Available online at www.sciencedirect.com

ScienceDirect

Journal of Economic Theory 167 (2017) 206–228

JOURNAL OF Economic Theory

www.elsevier.com/locate/jet

Steady states in matching and bargaining

Mihai Manea [1]

Department of Economics, MIT, United States

Received 22 June 2015; final version received 15 October 2016; accepted 17 October 2016

Available online 3 November 2016

Abstract

We establish the existence of steady states in two classic matching and bargaining models with general trader asymmetries, search processes, and production functions.

JEL classification: C78

Keywords: Steady states; Existence; Bargaining; Search; Random matching; Entry costs

1. Introduction

The steady state assumption is ubiquitous in matching and bargaining models with heterogeneous agents and search frictions (Diamond and Maskin, 1979; Rubinstein and Wolinsky, 1985; Gale, 1987; Shimer and Smith, 2000; Duffie et al., 2005; Atakan, 2006; Satterthwaite and Shneyerov, 2007; Manea, 2011; Lauermann, 2013; Lauermann and Noldeke, 2015). From an applied perspective, steady states provide a natural description of markets in which the relevant variables are stable over time. From a theoretical perspective, stationary models are tractable and provide insights into important market forces. Most of the existing research focuses on equilibrium outcomes contingent on a given stationary distribution of traders in the market and does not explain how such steady states are maintained or provide conditions under which steady

E-mail address: manea@mit.edu.

[1] I thank Dilip Abreu, Jernej Copic, Glenn Ellison, Faruk Gul, Michael Ostrovsky, Robert Shimer, and Lones Smith for comments and Aubrey Clark and Gabriel Kreindler for proofreading.

http://dx.doi.org/10.1016/j.jet.2016.10.003
0022-0531/

states exist. In this paper, we develop a unified approach for establishing the existence of steady states in two standard matching and bargaining models with general random search processes and production functions. The first model contributes to the literature on bargaining in markets and encompasses the stationary settings of Rubinstein and Wolinsky (1985), Gale (1987), and Manea (2011). The second model pertains to search theory, extending the framework of Shimer and Smith (2000) in several directions. We refer to the former model as the bargaining game and to the latter as the matching model.

Following convention, search proceeds in discrete periods in the bargaining game and in continuous time in the matching model, with an infinite horizon in both cases. We assume that there is a finite set of player types in either setting. This assumption is common in the bargaining literature but constitutes a key technical departure from the matching model of Shimer and Smith (2000). Players of each type have a common discount rate. A continuum of players is active in the market at every date. Active players meet available bargaining partners according to a stochastic search process. Each player meets at most one partner at a time, and partners observe each other's type. A production function determines the output that each pair of types can create and share upon meeting. In a steady state, the distribution of player types is constant over time. Steady states are sustained by means of endogenous entry decisions in the bargaining game and through exogenous match dissolutions in the matching model.

For simplicity, we integrate the bargaining protocol with the search process by modeling meetings as ordered pairs in which the first player assumes the role of the proposer and the second acts as the responder. A search process specifies the frequency of meetings for every pair of types as a function of the distribution of types among active players in the market. We assume that the search technology satisfies some minimal regularity conditions. The main assumption is that meeting frequencies vary continuously with the market composition.

The bargaining game is specified as follows. In every period, meetings take place with probabilities determined by the search technology and the distribution of active players in the market. In each meeting, the designated proposer makes an offer to the responder stipulating a division of the output available to the pair. If the responder accepts the offer, then the two players permanently exit the game with the shares agreed upon. If the proposal is rejected, then the two players part ways and search for new bargaining partners independently thereafter. We focus on stationary economies in which the distribution of active players is constant over time. To maintain a stationary market path, we assume that a constant measure of new players of each type decides whether to enter the game at the beginning of every period. New players who choose to join the market (as well as initial market participants) incur a one-time, type-dependent entry cost. Entry decisions then hinge on how entry costs compare to the payoffs achievable in the bargaining game. Equilibrium payoffs and incentives for agreements in the bargaining game depend on the underlying market distribution and the meeting probabilities it generates. Hence, both the inflows of players joining the market and the outflows of players reaching agreements are endogenous in the model. In a steady state, the inflows must balance the outflows for every player type. We establish that the bargaining game admits a steady state for every configuration of small entry costs.

In a companion paper (Manea, 2013), we study market dynamics in a version of the bargaining game considered here with exogenous non-stationary inflows. One result in that paper shows that if players engage in meetings with exogenous and possibly non-stationary frequencies over time, then the game can be solved using iterated conditional dominance. In the present steady state setting, the result implies that the market distribution and the induced meeting probabilities uniquely determine the balance of bargaining power and the equilibrium payoffs in the game.

Thus, any candidate steady state market composition is strategically consistent with a single payoff vector. Market inflows and outflows depend in turn on the payoffs, as discussed above. Hence, all steady state constraints can be expressed in terms of the conjectured steady state market composition. The balance equation leads to a characterization of steady states as fixed points of a certain correspondence.

The correspondence does not have a closed graph due to the individual rationality condition for entry decisions of initial market participants. We eliminate discontinuities by simultaneously discarding the individual rationality condition and perturbing the balance equation in the definition of the correspondence. Fixed points of the resulting family of perturbed correspondences satisfy individual rationality without invoking this condition explicitly. We prove that the perturbed correspondences have fixed points and derive a steady state market distribution as a limit of such points.

To apply Kakutani's fixed point theorem, we need to restrict every perturbed correspondence to a compact and convex domain such that its value at any point in the domain has a non-empty intersection with the domain. Small markets should be excluded from the domain because the search process does not typically extend by continuity to the degenerate "empty" market. The domain cannot contain arbitrarily large markets due to the compactness desideratum. The core of our fixed-point argument identifies appropriate constraints on market size ensuring that the restriction of the perturbed correspondence has the desired property. We develop lower and upper bounds on the unique payoffs consistent with a steady state market composition that translate into constraints on inflows and outflows in the candidate steady state. These constraints enable us to chart the extreme points of the perturbed correspondence and to ultimately construct a suitable domain restriction.

Our existence theorem complements the work of Gale (1987) and Manea (2011). That research borrows the "cloning" assumption of Rubinstein and Wolinsky (1985), whereby players who reach agreement are replaced by new players in the next period. The corresponding results characterize equilibrium outcomes for a known distribution of player types in the steady state, abstracting away from existence issues. The present analysis sheds light on the endogenous mechanics of steady states and provides foundations for the cloning assumption.

We apply similar ideas to the steady state analysis of the matching model, which represents a finite-type version of the Shimer and Smith (2000) setting with general search processes, bargaining protocols, and production functions. A finite measure of players of each type participates in the economy over the infinite time frame. At any instant in continuous time, every player is either single or matched with a partner. Matched pairs create a flow output which is endogenously shared between the parties. Only unmatched players search for potential partners. In a steady state, the distribution of unmatched players is constant over time. Every match is dissolved randomly and independently at a constant Poisson rate, and separated partners reenter the search process. Bargaining opportunities for unmatched players arrive with Poisson rates that depend continuously on the composition of the unmatched pool. In every meeting, the proposer suggests a sharing arrangement for the attainable output over the stochastic duration of the match. If the responder accepts the offer, then the match forms and the two parties start enjoying their agreed shares of flow output. Otherwise, the two players remain unmatched and continue their search independently.

