Reserve Design: Unintended Consequences and
The Demise of Boston’s Walk Zones∗

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Abstract
Admissions policies often use reserves to grant certain applicants higher priority for some (but not all) available seats. Boston’s school choice system, for example, reserved half of each school’s seats for local, “neighborhood” applicants, while leaving the other half for open competition. This paper show that in the presence of reserves, the precedence order, i.e. the order in which different types of seats are filled, has effects on distributional objectives comparable to the effects of adjusting reserve sizes. Either lowering the precedence order positions of reserve seats at a school or increasing the number of reserve seats weakly increase reserve-group assignment at the school. Using data from Boston, we show that reserve and precedence adjustments have similar quantitative effects. Our results illustrate that policies about precedence, heretofore under-explored, are inseparable from other aspects of admissions policies. Moreover, our findings explain the puzzling empirical fact that despite careful attention to the importance of neighborhood priority, the outcome of Boston’s implementation of its 50-50 reserve–open seat split was nearly identical to the outcome of a counterfactual system without any reserves. Transparency about these issues—in particular, how precedence unintentionally undermined the intended admissions policy—led to the elimination of Boston’s walk zones.

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1 Introduction

Admissions policies in school systems are often shaped by historical circumstances and modern-day compromises between competing interest groups. At many publicly-funded Indian engineering colleges, for example, seats are reserved for applicants from disadvantaged caste and gender groups (see Bagde, Epple, and Taylor (2016)). In the Indian system, an applicant from a disadvantaged group who qualifies for a school without invoking caste/gender priority is assigned one of the school’s regular seats instead of a reserve seat; the reserve seats are held for students who otherwise would not be able to gain admission. The public school administration in Boston also devised a reserve scheme, but based on neighborhood boundaries rather than student type. The Boston policy came after 1970s-era court ordered desegregation divided the city into geographically segregated communities. At each school in Boston, half of the seats at each school were made open to all applicants, while the other half prioritized applicants from the local neighborhood. Unlike in the Indian system, the Boston system filled reserved seats ahead of open seats.

Indian engineering admissions are decentralized in some states, while Boston’s school choice program is centralized. Under both systems, however, there are two types of seats at each school—reserved seats and open seats—and it is not uncommon for a given applicant to be qualified for both types of seat. When a student can be admitted to a school via multiple routes, an admissions policy must specify the precedence of different admissions tracks; in the cases of India and Boston, this means that policy must account for the orders in which reserve and open seats are processed. In this paper, we formally show that precedence plays a central role in achieving distributional objectives, qualitatively and quantitatively similar to the impact of reserve priorities. We then relate these results to a recent policy discussion in Boston, showing how an oversight leading to the wrong precedence policy completely undermined the city’s stated objectives in a subtle way.

Boston’s 50-50 reserve–open seat split emerged out of a city-wide discussion following the end of racial and ethnic criteria for school placement in 1999. Many advocated abandoning school choice and returning to neighborhood schooling, but the school committee chose instead to maintain school choice while making neighborhood, i.e. “walk-zone” priority, apply at 50% of each school’s seats (Appendix C excerpts the official policy). In popular accounts, the 50-50 slot split was described as “striking an uneasy compromise between neighborhood school advocates and those who want choice,” while the Superintendent hoped that the “plan would satisfy both factions, those who want to send children to schools close by and those who want choice” (Daley 1999).

The fragile compromise between the pro-neighborhood and pro-school choice factions has resurfaced in debates on admissions policies in Boston numerous times.1 Most recently, Boston Mayor Thomas Menino’s 2012 State of the City Address forcefully argued in favor of assigning students to schools closer to home (Menino 2012).2 Proposals from Boston Public Schools (BPS) and other

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1 A December 2003 community engagement process in Boston considered six different proposals for alternative neighborhood zone definitions. However, the only recommendation adopted by the school committee was to switch the assignment algorithm (Landsmark and Dajer 2004, Abdulkadiroğlu, Pathak, Roth, and Sönmez 2005). The discussion was renewed with a proposal for a five-zone plan in 2009, that was eventually rejected (Vaznis 2009).

2 Constituents had long believed that students were traveling too far to attend schools and sought to alter the plan
community members became the center of a year-long, city-wide discussion on school choice featuring over seventy public meetings and input from more than three thousand parents.3

Boston’s decision to revisit its reserve policy was in part motivated by a persistent empirical puzzle: While 50% of seats at each school were reserved for students living in the neighborhood/walk-zone, the fraction of neighborhood students assigned to popular schools consistently hovered around 50%. With half the seats reserved for neighborhood students and the other half open to everyone, one would expect more than 50% neighborhood assignment—as Boston’s stated policy suggests (see Appendix C).

In this paper, we show that Boston’s assignment puzzle was an unintended consequence of the chosen implementation: an accidental choice of a precedence order in which reserve seats were filled before open seats completely subverted the 50-50 compromise, resulting in allocation almost indistinguishable from a counterfactual setting without any reserve seats at all. Our first formal result shows that reserves and precedence are policy tools with similar qualitative effects for a given school. For any precedence order, replacing an open slot with a reserve slot weakly increases the assignment of reserve-eligible applicants. Similarly, given a reserve size, swapping the precedence of a reserve slot with that of a subsequent open slot weakly increases the assignment of reserve-eligible applicants.

Next, we investigate how our within-school results extend to centralized assignment systems based on the deferred acceptance algorithm. We find that for a given school, increasing the number of reserved seats (relative to open seats) or raising the precedence of open seats (relative to reserve seats) leads to increased admission of reserve-eligible applicants under the deferred acceptance algorithm. This result is, to our knowledge, the first-ever comparative static result for multi-agent priority improvements in matching models. The comparative statics we find do not necessarily extend to an aggregate increase in assignment of reserve-eligible applicants across all schools, because of interactions across schools in the deferred acceptance algorithm. However, the possibility of these cross-school interactions are not relevant in practice: Our comparative statics extend to the whole market in a two-school model, and we can also bound the worst case when reserves privilege the same group throughout the school system. Moreover, our theoretical analysis closely matches the empirical patterns observed in Boston; we show that Boston’s implementation of its intended 50-50 reserve–open compromise was in practice closer to a 10-90 system.

This paper contributes to a broader agenda, examined in a number of recent papers, that introduces concerns for diversity into the literature on school choice mechanism design (see, e.g., Budish, Che, Kojima, and Milgrom (2013), Echenique and Yenmez (2012), Erdil and Kumano (2012), Hafalir, Yenmez, and Yildirim (2012), Kojima (2012), Kominers and Sönmez (2013), and to assign students closer to home (Landsmark 2009).

3 For more on this debate, see the materials available at http://bostonschoolchoice.org and accounts by Goldstein (2012), Handy (2012), and Seelye (2012). In Fall 2012, BPS proposed five different plans which all restricted participant choice by reducing the number of schools students could rank; the idea behind each of these plans was to reduce the fraction of non-neighborhood applicants in competition for seats at each school. (The initial plans suggested dividing the city into 6, 9, 11, or 23 zones, or doing away with school choice entirely and reverting to assignment based purely on neighborhood.)
Kominers and Sönmez (2016)). When an applicant ranks a school with seats that employ different admissions criteria, it is as if she is indifferent between those school’s seats. Therefore, our work parallels investigations of indifferences in school choice problems (Erdil and Ergin 2008, Abdulkadiroğlu, Pathak, and Roth 2009, Pathak and Sethuraman 2011). However, results about school-side indifferences do not extend to those for indifferences in student preferences. Finally, our focus here is on establishing comparative static results, motivated by Boston’s policy developments. In subsequent work, Dur, Pathak, and Sönmez (2016) characterize optimal admissions policies motivated by Chicago’s placed-based affirmative action system.

Our paper proceeds as follows. Section 2 describes the Boston puzzle in more detail. Section 3 formally studies admissions policies in which applicants can be admitted via multiple routes. Section 4 examines how a school’s admissions policy interacts with a centralized admissions system based on deferred acceptance. Section 5 reports on data from Boston, while Section 6 concludes. Proofs are in the Appendix.

2 Motivation

2.1 Boston’s “50-50 Puzzle”

Despite widespread perception and policy intent that walk-zone applicants had been advantaged in the BPS school choice program since 1999, they appear to have had little advantage in practice. Even though 50% of seats at each school were reserved for walk-zone students, the assignment outcomes in Boston were close to those that would have arisen under a system without any walk-zone reserve. To see this, we compute the fraction of students assigned to walk-zone schools in Boston for the extreme case with no walk-zone priority—the 0% Walk system. Table 1 shows that despite the 50% walk-zone reserve, assignment outcomes under BPS’s system are nearly identical to those under 0% Walk; they differ for only 3% of Grade K1 students.

One might suspect that similarity between the BPS outcome and 0% Walk is driven by a strong preference for neighborhood schools among applicants, which would bring the outcomes of the two policies close together. However, this is not the case. We compare the BPS outcome to a 100% Walk counterfactual in which walk-zone priority applies at all slots. Under 100% Walk, 19% of Grade K1 students obtain a different assignment. Thus, the remarkable proximity of the BPS outcome to the 0% Walk ideal of school choice proponents does not suggest (or reflect) negligible

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4To compute counterfactual assignments here, we use internal preference data from BPS, and the same lottery numbers BPS used to break ties in its own assignment system. It is worth noting that strategy-proofness (i.e., truthfulness) of the assignment mechanism used in Boston justifies re-computing the assignment without modeling how applicants might submit preferences in these scenarios (see Abdulkadiroğlu, Pathak, Roth, and Sönmez (2006), Pathak and Sönmez (2008), Abdulkadiroğlu, Pathak, and Roth (2009) and Agarwal, Abdulkadiroğlu, and Pathak (2015)).

5We have repeated this calculation for 500 different random lottery draws. Under 0% Walk, the average difference is 3%, 4% and 1% for Grades K1, K2, and Grade 6, respectively. Under 100% Walk, the corresponding average difference is 20%, 18%, and 10%.
stakes in school choice. Rather, it presents a puzzle: Why does Boston’s assignment mechanism result in an assignment so close to one without any neighborhood priority, even though half of each school’s seats prioritize neighborhood students? Or more qualitatively, why didn’t Boston’s 50% reserve have much of an impact in practice, instead of, say, resulting in an outcome 50% of the way between 0% Walk and 100% Walk? To obtain an intuition about the answer, we turn to a simple example that illustrates Boston’s 50-50 split as implemented in the context of a single school.

2.2 A One-School Example

Consider a single school with 100 seats. Suppose there are 100 applicants with walk-zone priority and 100 applicants who do not have walk-zone priority. The tie-breaking lottery is such that, of the 100 best lottery applicants, 50 are from the walk zone and 50 are not. Figure 1 illustrates the situation, with both walk-zone applicants (in blue) and non walk-zone applicants (in orange) ordered by the random tie-breaker.

In Panel (a) of Figure 1, there is no walk-zone priority at the school, so students are admitted solely based on the random tie-breaker. Given the lottery numbers, the school admits an equal number of students from both groups. That is, 50 students from the walk zone and 50 students from outside the walk zone obtain admissions.

Figure 1: (a) Assignment without walk-zone priority and (b) Assignment with 50/50 policy

In Panel (b) of Figure 1, half of the seats use walk-zone priority and the other half are open. Under Boston’s school choice system, students from both groups first apply to the walk-zone half. At the walk-zone half, students who have walk-zone priority are admitted ahead of students who do not, and the admitted walk-zone students are those with the most favorable random tie-breakers.

6The patterns we observe are similar for grades above K1, with the smaller differences between 0% Walk and 100% Walk at higher grades driven by a larger share of continuing students who obtain guaranteed priority at higher grades.
Therefore, 50 students from the walk zone with the highest lottery numbers take up all of the seats in the walk half. Next, the remaining (and less favorable random tie-breaker) applicants from the walk zone apply to the open half of the school, together with all of the applicants from outside of the walk zone. For the open seats, students are admitted based only on the random tie-breaker. But at this point, walk-zone applicants are disadvantaged due to their unfavorable tie-breakers, so that only non walk-zone applicants are assigned to the 50 seats in the open half. The final allocation results in half of the school’s seats being assigned to walk-zone applicants, with the remaining half assigned to applicants from outside the walk zone.

Figure 1 shows how the 50-50 compromise can result in the same outcome as in a situation without any walk-zone priority. This example is stylized in a few ways. There are an equal number of applicants with walk-zone priority and without walk-zone priority, and the tie-breaking lottery has an equal number of students from each group among the top 100. However, it captures the main intuition for the phenomenon documented in Table 1.

This example shows that the precedence order in which seats are processed significantly affects the outcome. Had applicants first applied to the open half, 75 walk-zone applicants and 25 non-walk zone applicants would have been assigned, holding fixed the 50-50 seat split. At the time of Menino’s 2012 speech, the dramatic role of precedence order in disadvantaging walk-zone students came as a surprise to many (including us) and motivated the formal analysis to follow.

3 Admissions Policies with Reserves

To formalize the intuition presented in the preceding section, we now develop a model of school admissions policies in which some seats at each school may be reserved for members of distinguished groups (e.g., disadvantaged castes or walk-zone students). We prove comparative statics illustrating that both (1) increasing number of reserve seats and (2) raising the precedence order positions of open seats will (weakly) increase the number of reserve-eligible students who are accepted.

3.1 Decentralized Model

There is a finite set $I$ of students and a school $a$ with a finite set of slots $S^a$. Each slot $s \in S^a$ has a linear priority order $\pi^a$ over students in $I$. This linear priority order captures the “property rights” of the students for this slot in the sense that the higher a student is ranked under $\pi^a$, the stronger claims he has for the slot $s$ of school $a$.

We are interested in situations in which slot priorities are heterogeneous across slots of a given school. A consequence of this within-school heterogeneity is that we must determine how slots are assigned when a student is “qualified” for multiple slots that have different priority rankings. For each school $a$, the slots in $S^a$ are ordered according to a (linear) order of precedence $\triangleright^a$. Given a

\cite{note1}

\cite{note2}
school $a$ and two of its slots $s, s' \in S^a$, the expression $s \triangleright^a s'$ means that slot $s$ is to be filled before slot $s'$ at school $a$ whenever possible.

Given a school $a$ with a set of slots $S^a$, a list of slot priorities $(\pi^s)_{s \in S^a}$, an order of precedence $\triangleright^a$ with

$$s_1^a \triangleright^a s_2^a \triangleright^a \cdots \triangleright^a s_{|S^a|}^a,$$

and a set of students $J \subseteq I$, the choice of school $a$ from the set of students $J$, denoted by $C^a(J)$, and is obtained as follows: Slots at school $a$ are filled one at a time following the order of precedence $\triangleright^a$. The highest-priority student in $J$ under $\pi^s_1$, say student $j_1$, is chosen for slot $s_1^a$ of school $a$; the highest-priority student in $J \setminus \{j_1\}$ under $\pi^s_2$ is chosen for slot $s_2^a$, and so on.

We are particularly interested in slot priority structures in which some of the slots are reserved for applicants of a particular type (the “reserve-eligible”), while the slots are remaining slots are open. Suppose there is a master priority order $\pi^0$ that is uniform across all schools. This master priority is often determined by a random lottery or by performance on an admissions exam or in previous grades. For school $a$, there is a set $I_a \subset I$ of reserve-eligible students. Students who are not reserve-eligible are called reserve-ineligible. There are two types of slots.

1. At open slots, $\pi^s = \pi^0$ for each open slot $s$.

2. At reserve slots, any reserve-eligible student $i \in I_a$ has priority over any reserve-ineligible student $j \in I \setminus I_a$, and the priority order within each group is determined according to the master priority $\pi^0$.

