
<thead>
<tr>
<th></th>
<th>u</th>
<th>Y</th>
<th>h</th>
<th>l</th>
<th>C</th>
<th>TFP</th>
<th>Y/h</th>
<th>Wh/Y</th>
<th>π</th>
<th>R</th>
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<tr>
<td></td>
<td>73.71</td>
<td>58.51</td>
<td>47.72</td>
<td>62.09</td>
<td>20.38</td>
<td>5.86</td>
<td>23.91</td>
<td>27.02</td>
<td>6.96</td>
<td>22.27</td>
</tr>
</tbody>
</table>
Main Business Cycle Shock: Alternative Targets

Interchangeable facets of the same shock!

Unemployment; Output; Hours Worked; Investment; Consumption; TFP; Labor Productivity; Labor Share; Inflation Rate; Nom. Int. Rate

\( u \) shock; \( Y \) shock; \( I \) shock; \( h \) shock; \( C \) shock; Shaded area: 68% HPDI.
## Main Business Cycle Shock: Alternative Targets

<table>
<thead>
<tr>
<th></th>
<th>$u$</th>
<th>$Y$</th>
<th>$h$</th>
<th>$l$</th>
<th>$C$</th>
<th>TFP</th>
<th>$Y/h$</th>
<th>$Wh/Y$</th>
<th>$\pi$</th>
<th>$R$</th>
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<tr>
<td>$u$</td>
<td>73.71</td>
<td>58.51</td>
<td>47.72</td>
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<td>23.91</td>
<td>27.02</td>
<td>6.96</td>
<td>22.27</td>
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<tr>
<td>$Y$</td>
<td>56.24</td>
<td>80.13</td>
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<td>70.45</td>
<td>47.99</td>
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<td>11.62</td>
<td>22.61</td>
<td>19.47</td>
<td>7.23</td>
<td>22.38</td>
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<tr>
<td>$l$</td>
<td>59.03</td>
<td>66.60</td>
<td>45.20</td>
<td>80.29</td>
<td>19.01</td>
<td>3.81</td>
<td>33.74</td>
<td>36.44</td>
<td>7.69</td>
<td>21.51</td>
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<tr>
<td>$C$</td>
<td>19.19</td>
<td>31.59</td>
<td>20.15</td>
<td>17.10</td>
<td>68.30</td>
<td>1.57</td>
<td>12.93</td>
<td>10.31</td>
<td>9.93</td>
<td>4.50</td>
</tr>
</tbody>
</table>
The Main Business Cycle Shock: Alternative Targets

![Graphs of Y Shock, I Shock, and h Shock]

- **Y Shock**
- **I Shock**
- **h Shock**
First Principal Component, Business Cycle Frequencies

<table>
<thead>
<tr>
<th></th>
<th>$u$</th>
<th>$Y$</th>
<th>$h$</th>
<th>$l$</th>
<th>$C$</th>
<th>TFP</th>
<th>$Y/h$</th>
<th>$wh/Y$</th>
<th>$\pi$</th>
<th>$R$</th>
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<td>Raw Data</td>
<td>75.33</td>
<td>92.26</td>
<td>81.24</td>
<td>99.80</td>
<td>60.19</td>
<td>6.10</td>
<td>17.73</td>
<td>3.02</td>
<td>2.33</td>
<td>12.27</td>
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<tr>
<td>VAR-Based</td>
<td>63.31</td>
<td>87.33</td>
<td>62.47</td>
<td>99.72</td>
<td>26.67</td>
<td>1.22</td>
<td>29.19</td>
<td>14.16</td>
<td>0.68</td>
<td>8.10</td>
</tr>
</tbody>
</table>

- Similar message about variance contributions: $\text{MBC} \approx \text{1st PC}$
- But our approach adds info about (i) IRFs and (ii) footprint on other frequencies
The Main Long-Run Shock

<table>
<thead>
<tr>
<th>Target</th>
<th>Y</th>
<th>l</th>
<th>C</th>
<th>TFP</th>
<th>Y / h</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>99.59</td>
<td>95.94</td>
<td>99.47</td>
<td>95.66</td>
<td>97.82</td>
</tr>
<tr>
<td>l</td>
<td>96.88</td>
<td>97.83</td>
<td>96.41</td>
<td>91.62</td>
<td>93.02</td>
</tr>
<tr>
<td>C</td>
<td>99.34</td>
<td>95.63</td>
<td>99.53</td>
<td>95.39</td>
<td>97.59</td>
</tr>
<tr>
<td>TFP</td>
<td>97.39</td>
<td>92.55</td>
<td>97.40</td>
<td>98.43</td>
<td>98.46</td>
</tr>
<tr>
<td>Y / h</td>
<td>98.52</td>
<td>93.36</td>
<td>98.67</td>
<td>97.70</td>
<td>99.25</td>
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</table>
Disconnect Between the Short Run and the Long Run