We prove that the matching model always has a steady state. As in the bargaining game, a unique payoff vector is strategically consistent with the stationary profile of meeting frequencies generated by the distribution of unmatched players in any candidate steady state. The mapping from unmatched distributions to equilibrium payoffs facilitates a concise characterization of steady states in terms of fixed points of a collection of correspondences. As in the existence proof for the bargaining game, the challenge is to identify a compact restriction of the correspondences satisfying the hypotheses of Kakutani's fixed point theorem.

The rest of the paper is organized as follows. Sections 2 and 3 provide the steady state analyses of the bargaining game and the matching model, respectively. In Section 4, we provide closing remarks and discuss directions for future research. Proofs appear in the Appendix.

2. The bargaining game

We analyze the steady states of a model of bargaining in markets that generalizes the stationary models of Rubinstein and Wolinsky (1985), Gale (1987), and Manea (2011). Consider an economy with a finite set N of *player types*. A *production function* f specifies the output $f_{ij} = f_{ji} \geq 0$ that each pair of players of types i and j can jointly create. The economy operates at discrete dates $t = 0, 1, \ldots$ Players of type i have a *discount factor* $\delta_i \in (0, 1)$. We focus on stationary economies in which there is a continuum of constant measure $\mu_i \geq 0$ of players of type i active in the market at every date t. The market size is positive, $\sum_{i \in N} \mu_i > 0$. The composition of the market is described by the *market distribution* $\mu = (\mu_i)_{i \in N}$, which is an element of $\mathcal{M} = [0, \infty)^N \setminus \{\mathbf{0}\}$ ($\mathbf{0}$ denotes the zero vector in $\mathbb{R}^N$).

Players meet bargaining partners, one at a time, according to a *random search process* π whose properties we discuss momentarily. At any date when $\mu \in \mathcal{M}$ is the market distribution, each player of type i gets the opportunity to propose a division of the output f_{ij} to a partner of type j (negative shares are allowed) with probability $\pi_{ij}(\mu)$.[2] We assume that players recognize the type of their partners. If j accepts i's offer, then the two players exit the game with the shares agreed upon.[3] If j rejects the offer, then i and j break up and search for new bargaining partners independently in the future. A player who never reaches an agreement obtains zero payoff. There is a measure $\lambda_i > 0$ of *potential entrants* (or *new players*) of type i in every period $t \geq 1$. New players of type i incur a one-time *entry cost* $c_i > 0$ if they decide to join the market. Entry decisions of new players are endogenous in the steady state.

The search technology is required to be *continuous* with respect to the market composition, that is, $\pi_{ij}(\cdot)$ is a continuous function on $\mathcal{M}$ for every $(i, j) \in N \times N$. We assume that some productive activity is feasible in every market,

$$\max_{i,j \in N} \pi_{ij}(\mu) f_{ij} > 0, \forall \mu \in \mathcal{M}. \tag{1}$$

For some $\mu \in \mathcal{M}$, it is possible that the maximum in the expression above is achieved only by pairs (i, j) with $\mu_i = 0$. Note that when $\mu_i = 0$, players of type i may meet bargaining partners with positive probability and obtain profits.

The measure of players i who propose to types j in market μ is given by $\pi_{ij}(\mu)\mu_i$. The search process is measurable and measure preserving: any subset of proposers i making offers to players j is measurable if and only if the corresponding set of responders j is measurable, and the two sets have equal measures in such cases. Therefore, the measure of players of type

[2] It is possible that $f_{ii}\pi_{ii}(\mu) > 0$, so players of type i can match and produce with one another.

[3] For brevity, we refer to a specific player of type i as "player i" or simply "i" when the identity of the player is clear from the context.

i receiving offers from types j is $\pi_{ji}(\mu)\mu_j$ in every period.[4] Manea (2013) defines an explicit search process that generates meeting probabilities with the desired properties.

Since no player has more than one bargaining partner at a time, the search process needs to satisfy the *feasibility constraint*

$$\sum_{j \in N} \left(\pi_{ij}(\mu)\mu_i + \pi_{ji}(\mu)\mu_j\right) \leq \mu_i, \forall i \in N, \mu \in \mathcal{M}. \tag{2}$$

Any profile $(N, f, \delta, \pi, \lambda, c)$ with the properties above defines a *bargaining game*.

The most prominent search process in the literature is *linear search*, which is specified by $\pi_{ij}(\mu) = p\mu_j / \sum_{k \in N} \mu_k$ for $i, j \in N$ and $\mu \in \mathcal{M}$, where p is a constant in the interval $(0, 1/2]$.[5] Under this process, every player of type i has the opportunity to propose a trade to a type j with a probability proportional to the fraction of players j in the market. Note that linear search cannot be extended by continuity at $\mu = \mathbf{0}$, which justifies the definition of search processes on the domain $\mathcal{M}$ and our focus on steady states in $\mathcal{M}$.

We now turn to the analysis of the bargaining game. A general version of the model—with possibly non-stationary market dynamics but exogenous costless entry—is formalized in Manea (2013). The reader is referred to that paper for a discussion of measurability issues and game theoretic solution concepts. Here we assume that the game has perfect information and characterize stationary subgame perfect equilibria. Let v_i denote the common *expected discounted payoff* of all players of type i present in the game at the beginning of any period $t \geq 0$ (prior to the meeting stage at date t).

Consider the event in which a player of type i is selected to make an offer to a player j in period t. Since j's continuation payoff in case of disagreement is $\delta_j v_j$, in equilibrium player j accepts any offer higher than $\delta_j v_j$ and rejects offers lower than $\delta_j v_j$. Thus, player i has no incentive to make offers greater than $\delta_j v_j$ to j. In equilibrium, player i has a strict incentive (not) to make an acceptable offer if the payoff $f_{ij} - \delta_j v_j$ that he can extract from an agreement with j is greater (smaller) than his own disagreement continuation payoff of $\delta_i v_i$. Standard bargaining arguments show that if $f_{ij} - \delta_j v_j > \delta_i v_i$, then player i should offer exactly $\delta_j v_j$ to j, and j must accept the offer with probability 1 in equilibrium. When $f_{ij} - \delta_j v_j = \delta_i v_i$, the two players are indifferent between reaching the (unique) mutually acceptable agreement and resuming search separately.

