GAMES WITH IDENTICAL SHAPLEY VALUES

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Abstract. We introduce three classes of cooperative games that yield Shapley value zero for every player. Each class spans the linear space of games with zero Shapley values and leads to a different characterization of games with identical Shapley values. The special games we construct deliver three intuitive axiomatizations of the Shapley value.

Fix a set N of $n \geq 2$ *players*. A *coalition* is any subset of players $S \subseteq N$. A game v with transferable payoffs, simply called a *game* henceforth, associates a real number $v(S)$ to any coalition S, which represents the *value* coalition S can create and share among its members ($v(\emptyset) = 0$). A *solution* ψ assigns a *payoff* $\psi_i(v)$ to each player $i \in N$ for every game v.

Shapley (1953) proposed the following solution ϕ:

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!}(v(S \cup \{i\}) - v(S)), \forall i \in N. \tag{1}$$

This solution, now known as the *Shapley value*, has the following intuitive interpretation. If players are ordered randomly (all orderings being equally likely), then $\phi_i(v)$ represents the expected marginal contribution of player i to the coalition formed by his predecessors. The Shapley value has many elegant properties. For a comprehensive treatment, the reader may consult the monograph edited by Roth (1988) and the textbooks of Moulin (1988) and Osborne and Rubinstein (1994). Here we discuss only some of its properties—most of which Shapley introduced in his original paper—necessary for our analysis. Since these properties have been used in the context of axiomatic characterizations of the Shapley value, we refer to them as *axioms*.

Some preliminary definitions are necessary for stating the classic axioms. Player i is a *dummy* in game v if $v(S \cup \{i\}) = v(S)$ for all coalitions S. Players i and j are *interchangeable* in v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all coalitions S disjoint from $\{i, j\}$. A game v is *inessential*

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if $v(S) = \sum_{i \in S} v(\{i\})$ for all coalitions S. Given the assumption that the empty coalition has value 0, we view games as column vectors in the linear (vector) space $\mathbb{R}^{2^N \setminus \{\emptyset\}}$, which has dimension $2^n - 1$. Likewise, we represent solutions $(\psi_i(v))_{i \in N}$ for specific games v as column vectors in $\mathbb{R}^N$. Hence, for any pair of games v and w and real number α, $v + \alpha w$ is the game in which the value of coalition S is given by $v(S) + \alpha w(S)$; similarly, $\psi(v) + \alpha\psi(w)$ denotes the vector $(\psi_i(v) + \alpha\psi_i(w))_{i \in N}$. We use the notation $\mathbf{0}$ for the zero vector in either $\mathbb{R}^{2^N \setminus \{\emptyset\}}$ or $\mathbb{R}^N$ (the dimension will be clear from the context). It is well-known that the Shapley value ϕ satisfies the following axioms.

Axiom (Dummy)**.** Solution ψ satisfies the *dummy axiom* if $\psi_i(v) = 0$ whenever player i is a dummy in game v.

Axiom (Linearity)**.** Solution ψ satisfies the *linearity axiom* (or is *linear*) if $\psi(v + \alpha w) = \psi(v) + \alpha\psi(w)$ for every pair of games v and w and real number α.

Axiom (Symmetry)**.** Solution ψ satisfies the *symmetry axiom* if $\psi_i(v) = \psi_j(v)$ whenever players i and j are interchangeable in game v.

Axiom (Inessential)**.** Solution ψ satisfies the *inessential axiom* if $\psi_i(v) = v(\{i\})$ for all $i \in N$ in every inessential game v.

Given the natural embedding of games and solutions in the corresponding linear spaces, the Shapley value can be expressed as $\phi(v) = Av$, where A is an $n \times (2^n - 1)$ matrix that reflects the coefficients from formula (1). For inessential games v, we have $Av = \phi(v) = (v(\{i\}))_{i \in N}$ because the Shapley value satisfies the inessential axiom. Since the space of vectors $(v(\{i\}))_{i \in N}$ derived from inessential games v has dimension n, matrix A must have full row rank equal to n. It follows that the set of games in which all players have Shapley value 0, $\mathcal{Z} := \{v | Av = \mathbf{0}\}$, is a linear subspace of $\mathbb{R}^{2^N \setminus \{\emptyset\}}$ of dimension $2^n - n - 1$.

In what follows, we construct three classes of games, each spanning a space of dimension $2^n - n - 1$, in which all players have Shapley value 0. Since $\mathcal{Z}$ has dimension $2^n - n - 1$ and contains each class of games, we conclude that every class spans the full space $\mathcal{Z}$.