In Indian affirmative action systems, the reserve-eligible students are those from disadvantaged castes. Aygın and Bo (2013) describe reserves for public universities in Brazil where the reserve-eligible are racial minorities, low income families, and applicants from public high schools. In Boston Public Schools, the reserve-eligible groups are students who live in the school’s walk-zone, and we refer to reserved slots as walk-zone seats.\footnote{BPS also uses sibling priority, but for our theoretical analysis, we consider a simplified priority structure which only depends on walk-zone status; using data from BPS in Section 5, we show that this is a good approximation.}

3.2 Priority and Precedence Changes as Substitutes

We first examine the effects of an increase in the reserve size given a precedence order $\triangleright$. Suppose that slot $s^*$ at school $a$ is an open slot under priority $\pi$, but is a reserve slot under priority $\tilde{\pi}$. Suppose that $\pi^s = \tilde{\pi}^s$ for all slots $s \neq s^*$. Let $C$ and $D$ respectively be the choice functions for $a$ induced by the priorities $\pi$ and $\tilde{\pi}$ under precedence order $\triangleright$.

**Proposition 1.** Suppose that $D$ is the choice function for school $a$ obtained from $C$ by changing an open slot of to a reserve slot (leaving all other slots and the precedence order unchanged). For any set of students $\bar{I} \subseteq I$:

(i) All reserve-eligible students at school $a$ that are chosen from $\bar{I}$ under choice function $C$ are chosen under choice function $D$.
(ii) All reserve-ineligible students at school \( a \) that are chosen from \( \bar{I} \) under choice function \( D \) are chosen under choice function \( C \).

This result states that when a school increases its reserve size, it admits weakly more reserve-eligible students and weakly fewer reserve-ineligible students. For Boston, this result suggests that increasing the walk-zone percentage beyond 50% may increase neighborhood assignment.

What is much less apparent is that swapping the precedence order of a reserve slot and a subsequent open slot has the same qualitative effect as increasing the reserve. Suppose now that \( s_r \) is a reserve slot that immediately precedes an open slot \( s_o \) under the precedence order \( \succ \). Suppose, moreover, that precedence order \( \succ' \) is obtained from \( \succ \) by swapping the positions of \( s_r \) and \( s_o \) and leaving the positions of all other slots unchanged. Let \( C \) and \( D \) respectively be the choice functions for \( a \) induced by the precedence orders \( \succ \) and \( \succ' \) under slot priorities \( \pi^a \). We obtain the following analog to Proposition 1.

\textbf{Proposition 2.} Suppose that \( D \) is the choice function for school \( a \) obtained from \( C \) by swapping the precedence of a reserve slot and a subsequent open slot (leaving all other slots and the rest of the precedence order unchanged). For any set of students \( \bar{I} \subseteq I \):

(i) All reserve-eligible students at school \( a \) that are chosen from \( \bar{I} \) under choice function \( C \) are chosen under choice function \( D \).

(ii) All reserve-ineligible students at school \( a \) that are chosen from \( \bar{I} \) under choice function \( D \) are chosen under choice function \( C \).

Propositions 1 and 2 together show how priority and precedence changes are substitute levers for influencing the assignment of reserve-eligible applicants. While the role of the number of reserve slots is quite apparent, the role of the order of precedence is much more subtle. Indeed, the choice of precedence order is often considered a minor technical detail—and, to our knowledge, precedence had never explicitly entered school choice policy discussions until we initiated them in Boston in parallel with the present work.

Qualitatively, the effect of decreasing the precedence order position of a reserve slot is similar to the effect of replacing an open slot with a reserve slot. While this may appear counterintuitive at first, the reason is simple: By decreasing the precedence of a reserve slot, a reserve-eligible student who has high enough master priority to be eligible for both open and reserve slots may now be assigned to an open slot. This in turn increases the competition for the open slots and decreases competition for reserve slots.

Our observation of how changing applicant processing orders influences access for reserve-eligible applicants did surface in debates on affirmative policies in India. India’s constitution stipulates reservations for disadvantaged groups for government-funded educational institutes and public sector jobs, including seats in parliament. A debate on applicant processing made its way to the Supreme Court in 1975, where a judge ruled that the “benefits of the reservation shall be snatched away by the top creamy layer of the backward class, thus leaving the weakest among the weak and
leaving the fortunate layers to consume the whole cake.”¹⁰ In the context of our model, if reserve seats have higher precedence than open seats, then priority goes to applicants who do not need it (the “creamy layer”), leaving the remaining reserve-eligible without an opportunity to obtain open seats since they are out-competed by the reserve-ineligible.

So far our results are for a single school with given choice function; this directly informs us about reserves implemented in decentralized admissions in India and elsewhere. Since many centralized systems can be seen as iterated applications of choice functions, our results so far also provide an approximation for those centralized systems. We next formally examine how our results extend to centralized systems based on the deferred acceptance algorithm.

4 Centralized Admissions Systems with Reserves

Suppose now that there is a set of schools \( A \). We use the notation \( a_0 \) to denote a “null school” representing the possibility of being unmatched; we assume that this option is always available to all students. Let \( S \equiv \bigcup_{a \in A} S^a \) denote the set of all slots (excluding those at the null school). Each student \( i \) has a strict preference relation over \( A \cup \{a_0\} \).

A matching \( \mu : I \to A \) is a function that assigns a school to each student such that no school is assigned to more students than its total number of slots. This model generalizes the school choice model of Abdulkadiroğlu and Sönmez (2003) in that it allows for heterogenous priorities across the slots of a given school. Nevertheless, a mechanism based on the celebrated student-proposing deferred acceptance algorithm (Gale and Shapley 1962) easily extends to this model given our earlier description of a school’s choice function.

For a given list of slot priorities \((\pi^s)_{s \in S}\) and an order of precedence \(\succ^a\) at each school \( a \in A \), the outcome of the student-proposing deferred acceptance mechanism (DA) can be obtained as follows:

**Step 1:** Each student \( i \) applies to her top choice school. Each school \( a \) with a set of Step 1 applicants \( J^a_1 \) tentatively holds the applicants in \( C^a(J^a_1) \), and rejects the rest.

**Step \( \ell \):** Each student rejected in Step \( \ell - 1 \) applies to her most-preferred school (if any) that has not yet rejected her. Each school \( a \) considers the set \( J^a_\ell \) consisting of the new applicants to \( a \) and the students held by \( a \) at the end of Step \( \ell - 1 \), tentatively holds the applicants in \( C^a(J^a_\ell) \), and rejects the rest.

The algorithm terminates after the first step in which no students are rejected, assigning students to the schools holding their applications.

4.1 Comparative Statics for Deferred Acceptance

In the context of DA, we consider the effect of replacing an open slot with a reserve slot and swapping the precedence of reserve and an open slot with lower precedence. Both changes weakly

¹⁰The court case is *State of Kerala vs. NM Thomas* (1974).
increase the number of reserve students assigned to that school.

**Proposition 3.** Consider centralized assignment under DA. Fix a school \( a \in A \).

(i) For any given precedence order of slots, replacing an open slot of school \( a \) with a reserve slot weakly increases the number of reserve-eligible students assigned to school \( a \).

(ii) Fix the set of reserve slots and the set of open slots at each school. Then, switching the precedence order position of a reserve slot of school \( a \) with the position of a subsequent open slot weakly increases the number of reserve-eligible students assigned to school \( a \).

While this result is analogous to the results for decentralized admission for a single school in Propositions 1 and 2, its proof is substantially more involved, as it is necessary to consider the cascading effects of the centralized algorithm. That is, when either the precedence order or reserve size changes, a different set of applicants may apply to the school initiating a sequence of applications to other schools at subsequent stages of the deferred acceptance algorithm—and these subsequent applications need to be tracked carefully. Indeed, neither comparative static result follows from earlier comparative static approaches used in simpler models (e.g., Balinski and Sönmez (1999)) because they involve simultaneous priority improvements for a large number of students. As a result, we must develop a new proof strategy, which may be of independent interest in settings involving multi-agent priority improvements in matching models.

### 4.2 Aggregate Comparative Statics

Our analysis so far focused on the assignment of reserve-eligible students at a particular school at which there is a change in the reserve or precedence. A natural question is whether increased reserve-eligible student assignment at a particular school always translates into an increase in overall reserve-eligible assignment across all schools. As usual in going from partial to general equilibrium analysis, this is not a foregone conclusion. In particular, it is well-known that in matching models, interactions across the market can lead to counterintuitive overall predictions. The next example shows that the results for a single school do not imply an aggregate increase in reserve-eligible assignment.

**Example 1.** There are three schools, \( A = \{k, l, m\} \). Schools \( k \) and \( m \) have two slots and school \( l \) has three slots. There are seven students \( I = \{i_1, i_2, i_3, i_4, i_5, i_6, i_7\} \). Let \( I_a \) be the reserve-eligible students at school \( a \in A \). Let \( I_k = \{i_1, i_7\} \), \( I_l = \{i_2, i_3, i_4\} \), and \( I_m = \{i_5, i_6\} \). The master priority \( \pi^0 \) orders the students as:

\[
\pi^0: i_7 \succ i_2 \succ i_5 \succ i_3 \succ i_1 \succ i_6 \succ i_4 \succ i_2.
\]

The preference profile is:
First consider the case where school k’s first and school l’s second slots are reserve slots and all other slots are open slots. The outcome of deferred acceptance for this case is:

$$\mu = (i_1 \ i_2 \ i_3 \ i_4 \ i_5 \ i_6 \ i_7)$$

Observe that, in addition to the two reserve slots assigned to reserve-eligible students i_3 and i_4, two of the open slots (namely those assigned to i_6 and i_7) are also assigned to reserve-eligible students. As such, four students are assigned to schools for which they are reserve-eligible.

Next we replace the open slot at school k with a reserve slot, so that both slots at school k are reserve slots. We keep the slot types and precedence for the other schools the same. The outcome of DA for the second case is:

$$\mu = (i_1 \ i_2 \ i_3 \ i_4 \ i_5 \ i_6 \ i_7)$$

Observe that, while all three reserve slots are assigned to reserve-eligible students (i.e. students i_1, i_3, i_7) none of the open slots at any school are assigned to their reserve-eligible students. That is, the total number of reserve-eligible student assignments decreases when the open slot at school k is replaced with a reserve slot.

The preceding example illustrates that the direct “first-order” effect of a change at a given school may be undone by the indirect effect on other schools. Moreover, it is easy to modify Example 1 to show that when the precedence of a reserve and subsequent open slot are swapped at a given school, the overall reserve-eligible student assignment need not increase. (Example 2 in Appendix A presents this modification.) These negative findings highlight the richness involved with distributional comparative statics in slot-specific matching models.

### 4.3 Aggregate Effects Under Uniform Reserve Priority

One important feature of Example 1 is that the set of reserve-eligible students differs by school. When reserves represent walk-zone seats, as in Boston, we would expect them to differ by school since families are dispersed geographically (and thus live in different walk-zones). However, in a case like India, in which the reservation is intended to remedy a disadvantage like membership in a particular caste, the set of reserve-eligible students is the same for each school.

If for any two schools a, a’ ∈ A, we have that $I_a = I_{a'}$, then we refer to this case as one of **uniform reserve priority**.\(^{11}\) In the uniform reserve priority case, it is still possible that reserve-

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\(^{11}\) Aygün and Turhan (2016) describe a centralized admissions procedure with uniform priority used for universities in the Indian state of Maharashtra with the same reservations for disadvantaged castes.
eligible assignment can decrease when an open slot is replaced with a reserve slot (and likewise for swapping the precedence position of a reserve and subsequent open slot, as can be shown with a slight modification of the example; see Example 3 in Appendix A). However, even in the worst case scenario, only one fewer reserve-eligible student can be assigned under the uniform reserve priority.

**Proposition 4.** Consider centralized assignment under deferred acceptance. Suppose that we have uniform reserve priority, i.e., that for any two schools \( a, a' \in A \), we have \( I_a = I_{a'} \). Then:

(i) For any given precedence order of slots, replacing an open slot of a given school with a reserve slot cannot decrease the total assignment of reserve-eligible students across all schools by more than 1.

(ii) Fix the set of reserve and open slots at each school. Then, switching the precedence order position of a reserve slot of a given school with the position of a subsequent open slot cannot decrease the total assignment of reserve-eligible students by more than 1.

In contrast, without uniform reserve priority, it is easy to extend Example 1 to show that when an open slot is replaced with a reserve slot, the number of reserve-eligible students assigned can reduce by more than 1 (see Appendix A).

### 4.4 Aggregate Effects in the Two-School Case

When there are two schools, the effects of the reserve and precedence order changes described in Section 4.2 can be sharpened: either change weakly increases total reserve-eligible assignment. For the next result, we assume that each student is reserve-eligible at exactly one school and that students rank both schools.

**Proposition 5.** Consider centralized assignment under deferred acceptance. Suppose that there are two schools, that each student is reserve-eligible at exactly one school, and that students rank both schools. Then:

(i) Replacing an open slot of either school with a reserve slot weakly increases total reserve-eligible assignment.

(ii) Switching the precedence order position of a reserve slot at either school with that of a subsequent open slot weakly increases total reserve-eligible assignment.

(iii) The number of students assigned to their top-choice schools is independent of both the number of reserve slots and the precedence order.

The first two parts of Proposition 5 show that when there are only two schools, the aggregate effects of reserve and precedence changes examined for a particular school in Proposition 3 extend across all schools. The last part of Proposition 5 shows that in the two-school case, changes in the reserve size or precedence are entirely distributional—both instruments leave the aggregate number of students obtaining their top-choice schools unchanged.
Proposition 5 suggests a method to compute the policies that have the minimum and maximum number of reserve assignments. In the two-school case, the minimum number of reserve assignments across all priority and precedence policies is obtained when all slots are open; the maximum number of reserve assignments across all priority and precedence policies is obtained when all slots are reserved.

The analysis from the two-school case suggests that the possibilities shown by our negative examples require an elaborate sequence of applications and rejections involving more than two schools. We next turn to data from Boston, and see that the results in Proposition 5 provide a better approximation for Boston’s implementation situation than what is suggested by our negative examples that critically depend on carefully constructed rejection chains.

5 Precedence and Reserves in Boston

Table 2 reports the number of walk-zone students assigned to schools for different walk-zone percentages using the same lottery number as BPS. For each grade, more students are assigned to schools in their walk zones when the reserve size is 100% Walk, relative to 0% Walk. For Grade K1, the range between the two reserve policies is 11.2% of all students, which corresponds to 938 students. The range is 9.3% for Grade K2 and 5.4% for Grade 6. Consistent with Proposition 5, a higher reserve size corresponds to greater walk-zone assignment.

The Walk-Open precedence order has all walk-zone seats precede open seats, while the Open-Walk precedence has all open seats precede walk-zone seats. Consistent with Proposition 5, Table 2 shows that with a 50-50 seat split, more walk-zone students are assigned under Open-Walk precedence than under Walk-Open precedence. Moreover, the outcome of the Actual BPS policy is nearly identical to Walk-Open. Table 2 also shows that the outcome of Walk-Open with 50% Walk is very close to 0% Walk, whereas the outcome of Open-Walk with 50% Walk is substantially different than 100% Walk. For Grade K1, the range between the two extremal precedence policies is 8.3%, or 691 students. This range corresponds to roughly three-quarters of the range from the two extreme reserve policies. For Grade K2, the precedence range is also three-quarters of the reserve range, while for Grade 6 it is about two-thirds. These empirical observations show that the maximal effect of precedence policies is nearly as large as those corresponding to maximal changes in reserve sizes.