To summarize, player i expects equilibrium payoffs different from $\delta_i v_i$ in period t only upon proposing to types j for which $f_{ij} - \delta_j v_j > \delta_i v_i$. Such events occur with probability $\pi_{ij}(\mu)$ and generate a conditional expected payoff of $f_{ij} - \delta_j v_j$ for player i. This analysis leads to the following system of *payoff equations*:

$$v_i = \sum_{j \in N} \pi_{ij}(\mu) \max\left(f_{ij} - \delta_j v_j, \delta_i v_i\right) + \left(1 - \sum_{j \in N} \pi_{ij}(\mu)\right)\delta_i v_i, \forall i \in N. \tag{3}$$

[4] It is not necessary to specify the frequency $\tilde{\pi}_{ij}(\mu)$ with which every player i receives offers from a type j. As is standard in bargaining with complete information, equilibrium agreements make the responder indifferent between accepting the offer and continuing the search. Thus, the rate at which players receive offers does not directly affect payoffs. The variables $\tilde{\pi}_{ij}(\mu)$ can also be eliminated from steady state accounting. Since meetings are measure preserving, we have the identity $\tilde{\pi}_{ij}(\mu)\mu_i = \pi_{ji}(\mu)\mu_j$. Moreover, if a (measurable) fraction α_{ji} of proposer-responder meetings of type (j, i) reaches agreement at some date, then the outflow of responders i who accept offers in the course of such meetings can be computed as $\alpha_{ji}\pi_{ji}(\mu)\mu_j$.

[5] The upper bound on p is implied by the feasibility constraint (2).

Let α_{ij} denote the (time-independent) fraction of proposer-responder pairs (i, j) that reach agreement at any date.[6] We have argued above that α_{ij} must be 0, 1, or a number in the interval [0, 1] depending on whether the expression $f_{ij} - \delta_i v_i - \delta_j v_j$ is smaller than, greater than, or equal to 0, respectively. Thus, *matching decisions* are characterized by

$$\alpha_{ij} \in Z\left(f_{ij} - \delta_i v_i - \delta_j v_j\right), \forall i, j \in N, \tag{4}$$

where the correspondence $Z : \mathbb{R} \rightrightarrows [0, 1]$ is defined as follows:

$$Z(a) = \begin{cases} \{0\} & \text{if } a < 0 \\ [0,1] & \text{if } a = 0 \\ \{1\} & \text{if } a > 0. \end{cases}$$

New players i optimally choose to enter the market if $v_i > c_i$ and to stay out if $v_i < c_i$. Entry decisions are not pinned down for types i with $v_i = c_i$. Therefore, the (time-independent) fraction β_i of the measure λ_i of new players i who decide to participate in the market must be 0, 1, or some number in [0, 1] contingent on the payoff v_i being smaller than, greater than, or equal to c_i, respectively. The optimality of *entry decisions* is thus succinctly captured by correspondence Z,

$$\beta_i \in Z(v_i - c_i), \forall i \in N. \tag{5}$$

Since initial market participants also bear entry costs, we need to impose a further *individual rationality* constraint. For any player type represented with positive measure in the steady state, the cost of entry should not exceed the expected payoff in the bargaining game,

$$\forall i \in N : \mu_i > 0 \;\Rightarrow\; v_i \geq c_i. \tag{6}$$

In a steady state, the outflow of players reaching agreements must match the inflow of new players joining the market in every period. Since the total measure of players of type i who trade and exit the market (in the role of proposer or responder) in any period is $\sum_{j \in N}(\alpha_{ij}\pi_{ij}(\mu)\mu_i + \alpha_{ji}\pi_{ji}(\mu)\mu_j)$ and the mass of new entrants of type i is $\beta_i\lambda_i$, we obtain the following *balance equations*:

$$\sum_{j \in N}\left(\alpha_{ij}\pi_{ij}(\mu)\mu_i + \alpha_{ji}\pi_{ji}(\mu)\mu_j\right) = \beta_i\lambda_i, \forall i \in N. \tag{7}$$

A *steady state of the bargaining game* $(N, f, \delta, \pi, \lambda, c)$ is a profile (μ, v, α, β) formed by a market distribution $\mu \in \mathcal{M}$, expected payoffs $v \in [0, \infty)^N$, agreement rates $\alpha \in [0, 1]^{N \times N}$, and entry decisions $\beta \in [0, 1]^N$ that satisfies conditions (3)–(7).

Example. Consider a market with two player types, $N = \{1, 2\}$, who have a common discount factor δ, in which only "mixed" meetings generate production possibilities: $f_{11} = f_{22} = 0$, $f_{12} > 0$. Assume that players meet bargaining partners according to the linear search process with parameter $p = 1/4$, that is, $\pi_{ij}(\mu) = \mu_j/(4(\mu_1 + \mu_2))$ for all $i, j \in N$, and that the measure of potential entrants of type 1 exceeds the one of type 2, $\lambda_1 > \lambda_2$. We determine the structure of steady states for all specifications of entry costs $c_1, c_2 > 0$.

6 Using the probability spaces constructed by Sun (2006) in which the exact law of large numbers holds for a continuum of random variables, we can assume that players of any given type use identical mixed strategies and interpret α_{ij} as the probability with which every proposer-responder pair (i, j) reaches agreement.

Suppose that (μ, v, α, β) constitutes a steady state in this setting. If $v_i < c_i$ for some $i \in N$, then individual rationality requires that $\mu_i = 0$, which leads to $\pi_{ji}(\mu) = 0$ and $v_j = 0$ for $j = 3 - i$. But $v_j = 0 < c_j$ and individual rationality imply that $\mu_j = 0$, contradicting the constraint $\mu \neq \mathbf{0}$. Thus, we have $v_1 \geq c_1 > 0$ and $v_2 \geq c_2 > 0$. Then only pairs of players of distinct types reach agreement, and matching decisions are given by $\alpha_{11} = \alpha_{22} = 0$ and $\alpha_{12} = \alpha_{21} = 1$. We obtain the following system of payoff equations:

$$v_1 = \frac{\mu_2}{4(\mu_1 + \mu_2)}(f_{12} - \delta v_2) + \left(1 - \frac{\mu_2}{4(\mu_1 + \mu_2)}\right)\delta v_1$$

$$v_2 = \frac{\mu_1}{4(\mu_1 + \mu_2)}(f_{12} - \delta v_1) + \left(1 - \frac{\mu_1}{4(\mu_1 + \mu_2)}\right)\delta v_2.$$

The unique solution is

$$v_1 = \frac{x f_{12}}{4 - 3\delta} \text{ and } v_2 = \frac{(1-x) f_{12}}{4 - 3\delta}, \text{ where } x := \frac{\mu_2}{\mu_1 + \mu_2}.$$

If $v_1 > c_1$, then $\beta_1 = 1$ and the balance equation for type 1 leads to $\lambda_1 = \mu_1\mu_2/(2(\mu_1 + \mu_2))$. Since the balance equation for type 2 implies that $\lambda_2 \geq \mu_1\mu_2/(2(\mu_1 + \mu_2))$, we obtain a contradiction with the assumption that $\lambda_1 > \lambda_2$. It follows that $v_1 = c_1$, which leads to

$$x = \frac{(4 - 3\delta)c_1}{f_{12}}.$$

Note that

$$0 \leq v_2 - c_2 = \left(1 - \frac{(4-3\delta)c_1}{f_{12}}\right)\frac{f_{12}}{4 - 3\delta} - c_2 = \frac{f_{12}}{4 - 3\delta} - c_1 - c_2,$$

so a steady state exists only if $c_1 + c_2 \leq f_{12}/(4 - 3\delta)$.