An *oligarchy* is any coalition that consists of at least two players. The members of an oligarchy are called *oligarchs*. Let $\mathcal{O}$ denote the set of oligarchies, $\mathcal{O} = \{O \subseteq N | |O| \geq 2\}$.

We define two games for every oligarchy O. The *dog eat dog game* $\underline{w}^O$ for oligarchy O is specified by

$$\underline{w}^O(S) = \begin{cases} 1 & \text{if } |S \cap O| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The *scapegoat game* $\bar{w}^O$ for oligarchy O is specified by

$$\bar{w}^O(S) = \begin{cases} 1 & \text{if } |S \cap O| = |O| - 1 \\ 0 & \text{otherwise.} \end{cases}$$

We say that a game is *dog eat dog or scapegoat* if it has the respective structure for some oligarchy.

In the two types of games constructed above, oligarchs have some power and are instrumental for value creation but the oligarchy is factious and cannot cooperate effectively to realize any value. In dog eat dog games, a coalition creates value only if it includes a single oligarch—the fierce "dog." In scapegoat games, a coalition generates value only if it contains all but one oligarch—the "scapegoat." We will prove that either family of games constitutes a linearly independent subset of $\mathbb{R}^{2^N \setminus \{\emptyset\}}$ and thus spans a space of dimension $2^n - n - 1$ equal to the number of oligarchies.

Every player has Shapley value 0 in either type of oligarchic game. For instance, consider the dog eat dog game $\underline{w}^O$ for oligarchy O. In $\underline{w}^O$, all players in $N \setminus O$ are dummies and must obtain Shapley value 0 since ϕ satisfies the dummy axiom. All oligarchs are interchangeable in $\underline{w}^O$ and should obtain the same Shapley value because ϕ satisfies the symmetry axiom. Since $\underline{w}^O(N) = 0$, the common Shapley value of the oligarchs is also 0.[1] Similar arguments apply to scapegoat games.[2] Since ϕ is linear and $\phi(w) = \mathbf{0}$ for all games w defined above, the Shapley value satisfies the following axioms.

[1]For a direct proof, note that in the random ordering interpretation of the Shapley value, an oligarch's marginal contribution to the coalition formed by preceding players is 1 if he is the first oligarch in the order, -1 if he is the second oligarch in the order, and 0 in all other cases. For every oligarch, the first two events occur with equal probability and hence the oligarch's expected marginal contribution is 0.

[2]Analogously, for any oligarchy O and every set $\mathcal{I} \subseteq \{1, 2, \ldots, |O| - 1\}$, the game w defined by

$$w(S) = \begin{cases} 1 & \text{if } |S \cap O| \in \mathcal{I} \\ 0 & \text{otherwise} \end{cases}$$

delivers Shapley value 0 to all players.

Axiom (Dog Eat Dog)**.** Solution ψ satisfies the *dog eat dog axiom* if $\psi(v) = \psi(v + \alpha w)$ for every game v, any dog eat dog game w, and all real numbers α.

Axiom (Scapegoat)**.** Solution ψ satisfies the *scapegoat axiom* if $\psi(v) = \psi(v + \alpha w)$ for every game v, any scapegoat game w, and all real numbers α.

The intuition for the two axioms is that changing the cooperation structure by adding factious oligarchies should not affect the division of payoffs. Note that a solution ψ satisfies the dog eat dog (scapegoat) axiom if and only if $\psi(v) = \psi(v + w)$ for every game v and all games w that are linear combinations of dog eat dog (scapegoat) games.

We next introduce the third class of games. A *synergy function* is a game π with the property that $\pi(\{i\}) = 0$ for all $i \in N$. The *paper tiger game* with synergy π is defined by

$$w^\pi(S) = \sum_{i \in N} (\pi(S \cup \{i\}) - \pi(S)) \;\; (= \sum_{i \in N \setminus S} (\pi(S \cup \{i\}) - \pi(S))).$$

The interpretation of this game is that every player i is by nature a solitary "tiger," which can add synergies to any group S that excludes him. However, the synergy of the expanded group $S \cup \{i\}$ supersedes the original synergy of S, rendering i a "paper tiger." Since only outsiders add value to coalitions, all synergies "wash out" for the grand coalition, $w^\pi(N) = 0$.