What policy implements BPS’s intended 50-50 compromise? To answer this question, it is worth returning to the example in Figure 1. In Panel (b) of Figure 1, walk-zone applicants who are rejected from the walk half and apply for open slots have systematically worse lottery numbers and are out-numbered by non-walk-zone students. This results in two biases: (1) the walk-zone students who remain have the lowest lottery numbers among walk-zone applicants, leaving them unlikely to be assigned ahead of non-walk-zone applicants, and (2) there are twice as many non-walk-zone applicants as walk-zone applicants in the residual pool for open seats. We refer to the

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12 Appendix E provides details on the sample.

13 Appendix E elaborates on the difference between Actual BPS and Open-Walk.
first phenomenon as random number bias and the second as processing bias.

To examine the random number bias (only), Table 3 investigates the effects of using separate lotteries for the walk-zone and open seats. Column 2 reports on the Walk-Open precedence order with two lottery numbers. Even with two lotteries, there is still processing bias, as the pool of applicants from the walk zone is still depleted by the time the open slots are filled. Walk-Open with two lottery numbers assigns 48.4% of students to walk-zone schools at Grade K1 and is still close to the 46.6% assigned when Walk-Open is used with only one lottery number. That is, Walk-Open with two lottery numbers is much closer to 0% Walk than 100% Walk; this suggests that random number bias accounts for only part of the reason Boston’s assignment outcome is not midway between the 0% Walk and 100% Walk extremes.

Although it eliminates the random number bias, the remedy of using two lottery numbers has an important drawback. It is well-known that using multiple lottery numbers across schools with the deferred acceptance algorithm may generate efficiency losses (Abdulkadiroğlu and Sönmez 2003, Abdulkadiroğlu, Pathak, and Roth 2009). Even though the two lottery numbers are within schools (and not across schools), the same efficiency consequence arises here. The Unassigned row in the table provides indirect evidence from this fact. Comparing Table 2 to Table 3, for each precedence policy under the 50-50 split, there are at least as many unassigned students (and sometimes more) with two lotteries.

Open-Walk eliminates both types of biases because neither the lottery number distribution nor the pool of applicants is affected by application processing at the open half. In the example in Figure 1, Open-Walk would result in 75 students being assigned from the walk zone. Therefore, distributional objectives may need to be accommodated by adjusting the share of reserve slots.

To return to the Boston policy discussion, we conclude our investigation by examining how far the Boston system was from a 50-50 compromise. Table 4 computes the adjustment to the reserve size under Open-Walk that would correspond to BPS’s implementation of the 50-50 reserve. Depending on the grade, BPS’s implementation corresponds to Open-Walk with roughly 5-10% walk-zone reserve share. For Grade K1, the actual BPS implementation gives 47.2% of students walk-zone assignments; this is just above the Open-Walk treatment with a 5% walk-zone reserve (46.9%), but below the Open-Walk treatment with 10% walk-zone reserve (47.6%). For Grade K2, the actual BPS implementation has 48.5% walk-zone assignment, a figure close to the Open-Walk treatment with a 10% walk-zone reserve. For Grade 6, the actual BPS implementation is bracketed by Open-Walk with 5% and 10% walk-zone reserve. An unbiased implementation of the BPS implementation would place it a substantial distance away from a 50-50 compromise and closer to a 10-90 compromise.

6 Conclusion

Admissions policies in which applicants can be admitted to more than one type of seat raise questions associated with how seats should be processed. We have shown how both reserves and precedence are policy tools that have qualitatively similar impacts on school admission outcomes.
We have also examined how those findings results generalize to centralized assignment systems.

Our analysis resolved a puzzle underlying a policy debate in Boston. Many groups in Boston felt that the BPS school assignment system did not put enough weight on children attending schools closer to their homes—seemingly at odds with the stated policy reserving half of each school’s seats for walk-zone applicants. The resolution of this puzzle hinges on the important and surprising role played by the precedence order.

In addition to our comparative static results, our empirical analysis shows how the chosen precedence order effectively undermined the policy goal of the 50-50 slot split in Boston. Moreover, our empirical results establish that in Boston, the precedence order (1) is an important lever for achieving distributional objectives, and (2) has quantitative impacts almost as large as changes in the size of the walk-zone reserve.

The impact of precedence order choice on admissions was not understood at the time of Boston’s 50-50 compromise, and it is clear that Boston did not intend to choose a precedence order that undermined the walk-zone reserve (EAC 2013). When our work first made the unintended consequences of Boston’s precedence choice clear, our findings were seized upon by interest groups on all sides of the Boston school choice debate. Neighborhood schooling advocates were upset to learn that the precedence order had rendered the walk-zone reserve ineffective. School choice proponents, by contrast, pushed to maintain the Walk-Open precedence order, or to eliminate the walk-zone reserve entirely. (For details on the policy discussions and the impact of our research, see Appendix D.) Central to our own view was the need to encourage transparency: it is not sufficient to express the reserve policy without also specifying the precedence order.

Boston Superintendent Carol Johnson (2013) proposed eliminating walk-zone priority entirely, since it had not been working as intended, and because the new choice menus (Shi 2013) baked in a form of geographic preference under which students could only apply to schools close to their home. The new BPS admissions policy took effect for placing elementary and middle school students in the 2013-14 school year.
## Table 1. Difference between the Current Boston Mechanism and Alternative Walk Zone Splits

<table>
<thead>
<tr>
<th>Grade</th>
<th>Difference relative to current BPS</th>
<th>Grade</th>
<th>Difference relative to current BPS</th>
<th>Grade</th>
<th>Difference relative to current BPS</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>0% Walk (1)</td>
<td>K2</td>
<td>0% Walk (4)</td>
<td>K1</td>
<td>0% Walk (7)</td>
</tr>
<tr>
<td></td>
<td>100% Walk (2)</td>
<td></td>
<td>100% Walk (5)</td>
<td></td>
<td>100% Walk (8)</td>
</tr>
<tr>
<td></td>
<td># students (1)</td>
<td></td>
<td># students (4)</td>
<td></td>
<td># students (7)</td>
</tr>
<tr>
<td>2009</td>
<td>1770</td>
<td>1715</td>
<td>28</td>
<td>2348</td>
<td>54</td>
</tr>
<tr>
<td></td>
<td>46% 3% 19%</td>
<td></td>
<td>343</td>
<td></td>
<td>205</td>
</tr>
<tr>
<td>2010</td>
<td>1977</td>
<td>1902</td>
<td>62</td>
<td>2308</td>
<td>41</td>
</tr>
<tr>
<td></td>
<td>68% 3% 20%</td>
<td></td>
<td>269</td>
<td></td>
<td>171</td>
</tr>
<tr>
<td>2011</td>
<td>2071</td>
<td>1821</td>
<td>90</td>
<td>2073</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>50% 2% 19%</td>
<td></td>
<td>293</td>
<td></td>
<td>225</td>
</tr>
<tr>
<td>2012</td>
<td>2515</td>
<td>2301</td>
<td>101</td>
<td>2057</td>
<td>24</td>
</tr>
<tr>
<td></td>
<td>88% 3% 20%</td>
<td></td>
<td>403</td>
<td></td>
<td>247</td>
</tr>
<tr>
<td>All</td>
<td>8333</td>
<td>7739</td>
<td>281</td>
<td>8786</td>
<td>123</td>
</tr>
<tr>
<td></td>
<td>252% 3% 19%</td>
<td></td>
<td>1308</td>
<td></td>
<td>848</td>
</tr>
</tbody>
</table>

Notes. Table reports fraction of applicants whose assignments differ between the mechanism currently employed in Boston and two alternative mechanisms: one with a priority structure without walk-zone priorities at any seats (0% Walk), and the other with a priority structure with walk-zone priorities at all seats (100% Walk).
Table 2. Number of Students Assigned to School in Walk Zone, One Lottery Number

<table>
<thead>
<tr>
<th></th>
<th>Priorities = 0% Walk</th>
<th>Priorities = 50% Walk Changing Precedence</th>
<th>Priorities = 100% Walk</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
</tr>
<tr>
<td><strong>Walk Zone</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3849</td>
<td>3879</td>
<td>3930</td>
</tr>
<tr>
<td></td>
<td>46.2%</td>
<td>46.6%</td>
<td>47.2%</td>
</tr>
<tr>
<td><strong>Outside Walk Zone</strong></td>
<td>2430</td>
<td>2399</td>
<td>2353</td>
</tr>
<tr>
<td></td>
<td>29.2%</td>
<td>28.8%</td>
<td>28.2%</td>
</tr>
<tr>
<td><strong>Unassigned</strong></td>
<td>2054</td>
<td>2055</td>
<td>2050</td>
</tr>
<tr>
<td></td>
<td>24.6%</td>
<td>24.7%</td>
<td>24.6%</td>
</tr>
<tr>
<td><strong>II. Grade K2</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Walk Zone</strong></td>
<td>3651</td>
<td>3685</td>
<td>3753</td>
</tr>
<tr>
<td></td>
<td>47.2%</td>
<td>47.6%</td>
<td>48.5%</td>
</tr>
<tr>
<td><strong>Outside Walk Zone</strong></td>
<td>2799</td>
<td>2764</td>
<td>2694</td>
</tr>
<tr>
<td></td>
<td>36.2%</td>
<td>35.7%</td>
<td>34.8%</td>
</tr>
<tr>
<td><strong>Unassigned</strong></td>
<td>1289</td>
<td>1290</td>
<td>1292</td>
</tr>
<tr>
<td></td>
<td>16.7%</td>
<td>16.7%</td>
<td>16.7%</td>
</tr>
<tr>
<td><strong>III. Grade 6</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Walk Zone</strong></td>
<td>3439</td>
<td>3476</td>
<td>3484</td>
</tr>
<tr>
<td></td>
<td>39.1%</td>
<td>39.6%</td>
<td>39.7%</td>
</tr>
<tr>
<td><strong>Outside Walk Zone</strong></td>
<td>4782</td>
<td>4750</td>
<td>4743</td>
</tr>
<tr>
<td></td>
<td>54.4%</td>
<td>54.1%</td>
<td>54.0%</td>
</tr>
<tr>
<td><strong>Unassigned</strong></td>
<td>565</td>
<td>560</td>
<td>559</td>
</tr>
<tr>
<td></td>
<td>6.4%</td>
<td>6.4%</td>
<td>6.4%</td>
</tr>
</tbody>
</table>

Notes. Table reports fraction of applicants assigned to walk-zone schools under several alternative assignment procedures from 2009-2012. 0% Walk implements the student-proposing deferred acceptance mechanism with no walk zone priority; 100% Walk implements the student-proposing deferred acceptance mechanism with all slots having walk-zone priority. Columns (2)-(4) hold the 50-50 school seat split fixed. Walk-Open implements the precedence order in which all walk-zone slots are ahead of open slots. Actual BPS implements BPS’s system. Open-Walk implements the precedence order in which all open slots are ahead of walk-zone slots.
<table>
<thead>
<tr>
<th></th>
<th>Priorities = 0% Walk</th>
<th>Priorities = 50% Walk Changing Precedence</th>
<th>Priorities = 100% Walk</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
</tr>
<tr>
<td>Walk Zone</td>
<td>3849</td>
<td>4034</td>
<td>4556</td>
</tr>
<tr>
<td></td>
<td>46.2%</td>
<td>48.4%</td>
<td>54.7%</td>
</tr>
<tr>
<td></td>
<td>61.3%</td>
<td>64.5%</td>
<td>72.7%</td>
</tr>
<tr>
<td>Outside Walk Zone</td>
<td>2430</td>
<td>2217</td>
<td>1709</td>
</tr>
<tr>
<td></td>
<td>29.2%</td>
<td>26.6%</td>
<td>20.5%</td>
</tr>
<tr>
<td>Unassigned</td>
<td>2054</td>
<td>2082</td>
<td>2068</td>
</tr>
<tr>
<td></td>
<td>24.6%</td>
<td>25.0%</td>
<td>24.8%</td>
</tr>
</tbody>
</table>

**Notes.** Table reports fraction of applicants assigned to walk-zone schools under several alternative assignment procedures from 2009-2012. 0% Walk implements the student-proposing deferred acceptance mechanism with no walk zone priority; 100% implements the student-proposing deferred acceptance mechanism with all slots having walk-zone priority. Columns (2)-(3) hold the 50-50 school seat split fixed. Walk-Open implements the precedence order in which all walk-zone slots are ahead of open slots, but uses two different lottery numbers for walk and open seats. Open-Walk implements the precedence order in which all open slots are ahead of walk-zone slots, but uses two different lottery numbers for walk and open. The same lottery numbers are used for each simulation.
### Table 4. What Policy Was Being Implemented in Boston?

<table>
<thead>
<tr>
<th>Priorities = 0%</th>
<th>Priorities = 5%</th>
<th>Priorities = 10%</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Walk Zone</strong></td>
<td><strong>Walk Zone</strong></td>
<td><strong>Walk Zone</strong></td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
</tr>
<tr>
<td>Walk Zone</td>
<td>3849</td>
<td>3651</td>
</tr>
<tr>
<td></td>
<td>46.2%</td>
<td>47.2%</td>
</tr>
<tr>
<td>Outside Walk Zone</td>
<td>2430</td>
<td>2799</td>
</tr>
<tr>
<td></td>
<td>29.2%</td>
<td>36.2%</td>
</tr>
<tr>
<td>Unassigned</td>
<td>2054</td>
<td>1289</td>
</tr>
<tr>
<td></td>
<td>24.6%</td>
<td>16.7%</td>
</tr>
</tbody>
</table>

#### I. Grade K1
- **Walk Zone**
  - Priorities = 0%: 3849, 46.2%
  - Priorities = 5%: 3651, 47.2%
  - Priorities = 10%: 3439, 39.1%
- **Outside Walk Zone**
  - Priorities = 0%: 2430, 29.2%
  - Priorities = 5%: 2799, 36.2%
  - Priorities = 10%: 4782, 54.4%
- **Unassigned**
  - Priorities = 0%: 2054, 24.6%
  - Priorities = 5%: 1289, 16.7%
  - Priorities = 10%: 565, 6.4%

#### II. Grade K2
- **Walk Zone**
  - Priorities = 0%: 3651, 47.2%
  - Priorities = 5%: 3692, 47.7%
  - Priorities = 10%: 3439, 39.1%
- **Outside Walk Zone**
  - Priorities = 0%: 2799, 36.2%
  - Priorities = 5%: 2757, 35.6%
  - Priorities = 10%: 4782, 54.4%
- **Unassigned**
  - Priorities = 0%: 3651, 24.6%
  - Priorities = 5%: 1289, 16.7%
  - Priorities = 10%: 565, 6.4%

#### III. Grade 6
- **Walk Zone**
  - Priorities = 0%: 3849, 46.2%
  - Priorities = 5%: 3906, 46.9%
  - Priorities = 10%: 3965, 47.6%
- **Outside Walk Zone**
  - Priorities = 0%: 2430, 29.2%
  - Priorities = 5%: 2369, 28.4%
  - Priorities = 10%: 2304, 27.6%
- **Unassigned**
  - Priorities = 0%: 2054, 24.6%
  - Priorities = 5%: 2058, 24.7%
  - Priorities = 10%: 2064, 24.8%

**Notes.** Table reports fraction of applicants assigned to walk-zone schools under several alternative assignment procedures from 2009-2012. 0% Walk implements the student-proposing deferred acceptance mechanism with no walk zone priority. Open-Walk implements the precedence order in which all open slots are ahead of walk-zone slots. The same lottery numbers are used for each simulation.
A Examples

Example 2 (Swapping the Precedence of Open and a Subsequent Reserve Slot). There are three schools $A = \{k, l, m\}$. School $k$ and $m$ have two slots and school $l$ has three slots. There are seven students $I = \{i_1, i_2, i_3, i_4, i_5, i_6, i_7\}$. Let $I_a$ be the set of reserve-eligible students $a \in A$. Let $I_k = \{i_1, i_7\}$, $I_l = \{i_2, i_3, i_4\}$ and $I_m = \{i_5, i_6\}$. The master priority $\pi^o$ orders the students as:

$\pi^o : i_7 \succ i_2 \succ i_5 \succ i_3 \succ i_1 \succ i_6 \succ i_4 \succ i_2$.