If $c_1 + c_2 < f_{12}/(4 - 3\delta)$, then $v_2 > c_2$ and hence $\beta_2 = 1$. The balance equations require that

$$\beta_1\lambda_1 = \lambda_2 = \frac{\mu_1\mu_2}{2(\mu_1 + \mu_2)} = \frac{x\mu_1}{2},$$

so

$$\mu_1 = \frac{2\lambda_2}{x} = \frac{2\lambda_2 f_{12}}{(4 - 3\delta)c_1} \text{ and } \beta_1 = \frac{\lambda_2}{\lambda_1}.$$

Then $\mu_2 = x\mu_1/(1 - x) = 2\lambda_2/(1 - x)$, where x was computed above. Hence, in this case the steady state is unique. In the steady state, all new players of type 2 enter the market, while only a fraction of new players 1 do. Players of type 1 are rationed out of the market in order to maintain an equal inflow of players of both types that matches the outflow of mixed pairs of players reaching agreement.

If $c_1 = c_2 \to 0$, then $x \to 0$, $\mu_1 \to \infty$, and $\mu_2 \to 2\lambda_2$. Since there are more potential entrants of type 1 than of type 2 and only meetings between distinct types generate agreements, stabilizing the market requires some players of type 1 not to enter. This is achieved by overcrowding the market with types 1, which lowers the frequency with which players 1 find bargaining partners to the point where their steady state payoffs match their low entry costs.

In the cutoff case $c_1 + c_2 = f_{12}/(4 - 3\delta)$, we have $x = c_1/(c_1 + c_2)$, $v_1 = c_1$, and $v_2 = c_2$. Entry decisions solve

$$\beta_1\lambda_1 = \beta_2\lambda_2 = \frac{\mu_1\mu_2}{2(\mu_1 + \mu_2)} = \frac{c_1\mu_1}{2(c_1 + c_2)},$$

so

$$\beta_1 = \frac{c_1\mu_1}{2\lambda_1(c_1+c_2)} \text{ and } \beta_2 = \frac{c_1\mu_1}{2\lambda_2(c_1+c_2)}.$$

It must be that $\beta_2 = \max(\beta_1, \beta_2) \le 1$, which holds if $\mu_1 \le 2\lambda_2(1+c_2/c_1)$. Any $\mu_1 \in (0, 2\lambda_2(1+c_2/c_1)]$ corresponds to a steady state in which $\mu_2 = x\mu_1/(1-x) = \mu_1 c_1/c_2$. In this case, we have a family of steady states parametrized by μ_1. In the steady state for any $\mu_1 \in (0, 2\lambda_2(1+c_2/c_1)]$, both player types are indifferent between entering the market and staying out, and inflows are rationed to balance the measure $c_1\mu_1/(2(c_1+c_2))$ of mixed matches in every period.

We have found that steady states exist in the example if and only if $c_1 + c_2 \le f_{12}/(4-3\delta)$. The next result establishes that the bargaining game admits a steady state whenever entry costs are sufficiently low.

Theorem 1. *For every profile $(N, f, \delta, \pi, \lambda)$, there exists $\kappa(N, f, \delta, \pi, \lambda) > 0$ such that any bargaining game $(N, f, \delta, \pi, \lambda, c)$ with $c_i < \kappa(N, f, \delta, \pi, \lambda)$ for all $i \in N$ has a steady state.*

The proof appears in the Appendix. Here we sketch the general approach. A contraction argument shows that the system of payoff equations (3) has a unique solution $v(\mu)$ for every $\mu \in \mathcal{M}$, which is continuous in μ.[7] The function $v(\cdot)$ allows us to express all steady state constraints in terms of the variable μ and facilitates a parsimonious characterization of steady states. Specifically, $\mu \in \mathcal{M}$ constitutes a steady state market composition if and only if μ is a fixed point of the correspondence $S : \mathcal{M} \rightrightarrows \mathbb{R}^N$ defined by

$$S(\mu) = \Big\{ (\tilde{\mu}_i)_{i\in N} \Big| \forall i \in N, v_i(\mu) < c_i \Rightarrow \tilde{\mu}_i = 0;$$
$$\exists \alpha_{ij} \in Z\left(f_{ij} - \delta_i v_i(\mu) - \delta_j v_j(\mu)\right) \;\&\; \beta_i \in Z\left(v_i(\mu) - c_i\right), \forall i, j \in N$$
$$\text{s.t. } \tilde{\mu}_i = \mu_i - \sum_{j\in N} \left(\alpha_{ij}\pi_{ij}(\mu)\mu_i + \alpha_{ji}\pi_{ji}(\mu)\mu_j\right) + \beta_i\lambda_i, \forall i \in N \Big\}.$$

Note that $S(\mu)$ is not always a product set in $\mathbb{R}^N$ since the components $\tilde{\mu}_i$ and $\tilde{\mu}_j$ are linked by variables α_{ij} and α_{ji} in the conditions above.

Let $\underline{\lambda} = \min_{i\in N} \lambda_i$ and $\underline{f} = \min\{f_{ij} \mid i, j \in N, f_{ij} > 0\}$. We prove that S has a fixed point whenever all entry costs are smaller than

$$\kappa = \min\left(\min_{\mu\in\mathcal{M}, \|\mu\|\in[\underline{\lambda}/2, 2\underline{\lambda}]} \max_{i\in N} v_i(\mu), \frac{\underline{f}}{6 - 4\min_{i\in N}\delta_i} \right),$$

where we use the notation $\|\mu\| = \sum_{i\in N} \mu_i$. Fix a cost profile c with $c_i < \kappa$ for all $i \in N$. Imposing the individual rationality condition (6) explicitly in the construction of S generates

[7] This is a special case of Theorem 1 of Manea (2013). More generally, that result shows that bargaining games with heterogeneous traders in which the path of meeting probabilities is exogenous and possibly non-stationary can be solved by iterated conditional dominance. In addition, the unique payoffs surviving iterated conditional dominance vary continuously with respect to meeting probabilities. One corollary of the former result is that if meeting probabilities are stationary as in our steady state framework, then the payoffs strategically consistent with the meeting probabilities inherit the stationarity. Thus, the payoff stationarity typically adopted by steady state analyses (including the present one) may be derived as an equilibrium implication in the underlying stationary game.

discontinuities. To avoid this problem, we perturb S to obtain the upper hemicontinuous correspondences $S^\rho : \mathcal{M} \rightrightarrows \mathbb{R}^N$ for $\rho \in [1/2, 1)$ as follows

$$S^\rho(\mu) = \Big\{ (\tilde{\mu}_i)_{i \in N} \Big| \exists \alpha_{ij} \in Z\left(f_{ij} - \delta_i v_i(\mu) - \delta_j v_j(\mu)\right) \ \&\ \beta_i \in Z\left(v_i(\mu) - c_i\right), \forall i, j \in N$$
$$\text{s.t. } \tilde{\mu}_i = \rho\Big(\mu_i - \sum_{j \in N} \left(\alpha_{ij}\pi_{ij}(\mu)\mu_i + \alpha_{ji}\pi_{ji}(\mu)\mu_j\right)\Big) + \beta_i \lambda_i, \forall i \in N \Big\}. \quad (8)$$

Fixed points of S^ρ describe steady states in an environment where the size of each population $i \in N$ decays by a factor of ρ at the end of every bargaining round (but incentives for agreements are governed by the disagreement payoffs $v(\cdot)$). The decay factor acts as a proxy for the individual rationality condition. Indeed, if μ is a fixed point of S^ρ for some $\rho \in [1/2, 1)$, then $\mu_i = 0$ for any $i \in N$ such that $v_i(\mu) < c_i$. We derive a fixed point for S as a limit of fixed points of S^ρ for $\rho \to 1^-$.