<table>
<thead>
<tr>
<th></th>
<th>u</th>
<th>Y</th>
<th>h</th>
<th>l</th>
<th>C</th>
<th>TFP</th>
<th>Y/h</th>
</tr>
</thead>
<tbody>
<tr>
<td>MBC shock → Long Run</td>
<td>20.83</td>
<td>4.64</td>
<td>5.45</td>
<td>5.16</td>
<td>4.13</td>
<td>4.09</td>
<td>3.88</td>
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<tr>
<td>LR TFP shock → Short Run</td>
<td>9.63</td>
<td>24.78</td>
<td>11.01</td>
<td>17.56</td>
<td>15.58</td>
<td>22.01</td>
<td>21.89</td>
</tr>
</tbody>
</table>

MBC shock → TFP at different horizons
MBC Shock: Main Properties and Prelim Lessons

- Explains bulk of BC volatility in key quantities
- Realistic business cycle, with $u$, $h$, $Y$, $I$, $C$ moving in tandem
- Interchangeability: same IRFs regardless of target
  - support for parsimonious theories
- $\approx 0$ comovement with TFP at BC frequencies
  - rules out technology and financial, uncertainty or other shocks that map to TFP fluctuations
- $\approx 0$ footprint on the Long Run (and conversely LR has small footprint on BC)
  - hard to reconcile with Beaudry & Portier (2006)
- Disconnect from inflation (coming soon)
More on News Shocks: a Semi-structural Exercise

- Could it be that disconnect between SR and LR reflects offsetting effects of
  (i) expansionary news shocks and (ii) contractionary unanticipated shocks?
- Semi-structural exercise using our anatomy:
  - recover these two shocks from reduced-form shocks that drive TFP in SR and LR
- Explore sensitivity to VAR size
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  - recover these two shocks from reduced-form shocks that drive TFP in SR and LR
- Explore sensitivity to VAR size

Variance Contribution of News Shock to Unemployment

\[ \text{VAR}_1 = \{u, \text{TFP}\}, \text{VAR}_2 = \text{VAR}_1 \cup \{I\}, \text{VAR}_3 = \text{VAR}_2 \cup \{Y, C, h\}, \text{VAR}_4 = \text{Baseline VAR}, \text{VAR}_5 = \text{VAR}_4 \cup \{\text{SP500}\}, \text{VAR}_6 = \text{VAR}_5 \cup \{\text{utilization}\}, \text{VAR}_7 = \text{VAR}_6 \cup \{\text{credit spread}\}. \]
Robust to

- More lags, VECM
- Varying the sample: Post vs Pre-Volcker era, w/o Great Recession/ZLB ...
- Adding variables: $SP$, $P^i/P^c$, financial variables ...
- ...
- Shifting to time domain rather than frequency domain
## MBC Shock: Robustness

### Short-Run Variance Contributions

<table>
<thead>
<tr>
<th></th>
<th>u</th>
<th>Y</th>
<th>h</th>
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<tr>
<td>[1] Benchmark</td>
<td>73.71</td>
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<td>47.72</td>
<td>62.09</td>
<td>20.38</td>
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<tr>
<td>[4] VECM(2)</td>
<td>64.85</td>
<td>54.99</td>
<td>48.82</td>
<td>53.78</td>
<td>44.93</td>
<td>12.17</td>
<td>19.51</td>
<td>29.71</td>
<td>11.29</td>
<td>19.51</td>
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<tr>
<td>[7] pre-Volcker</td>
<td>74.23</td>
<td>56.75</td>
<td>43.21</td>
<td>61.50</td>
<td>23.43</td>
<td>6.82</td>
<td>30.69</td>
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<td>27.60</td>
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<td>[8] post-Volcker</td>
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<td>50.37</td>
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<td>23.01</td>
<td>4.65</td>
<td>15.05</td>
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<td>[10] Financial</td>
<td>68.57</td>
<td>57.56</td>
<td>46.84</td>
<td>59.95</td>
<td>25.94</td>
<td>7.04</td>
<td>27.20</td>
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<tr>
<td>[11] Chained-Type C&amp;I</td>
<td>81.41</td>
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## MBC Shock: Robustness

### Long-Run Variance Contributions

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MBC Shock: Robustness

Robustness of IRFs

Baseline; --- VECM$_1$; 1960-2007; --- Extended; Financial
MBC as a Demand Shock along a Philips curve?