The preference profile is:

\[
\begin{array}{ccccccc}
p^{i_1} & p^{i_2} & p^{i_3} & p^{i_4} & p^{i_5} & p^{i_6} & p^{i_7} \\
 k & m & l & l & k & l & k \\
 l & m & l & m & & & \\
\end{array}
\]

First consider the case where school $k$’s first and school $l$’s second slots are reserve slots and all other slots are open.

The outcome of DA for this case is:

$\mu_1 = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 \\ l & m & l & l & k & m & k \end{pmatrix}$.

Observe that, in addition to the two reserve slots assigned to reserve-eligible students $i_3$ and $i_4$, two of the open slots (namely those assigned to students $i_6$ and $i_7$) are also assigned to reserve-eligible students. As such, four reserve-eligible students are assigned to schools for which they obtain a reserve.

Next change the order of precedence at school $k$ so that its open slot has higher precedence than a reserve slot. We keep the slot types and precedence for the other schools the same.

The outcome of DA for the second case is:

$\mu_2 = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 \\ k & m & l & m & l & l & k \end{pmatrix}$.

Observe that, while all three reserve slots are assigned to reserve-eligible students (i.e., students $i_1, i_3, i_7$), none of the open slots at any school are assigned to reserve-eligible students of that school. That is, the total number of reserve-eligible student assignments decreases when the open slot at school $k$ precedes the reserve slot.

Example 3 (Uniform Priority). There are three schools $A = \{a, b, c\}$. School $a$ and $b$ have two available slots and school $l$ has one slot. There are seven students $I = \{i_1, i_2, i_3, i_4, i_5, i_6\}$. We will say that the reserve-eligible students are $I_a = I_b = I_l = \{i_1, i_2, i_3, i_4\}$, while students $i_5$ and $i_6$ are reserve-ineligible. The master priority is $\pi^o : i_1 \succ i_5 \succ i_2 \succ i_3 \succ i_6 \succ i_4$. The preference profile is:
First, consider the case where school $a$'s first slot is reserve and second slot is open, school $b$'s first slot is open and second slot is reserve, and school $c$'s slot is open. The outcome of DA is:

$$\mu_1 = \left( i_1, i_2, i_3, i_4, i_5, i_6 \right).$$

Next, suppose that school $a$ second slot is reserve. The outcome of DA is:

$$\mu_2 = \left( i_1, i_2, i_3, i_4, i_5, i_6 \right).$$

Alternatively, suppose that school $a$'s first slot is open and the second slot is reserve. This corresponds to a swap of the precedence compared to the first case. The outcome of DA is:

$$\mu_3 = \left( i_1, i_2, i_3, i_4, i_5, i_6 \right).$$

Observe that in the second and third case, there is one fewer reserve-eligible student assigned (student $i_4$) when either an open slot is replaced by a reserve slot or the precedence position of a reserve slot is replaced by an open slot. This example shows that it is possible that reserve-eligible assignment can decrease with these changes even under uniform priority. □

Without a uniform priority, it is possible that when a reserve slot is replaced with an open slot, the total reserve-eligible assignment decreases by more than one. To see this fact, suppose that there are two more schools, $d$ and $e$ each with one seat, and two more students 7 and 8. Suppose that $I_a = i_1, i_2$, $I_b = i_3, i_4$, $I_c = i_5, i_8$, $I_d = i_6$, and $I_e = i_7$. Let the master priority be:

$$\pi^0 : i_1 \succ i_5 \succ i_2 \succ i_3 \succ i_6 \succ i_7 \succ i_8 \succ i_4.$$

Suppose that both school $d$ and $e$ have open slots. Modify the preference of student $i_6$ and specify the preference of student $i_7$ and $i_8$ as follows:

<table>
<thead>
<tr>
<th>$P_{i_6}$</th>
<th>$P_{i_7}$</th>
<th>$P_{i_8}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>$d$</td>
<td>$e$</td>
</tr>
<tr>
<td>$a$</td>
<td>$e$</td>
<td></td>
</tr>
<tr>
<td>$b$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$e$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
First consider the case where school $k$'s first and school $l$'s second slots are reserve slots and all other slots are open slots. The outcome of DA for this case is:

$$\mu_1 = \left(\begin{array}{cccccccc} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 & i_8 \\ a & c & b & b & a & \emptyset & e & \emptyset \end{array}\right).$$

Observe that five reserve-eligible students are assigned to schools for which they obtain a reservation.

Next we replace the open slot at school $k$ with a reserve slot, so that both slots at school $k$ are reserve slots. We keep the slot types and precedence for the other schools the same. The outcome of DA for this case is:

$$\mu_2 = \left(\begin{array}{cccccccc} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 & i_8 \\ a & a & b & \emptyset & b & c & d & e \end{array}\right).$$

Observe that three reserve-eligible students are assigned to schools for which they obtain a reservation.

**B Proofs**

**B.1 Proof of Proposition 1 and Preliminaries for Proposition 3.(i)**

We prove the following lemma which implies Proposition 1; this lemma is then used in the proof of Proposition 3.(i).

**Lemma 1.** For any set of students $\bar{I} \subseteq I$, as pictured in Figure 2:

1. All reserve-eligible students of $a$ that are chosen from $\bar{I}$ under choice function $C^a$ are chosen under choice function $D^a$ (i.e. $[(C^a(\bar{I})) \cap I_a] \subseteq [(D^a(\bar{I})) \cap I_a]$). Moreover, $|[\{(D^a(\bar{I})) \cap I_a\} \backslash [(C^a(\bar{I})) \cap I_a]| \leq 1$.

2. All reserve-ineligible students of $a$ that are chosen from $\bar{I}$ under choice function $D^a$ are chosen under choice function $C^a$ (i.e. $[(D^a(\bar{I})) \cap (I \setminus I_a)] \subseteq [(C^a(\bar{I})) \cap (I \setminus I_a)]$). Moreover, $|[\{(C^a(\bar{I})) \cap (I \setminus I_a)\} \backslash [(D^a(\bar{I})) \cap (I \setminus I_a)]| \leq 1$.

**Proof.** We proceed by induction on the number $q_a$ of slots at $a$. The base case $q_a = 1$ is immediate, as then $S^a = \{s\}$ and $C^a(\bar{I}) \neq D^a(\bar{I})$ if and only if a reserve-eligible student of $a$ is assigned to $s$ under $D$, but a reserve-ineligible student is assigned to $s$ under $C$, that is, if $D^a(\bar{I}) \subseteq I_a$ while $C^a(\bar{I}) \subseteq I \setminus I_a$. It follows immediately from this observation that $|[\{C^a(\bar{I}) \cap I_a\} \subseteq [(D^a(\bar{I})) \cap I_a]$,

$$\|[D^a(\bar{I})] \cap I_a\] \backslash [(C^a(\bar{I})) \cap I_a] \leq 1,$$ 

$$|[D^a(\bar{I}) \cap (I \setminus I_a)] \subseteq [(C^a(\bar{I})) \cap (I \setminus I_a)],$$

and $\|[C^a(\bar{I})] \cap (I \setminus I_a)] \subseteq 1$.

Now, given the result for the base case $q_a = 1$, we suppose that the result holds for all $q_a < \ell$ for some $\ell \geq 1$; we show that this implies the result for $q_a = \ell$. We suppose that $q_a = \ell$, and let $\bar{s} \in S^a$ be the slot which is minimal (i.e., processed/filled last) under the precedence order $\succeq^a$. A
∀I ⊆ I,

\[
\begin{array}{c|c|c}

I_a & I \setminus I_a & \bar{I} \\
\hline
D^a(I) & C^a(I) & \\
\end{array}
\]

Figure 2: Comparison of \(C^a(I)\) and \(D^a(I)\), as described formally in the lemma.

student eligible for one type of slot is also eligible for the other, and hence \(\bar{s}\) is either full in both cases or empty in both cases. Moreover, the result follows directly from the inductive hypothesis in the case if no student is assigned to \(\bar{s}\) (under either priority structure); hence, we assume that

\[
|C^a(\bar{I})| = |D^a(\bar{I})| = q_a = \ell. \tag{1}
\]

If \(\bar{s} = s\), then the result follows just as in the base case: It is clear from the algorithms defining \(C^a\) and \(D^a\) that in the computations of \(C^a(\bar{I})\) and \(D^a(\bar{I})\), the same students are assigned to slots \(s\) with higher precedence than \(s = \bar{s}\) (i.e., slots \(s\) with \(s \triangleright^a s = \bar{s}\)), as those slots’ priorities and relative precedence ordering are fixed. Thus, as in the base case, \(C^a(\bar{I}) \neq D^a(\bar{I})\) if and only if a reserve-eligible student of \(a\) is assigned to \(s\) under \(D\), but a reserve-ineligible student is assigned to \(s\) under \(C\).

If \(\bar{s} \neq s\), then \(s \triangleright^a \bar{s}\). We let \(J \subseteq \bar{I}\) be the set of students assigned to slots in \(S^a \setminus \{\bar{s}\}\) in the computation of \(C^a(\bar{I})\), and let \(K \subseteq \bar{I}\) be the set of students assigned to slots in \(S^a \setminus \{\bar{s}\}\) in the computation of \(D^a(\bar{I})\).

The first \(q_a - 1 = \ell - 1\) slots of \(a\) can be treated as a school with slot-set \(S^a \setminus \{\bar{s}\}\) (under the precedence order induced by \(\triangleright^a\)). Thus, the inductive hypothesis (in the case \(\ell - 1\)) implies:

\[
|K \cap I_a| \subseteq [K \cap I_a]; \tag{2}
\]

\[
|[J \cap I_a] \setminus [K \cap I_a]| \leq 1; \tag{3}
\]

\[
[K \cap (I \setminus I_a)] \subseteq [J \cap (I \setminus I_a)]; \tag{4}
\]

\[
|[K \cap (I \setminus I_a)] \setminus [J \cap (I \setminus I_a)]| \leq 1. \tag{5}
\]
If we have equality in (2) and (4), then the set of students available to be assigned to $\bar{s}$ in the computation of $C^a(\bar{I})$ is the same as in the computation of $D^a(\bar{I})$. Thus, as $\pi^s = \bar{\pi}^s$ by assumption, we have $C^a(\bar{I}) = D^a(\bar{I})$; hence, the desired result follows immediately.\footnote{As $|J| = |K|$, by (1), equality holds in one of (2) and (4) if and only if it holds for both inclusions (2) and (4).

If instead the inclusions in (2) and (4) are strict, then by (3) and (5), respectively, we have a unique student $k \in [K \cap I_a] \setminus [J \cap I_a]$ and a unique j student $j \in [J \cap (I \setminus I_a)] \setminus [K \cap (I \setminus I_a)]$. Here, $k$ is reserve-eligible at $a$ and is assigned to a slot $s$ with higher precedence than $\bar{s}$ (i.e. a slot $s$ with $s \equiv^a \bar{s}$) in the computation of $D^a(\bar{I})$, but is not assigned to such a slot in the computation of $C^a(\bar{I})$. Likewise, $j$ is reserve-ineligible at $a$, is assigned to a slot $s$ with $s \equiv^a \bar{s}$ in the computation of $C^a(\bar{I})$, and is not assigned to such a slot in the computation of $D^a(\bar{I})$. By construction, $k$ must be the $\pi^o$-maximal student in $[\bar{I} \setminus J] \cap I_a$ and $j$ must be the $\pi^o$-maximal student in $[\bar{I} \setminus K] \cap (I \setminus I_a)$ (indeed, $j$ is $\pi^a$-maximal in $\bar{I} \setminus K$).

Now:

- If $\bar{s}$ is a reserve slot, then $k$ is assigned to $\bar{s}$ in the computation of $C^a(\bar{I})$; hence, $C^a(\bar{I}) = J \cup \{k\}$. Thus, as $k \in [K \cap I_a] \setminus [J \cap I_a]$, we have $[(C^a(\bar{I})) \cap I_a] \subseteq [(D^a(\bar{I})) \cap I_a]$ by (2), and $|[D^a(\bar{I})] \cap I_a| \leq 1$ by (3). In the computation of $D^a(\bar{I})$, meanwhile, if a reserve-eligible student is not assigned to $\bar{s}$, then $j$ must be assigned to $\bar{s}$, as $j$ is $\pi^o$-maximal among students in $[\bar{I} \setminus K] \cap (I \setminus I_a)$. It follows that $|[D^a(\bar{I})] \cap (I \setminus I_a)| \subseteq |(C^a(\bar{I})) \cap (I \setminus I_a)| \subseteq |D^a(\bar{I})| \cap (I \setminus I_a)|$ (by (4)), and $|[C^a(\bar{I})] \cap (I \setminus I_a)| \subseteq |D^a(\bar{I})| \cap (I \setminus I_a)| \leq 1$ (by (5)).

- If $\bar{s}$ is an open slot, then $j$ is assigned to $\bar{s}$ in the computation of $D^a(\bar{I})$; hence, $D^a(\bar{I}) = K \cup \{j\}$. Thus, as $j \in [J \cap (I \setminus I_a)]$, we have $|[D^a(\bar{I})] \cap (I \setminus I_a)| \subseteq |(C^a(\bar{I})) \cap (I \setminus I_a)|$ by (4), and $|[C^a(\bar{I})] \cap (I \setminus I_a)| \subseteq |D^a(\bar{I})| \cap (I \setminus I_a)| \leq 1$ by (5). Meanwhile, if a reserve-eligible student is assigned to $\bar{s}$ in the computation of $C^a(\bar{I})$, then it must be $k$, as $k$ is $\pi^a$-maximal among students in $[\bar{I} \setminus J] \cap I_a$. It follows that $|(C^a(\bar{I})) \cap I_a| \subseteq |D^a(\bar{I})| \cap I_a| \subseteq |(C^a(\bar{I})) \cap I_a| \subseteq |D^a(\bar{I})| \cap I_a| \subseteq |D^a(\bar{I})| \cap I_a| \leq 1$ by (3).

These observations complete the induction.

\[\Box\]

### B.2 Proof of Proposition 2 and Preliminaries for Proposition 3.(ii)

We prove the following lemma which implies Proposition 2; this lemma is then used in the proof of Proposition 3.(ii).

**Lemma 2.** For any set of students $\bar{I} \subseteq I$, as pictured in Figure 2:

1. All reserve-eligible students at a that are chosen from $\bar{I}$ under choice function $C^a$ are chosen under choice function $D^a$ (i.e. $|(C^a(\bar{I})) \cap I_a| \subseteq |(D^a(\bar{I})) \cap I_a|$). Moreover, $|[D^a(\bar{I})] \cap I_a| \subseteq |(C^a(\bar{I})) \cap I_a| \subseteq |D^a(\bar{I})| \cap I_a| \leq 1$.

\[\footnote{Note that as $C^a(\bar{I}) = D^a(\bar{I})$, we have $|[C^a(\bar{I})] \cap I_a| \subseteq |D^a(\bar{I})| \cap I_a| = 0 \leq 1$, and $|[D^a(\bar{I})] \cap I_a| \subseteq |C^a(\bar{I})| \cap (I \setminus I_a)| = 0 \leq 1$.} \]

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2. All reserve-ineligible students at a that are from $\bar{I}$ chosen under choice function $D^a$ are chosen under choice function $C^a$ (i.e. $[(D^a(\bar{I})) \cap (I \setminus I_a)] \subseteq [(C^a(\bar{I})) \cap (I \setminus I_a)]$). Moreover, $|[(C^a(\bar{I})) \cap (I \setminus I_a)] \setminus [(D^a(\bar{I})) \cap (I \setminus I_a)]| \leq 1$.