The main part of our proof establishes that S^ρ has a fixed point for any $\rho \in [1/2, 1)$. Aiming to apply Kakutani's fixed point theorem, we construct a restriction of S^ρ to a common domain and range $\mathcal{C} \subset \mathcal{M}$, which is compact and convex. The restricted correspondence $\bar{S}^\rho : \mathcal{C} \rightrightarrows \mathcal{C}$ is defined by $\bar{S}^\rho(\mu) = S^\rho(\mu) \cap \mathcal{C}$ for all $\mu \in \mathcal{C}$. Extremely small markets should be excluded from $\mathcal{C}$ because the search technology does not generally extend by continuity to the degenerate market $\mu = \mathbf{0}$. Moreover, $\mathcal{C}$ cannot contain arbitrarily large markets due to the compactness desideratum. Most importantly, the boundaries of $\mathcal{C}$ must be carefully chosen to ensure that the resulting correspondence $\bar{S}^\rho$ is non-empty valued.

The crux of our fixed-point argument shows that the set

$$\mathcal{C} = \left\{ \mu \in \mathcal{M} \middle| \|\mu\| \geq \underline{\lambda}/2 \ \&\ \mu_i \leq \lambda_i \left(1 + \frac{\max_{j \in N} f_{ij}}{c_i(1 - \delta_i)}\right), \forall i \in N \right\}$$

constitutes a suitable domain restriction. To prove that $S^\rho(\mu) \cap \mathcal{C}$ is non-empty for $\mu \in \mathcal{C}$, we develop lower and upper bounds on $v(\mu)$. These bounds translate into constraints on inflows and outflows for the candidate steady state μ, which are informative about the location of extreme points of $S^\rho(\mu)$ in relation to $\mathcal{C}$.

We can check that $\bar{S}^\rho$ satisfies the remaining hypotheses of Kakutani's theorem and conclude that S^ρ has a fixed point $\mu^\rho \in \mathcal{C}$. Since $\mathcal{C}$ is sequentially compact, there exists a sequence of ρ's approaching 1 along which μ^ρ converges to some limit μ. Then we exploit the continuity of $\pi(\cdot)$ and $v(\cdot)$ and the closed graph property of Z to argue that μ is a fixed point of S and hence constitutes a steady state market distribution.

3. The matching model

We consider a version of the Shimer and Smith (2000) matching model adapted to our framework with finite types. The universe consists of a continuum of measure $\theta_i > 0$ of players of every *type* i from a finite set N. Time runs continuously in the interval $[0, \infty)$. Each player is endogenously either *matched* or *unmatched* at any instant $t \in [0, \infty)$. A *match* between types i and j creates a *flow output* $f_{ij} = f_{ji} \geq 0$. Players of type i have a *discount rate* $r_i > 0$. In a steady state of the economy, the distribution of unmatched players is constant over time. Let μ_i denote the measure of unmatched players i in a candidate steady state. The *unmatched market distribution* $\mu = (\mu_i)_{i \in N}$ is an element of the set $\mathcal{M} = \prod_{i \in N} [0, \theta_i] \setminus \{\mathbf{0}\}$.

We assume that matches entail time-consuming activity that prevents matched players from meeting new partners. Hence only unmatched players engage in search. Unmatched players seek matches according to a *search process* analogous to the one from the discrete-time bargaining game. An unmatched player i gets the opportunity to propose a match and a fixed division of the flow payoff f_{ij} to a type j with Poisson rate $\pi_{ij}(\mu)$. If j accepts the proposal, then the match forms and the two parties start enjoying their agreed shares of flow output. Otherwise the two players remain unmatched and continue their search independently. The flow payoff of unmatched players is normalized to zero. In this stationary environment, matching terms that are initially acceptable are worth retaining, so players do not have incentives to quit matches. Consequently, we assume that matched players do not break up with their partners intentionally. Instead, the pool of unmatched players is replenished by exogenous match dissolutions. Each match *ends* randomly and independently at a Poisson rate $\omega > 0$. Separated partners reenter the search process.

As in the bargaining model, we assume that $\pi_{ij}(\mu)$ is continuous in $\mu \in \mathcal{M}$ for every $(i, j) \in N \times N$. We also maintain the assumption that the search process is measurable and measure preserving. We additionally require that in any (short) time interval, the total measure of unmatched players who find potential partners vanishes as the market becomes small,

$$\lim_{\mu \to \mathbf{0}} \sum_{i,j \in N} \pi_{ij}(\mu)\mu_i = 0. \tag{9}$$

A sufficient condition for this regularity assumption is that all meeting intensities $\pi_{ij}(\mu)$ be uniformly bounded. Every profile (N, f, r, π, ω) with the properties stated above defines a *matching model*.

Shimer and Smith (2000) restrict their analysis to the *quadratic search* process, which takes the form $\pi_{ij}(\mu) = p\mu_j$ for some parameter $p > 0$ in our discrete-type framework. The term "quadratic" hints at the fact that the total measure of unmatched players meeting potential matches per unit of time under this search process is a quadratic function of market size. For this reason, there is no analogue of quadratic search for large markets in the discrete-time setting of the previous section. Indeed, if $\pi_{ij}(\mu) = p\mu_j$, then the feasibility constraint (2) becomes $2p\mu_i \sum_{j \in N} \mu_j \leq \mu_i$ for all $i \in N$ and $\mu \in \mathcal{M}$, which is violated whenever $\sum_{j \in N} \mu_j > 1/(2p)$. As Diamond and Maskin (1979) observe, "with a high density of potential partners, the quadratic technology seems a poor approximation." Noldeke and Troger (2009) argue that the *linear search* technology, which in the continuous-time setting is specified by $\pi_{ij}(\mu) = p\mu_j / \sum_{k \in N} \mu_k$ for a constant $p > 0$, is more plausible and predominates in the search literature. While it may be helpful to interpret our existence result in the context of simple search processes such as linear and quadratic search, it should be emphasized that our analysis applies for any continuous search technology that satisfies the minimal regularity condition (9).