Challenge #1: tiny signal-to-noise ratio (negligible $R^2$)

<table>
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<tr>
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<th>$u$</th>
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Challenge #2: magnitude

![Graph showing actual and predicted inflation responses]

Actual inflation response; Predicted, textbook NKPC.
Bottom Line So Far

- Supports parsimonious models with dominant shock/propagation mechanism
- Rules out following candidates for that role
  - technology shocks
  - financial, uncertainty, or other shocks that map to TFP fluctuations
  - news about future TFP
  - inflationary demand shocks of textbook variety
- Remaining possibilities
  - demand shocks of DSGE variety (extremely flat Philips curve)
  - demand shocks without sticky prices/Philips curves
  - ...
Evaluating DSGE Models
Evaluating Two DSGE Models

- **JPT** (Justiniano, Primiceri & Tambalotti, 2010)
  - Same as CEE, SW (but estimation more suitable for our purposes)
  - Sticky prices, Sticky wages, Monetary Policy
  - Standard Bells and Whistles (Habit, Invt Adj Costs, Utilization)
  - Multiple shocks (but 1 shock is most important)

- **ACD** (Angeletos, Collard & Dellas, 2018)
  - RBC with variation in “confidence”
  - Waves of optimism and pessimism about SR economic outlook
  - Example of literature on demand shocks without sticky prices/Phillips curves

Q: Do these models match MBC template from the data?
A: Only second meets interchangeability property
Evaluating Two DSGE Models

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- Q: Do these models match MBC template form the data?
  - A: Only second meets interchangeability property
MBC facets interchangeable in ACD model (as in data), less so in JPT

\[ \implies \text{JPT/CEE/SW lacks the “right” propagation mechanism} \]
JPT and ACD: Interchangeability of MBC Facets

- Measure of Interchangeability: \( D_v = \frac{1}{4} \sum_{f \in F} \sqrt{\sum_{k=0}^{20} (Z_{v,k}^f - \bar{Z}_{v,k})^2} \)

- Smaller numbers mean more interchangeability

<table>
<thead>
<tr>
<th></th>
<th>Y</th>
<th>C</th>
<th>I</th>
<th>h</th>
<th>Average</th>
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<td>0.28</td>
<td>0.64</td>
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<td>JPT</td>
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<td>3.19</td>
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<td>ACD</td>
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<td>0.56</td>
<td>1.56</td>
<td>0.22</td>
<td>0.75</td>
</tr>
</tbody>
</table>

- Ranking robust to re-estimating both models on the basis of our factors
## Contribution of Theoretical Shocks to Factors

<table>
<thead>
<tr>
<th>Factor</th>
<th>JPT</th>
<th>ACD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>31%</td>
<td>88%</td>
</tr>
<tr>
<td>I</td>
<td>0%</td>
<td>80%</td>
</tr>
<tr>
<td>C</td>
<td>33%</td>
<td>93%</td>
</tr>
<tr>
<td>h</td>
<td>0%</td>
<td>99%</td>
</tr>
</tbody>
</table>

In JPT, “A shock” a permanent technology shock, “I shock” a transitory investment-specific demand shock, “C shock” a transitory discount-factor; “other” include monetary policy, price, wage markup shocks. In ACD, “beliefs” a transitory shock to higher-order beliefs; “other” include both transitory and permanent technology shocks, news shocks, and I and C shocks of JPT.

- JPT and many other DSGE models: specialized shocks ⇒ poor interchangeability
- ACD: “shotgun” shock ⇒ great interchangeability
JPT and ACD: Theoretical Shocks vs MBC in Data

JPT: $A, I, \text{ and } C$ shocks

\[ \text{JPT (and many other models): No individual shock resembles the MBC shock in the data; } \]
ACD: the confidence shock does

- needless to say, this doesn’t mean that ours is the “right” model
- but illustrates what the current paradigm misses and what it takes to match MBC template
(a) Data (1985-2011)
\[ Y \text{ factor; } h \text{ factor; } I \text{ factor; } C \text{ factor.} \]

- Interchangeability: great in terms of $Y$, $h$, $I$, worse in terms of $C$
- Real-financial nexus: misses dynamics of credit spread and credit level
• Simple and flexible method for dissecting the macroeconomic dynamics
• Supports hypothesis of dominant propagation mechanism
• Provides an empirical template for it ⇒ looks like a non-inflationary AD shock
• Detects defects in propagation dynamics of DSGE models fitted to the data
• Perhaps resolution rests on accommodating demand-driven cycles even without sticky prices
### Business-Cycle Moments