Proof. We proceed by induction on the number $q_a$ of slots at $a$.

First, we prove the base case $q_a = 2$. We denote by $i_{s_r}$ and $i_{s_o}$ (resp. $j_{s_r}$ and $j_{s_o}$) the students respectively assigned to slots $s_r$ and $s_o$ in the computation of $C^a(\bar{I})$ (resp. $D^a(\bar{I})$).

Now:

- If $\{i_{s_r}, i_{s_o}\} \subset I_a$, then the ordering under $\pi^o$ must rank $i_{s_r}$ highest among all students $i \in \bar{I}$, and rank $i_{s_o}$ second-highest among all students $i \in \bar{I}$, as otherwise some student $i \in \bar{I}$ with $i \neq i_{s_r}$ would have higher rank than $i_{s_o}$ under $\pi^o$, and would thus have higher claim than $i_{s_o}$ for (open) slot $s_o$ under precedence order $p^a$. But then, $i_{s_r}$ is the $\pi^o$-maximal student in $\bar{I}$ and $i_{s_o}$ is the $\pi^o$-maximal reserve-eligible student in $\bar{I} \setminus \{i_{s_r}\}$; hence, we must have $j_{s_o} = i_{s_r}$ and $j_{s_r} = i_{s_o}$, so that $D^a(\bar{I}) = C^a(\bar{I})$. In this case, $|[(C^a(\bar{I})) \cap (I \setminus I_a)] \setminus [(D^a(\bar{I})) \cap (I \setminus I_a)]| = 0 \leq 1$ and $|[(D^a(\bar{I})) \cap I_a] \setminus [(C^a(\bar{I})) \cap I_a]| = 0 \leq 1$.

- If $\{i_{s_r}, i_{s_o}\} \subset (I \setminus I_a)$, then $\bar{I}$ contains no reserve-eligible students at $a$ (i.e. $\bar{I} \cap I_a = \emptyset$) and $i_{s_r}$ and $i_{s_o}$ are then just the $\pi^o$-maximal reserve-ineligible student in $\bar{I}$. In this case, we find that $j_{s_o} = i_{s_r}$ and $j_{s_r} = i_{s_o}$; hence, $D^a(\bar{I}) = C^a(\bar{I})$. Once again, we have $|[(C^a(\bar{I})) \cap (I \setminus I_a)] \setminus [(D^a(\bar{I})) \cap (I \setminus I_a)]| = 0 \leq 1$ and $|[(D^a(\bar{I})) \cap I_a] \setminus [(C^a(\bar{I})) \cap I_a]| = 0 \leq 1$.

- If $i_{s_r} \in I_a$ and $i_{s_o} \in (I \setminus I_a)$, then $i_{s_r}$ is the $\pi^o$-maximal reserve-eligible student at $a$ in $\bar{I}$. If $i_{s_r}$ is also $\pi^o$-maximal among all students in $\bar{I}$, then we have $j_{s_o} = i_{s_r}$. Moreover, in this case either

1. $j_{s_r} \in I_a$, or
2. $i_{s_r}$ is the only reserve-eligible student at $a$ in $\bar{I}$, so that $j_{s_r} = i_{s_o}$.

Alternatively, if $i_{s_r}$ is not $\pi^o$-maximal among all students in $\bar{I}$, then $i_{s_o}$ must be $\pi^o$-maximal among all students in $I_a$, so that $j_{s_o} = i_{s_o}$ and $j_{s_r} = i_{s_r}$.

We therefore find that

$$[(C^a(\bar{I})) \cap I_a] = \{i_{s_r}\} \subseteq [(D^a(\bar{I})) \cap I_a];$$

hence, $|[(C^a(\bar{I})) \cap (I \setminus I_a)] \setminus [(D^a(\bar{I})) \cap (I \setminus I_a)]| \leq 1$. Additionally, we have

$$[(D^a(\bar{I})) \cap (I \setminus I_a)] \subseteq \{i_{s_o}\} = [(C^a(\bar{I})) \cap (I \setminus I_a)],$$

so that $|[(D^a(\bar{I})) \cap I_a] \setminus [(C^a(\bar{I})) \cap I_a]| \leq 1$.

- We cannot have $i_{s_r} \in (I \setminus I_a)$ and $i_{s_o} \in I_a$, as $s_r$ is a reserve slot (and thus gives all students in $I_a$ higher priority than students in $I \setminus I_a$) and $s_r \succ^a s_o$.

\[\text{Note that the setup requires at least two distinct slots of } a, \text{ so } q_a = 2 \text{ a priori.}\]
The preceding four cases are exhaustive and the desired result holds in each; thus, we have the base case for our induction.

Now, given the result for the base case \( q_a = 2 \), we suppose that the result holds for all \( q_a < \ell \) for some \( \ell \geq 2 \); we show that this implies the result for \( q_a = \ell \). Thus, we suppose that \( q_a = \ell \). We let \( \bar{s} \in S^a \) be the slot which is minimal under the precedence order \( \triangleright^a \). A student eligible for one type of slot is also eligible for the other, and hence \( \bar{s} \) is either full in both cases or empty in both cases. Moreover, the result follows directly from the inductive hypothesis in the case if no student is assigned to \( \bar{s} \) (under either priority structure); hence, we assume that

\[
|C^a(\bar{I})| = |D^a(\bar{I})| = q_a = \ell. \tag{6}
\]

If \( \bar{s} = s_o \), then the result follows just as in the base case, as it is clear from the procedures defining \( C^a \) and \( D^a \) that the same students are assigned to slots \( s \) with higher precedence than \( s_r \) (i.e. slots \( s \) with \( s \triangleright^o s_r \triangleright^a s_o = \bar{s} \)) in the computations of \( C^a(\bar{I}) \) and \( D^a(\bar{I}) \).

If \( \bar{s} \neq s_o \), then \( s_r \triangleright^o s_o \triangleright^a \bar{s} \). We let \( J \subseteq \bar{I} \) be the set of students assigned to slots in \( S^a \setminus \{\bar{s}\} \) in the computation of \( C^a(\bar{I}) \), and let \( K \subseteq \bar{I} \) be the set of students assigned to slots in \( S^a \setminus \{\bar{s}\} \) in the computation of \( D^a(\bar{I}) \). The first \( \ell - 1 \) slots of \( a \) can be treated as a school with slot-set \( S^a \setminus \{\bar{s}\} \) (under the precedence order induced by \( \triangleright^a \)). Thus, the inductive hypothesis (in the case \( q_a = \ell - 1 \)), implies:

\[
|J \cap I_a| \subseteq |K \cap I_a|; \tag{7}
\]
\[
|[J \cap I_a] \setminus [K \cap I_a]| \leq 1; \tag{8}
\]
\[
|K \cap (I \setminus I_a)| \subseteq |J \cap (I \setminus I_a)|; \tag{9}
\]
\[
|[K \cap (I \setminus I_a)] \setminus [J \cap (I \setminus I_a)]| \leq 1. \tag{10}
\]

If we have equality in (7) and (9),\(^{17}\) then the set of students available to be assigned to \( \bar{s} \) in the computation of \( C^a(\bar{I}) \) is the same as in the computation of \( D^a(\bar{I}) \). Thus, as \( \pi^s = \bar{\pi}^s \) by assumption, we have \( C^a(\bar{I}) = D^a(\bar{I}) \); hence, the desired result follows immediately.\(^{18}\)

If instead the inclusions in (7) and (9) are strict, then by (8) and (10), respectively, we have a unique student \( k \in [K \cap I_a] \setminus [J \cap I_a] \) and a unique student \( j \in [J \cap (I \setminus I_a)] \setminus [K \cap (I \setminus I_a)] \). Here, \( k \) is reserve-eligible at \( a \) and is assigned to a slot \( s \) with higher precedence than \( \bar{s} \) (i.e. a slot \( s \) with \( s \triangleright^o \bar{s} \) in the computation of \( D^a(\bar{I}) \)) but is not assigned to such a slot in the computation of \( C^a(\bar{I}) \). Likewise, \( j \) is reserve-ineligible at \( a \), is assigned to a slot \( s \) with higher precedence than \( \bar{s} \) (i.e. a slot \( s \) with \( s \triangleright^o \bar{s} \)) in the computation of \( C^a(\bar{I}) \), and is not assigned to such a slot in the computation of \( D^a(\bar{I}) \). By construction, \( k \) must be the \( \pi^o \)-maximal student in \( [\bar{I} \setminus J] \cap I_a \) and \( j \) must be the \( \pi^o \)-maximal student in \( [\bar{I} \setminus K] \cap (I \setminus I_a) \) (indeed, \( j \) is \( \pi^o \)-maximal in \( \bar{I} \setminus K \)).

Now:

- If \( \bar{s} \) is a reserve slot, then \( k \) is assigned to \( \bar{s} \) in the computation of \( C^a(\bar{I}) \); hence, \( C^a(\bar{I}) = J \cup \{k\} \). Thus, as \( k \in [K \cap I_a] \), we have \( [(C^a(\bar{I})) \cap I_a] \subseteq [(D^a(\bar{I})) \cap I_a] \) by (7), and \( |[(D^a(\bar{I})) \cap I_a]| \leq 1 \).
to schools is completely novel, is construction of a copy economy to DA but easier to analyze in our framework. The second, which to the best of our knowledge (Milgrom 2005, Hatfield and Kojima 2010), a stable matching algorithm which is outcome-equivalent to DA.

In this section, we prove Proposition 3.(i) and 3.(ii) using a completely parallel argument for the two results. Our proof makes use two technical machinery components. The first, which is well-known in the matching literature, is the cumulative offer process (Kelso and Crawford 1982, Hatfield and Milgrom 2005, Hatfield and Kojima 2010), a stable matching algorithm which is outcome-equivalent to DA but easier to analyze in our framework. The second, which to the best of our knowledge is completely novel, is construction of a copy economy, a setting in which each student $i$ who is rejected by school $a$ are replaced by two “copies”—a top copy $i^a$ who takes the role of $i$ in applying to schools $i$ weakly prefers to $a$, and a bottom copy $i^b$ who takes the role of $i$ in applying to schools $i$ likes less than $a$.

Because bottom copies $i^b$ can act independently of top copies $i^a$ and the cumulative offer process is independent of proposal order, constructing copies enables us to track how the market responds to rejection of students $i$ by $a$ before $i$ (or rather, $i^a$) applies to $a$.

### B.3.1 The Cumulative Offer Process

**Definition.** In the cumulative offer process under choice functions $\tilde{C}$, students propose to schools in a sequence of steps $\ell = 1, 2, \ldots$:

- **Step 1.** Some student $i^1 \in I$ proposes to his favorite school $a^1$. Set $\tilde{A}^2_{a^1} = \{i^1\}$, and set $\tilde{A}^2_a = \emptyset$ for each $a \neq a^1$; these are the sets of students available to schools at the beginning of Step 2. Each school $a \in A$ holds $\tilde{C}^a(\tilde{A}^2_a)$ and rejects all other students in $\tilde{A}^2_a$.

- **Step $\ell$.** Some student $i^\ell \in I$ not currently held by any school proposes to his most-preferred school that has not yet rejected him, $a^\ell$. Set $\tilde{A}^{\ell+1}_{a^\ell} = \tilde{A}^\ell_{a^\ell} \cup \{i^\ell\}$, and set $\tilde{A}^{\ell+1}_a = \tilde{A}^\ell_a$ for each $a \neq a^\ell$. Each school $a \in A$ holds $\tilde{C}^a(\tilde{A}^{\ell+1}_a)$ and rejects all other students in $\tilde{A}^{\ell+1}_a$. 

These observations complete the induction. \qed
If at any Step $\ell + 1$ no student is able to propose—that is, if all students not on hold have proposed to all schools they find acceptable—then the process terminates. The \textit{outcome of the cumulative offer process} is the match $\bar{\mu}$ that assigns each school $a \in A$ the students it holds at the end of the last step before termination: $\bar{\mu}_a = \bar{C}^a(\bar{A}_{\ell+1})$.

In our context, the cumulative offer process outcome is independent of the proposal order and is equal to the outcome of the student-optimal stable mechanism (see Hatfield and Kojima (2010) and Kominers and Sönmez (2013)).

\subsection*{B.3.2 Copy Economies}

We denote by $\bar{\mu}$ the outcome of cumulative offer process under choice functions $\bar{C}$ (associated to priorities $\bar{\pi}$). For a fixed school $a \in A$, we let $R_a \subseteq I$ be the set of students rejected by $a$ during the cumulative offer process under choice functions $\bar{C}^a$. Formally, we have $R_a = \{ i \in I : a P^i \bar{\mu}_i \}$. We fix some $\hat{R}_a \subseteq R_a$ and construct a \textit{copy economy} with set of schools $A$, set of slots $S$, and precedence order profile $\triangleright$. The set of students in the copy economy, denoted $\hat{I}$, is obtained by replacing each student $i \in \hat{R}_a$ with a \textit{top copy} $i^t$ and a \textit{bottom copy} $i^b$:

$$\hat{I} = (I \setminus \hat{R}_a) \cup \left( \bigcup_{i \in \hat{R}_a} \{ i^t, i^b \} \right).$$

The relationship between the set $\hat{I}$ of students in the copy economy and the set $I$ of students in the original economy is pictured in Figure 3. Note that we suppress the dependence of $\hat{I}$ on $\hat{R}_a$, as the set $\hat{R}_a$ under consideration is clearly identified whenever we undertake a copy economy construction. For a copy $i^c$ of agent $i$ (where $c \in \{ t, b \}$), we say that $i$ is the agent \textit{underlying} $i^c$.

Copies’ preferences correspond to specific (post- and pre-)truncations of their underlying agents’ preferences, as pictured in Figure 4. The preference relation $P^{i^t}$ of the top copy of $i$ is defined so that

- $a P^{i^t} a' \iff a P^i a'$ for all $a, a' \in A \setminus \{ a_0 \}$, and
- $a_0 P^{i^t} a \iff a P^i a$ for all $a \in A$.

That is, $P^{i^t}$ is the “top part” of $P^i$ that ranks all the schools which $i$ (weakly) prefers to $a$.\footnote{This observation follows from the facts—proven respectively in Proposition 3 and Lemma D.1 of Kominers and Sönmez (2013).}

The preference relation $P^{i^b}$ of the bottom copy of $i$ is defined similarly, with

- $a R^{i^b} a \implies a_0 P^{i^b} a$ for all $a \in A$.
- $a P^{i^b} a' a \implies a P^{i^b} a'$ for all $a, a' \in A$.

That is, $P^{i^b}$ is the “bottom part” of $P^i$ that ranks all the schools which $i$ likes (strictly) less than $a$.\footnote{This observation follows from the facts—proven respectively in Proposition 3 and Lemma D.1 of Kominers and Sönmez (2013).}

The priorities $\bar{\pi}^s$ in the copy economy (for each $s \in S$) are constructed by replacing agents in $\hat{R}_a$ with their copies as follows:
Figure 3: Relationship between the set $\hat{I}$ of students in the copy economy and the set $I$ of students in the original economy.

$P^i : a^1 \succ \cdots \succ a^\ell \succ a \succ a^\ell+1 \succ \cdots$

Figure 4: Construction of copies’ preference relations: $P^i_t$ is the “top part” of $P^i$ that ranks all the schools which $i$ (weakly) prefers to $a$, while $P^i_b$ is the “bottom part” of $P^i$ that ranks all the schools which $i$ likes (strictly) less than $a$. (Note that we must insert $a_0$ at the end of $P^i_t$.)