We now proceed to the analysis of steady states in the matching model. Suppose that the composition of the pool of unmatched players is given by μ at every date. Denote by V_i the *expected discounted payoff* of an unmatched player of type i and by V_{ij} his expected payoff at the instant when he proposes a match to a player of type j in the steady state. We derive Bellman equations for these values building on the arguments of Shimer and Smith (2000).

When a player of type i proposes a match to a type j, the latter player accepts offers of flow payoffs greater than his own unmatched flow value $r_j V_j$ and rejects offers smaller than $r_j V_j$. Hence, player i has no incentive to offer j flow payoffs in excess of $r_j V_j$. By making an offer just above $r_j V_j$, player i can secure a flow payoff arbitrarily close to $f_{ij} - r_j V_j$ in a match with j. Thus, player i has a strict incentive to make an acceptable offer to j if and only if his profit from matching with j exceeds his opportunity cost of suspending search, i.e., $f_{ij} - r_j V_j > r_i V_i$. As in the bargaining game, when this inequality holds, player i offers j a flow payoff of exactly $r_j V_j$ and j accepts the match in equilibrium.

Therefore, when $f_{ij} - r_j V_j > r_i V_i$, players i and j reach an agreement whereby proposer i receives a flow payoff of $f_{ij} - r_j V_j$ while the match (i, j) lasts. The match breaks up at rate ω, and player i suffers a capital loss of $V_{ij} - V_i$ at the time of separation. We obtain the following Bellman equation for i's expected payoff conditional on proposing an agreeable match to a type j:

$$r_i V_{ij} = f_{ij} - r_j V_j - \omega \left(V_{ij} - V_i\right).$$

This equation is equivalent to

$$V_{ij} - V_i = \frac{f_{ij} - r_i V_i - r_j V_j}{r_i + \omega}.$$

If $f_{ij} - r_j V_j \leq r_i V_i$ instead, then proposer i cannot extract a dividend above his baseline unmatched flow payoff by matching with j, so $V_{ij} = V_i$. The analysis hereto leads to the formula

$$V_{ij} - V_i = \frac{\max(f_{ij} - r_i V_i - r_j V_j, 0)}{r_i + \omega}. \tag{10}$$

The preceding arguments show that upon receiving an offer, a responder of type i expects a discounted payoff equal to his unmatched value V_i. When given the opportunity to make an offer to a player of type j, a proposer of type i enjoys capital gains of $V_{ij} - V_i$. For each unmatched player i, this opportunity arises at rate $\pi_{ij}(\mu)$. Hence type i's unmatched expected payoff V_i solves the Bellman equation

$$r_i V_i = \sum_{j \in N} \pi_{ij}(\mu) \left(V_{ij} - V_i\right). \tag{11}$$

Substituting (10) into (11), we obtain the following formula for unmatched expected payoffs:

$$r_i V_i = \frac{1}{r_i + \omega} \sum_{j \in N} \pi_{ij}(\mu) \max\left(f_{ij} - r_i V_i - r_j V_j, 0\right).$$

Hence the unmatched *flow payoffs* $v_i := r_i V_i$ solve the system of *payoff equations*

$$v_i = \frac{1}{r_i + \omega} \sum_{j \in N} \pi_{ij}(\mu) \max\left(f_{ij} - v_i - v_j, 0\right), \forall i \in N. \tag{12}$$

These formulae boil down to the continuous-time limit of the payoff equations (3) if matches are permanent ($\omega = 0$), as assumed in the bargaining game of the previous section.

We now turn to *matching decisions*. As argued above, players of types i and j have strict incentives (not) to form a match upon meeting if $f_{ij} - v_i - v_j$ is positive (negative). When $f_{ij} - v_i - v_j = 0$, players i and j are indifferent between matching with each other and resuming the search. Using the correspondence Z defined in Section 2, the fraction of proposer-responder pairs (i, j) that agree to match, which we denote by α_{ij}, satisfies

$$\alpha_{ij} \in Z\left(f_{ij} - v_i - v_j\right), \forall i, j \in N. \tag{13}$$

In a steady state, the rates at which matches form and dissolve must balance for every player type. The measure of unmatched players of type i who find an agreeable match (in the role of proposer or responder) per unit of time is given by $\sum_{j \in N} \left(\alpha_{ij}\pi_{ij}(\mu)\mu_i + \alpha_{ji}\pi_{ji}(\mu)\mu_j\right)$. At every date, there is a stock of size $\theta_i - \mu_i$ of matched players i. Since matched players lose their partners at the exogenous rate ω, we obtain the following *balance equations*:

$$\sum_{j \in N} \left(\alpha_{ij}\pi_{ij}(\mu)\mu_i + \alpha_{ji}\pi_{ji}(\mu)\mu_j\right) = \omega(\theta_i - \mu_i), \forall i \in N. \quad (14)$$

A *steady state of the matching model* (N, f, r, π, ω) is a profile (μ, v, α) that consists of an unmatched distribution $\mu \in \mathcal{M}$, flow payoffs $v \in [0, \infty)^N$, and agreement rates $\alpha \in [0, 1]^{N \times N}$ satisfying conditions (12)–(14). The next result establishes the existence of steady states in the matching model.

Theorem 2. *Every matching model has a steady state.*

The proof can be found in the Appendix. We first prove that, as in the bargaining game, the system of payoff equations (12) has a unique solution $v(\mu)$, which varies continuously in μ. The function $v(\cdot)$ delivers a characterization of steady states that relies exclusively on the variable μ. For $\rho > 0$, define the correspondence $S^\rho : \mathcal{M} \rightrightarrows \mathbb{R}^N$ as follows

$$\begin{aligned} S^\rho(\mu) = \Big\{ (\tilde{\mu}_i)_{i \in N} \Big| & \exists \alpha_{ij} \in Z\left(f_{ij} - v_i(\mu) - v_j(\mu)\right), \forall i, j \in N \\ & \text{s.t. } \tilde{\mu}_i = \mu_i + \rho\Big(\omega(\theta_i - \mu_i) - \sum_{j \in N} \left(\alpha_{ij}\pi_{ij}(\mu)\mu_i + \alpha_{ji}\pi_{ji}(\mu)\mu_j\right)\Big), \forall i \in N \Big\}. \end{aligned} \quad (15)$$

Then μ constitutes a steady state unmatched distribution if and only if μ is a fixed point of S^ρ for some $\rho > 0$.[8]

As in the proof of Theorem 1, the challenge is to identify a restriction of the domain and range of S^ρ to a compact and convex set. In general, it is difficult to constrict the range of S^ρ because the matching intensities $\pi_{ij}(\mu)$ may be unbounded. However, we use the flexibility of choosing $\rho > 0$ arbitrarily small to scale down these variables and construct a non-empty, compact, and convex set $\mathcal{C} \subset \mathcal{M}$ with the property that $S^\rho(\mathcal{C}) \subseteq \mathcal{C}$.