<table>
<thead>
<tr>
<th>Measure</th>
<th>Data</th>
<th>Experiment 1</th>
<th>Experiment 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{st.dev}(y_t) )</td>
<td>1.41</td>
<td>1.39</td>
<td>1.01</td>
</tr>
<tr>
<td>( \text{st.dev}(\pi_t) )</td>
<td>0.21</td>
<td>0.30</td>
<td>0.25</td>
</tr>
<tr>
<td>( \text{corr}(y_t, y_{t-1}) )</td>
<td>0.92</td>
<td>0.91</td>
<td>0.89</td>
</tr>
<tr>
<td>( \text{corr}(y_t, y_{t-2}) )</td>
<td>0.70</td>
<td>0.67</td>
<td>0.61</td>
</tr>
<tr>
<td>( \text{corr}(\pi_t, \pi_{t-1}) )</td>
<td>0.91</td>
<td>0.89</td>
<td>0.86</td>
</tr>
<tr>
<td>( \text{corr}(\pi_t, \pi_{t-2}) )</td>
<td>0.67</td>
<td>0.61</td>
<td>0.49</td>
</tr>
<tr>
<td>( \text{corr}(y_t, \pi_{t-2}) )</td>
<td>-0.11</td>
<td>0.11</td>
<td>-0.08</td>
</tr>
<tr>
<td>( \text{corr}(y_t, \pi_{t-1}) )</td>
<td>0.06</td>
<td>0.18</td>
<td>-0.15</td>
</tr>
<tr>
<td>( \text{corr}(y_t, \pi_{t}) )</td>
<td>0.22</td>
<td>0.22</td>
<td>-0.17</td>
</tr>
<tr>
<td>( \text{corr}(y_t, \pi_{t+1}) )</td>
<td>0.34</td>
<td>0.20</td>
<td>-0.13</td>
</tr>
<tr>
<td>( \text{corr}(y_t, \pi_{t+2}) )</td>
<td>0.43</td>
<td>0.13</td>
<td>-0.07</td>
</tr>
</tbody>
</table>

Moments obtained from bandpass-filtered series (6-32 Quarters). The two model-based experiments are those described in the text.
Technicalities

- Consider the VAR

\[ A(L)X_t = u_t, \]

with \( A(L) \equiv \sum_{\tau=0}^{p} A_\tau L^\tau \), \( A(0) = I \) and \( E(u_t u_t') = \Sigma \);

- Orthogonalize the residuals as \( u_t = S\epsilon_t \) where \( E(\epsilon_t \epsilon_t') = I \);

- Rewrite \( S \) as \( S = \tilde{S}Q \), where \( \tilde{S} \) is the Cholesky decomposition of \( \Sigma \), and \( Q \) is an orthonormal matrix \( (QQ' = I) \)

\[ \Rightarrow \epsilon_t = S^{-1}u_t = Q'\tilde{S}^{-1}u_t \]

\[ \Rightarrow \text{Each } \epsilon_t \text{ is associated to a column of } Q. \]
Technicalities

- Let us write the VMA(∞) representation of the VAR

\[ X_t = B(L)u_t \]

where \( B(L) = A(L)^{-1} \) is an infinite matrix polynomial of the form \( B(L) = \sum_{\tau=0}^{\infty} B_{\tau} L^{\tau} \).

- Replace \( u_t = \tilde{S}Q\varepsilon_t \),

\[ X_t = C(L)Q\varepsilon_t = \Gamma(L)\varepsilon_t, \]

where \( C(L) = B(L)\tilde{S} \) and \( \Gamma(L) = C(L)Q \) are infinite matrix polynomials.

- The contribution of shock \( j \) to the spectral density of variable \( k \) over the frequency band \([\omega, \bar{\omega}]\) is given by

\[ \Upsilon(q; k, \omega, \bar{\omega}) \equiv \int_{\omega \in [\omega, \bar{\omega}]} (C[k](e^{-i\omega})q C[k](e^{-i\omega})q) d\omega = q' \left( \int_{\omega \in [\omega, \bar{\omega}]} C[k](e^{-i\omega}) C[k](e^{-i\omega}) d\omega \right) \]

- \( q \) is then determined by maximizing the latter quantity \( \longrightarrow \) Standard eigenvalue problem.