- $i^t \pi^s i^b$ for all $i \in \hat{R}_a$;
- $i, i^t \pi^s \tilde{i} \Rightarrow i, \tilde{i} \in I \setminus \hat{R}_a$;
- $i^s i^t \pi^s \tilde{i}^t \pi^s \tilde{i}^b$ for all $i \in I \setminus \hat{R}_a$ and $\tilde{i} \in \hat{R}_a$;
- $\tilde{i}^t \pi^s \pi^s i^b$ for all $i \in I \setminus \hat{R}_a$ and $\tilde{i} \in \hat{R}_a$; and
- $i^t \pi^s \tilde{i}^t \pi^s \tilde{i}^b$ for all $i, \tilde{i} \in \hat{R}_a$.

This construction is illustrated in Figure 5. We write $\hat{C}$ for the choice functions induced by these priorities.

We say that a set of students $\tilde{I} \subseteq I$ is *equivalent up to copies* to a set of students $\hat{I} \subseteq \hat{I}$ if $\tilde{I}$ and $\hat{I}$ share the same set of students in $I \setminus \hat{R}_a$, and the set of students underlying the set of copy students in $\hat{I}$ exactly equals $\tilde{I} \cap \hat{R}_a$. That is, $\tilde{I} \subseteq I$ is equivalent to $\hat{I} \subseteq \hat{I}$ up to copies if

- $[\tilde{I} \cap (I \setminus \hat{R}_a)] = [\hat{I} \cap (I \setminus \hat{R}_a)]$ and

Sönmez (2013)—that the choice functions in our setting satisfy both the substitutability condition of Hatfield and Milgrom (2005) and the irrelevance of rejected contracts condition of (Aygün and Sönmez 2013).

20 Here, $a$ need not be as defined in Sections B.1 and B.2, although it will be in Section B.3.3.

21 Note that $P^i_t$ could be empty, in the case that $a$ is the first choice of student $i$.

22 Note that $P^i_t$ could be empty, in the case that $a$ is least-preferred acceptable school of $i$. 
Figure 5: Construction of priorities in the copy economy: \( \pi^s \) is constructed so that each instance of an agent \( i \in \hat{R}_a \) in priority order \( \pi^s \) is replaced with the “subrelation” \( i^\pi_s i^b \).

- \([\hat{I} \cap \hat{R}_a] = \{ i \in \hat{R}_a \subseteq I : i^c \in \hat{I} \text{ for some } c \in \{t, b\} \} \).

In this case, we write \( \hat{I} \equiv \hat{I} \).

Because the priorities \( \pi^s \) are constructed so that each instance of an agent \( i \in \hat{R}_a \) in priority order \( \pi^s \) is replaced with the “subrelation” \( i^\pi_s i^b \), the following lemma is immediate.

**Lemma 3.** Suppose that \( \hat{I} \equiv \hat{I} \), and suppose moreover that for each \( i \in \hat{R}_a \), there is at most one copy of \( i \) in \( \hat{I} \). Then, for each school \( a \in A \), we have \( \hat{C}^a(\hat{I}) \equiv \hat{C}^a(\hat{I}) \).

We let \( \hat{\mu} \) be the outcome of the cumulative offer process in the copy economy. We now show that \( \hat{\mu} \) in a certain sense corresponds to \( \bar{\mu} \) under copy equivalence. Specifically, we show that under \( \hat{\mu} \) (in the copy economy):

- students in \( i \in I \setminus \hat{R}_a \) have the same assignments under \( \bar{\mu} \);
- all the top copies of students in \( \hat{R}_a \) are assigned to \( a_0 \); and
- all the bottom copies of students \( i \in \hat{R}_a \) receive the assignment received by \( i \) under \( \bar{\mu} \).

**Lemma 4.** We have:

- \( \hat{\mu}_i = \bar{\mu}_i \) for any \( i \in (I \setminus \hat{R}_a) \);
- \( \hat{\mu}_t = a_0 \) for any \( i \in \hat{R}_a \); and
- \( \hat{\mu}_b = \bar{\mu}_i \) for any \( i \in \hat{R}_a \).

Consequently, for each \( a \in A \), we have \( \hat{\mu}_a \equiv \bar{\mu}_a \).

**Proof.** We let \( \hat{\Sigma} = \langle (i^1 \rightarrow a^1), (i^2 \rightarrow a^2), \ldots, (i^L \rightarrow a^L) \rangle \) be a (full) proposal sequence that can arise in the cumulative offer process under choice functions \( \bar{C} \) (i.e. in the original economy).\(^{23}\) Now, for each \( \ell \) for which \( i^\ell \in \hat{R}_a \), we let

\[ \hat{i}^\ell = \begin{cases} i^\ell \in \hat{R}_a \rightarrow a^\ell \hat{R}_a^a \rightarrow a \\ \ell^b \rightarrow \hat{a}P_i^\ell a^\ell \end{cases} \]

that is, \( \hat{i}^\ell \) is the copy of \( i^\ell \) who finds school \( a^\ell \) acceptable. For each \( \ell \) for which \( i^\ell \in (I \setminus \hat{R}_a) \), we let \( \hat{i}^\ell = i^\ell \).

\(^{23}\)Since the cumulative offer process outcome is independent of the proposal order, we can analyze an arbitrary proposal sequence for the original economy.
Claim. The proposal sequence

\[ \hat{\Sigma} = \left( (i^1 \to a^1), (i^2 \to a^2), \ldots, (i^\ell \to a^\ell), \ldots, (i^L \to a^L) \right) \oplus \left( (i^1 \to a_0) : i \in \hat{R}_a \right) \]  

(11)

is a valid sequence of proposals for the cumulative offer process under choice functions \( \hat{C} \) (in the copy economy). \(^{24}\)

Proof. For each \( \ell \) and \( a \in A \), we let \( \bar{A}_a^\ell \) denote the sets of available students arising in the cumulative offer process (in the original economy) under proposal order \( \bar{\Sigma} \), and let \( \hat{A}_a^\ell \) denote the sets of available students arising in the cumulative offer process (in the copy economy) under proposal order \( \hat{\Sigma} \). \(^{25}\)

We proceed by induction, showing that

1. \( (i^\ell \to a^\ell) \) is a valid proposal in step \( \ell \) of the cumulative offer process (in the copy economy), and

2. for each \( \ell \leq L \) and \( a \in (A \setminus \{a_0\}) \), we have

\[ \bar{A}_a^{\ell+1} \oplus \hat{A}_a^{\ell+1}. \]  

(12)

Both hypotheses are clearly true in the base case \( \ell = 1 \), so we assume that the hold up to \( \ell \), and show that this implies them in the case \( \ell + 1 \).

The proposal \( (i \to a) \) occurs in the sequence \( \bar{\Sigma} \) at most once, as no student ever proposes to the same school twice in the cumulative offer process. Thus, by our construction of \( \hat{\Sigma} \), we see that there is no student \( i \) for whom two distinct copies propose to some (nonnull) school \( a \in (A \setminus \{a_0\}) \).

It follows that for each \( i \in \hat{R}_a \), there is at most one copy of \( i \) in \( \hat{A}_a^{\ell+1} \). Thus, the conclusion of Lemma 3 applies:

\[ \hat{C}^a(\hat{A}_a^{\ell+1}) \oplus \hat{C}^a(\bar{A}_a^{\ell+1}) \]  

(13)

for each school \( a \in (A \setminus \{a_0\}) \).

Now, if \( i^{\ell+1} \in (I \setminus \hat{R}_a) \), then \( i^{\ell+1} \) is not held by any school \( a \in A \) at the end of step \( \ell \) of the cumulative offer process in the original economy, i.e. \( i^{\ell+1} \notin (\cup_{a \in (A \setminus \{a_0\})} \hat{C}^a(\bar{A}_a^{\ell+1})) \), and \( i^{\ell+1} \) has not proposed to school \( a^{\ell+1} \) by the end of step \( \ell \) of that process. We then see immediately from (13) that \( (i^{\ell+1} \to a^{\ell+1}) = (i^{\ell+1} \to a^{\ell+1}) \) is a valid proposal at step \( \ell + 1 \) of the cumulative offer process in the copy economy. Moreover, it follows from (12) that

\[ \bar{A}_a^{\ell+2} = (\bar{A}_a^{\ell+1} \cup \{i^{\ell+1}\}) \oplus (\hat{A}_a^{\ell+1} \cup \{i^{\ell+1}\}) \oplus (\bar{A}_a^{\ell+1} \cup \{i^{\ell+1}\}) = \bar{A}_a^{\ell+2} \]

for each \( a \in (A \setminus \{a_0\}) \), as desired.

If instead \( i^{\ell+1} \in \hat{R}_a \), then \( i^{\ell+1} \) is not held by any school \( a \in A \) at the end of step \( \ell \) of the cumulative offer process in the original economy, i.e. \( i^{\ell+1} \notin (\cup_{a \in (A \setminus \{a_0\})} \hat{C}^a(\bar{A}_a^{\ell+1})) \), and \( i^{\ell+1} \) has

---

\(^{24}\) Here, \( \oplus \) denotes the concatenation of sequences. Note that the ordering of the proposals in the appended subsequence \( \left\langle (i^1 \to a_0) : i \in \hat{R}_a \right\rangle \) can be arbitrary.

\(^{25}\) Formally, the sets \( \bar{A}_a^\ell \) are given, and we construct the sets \( \hat{A}_a^\ell \) inductively, as we show by induction that \( \hat{\Sigma} \) is a valid cumulative offer process proposal order.
not proposed to school \(a^{\ell+1}\) by the end of step \(\ell\) of that process. From (13), then, we see that no copy of \(i^{\ell+1}\) is held by any school \(a \in (A \setminus \{a_0\})\) at the end of step \(\ell\) of the cumulative offer process in the copy economy. Noting that no top copy \(i^t\) proposes to \(a_0\) until after proposal \((\vec{i}^L \to a^L)\), we see that both the top and bottom copies of \(i^{\ell+1}\) are available to propose at the beginning of step \(\ell + 1\) of the cumulative offer process in the copy economy; hence, \(\vec{i}^{\ell+1}\) is available to propose at step \(\ell + 1\) irrespective of which copy of \(i^{\ell+1}\) he or she is. Combining the preceding observations, we see that \((\vec{i}^{\ell+1} \to a^{\ell+1})\) is a valid proposal at step \(\ell + 1\) of the cumulative offer process in the copy economy. As in the prior case, we then have from (12) that

\[
A_a^{\ell+2} = (A_a^{\ell+1} \cup \{i^{\ell+1}\}) \cap (A_a^{\ell+1} \cup \{i^{\ell+1}\}) = (A_a^{\ell+1} \cup \{i^{\ell+1}\}) = A_a^{\ell+2}.
\]

The preceding observations show that

\[
\left\langle (i^1 \to a^1), (i^2 \to a^2), \ldots, (i^\ell \to a^\ell), \ldots, (i^L \to a^L) \right\rangle
\]

is a valid sequence of proposals for the cumulative offer process in the copy economy. Now, we note that following these proposals, all students in \((I \setminus \hat{R}_a) \cup \{i^b : i \in \hat{R}_a\}\) are held by (possibly null) schools, and all students in \(\{i^t : i \in \hat{R}_a\}\) are available to propose again.

The final proposal of each top copy \(\hat{i}^t \in \{i^t : i \in \hat{R}_a\}\) is \((\hat{i}^t \to a)\), by construction of (14). As only the top copies \(\hat{i}^t \in \{i^t : i \in \hat{R}_a\}\) are available to propose in step \(L + 1\) and those students' preference relations terminate after ranking \(a\), the cumulative offer process in the copy economy is completed by running the sequence of proposals \(\left\langle (i^t \to a^t) : i \in \hat{R}_a \right\rangle\) (in any order).

Now, we observe that under proposal sequence \(\hat{\Sigma}\) as defined by (11):

O1. The last school each student \(i \in (I \setminus \hat{R}_a)\) proposes to is the school that \(i\) proposes to last in the cumulative offer process in the original economy (under proposal sequence \(\Sigma\)).

O2. For each \(i \in \hat{R}_a\), the last school \(i^t\) proposes to is \(a_0\).

O3. For each \(i \in \hat{R}_a\), the last school \(i^b\) proposes to is the school that \(i\) proposes to last in the cumulative offer process in the original economy (under proposal sequence \(\Sigma\)).

Now, any valid cumulative offer process proposal sequence in the copy economy yields the outcome \(\hat{\mu}\). Thus, in particular, we see that \(\hat{\mu}\) is the outcome of the cumulative offer process in the copy economy under proposal sequence \(\hat{\Sigma}\). The desired result then follows from observations O1–O3. □

### B.3.3 Main Argument

In the sequel, we assume the setup of either Section B.1 or Section B.2, let \(C' = D'\) for all schools \(a' \neq a\), and let \(\mu\) and \(\nu\) respectively denote the cumulative offer process outcomes under the choice functions \(C\) and \(D\). We make use of an Adjustment Lemma, which is Lemma 1 for the case of Proposition 3.(i) and Lemma 2 for the case of Proposition 3.(ii). We denote by \(n_a(\hat{\mu}) \equiv |\hat{\mu}_a \cap I_a|\) the number of reserve-eligible students (of \(a\)) assigned to \(a\) in matching \(\hat{\mu}\).

First, we note the following immediate corollary of the Adjustment Lemma.
Lemma 5. For \( \bar{I} \subseteq I \), if \(|\bar{I}| > q_a\), then \(|\bar{I} \setminus (C^a(\bar{I}))| \cap |\bar{I} \setminus (D^a(\bar{I}))| \geq |\bar{I}| - q_a - 1\). \(^{26}\)

Now, we let \( \hat{R}_a \subseteq I \) be the set of students who are rejected from \( a \) in both the cumulative offer process under choice functions \( C \) and the cumulative offer process under choice functions \( D \). As the students in \( \hat{R}_a \) are rejected in the cumulative offer process under both choice functions \( C \) and \( D \), we may consider the copy economy associated to the original economy by the construction introduced in Section B.3.2, for both choice function profiles. We denote by \( \bar{I} \) the set of students in this economy, and denote by \( \hat{C} \) and \( \hat{D} \) the copy economy choice functions associated to \( C \) and \( D \), respectively.

We denote by \( \mu \) and \( \nu \) the cumulative offer process outcomes under choice functions \( C \) and \( D \), respectively. Analogously, we denote by \( \hat{\mu} \) and \( \hat{\nu} \) the cumulative offer process outcomes under choice functions \( \hat{C} \) and \( \hat{D} \). By Lemma 4, we have

- \( \hat{\mu}_i = \mu_i \) and \( \hat{\nu}_i = \nu_i \) for any \( i \in (I \setminus \hat{R}_a) \);
- \( \hat{\mu}_\emptyset = a_0 \) and \( \hat{\nu}_\emptyset = a_0 \) for any \( i \in \hat{R}_a \); and
- \( \hat{\mu}_i = \mu_{i^b} \) and \( \hat{\nu}_i = \nu_{i^b} \) for any \( i \in \hat{R}_a \).