Designing the domain of the steady state correspondence is more complicated for the bargaining game than for the matching model due to key strategic differences between the two models. Since entry decisions in the bargaining game are endogenous, the proof of Theorem 1 develops lower and upper bounds on payoffs that translate into constraints on inflows and outflows in the candidate steady state. By contrast, entrance into the unmatched market is governed by exogenous match dissolutions in the matching model, and a detailed analysis of payoffs and inflows is not necessary for the proof of Theorem 2. Another important step in the fixed-point argument of Theorem 1 identifies the upper bounds $\mu_i \leq \lambda_i(1 + \max_{j \in N} f_{ij}/(c_i(1 - \delta_i)))$ on the domain of the steady state correspondence for the bargaining game. The example from Section 2 demonstrates that small entry costs may imply arbitrarily large steady state markets, so the domain specification necessarily depends on entry costs. The steady state analysis of the matching model does not involve an analogous step because the bound $\mu_i \leq \theta_i$ is explicitly built into the model.

[8] Recall that in the context of the discrete-time bargaining model, the set $S(\mu)$ contains all markets that may arise in the next period from a current market distribution μ, given the rationality constraints on inflows and agreements implied by a strategic environment in which players face the market distribution μ in perpetuity. Similarly, in the continuous-time matching model, $S^\rho(\mu)$ consists of unmatched market distributions that can emerge from market μ in a time interval of length ρ (for small ρ) if players expect matching opportunities to arrive with stationary frequencies $\pi(\mu)$.

Theorem 2 does not extend to search processes that violate the regularity condition (9). For instance, assume that there is a single player type i, with $f_{ii} > 0$ and $\pi_{ii}(\mu) = \omega\theta_i/\mu_i$. This search technology is continuous on $\mathcal{M}$ but does not satisfy (9). In a steady state (μ, v, α), it must be that $v_i < f_{ii}/2$ and $\alpha_{ii} = 1$. Then the left-hand side of the balance equation (14) simplifies to $2\omega\theta_i$, which is greater than the right-hand side $\omega(\theta_i - \mu_i)$ for all μ_i. Hence, no steady state exists in this example.

Related literature. Shimer and Smith (2000) establish the existence of steady states in a version of the matching model with an interval of player types under the following assumptions: the search technology takes the special quadratic form; agreements are determined according to the Nash bargaining solution; and the production function f is either strictly super- or sub-modular. Our proofs differ in the choice of primitive variables employed to characterize steady states as fixed points. In a steady state (μ, v, α), the measures of unmatched players $(\mu_i)_{i\in N}$ solve the system of balance equations (14), while the payoffs $(v_i)_{i\in N}$ are simultaneously determined by the payoff equations (12). The matching decisions α can be eliminated from the analysis using their simple relation to payoffs captured by (13). The two key systems of equations (12) and (14) can be "solved" in different orders, leading to distinct approaches for computing steady states. The approach designed by Shimer and Smith (2000) expresses all steady state constraints in terms of the payoffs v. This approach boils down to solving for μ in (14) as a function of α and substituting the solution for the arguments of π in (12). An alternative approach, which we develop in the proof of Theorem 2, relies on the "free variable" μ. Our construction of the steady state correspondence S^ρ effectively solves for v in (12) and plugs the solution in (14) via the variables α. A third approach, due to Lauermann and Noldeke (2015), treats all of μ, v, and α as free variables and analyzes steady states as fixed points of a correspondence defined on the domain of profiles (μ, v, α).

To apply a fixed-point theorem, the former approach entails showing that for any profile of matching decisions α, the system of balance equations (14) has a unique solution μ, which varies continuously with α. Shimer and Smith (2000) prove this fact, coined the *fundamental matching lemma*, for the quadratic search technology. Noldeke and Troger (2009) extend the conclusion of the matching lemma to the more intuitive but less tractable case of linear search. Either proof is closely tailored to the particular search technology. Even though theoretically interesting, the fundamental matching lemma is unnecessarily strong for the proof of existence of steady states. Indeed, standard fixed-point theorems imply that the balance equations (14) have a solution μ for every α. The challenge in proving the matching lemma is the uniqueness of the solution, which is a sufficient condition for the existence of a continuous selection. While continuity is an important ingredient for many fixed-point results, the availability of a solution μ to (14) that is continuous in α is not directly relevant for the existence of steady states. Furthermore, the conclusion of the matching lemma regarding uniqueness does not hold for many continuous search processes.

Neither π nor the implicit function expressing the dependence of μ on α in the matching lemma is typically linear. This means that the composition of mappings characterizing steady states in terms of fixed points v is not guaranteed to be convex-valued at any point v where (13) does not pin down the matching decisions α, rendering many fixed-point results inapplicable. However, Shimer and Smith manage to eliminate multi-valued mappings from their analysis. Relying on the assumptions of an atomless distribution of player types and a strictly super- or sub-modular production function, they show that the set of pairs of players that are indifferent between accepting and rejecting mutual matches has measure zero for any specification of (disagreement) payoffs v.

Our approach circumvents the matching lemma by employing the distribution of unmatched players μ, rather than the payoff vector v, as the free variable in the fixed-point construction. The advantage of this approach is that the system of payoff equations (12) is better behaved than the system of balance equations (14). Our contraction argument establishes that the payoff equations have a unique solution $v(\mu)$ for any $\mu \in \mathcal{M}$ that varies continuously in μ. This approach recognizes the simplicity of translating payoff equations and matching decisions into constraints on v, and then α, in terms of μ. Combining these constraints with the balance equations, we obtain a characterization of steady states that relies exclusively on μ. Since α enters the balance equations linearly, the role of μ as the ultimate free variable in our fixed-point argument also ensures that the underlying correspondence is convex-valued.

Lauermann and Noldeke's (2015) proof does not rely on a detailed analysis of either the payoff or the balance system of equations. Their insight is that "solving" the two systems "in parallel" bypasses the intermediate step involved in the "sequential" methods outlined above that establishes existence, uniqueness, and continuity of the solution for one of the two systems. Accordingly, Lauermann and Noldeke define a correspondence on the domain of complete profiles of steady state variables, including payoffs and unmatched distributions, whose fixed points capture all steady state conditions explicitly.[9]

We finally comment on the technical challenges involved in extending our methods to settings with a continuum of types $i \in [0, 1]$ and a continuous steady state density μ of unmatched players. As explained above, Shimer and Smith show that in such settings the assumption of a strictly super- or sub-modular production function f implies that the matching probabilities α_{ij} must be either 0 or 1 for almost all pairs of types (i, j). Then the analogue of the correspondence S^ρ from our proof is a function, and the density $S^\rho(\mu)$ evaluated at type i depends on an integral over $j \in [0, 1]$ that contains the term α_{ij}. The standard application of Schauder's theorem that would establish the existence of a fixed point for S^ρ entails proving that the range of S^ρ is a set of equicontinuous densities, which is related to the smoothness of the boundary of the *matching set* $\{(i, j)|\alpha_{ij} = 1\}$. The examples from Figure 3 of Shimer and Smith's paper suggest that the boundary of the matching set is not generally characterized by smooth parameterizations. The set of types j with which type i agrees to match ($\alpha_{ij} = 1$) is not necessarily convex, so the boundary of the matching set cannot always be parameterized using "cutoff" types. Shimer and Smith discover that either super- or sub-modularity of the production function as well as of the logarithm of its first- and cross-derivatives guarantees the convexity of matching sets. The restrictiveness and complexity of their sufficient conditions for convexity cast some pessimism on the more detailed topological analysis of matching set boundaries needed for this path of proof.