The set of paper tiger games constitutes a linear subspace of $\mathbb{R}^{2^N \setminus \{\emptyset\}}$ that has dimension at most $2^n - n - 1$ because each component of any element $(w^\pi(S))_{S \in 2^N \setminus \{\emptyset\}}$ is a linear function of the $2^n - n - 1$ variables $(\pi(S))_{S \in \mathcal{O}}$. Note that for any oligarchy O and the synergy function π specified by

$$\pi(S) = \begin{cases} 1 & \text{if } O \subseteq S \\ 0 & \text{otherwise,} \end{cases}$$

the paper tiger game w^π is identical to the scapegoat game $\bar{w}^O$. Thus, the space of paper tiger games contains the linear space spanned by scapegoat games. We will show that the latter space has dimension $2^n - n - 1$, which implies that the space of paper tiger games has dimension $2^n - n - 1$ and coincides with the space spanned by scapegoat games. Hence, every paper tiger game is a linear combination of scapegoat games. The linearity of the Shapley value, along with the fact that $\phi(\bar{w}^O) = \mathbf{0}$ for all scapegoat games $\bar{w}^O$, implies that $\phi(w^\pi) = \mathbf{0}$ for every paper tiger game w^π. Therefore, the Shapley value satisfies the following axiom, which captures the "paper tiger" metaphor.

Axiom (Paper Tiger)**.** Solution ψ satisfies the *paper tiger axiom* if $\psi(v) = \psi(v + w)$ for every game v and any paper tiger game w.

The arguments above lead to three alternative characterizations of the space $\mathcal{Z}$ of games in which all players obtain Shapley value 0.

Theorem 1. *The condition that game v delivers Shapley value 0 to all players is equivalent to each of the following properties:*

(1) *v is a linear combination of dog eat dog games;*
(2) *v is a linear combination of scapegoat games;*
(3) *v is a paper tiger game.*

The proof of the theorem appears at the end of the paper. We next provide two corollaries that invoke paper tiger games. In light of Theorem 1, we can restate either corollary using a linear combination of each type of oligarchic game in lieu of the paper tiger game. The first corollary follows from the linearity of the Shapley value.

Corollary 1. *Games v and w yield identical Shapley values if and only if their difference $v - w$ is a paper tiger game.*

Fix a game v. The *Shapley inessential game w of v* is defined by $w(S) = \sum_{i \in S} \phi_i(v)$ for all coalitions S. Since the Shapley value satisfies the inessential axiom, we have that $\phi_i(w) = w(\{i\}) = \phi_i(v)$ for all $i \in N$. Then the linearity of the Shapley value implies that $\phi(v - w) = \mathbf{0}$. Thus, game v can be decomposed into its Shapley inessential game w and the game $v - w$ in $\mathcal{Z}$. This conclusion leads to another corollary of Theorem 1.

Corollary 2. *Every game is the sum of its Shapley inessential game and a paper tiger game.*

If a solution ψ is pinned down for inessential games by the inessential axiom, and the addition of games in $\mathcal{Z}$ does not affect the solution as implied by any of the dog eat dog, scapegoat, or paper tiger axioms, then ψ must coincide with the Shapley value ϕ. These observations, along with Theorem 1 and Corollary 2, lead to three axiomatizations of the Shapley value.

Theorem 2. *A solution is the Shapley value if and only if it satisfies the inessential axiom and any one of the dog eat dog, scapegoat, and paper tiger axioms.*

We finally comment on a connection between our paper tiger axiom and an axiom due to Hamiache (2001). Derive a synergy function π from a game v and a real number λ as follows:

$$\pi(S) = \lambda(v(S) - \sum_{i \in S} v(\{i\})). \tag{2}$$

Let w^π denote the paper tiger game with synergy π and define the game $v_\lambda := v + w^\pi$. Simple algebra shows that

$$v_\lambda(S) = v(S) + \lambda \sum_{i \in N \setminus S} \left(v(S \cup \{i\}) - v(S) - v(\{i\})\right), \forall S \subseteq N.$$

Since w^π is a paper tiger game, Theorem 2 implies that the games v and v_λ have the same Shapley values. Hamiache (2001) uses this property, coined *associated consistency*, to develop a characterization of the Shapley value. In addition to the inessential axiom, his characterization requires a continuity axiom because associated consistency is a weaker version of our paper tiger axiom that applies only to pairs of games (v, w^π) for which the synergy function π has the special relation to v described by formula (2).