As every student in \( \hat{R}_a \) is rejected from \( a \) in the cumulative offer process under choice functions \( C \), we have

\[
[\mu_a \cap I_a] = [(\mu_a \cap I_a) \setminus \hat{R}_a] = [\mu_a \cap (I_a \setminus \hat{R}_a)] = [\hat{\mu}_a \cap (I_a \setminus \hat{R}_a)] = [\hat{\mu}_a \cap I_a],
\]

where the second-to-last equality follows from the fact that \( \hat{\mu}_i = \mu_i \) for each \( i \in (I_a \setminus \hat{R}_a) \), and the last equality follows because \( [\hat{\mu}_a \cap \hat{R}_a] = \emptyset \). It follows that

\[
n_a(\mu) = |\mu_a \cap I_a| = |\hat{\mu}_a \cap I_a|.
\]

Analogously, we find that

\[
n_a(\nu) = |\nu_a \cap I_a|.
\]

Thus, to show our proposition it suffices to prove that weakly more reserve-eligible students of \( a \) are assigned to \( a \) under \( \hat{\nu} \) than under \( \hat{\mu} \). To show this, we recall that cumulative offer processes are always independent of proposal order, and consider particular orders for the cumulative offer processes under choice functions \( \hat{C} \) and \( \hat{D} \).

Under each process, we first execute as many proposals \((i \rightarrow a)\) as possible with \( i \in \bar{I} \) and \( aP^i a \). Since \( \hat{C} \) and \( \hat{D} \) differ only with respect to \( \hat{C}^a \) and \( \hat{D}^a \), we can use the exact same order of proposals in each cumulative offer processes, in the initial sequence of proposals. Once such proposals are completed, each student in \( \bar{I} \) either

- is on hold at some (possibly null) school \( a \neq a \), or

\(^{26}\)Formally, this is also true in the case that \(|\bar{I}| < q_a + 1\), as then \(|\bar{I} \setminus (C^a(\bar{I}))| \cap |\bar{I} \setminus (D^a(\bar{I}))| < q_a + 1\), so that \(|\bar{I}| - q_a - 1 < 0\).
has proposed to all schools he prefers to \( a \) and is available to propose to \( a \);

we let \( \bar{J} \) be the set of students in the latter of these two categories. By construction, at this stage, the sets of students on hold at schools \( a' \neq a \) are the same in both processes. Also, \( \bar{J} \) contains no bottom copies of any agent \( i \in \bar{R}_a \), as bottom copies do not find \( a \) acceptable.

We continue the cumulative offer processes by having the students in \( \bar{J} \) propose to \( a \) in uninterrupted sequence. Following these proposals, the set of students available to \( a \) is exactly \( \bar{J} \).

If \( |\bar{J}| \leq q_a \), then all students in \( \bar{J} \) are held by \( a \), and both processes terminate after all the students in \( \bar{J} \) have proposed to \( a \). In this case, we have \( \tilde{\mu}_a = \tilde{\nu}_a \) (indeed, \( \tilde{\mu} = \tilde{\nu} \)); hence (16) and (17) together show that \( n_a(\mu) = n_a(\nu) \).

If instead \( |\bar{J}| > q_a \), then we examine the set

\[
\bar{K} \equiv [(\bar{J} \setminus \tilde{C}^a(\bar{J})) \cap (\bar{J} \setminus \tilde{D}^a(\bar{J}))]
\]

of students rejected under both \( \tilde{C}^a \) and \( \tilde{D}^a \) when (exactly) the set of students \( \bar{J} \) is available. By construction, \( \bar{K} \) is copy-equivalent to a subset of \( \bar{R}_a \). Thus, we see that \( \bar{K} \) must consist entirely of top copies of students in \( \bar{R}_a \), as represented in the exterior box of Figure 6. All such copies rank \( a_0 \) directly below \( a \). Hence, we may continue the cumulative offer processes by having all of these students propose to \( a_0 \); we execute all such proposals.

Recall that up to this point, we have executed the cumulative offer processes under choice functions \( \tilde{C} \) and \( \tilde{D} \) in complete, step-by-step parallel. The sets of students held by at each school \( a \neq a \) (including \( a_0 \)) are exactly the same; meanwhile, \( a \) holds \( \tilde{C}^a(\bar{J}) \) in the process under choice functions \( \tilde{C} \) and holds \( \tilde{D}^a(\bar{J}) \) in the process under choice functions \( \tilde{D} \).

Using Lemma 5 and the fact that \( a \) fills all its slots when possible, we can bound the size of \( \bar{K} \): we have

\[ |\bar{J}| - q_a \geq |\bar{K}| \geq |\bar{J}| - q_a - 1, \]

as pictured in the interior of Figure 6.

- If \( |\bar{J}| - q_a = |\bar{K}| \), then (again because \( a \) fills all its slots when possible) we must have \( \tilde{C}^a(\bar{J}) = \tilde{D}^a(\bar{J}) \), as pictured in Figure 7. In this case, the cumulative offer process terminates after the final proposals of students in \( \bar{K} \) (which can be processed in the same order under choice functions \( \tilde{C} \) and \( \tilde{D} \)); we then have \( \tilde{\mu}_a = \tilde{\nu}_a \) (indeed, \( \tilde{\mu} = \tilde{\nu} \)), which again shows that \( n_a(\mu) = n_a(\nu) \).

- If instead \( |\bar{K}| = |\bar{J}| - q_a - 1 \), then \( \tilde{C}^a(\bar{J}) \neq \tilde{D}^a(\bar{J}) \). By the Adjustment Lemma, we see that

\[ |(\tilde{D}^a(\bar{J}) \cap I_a)| > |(\tilde{C}^a(\bar{J}) \cap I_a)|. \tag{18} \]

Moreover, the Adjustment Lemma shows that there is a unique student \( \tilde{i} \in [(\tilde{C}^a(\bar{J}) \cap (I \setminus I_a)) \setminus (\tilde{D}^a(\bar{J}) \cap (I \setminus I_a))] \) and a unique student \( \tilde{j} \in [(\tilde{D}^a(\bar{J}) \cap I_a) \setminus (\tilde{C}^a(\bar{J}) \cap I_a)] \), as pictured in Figure 8.
Figure 6: The structure of the choices of $a$ from $\tilde{J}$ under choice functions $\tilde{D}^a$ and $\tilde{C}^a$, as implied by the Adjustment Lemma.

Figure 7: Case 1: $|\tilde{K}| = |\tilde{J}| - q_a$ and $\tilde{C}^a(\tilde{J}) = \tilde{D}^a(\tilde{J})$. 
First, we suppose that $\tilde{J} \setminus \tilde{K}$ contains no copies of students in $\hat{R}_a$. Then, in particular, neither $\tilde{i}$ nor $\tilde{j}$ is a copy of a student in $\hat{R}_a$. In this case, we know that $\tilde{i}, \tilde{j} \in (\tilde{I} \setminus \hat{R}_a) = (I \setminus \hat{R}_a)$.

As $\tilde{i}$ is rejected by $a$ in the cumulative offer process under choice functions $\hat{D}$, we know that $aP^i \tilde{\nu}_i = \nu_i$; it follows that $\tilde{i}$ is rejected in the cumulative offer process under choice functions $D$. But $\tilde{i} \notin \hat{R}_a$, so we know that $\tilde{i}$ is not rejected in the cumulative offer process under choice functions $C$. Thus, $\mu_i R^\tilde{i} a$. As $\tilde{\nu}_i = \mu_i$ and $\tilde{i}$ proposes to $a$ in the cumulative offer process under choice functions $\tilde{C}$, we find that we must have $\tilde{\nu}_i = a$, that is, $\tilde{i}$ is not rejected by $a$ in the remainder of the cumulative offer process under choice functions $\tilde{C}$. By our choice of $\tilde{i}$, however, we know that $\tilde{i} \in (I \setminus I_a)$ and that $\tilde{i}$ has the lowest rank under $\pi^a$ among all reserve-ineligible students in $\tilde{C}^a(\tilde{J})$. It follows that the number of reserve-ineligible students assigned to $a$ weakly increases throughout the remainder of the cumulative offer process under choice functions $\tilde{C}$, that is,

$$|\tilde{\mu}_a \cap (I \setminus I_a)| \geq |(\tilde{C}^a(\tilde{J})) \cap (I \setminus I_a)|.$$

This implies that

$$|\tilde{\mu}_a \cap I_a| \leq |(\tilde{C}^a(\tilde{J})) \cap I_a|.$$  \hspace{1cm} (19)

Analogously, we find that $\tilde{\nu}_j = a$, which implies that

$$|\tilde{\nu}_a \cap I_a| \geq |(\tilde{D}^a(\tilde{J})) \cap I_a|.$$  \hspace{1cm} (20)
Now, we find that

\[ n_a(\nu) = |\bar{\nu}_a \cap I_a| \]  
\[ \geq |(\bar{D}^a(J)) \cap I_a| \]  
\[ > |(\bar{C}^a(J)) \cap I_a| \]  
\[ \geq |\bar{\mu}_a \cap I_a| \]  
\[ = n_a(\mu), \]

where (21) follows from (17), (22) follows from (20), (23) follows from (19), (24) follows from (18), and (25) follows from (16).

Finally, we consider the case in which \( \bar{J} \setminus \bar{K} \) contains at least one copy of a student in \( \bar{R}_a \). By construction, any such copy must be a top copy.

**Claim.** If some copy \( i^c \notin \{i, j\} \) is in \( \bar{J} \setminus \bar{K} \) for some \( i \in \bar{R}_a \), then at least one of \( i, j \) must be a copy, as well.

**Proof.** We suppose first that the agent \( i \) underlying \( i^c \) is reserve-eligible at \( a \), i.e. \( i \in I_a \), but that \( j \) is not a copy. Then, since \( i^c \in ((\bar{J} \setminus \bar{K}) \setminus \{i, j\}) \), we know that \( i^c \in (\bar{C}^a(J)) \); in particular, \( i \) has higher rank under \( \pi^o \) than \( j \). As \( i \in \bar{R}_a \), we know that \( i \) is rejected in the cumulative offer processes under both choice functions \( C \) and \( D \). It follows that the lower-\( \pi^o \)-ranked student \( j \) must also be rejected in the cumulative offer processes under both choice functions \( C \) and \( D \), as he proposes to \( a \) in each of those processes; hence, we must have \( j \in \bar{R}_a \). This is impossible, since we assumed that \( j \) is not a copy, and all students in \( \bar{R}_a \) are represented by copies in the copy economy. An analogous argument shows that if the agent \( i \) underlying \( i^c \) is reserve-ineligible at \( a \), then \( i^c \) must be a (top) copy.

**Claim.** There is at most one (top) copy in \( \bar{J} \setminus \bar{K} \).

**Proof.** We suppose there are at least two (top) copies in \( \bar{J} \setminus \bar{K} \). By the preceding claim, either \( \tilde{i} \) of \( \tilde{j} \) is a (top) copy. We assume the former case (\( \tilde{i} \) is a copy); the argument in the latter case is analogous. We let \( \bar{i}^c \) be a (top) copy in \( \bar{J} \setminus \bar{K} \) with \( \bar{i}^c \neq \bar{i} \).

As \( \bar{i} \notin D^a(J) \), we may continue the cumulative offer process under choice functions \( D \) (after having all students in \( \bar{K} \) propose to \( a_0 \)) by having \( \bar{i} \) apply to his next-most-preferred school after \( a \)—since \( \bar{i} \) is a top copy, this school is \( a_0 \). At this point, \( \bar{i} \) is held by \( a_0 \). At the end of this cumulative offer process step, \( a \) must hold all the students in \( (\bar{J} \setminus \bar{K}) \setminus \{\bar{i}\} \), or else \( a \) holds fewer than \( q_a \) students. But this means that the process terminates, as all students are held by schools. We therefore have \( \hat{\nu}_a = a \). This contradicts the fact that we must have \( \hat{\nu}_a = a_0 \) (by Lemma 4), as \( \bar{i}^c \) is a top copy.

The preceding claims show that there is exactly one (top) copy in \( \bar{J} \setminus \bar{K} \), and that it is either \( \bar{i} \) or \( \tilde{j} \). We assume the former case (\( \bar{i} \) is a copy); the argument in the latter case is analogous. We
may continue the cumulative offer process under choice functions \( \hat{D} \) (after having all students in \( \hat{K} \) propose to \( a \)) by having \( i \) propose to his next-most-preferred school after \( a \)—since \( i \) is a top copy, this school is \( a_0 \), and the process terminates after the \((i \rightarrow a_0)\) proposal. Then, we have \( \hat{\nu}_a = \hat{D}^a(\hat{J}) \), so

\[
|\hat{\nu}_a \cap (I \setminus I_a)| = |(\hat{D}^a(\hat{J})) \cap (I \setminus I_a)| = |(\hat{C}^a(\hat{J})) \cap (I \setminus I_a)| - 1. \tag{26}
\]

Meanwhile, the student underlying \( i \) has lower rank under \( \pi^a \) than any reserve-ineligible student in \( \hat{J} \setminus \hat{K} \). It follows that \( i \) will be the first student in \( (\hat{J} \setminus \hat{K}) \setminus I_a \) rejected from \( a \)—since \( i \) is a top copy, this school is \( a_0 \) and the process terminates after the \((i \rightarrow a_0)\) proposal. Thus, we see that

\[
|\hat{\mu}_a \cap (I \setminus I_a)| \geq |(\hat{C}^a(\hat{J})) \setminus \{i\} \cap (I \setminus I_a)| = |(\hat{C}^a(\hat{J})) \cap (I \setminus I_a)| - 1. \tag{27}
\]

Combining (26) and (27), we see that

\[
|\hat{\nu}_a \cap (I \setminus I_a)| = |(\hat{D}^a(\hat{J})) \cap (I \setminus I_a)| - 1 \leq |\hat{\mu}_a \cap (I \setminus I_a)|;
\]

it follows that

\[
|\hat{\nu}_a \cap I_a| \geq |\hat{\mu}_a \cap I_a|. \tag{28}
\]

Combining (28) with (16) and (17), we find that \( n_a(\nu) \geq n_a(\mu) \).

### B.4 Proof of Proposition 4

We prove Proposition 4 by further analyzing the proof of Proposition 3—particularly, the end of Section B.3.3—in the case that the set of reserve-eligible students is uniform across schools. As such, we maintain the notations and conventions of Section B.3, and add the additional hypothesis that for any two schools

\[
a', a'' \in A, \text{ we have } I_{a'} = I_{a''}. \tag{29}
\]

By (29), the number of reserve-eligible students matched under any matching \( \hat{\mu} \) is exactly

\[
\left| \bigcup_{a' \in A} (\hat{\mu}_{a'} \cap I_a) \right| ;
\]

hence, in order to prove the result we just need to show that at most one more student in \( I_a \) is matched in \( \mu \) (that is, under \( \hat{C} \)) than in \( \nu \) (that is, under \( \hat{D} \)).

We revisit the construction of \( \hat{J} \) (presented on page 34) and the subsequent construction of \( \hat{K} \) (page 34). If \( |\hat{J}| \leq q_a \) or \( |\hat{K}| = |\hat{J}| - q_a \), then \( \hat{\mu} = \hat{\nu} \) (see page 34), so we have \( \mu = \nu \) by Lemma 4, and the result is immediate.

Now, we consider the case in which \( |\hat{K}| = |\hat{J}| - q_a - 1 \). As all the students in \( \hat{K} \) are top copies, we know that they are assigned to \( a_0 \) in the cumulative offer processes under both \( \hat{C} \) and \( \hat{D} \). Hence,
the cumulative offer process under $\tilde{C}$ will terminate whenever some student applies to either $a_0$ or a school that is not filled to capacity; likewise, the cumulative offer process under $\tilde{D}$ will terminate whenever some student proposes to either $a_0$ or a school that is not filled to capacity.

If the cumulative offer process under $\tilde{D}$ terminates with some student proposing to a school that is not filled to capacity, then all students matched under $\tilde{\mu}$ are matched under $\tilde{\nu}$—the switch from $\tilde{C}$ to $\tilde{D}$ just affects the distribution of students across schools. In this case, by Lemma 4, the number of reserve-eligible students matched in $\mu$ is that same as in $\nu$. If the cumulative offer process under $\tilde{D}$ terminates with a reserve-ineligible student proposing to $a_0$, then all reserve-eligible students matched under $\tilde{\mu}$ are matched under $\tilde{\nu}$, and—again by Lemma 4—weakly more reserve-eligible students are matched in $\nu$ than under $\mu$. Finally, if the cumulative offer process under $\tilde{D}$ terminates with a reserve-eligible student proposing to $a_0$, we have two cases to consider: If the cumulative offer process under $\tilde{C}$ terminates with a reserve-ineligible student proposing to $a_0$ or to a school that is not filled to capacity, then there is at most one more reserve-eligible student matched under $\tilde{\mu}$ than under $\tilde{\nu}$ (and hence, at most one more reserve-eligible student matched under $\mu$ than under $\nu$). If the cumulative offer process under $\tilde{C}$ terminates with a reserve-eligible student proposing to $a_0$ or to a school that is not filled to capacity, then again there is at most one more reserve-eligible student matched under $\tilde{\mu}$ than under $\tilde{\nu}$ (and hence, at most one more reserve-eligible student matched under $\mu$ than under $\nu$).

**B.5 Proof of Proposition 5**

Matchings $\mu$ and $\nu$ are obtained as in Appendix B.3: Either one of the open slots is replaced with a reserve slot, or the precedence position of a reserve slot is switched with that of a subsequent open slot to obtain $D$ from $C$, and $\nu$ and $\mu$ are, respectively, the associated cumulative offer process outcomes.

**Lemma 6.** We have $|\nu_a| = |\mu_a|$ and $|\nu_b| = |\mu_b|$. That is, the number of slots filled at each school is the same under $\mu$ as under $\nu$.

**Proof.** If both of the schools $a$ and $b$ have an empty slot under either matching, stability implies that all students get their first choices under each matching; hence $\nu = \mu$ and the result holds immediately. Likewise, if neither school has an empty slot under either matching, the result holds immediately since then $|\nu_a| = |\mu_a| = |S^a|$ and $|\nu_b| = |\mu_b| = |S^b|$. Hence the only non-trivial case is when, under one of the matchings, one school is full but the other is not.

Without loss of generality, we suppose that under matching $\mu$, school $a$ has an empty slot whereas school $b$ has all its slots full. Then not only does each student who is assigned a slot at school $b$ under matching $\mu$ prefer school $b$ to school $a$, but also there are at least as many students with a first choice of school $b$ as the number of slots at school $b$. Thus by stability school $b$ must fill all its slots under matching $\nu$ as well; hence, $|\nu_b| = |\mu_b| = |S^b|$. By assumption,

- there are at least as many slots as students, and
- all students find both schools acceptable;
therefore, we see that

\[ |\nu_a| = |I| - |\nu_b| = |I| - |\mu_b| = |\mu_a|. \]

This observation completes the proof.

B.5.1 Proof of Proposition 5.(i) and Proposition 5.(ii)

We prove Propositions 5.(i) and 5.(ii) using a completely parallel argument for the two results. We make use of an Adjustment Proposition, which is is Proposition 3.(i) for the case of Proposition 5.(i), and Proposition 3.(ii) for the case of Proposition 5.(ii).

Proposition 6. There is weakly more reserve-eligible assignment under \( \nu \) than under \( \mu \), that is,

\[ n_a(\nu) + n_b(\nu) \geq n_a(\mu) + n_b(\mu). \]

Proof. Without loss of generality, we assume the priority structure of school \( a \) has changed (i.e. that \( a = a \) in the setup of Appendix B.3).

If \( \nu_a = \mu_a \), then we have

\[ \nu_b = I \setminus \nu_a = I \setminus \mu_a = \mu_b, \]

as by assumption

- there are at least as many slots as students, and
- all students find both schools acceptable.

Thus, in this case the result is immediate.

If \( \nu_a \neq \mu_a \),

\[ |\nu_a \cap I_a| = n_a(\nu) > n_a(\mu) = |\mu_a \cap I_a| \]

by the Adjustment Proposition. Therefore Lemma 6 implies that

\[ |\nu_a \cap (I \setminus I_a)| = |\nu_a| - |\nu_a \cap I_a| < |\mu_a| - |\mu_a \cap I_a| = |\mu_a \cap (I \setminus I_a)|, \]

which in turn implies that

\[ |\nu_a \cap I_b| < |\mu_a \cap I_b| \]

as \( I \setminus I_a = I_b \) by assumption. Thus, we see that

\[ n_b(\nu) = |\nu_b \cap I_b| = |I_b| - |\nu_a \cap I_b| > |I_b| - |\mu_a \cap I_b| = |\mu_b \cap I_b| = n_b(\mu) \]

as all students (and in particular all students in \( I_b \)) are matched under both \( \mu \) and \( \nu \). Hence in this case

\[ n_a(\nu) + n_b(\nu) > n_a(\mu) + n_b(\mu); \]

this completes the proof. \( \square \)
B.5.2 Proof of Proposition 5.(iii)

Let $r_a^1$ denote the number of students who rank school $a$ as first choice, and let $r_b^1$ denote the number of students who rank school $b$ as first choice.

We can obtain the outcome of the SOSM by either the student proposing deferred acceptance algorithm or the cumulative offer process. We utilize the former in this proof.

By assumption, $|S_a| + |S_b| \geq |I|$. Thus, as each student has a first choice,

$$|S_a| + |S_b| \geq r_a^1 + r_b^1.$$ 

Hence, either:

1. $|S_a| \geq r_a^1$ and $|S_b| \geq r_b^1$, or
2. $|S_a| > r_a^1$ and $|S_b| < r_b^1$, or
3. $|S_a| < r_a^1$ and $|S_b| > r_b^1$.

In the first case, the student proposing deferred acceptance algorithm terminates in one step and all students receive their first choices under both $\mu$ and $\nu$. Thus, the result is immediate in this case.

The analyses of the second and third cases are analogous, so it suffices to consider the case that $|S_a| > r_a^1$ and $|S_b| < r_b^1$.

Claim. For this case, under both $\mu$ and $\nu$,

- the number of students receiving their first choices is equal to $|S_b| + r_a^1$, and
- the number of students receiving their second choices is equal to $r_b^1 - |S_b|$.

Proof. We consider the construction of either $\mu$ or $\nu$ through the student proposing deferred acceptance algorithm, and observe that school $b$ receives $r_b^1 > |S_b|$ offers in Step 1, holding $|S_b|$ of these while rejecting $r_b^1 - |S_b|$. School $a$, meanwhile, receives $r_a^1 < |S_a|$ offers and holds all of them. In Step 2, all students rejected by school $b$ apply to school $a$, bringing the total number of applicants at school $a$ to $r_a^1 + (r_b^1 - |S_b|)$. As $r_a^1 + (r_b^1 - |S_b|) \leq |S_a|$ by assumption, no student is rejected by school $a$, and the algorithm terminates in Step 2. Hence, under both $\mu$ and $\nu$,

- $|S_b|$ students are assigned to school $b$ as their first choice,
- $r_a^1$ students are assigned to school $a$ as their first choice, and
- $r_b^1 - |S_b|$ students are assigned to school $a$ as their second choice.

These observations show the claim.

The preceding claim shows the result for the second case; since an analogous argument shows the result for the third case, this completes the proof. □
C Official BPS 50-50 Policy

The official document describing the 50-50 policy states (BPS 1999):

Fifty percent walk zone preference means that half of the seats at a given school are subject to walk zone preference. The remaining seats are open to students outside of the walk zone.

RATIONALE: One hundred percent walk zone preference in a controlled choice plan without racial guidelines could result in all available seats being assigned to students within the walk zone. The result would limit choice and access for all students, including those who have no walk zone school or live in walk zones where there are insufficient seats to serve the students residing in the walk zone.

Patterns of parent choice clearly establish that many choose schools outside of their walk zone for many educational and other reasons. […] One hundred percent walk zone preference would limit choice and access for too many families to the schools they want their children to attend. On the other hand, the policy also should and does recognize the interests of families who want to choose a walk zone school.

Thus, I believe fifty percent walk zone preference provides a fair balance.

D Excerpts from Boston Policy Discussion

After a preliminary version of our research became available, Pathak and Sönmez interacted with BPS’s staff. Parts of our research were presented to the Mayor’s twenty-seven-member Executive Advisory Committee (EAC), which was charged with recommending amendments to the BPS school choice program. We explained that the BPS walk-zone priority was not having its intended impact because of the chosen precedence order. The EAC meeting minutes summarized the discussion (EAC 2013):

“A committee member stated that the walk-zone priority in its current application does not have a significant impact on student assignment. The committee member noted that this finding was consistent with anecdotal evidence that the committee had heard from parents.”

Following the presentation, the EAC immediately recommended that BPS switch to a “Compromise” precedence order, which first fills half of the walk-zone slots, then fills all the open slots, and then the fills the second half of the walk-zone slots. The Compromise precedence order attempts to even out the treatment of walk-zone applicants through changes in the order of slots. Initially, when the first few open slots are processed, the walk-zone applicant pool has adversely selected lottery numbers, but this bias becomes less important by the time the last open slots are processed. The meeting minutes state:

“BPS’s recommendation is to utilize the [C]ompromise method in order to ensure that the walk-zone priority is not causing an unintended consequence that is not in stated policy.”
Part of the initial appeal of the Compromise method is the anticipated difficulty of describing a system employing two lottery numbers. The switch to the Compromise treatment would lead to an increase in the number of students assigned to their walk-zone schools. This change, together with the proposals to shrink zones or adopt a plan with smaller choice menus, raised concerns that the equity of access would decrease.

Our discovery about the role of precedence proved so significant that it became part of the fight between those favoring neighborhood assignment and those favoring increased choice. Proponents of neighborhood assignment interpreted our findings as showing that the (unintentional) improper implementation of the 50-50 school split caused hundreds of students to be shut out of their neighborhood schools. They argued that a change in the precedence order would be the only policy consistent with the School Committee’s 1999 policy goals.

School choice proponents seized on our findings for multiple reasons. Some groups, such as the activist Metropolitan Area Planning Council, fought fiercely to keep the 50-50 seat split with the existing precedence order (MAPC 2013):

"The assignment priority given to walk-zone students has profound impacts on the outcomes of any new plan. The possible changes that have been proposed or discussed include increasing the set-aside, decreasing the set-aside, changing the processing order, or even reducing the allowable distance for walk zone priority to less than a mile. Actions that provide additional advantage to walk-zone students are likely to have a disproportionate negative impact on Black and Hispanic students, who are more reliant on out-of-walk-zone options for the quality schools in their basket."

The symbolism of the 50-50 split, combined with BPS’s precedence order, resounded with sophisticated choice proponents because it created the impression that they were giving something away to neighborhood proponents even though they really were not.

Confirming the counterintuitive nature of our results, other parties expressed skepticism on how walk-zone priority as implemented did not have large implications for student assignment. For instance, the City Councillor in charge of education publicly testified (Connolly 2013):

"MIT tells us that so many children in the walk zones of high demand schools ‘flood the pool’ of applicants, and that children in these walk zones get in in higher numbers, so walk zone priority doesn’t really matter."

"Maybe, that is true. But if removing the walk zone priority doesn’t change anything, why change it all?"

In response to this and similar questions, we argued that moving away from the BPS priority/precedence structure would improve transparency and thus make it easier for BPS to target adequate implementation of its policy goals.

Choice proponents also interpreted our findings as an argument for removing walk-zone priority entirely. Indeed, given that walk-zone priority plays a relatively small role (as implemented by
BPS relative to 0% Walk), simply eliminating it might increase transparency about how the system works. Getting rid of walk zone priority altogether avoids the (false) impression that applicants from the walk zone are receiving a boost under the mechanism.

In March 2013, the Boston school committee voted to adopt a new “Home-Based system” proposed by Peng Shi, under which each student receives an individualized choice menu based on his or her home address (Shi 2013). The new plan reduced the number of schools that applicants could rank, but ensured that each applicant was able to rank a sufficient number of highly-rated schools.27 The reduction in the size of the choice menu under the Home-Based system, together with the subtle implementation issues in the role of walk zone in Boston’s historic 50-50 split, led Boston Superintendent Carol Johnson to support the idea of transparency. On March 13, 2013, Dr. Johnson announced (Johnson 2013):

“After viewing the final MIT and BC presentations on the way the walk zone priority actually works, it seems to me that it would be unwise to add a second priority to the Home-Based model by allowing the walk zone priority be carried over.”

...  
“Leaving the walk zone priority to continue as it currently operates is not a good option. We know from research that it does not make a significant difference the way it is applied today: although people may have thought that it did, the walk zone priority does not in fact actually help students attend schools closer to home. The External Advisory Committee suggested taking this important issue up in two years, but I believe we are ready to take this step now. We must ensure the Home-Based system works in an honest and transparent way from the very beginning."

The new plan went into affect for elementary and middle schools in Fall, 2013.

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27 For additional research related to the new plan, see Ashlagi and Shi (2014), Pathak and Shi (2014), and Shi (2014). Seelye (2013) provides a popular account of the public policy debate surrounding assignment zones in Boston.
E Boston Data Appendix

Relative to our two-priority-type model, BPS has three additional priority groups:

1. **guaranteed applicants**, who are typically continuing on at their current schools,

2. **sibling-walk applicants**, who have siblings currently attending a school and live in the walk zone, and

3. **sibling applicants**, who have siblings attending a school and live outside the walk zone.

Under BPS's slot priorities, applicants are ordered as follows:

<table>
<thead>
<tr>
<th>Walk-Zone Slots</th>
<th>Open Slots</th>
</tr>
</thead>
<tbody>
<tr>
<td>Guaranteed</td>
<td>Guaranteed</td>
</tr>
<tr>
<td>Sibling-Walk</td>
<td>Sibling-Walk, Sibling</td>
</tr>
<tr>
<td>Sibling</td>
<td></td>
</tr>
<tr>
<td>Walk</td>
<td>Walk, No Priority</td>
</tr>
<tr>
<td>No Priority</td>
<td></td>
</tr>
</tbody>
</table>

A single random lottery number is used to order students within priority groups, and this number is the same for both types of slots.

We use data covering four years from 2009–2012, when BPS employed a mechanism based on the student-proposing deferred acceptance algorithm. Students interested in enrolling in or switching schools are asked to list schools each January for the first round. Students entering kindergarten can either apply for elementary school at Grade K1 or Grade K2 depending on whether they are four or five years old. Since the mechanism is based on the student-proposing deferred acceptance algorithm and there is no restriction on the number of schools that can be ranked, the assignment mechanism is strategy-proof\(^\text{28}\); BPS informs families of this property on the application form, advising:

> List your school choice in your true order of preference. If you list a popular school first, you won’t hurt your chances of getting your second choice school if you don’t get your first choice (BPS 2012).

Since the BPS mechanism is strategy-proof, we can isolate the effects of changes in priorities and precedence by holding submitted preferences fixed.\(^\text{29}\)

\(^\text{28}\)For analysis of the effects of restricting the number of choices which can be submitted, see the work of Haeringer and Klijn (2009), Calsamiglia, Haeringer, and Klijn (2010), and Pathak and Sönmez (2013).

\(^\text{29}\)As a check on our understanding of the data, we verified that we can re-create the assignments produced by BPS. Across four years and three applicant grades, we can match 98% of the assignments. Based on discussions with BPS, we learned that the reason why we do not exactly re-create the BPS assignment is that we do not have access to BPS’s exact capacity file, and instead must construct it \textit{ex post} from the final assignment. There are small
In the Actual BPS policy (shown in Table 2), applicants with sibling priority who live outside the walk-zone apply to open slots before applying to walk-zone slots. Applicants with sibling priority who live in the walk-zone apply to walk zone slots before applying to open slots, as they would in Walk-Open.
References