Proof of Theorem 1

We have argued that $\mathcal{Z} = \{v | \phi(v) = \mathbf{0}\}$ is a linear subspace of $\mathbb{R}^{2^N \setminus \{\emptyset\}}$ of dimension $2^n - n - 1$ that contains all dog eat dog games. Since there are exactly $2^n - n - 1$ dog eat dog games $(\underline{w}^O)_{O \in \mathcal{O}}$, every element of $\mathcal{Z}$ is a linear combination of such games if and only if the set $\{\underline{w}^O | O \in \mathcal{O}\}$ is linearly independent. To establish the linear independence of $\{\underline{w}^O | O \in \mathcal{O}\}$, we proceed by contradiction. Assume that $\sum_{O \in \mathcal{O}} \alpha^O \underline{w}^O = \mathbf{0}$ for a profile $(\alpha^O)_{O \in \mathcal{O}}$ in which not all entries are 0. In this case,

$$\sum_{O \in \mathcal{O}} \alpha^O \underline{w}^O(S) = 0, \forall S \subseteq N,$$

which means that

$$\sum_{O \in \mathcal{O} : |S \cap O| = 1} \alpha^O = 0, \forall S \subseteq N. \tag{3}$$

We show by induction on $|S|$ that

$$\sum_{O \supseteq S} \alpha^O = 0 \tag{4}$$

for all $S \subseteq N$. For the base case $|S| = 1$, the condition $O \supseteq S$ is equivalent to $|S \cap O| = 1$, so (4) is a consequence of (3). We set out to prove the inductive claim for a coalition $S \subseteq N$ assuming that it is true for all smaller coalitions.

We first establish the identity

$$\sum_{O \in \mathcal{O}: |S \cap O| = 1} \alpha^O = \sum_{S' \subseteq S} (-1)^{|S'|-1} |S'| \sum_{O \in \mathcal{O}: O \supseteq S'} \alpha^O. \tag{5}$$

Indeed, if $|S \cap O| = k$, then the sum of coefficients of α^O in the expression on the right-hand side of the equation above is

$$\sum_{S' \subseteq S \cap O} (-1)^{|S'|-1} |S'| = \sum_{s=1}^{k} (-1)^{s-1} s \binom{k}{s} = k \sum_{s=1}^{k} (-1)^{s-1} \binom{k-1}{s-1}. \tag{6}$$

The first equality uses the fact that there are $\binom{k}{s}$ sets $S' \subseteq S \cap O$ with $|S'| = s$ if $|S \cap O| = k$, while the second relies on the equations

$$s\binom{k}{s} = s\frac{k!}{s!(k-s)!} = k\frac{(k-1)!}{(s-1)!(k-s)!} = k\binom{k-1}{s-1}.$$

Identity (5) then follows from the observation that the summand in the last term of (6) represents the binomial expansion of $(1-1)^{k-1}$, which is 1 for $k = 1$ and 0 for $k \geq 2$.

Equations (3) and (5) imply that

$$\sum_{S' \subseteq S} (-1)^{|S'|-1} |S'| \sum_{O \in \mathcal{O}: O \supseteq S'} \alpha^O = 0. \tag{7}$$

By the inductive hypothesis, we have that $\sum_{O \in \mathcal{O}: O \supseteq S'} \alpha^O = 0$ for all $S' \subset S$. Then (7) implies that $\sum_{O \in \mathcal{O}: O \supseteq S} \alpha^O = 0$, which establishes the inductive step.

We are now prepared to complete the proof by contradiction. Let $\tilde{O}$ be one of the largest (cardinality) oligarchies with the property that $\alpha^{\tilde{O}} \neq 0$. Then $\alpha^O = 0$ for all $O \supset \tilde{O}$. Applying (4) for $S = \tilde{O}$, we obtain $\alpha^{\tilde{O}} = 0$, a contradiction.

To establish the second equivalence stated by the theorem, we must similarly show that the set of scapegoat games $\{\bar{w}^O | O \in \mathcal{O}\}$ is linearly independent. Note that

$$\bar{w}^O(S) = \begin{cases} 1 & \text{if } |(N \setminus S) \cap O| = 1 \\ 0 & \text{otherwise,} \end{cases}$$

so the scapegoat and the dog eat dog games for oligarchy O are related by the formula $\bar{w}^O(S) = \underline{w}^O(N \setminus S)$ for all $S \subseteq N$. Then the linear independence of $\{\bar{w}^O | O \in \mathcal{O}\}$ follows from the arguments for dog eat dog games.

Finally, the conclusion that the space of paper tiger games is identical to $\mathcal{Z}$ follows from the finding that $\mathcal{Z}$ spans the set of scapegoat games and the arguments provided after the definition of paper tiger games. □

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