4. Conclusion

This paper provides foundations for steady state economies. We established the existence of steady states in two classic matching and bargaining models with general search processes and production functions. Our methods may prove useful in other stationary settings.

In future research, it would be interesting to explore whether convergence to a steady state obtains for any initial market distribution in the context of non-stationary versions of the models considered here. In the case of the bargaining model, the dynamic game of Manea (2013)

[9] Their approach accommodates settings with nontransferable matching payoffs in addition to the transferable case studied here. However, it offers a limited understanding of the properties and the structure of steady states because it "black-boxes" all steady state constraints into the fixed-point argument.

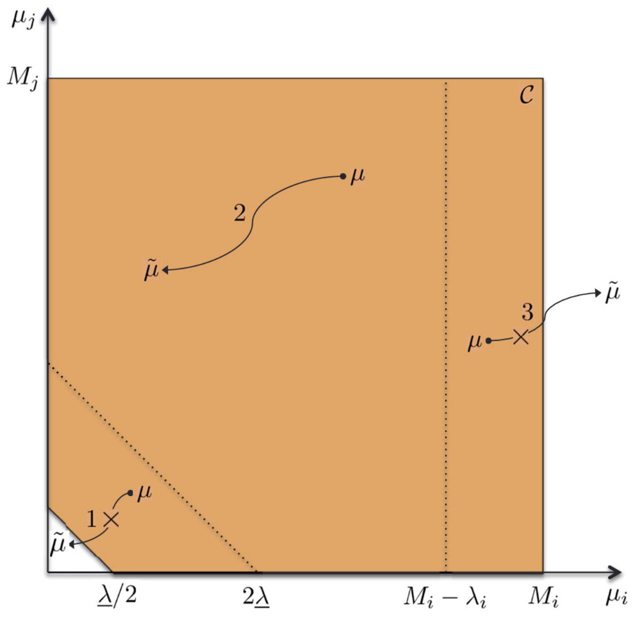

Fig. 1. Step 1.3 proves that if $\|\mu\| \in [\underline{\lambda}/2, 2\underline{\lambda}]$ then $\|\tilde{\mu}\| > \underline{\lambda}/2$ for all $\tilde{\mu} \in S^\rho(\mu)$ (crossed arrow 1) and if $\|\mu\| > 2\underline{\lambda}$ then there exists $\tilde{\mu} \in S^\rho(\mu)$ with $\|\tilde{\mu}\| > \underline{\lambda}/2$ (arrow 2). M_i and $\mathcal{C}$ are defined in Step 1.4. Step 1.5 shows that $\mu_i \le M_i$ and $\tilde{\mu} \in S^\rho(\mu)$ imply that $\tilde{\mu}_i \le M_i$ (crossed arrow 3).

provides a framework for investigating this question. Another open problem concerns the possibility of extending our results to settings with a continuum of types in order to obtain an exact generalization of Shimer and Smith's (2000) existence theorem. While this is an engrossing technical puzzle, it is not clear that the continuum-type setting bears more relevance for applications. Finite-type models provide the advantage of computational tractability and are better suited for empirical work.

Appendix. Proofs

Proof of Theorem 1. Consider a bargaining game $(N, f, \delta, \pi, \lambda, c)$. We shall argue that for any $\mu \in \mathcal{M}$, the system of payoff equations (3) has a unique solution $v(\mu)$. Formula (8) from Section 2 relies on the function $v(\cdot)$ to define a correspondence S^ρ. We seek to prove that there exists $\kappa(N, f, \delta, \pi, \lambda) > 0$ such that S^ρ has a fixed point if $\rho \in [1/2, 1)$ and $c_i < \kappa(N, f, \delta, \pi, \lambda)$ for all $i \in N$. We suppress the parameters of κ for the rest of the proof. The core of our fixed-point argument identifies a restriction on the domain and range of S^ρ to a compact and convex set $\mathcal{C} \subset \mathcal{M}$ for which we can apply Kakutani's theorem. Then we derive a fixed point of S (the correspondence characterizing steady states defined in Section 2) as a limit of fixed points of S^ρ for $\rho \to 1^-$. The proof proceeds in several steps. Step 1.1 establishes the uniqueness of the solution to the system of payoff equations (3) for every $\mu \in \mathcal{M}$ and the continuity of the solution with respect to μ. Step 1.2 defines κ. The remaining steps assume that $c_i \in (0, \kappa)$ for $i \in N$. Steps 1.3–1.6 consider a fixed $\rho \in [1/2, 1)$. Fig. 1 illustrates some of the main ideas.

Step 1.1. Uniqueness and continuity of the solution for the payoff equations.

Fix $\mu \in \mathcal{M}$ and $\varepsilon \geq 0$. Consider $\mu' \in \mathcal{M}$ with the property that

$$\sum_{j \in N} \left| \frac{\pi_{ij}(\mu)}{1 - \delta_i \left(1 - \sum_{k \in N} \pi_{ik}(\mu)\right)} - \frac{\pi_{ij}(\mu')}{1 - \delta_i \left(1 - \sum_{k \in N} \pi_{ik}(\mu')\right)} \right| \leq \frac{1 - \max_{k \in N} \delta_k}{\max_{k,l \in N} f_{kl}} \varepsilon \qquad (16)$$

for all $i \in N$. Exclusively for the purpose of this step, we simplify notation by omitting the arguments μ and μ' in the function $\pi(\cdot)$ and writing π_{ij} and π'_{ij} for $\pi_{ij}(\mu)$ and $\pi_{ij}(\mu')$, respectively. We use the notation $\|v - v'\|_\infty = \max_{i \in N} |v_i - v'_i|$. Suppose that the vectors v and v' solve the payoff equations (3) corresponding to markets μ and μ', respectively. We rewrite the payoff equations as follows:

$$v_i = \sum_{j \in N} \frac{\pi_{ij}}{1 - \delta_i \left(1 - \sum_{k \in N} \pi_{ik}\right)} \max\left(f_{ij} - \delta_j v_j, \delta_i v_i\right)$$

$$v'_i = \sum_{j \in N} \frac{\pi'_{ij}}{1 - \delta_i \left(1 - \sum_{k \in N} \pi'_{ik}\right)} \max\left(f_{ij} - \delta_j v'_j, \delta_i v'_i\right).$$

Subtracting the equalities above, we obtain

$$\begin{aligned}
|v_i - v'_i| &= \Bigg| \sum_{j \in N} \frac{\pi_{ij}}{1 - \delta_i \left(1 - \sum_{k \in N} \pi_{ik}\right)} \left(\max\left(f_{ij} - \delta_j v_j, \delta_i v_i\right) - \max\left(f_{ij} - \delta_j v'_j, \delta_i v'_i\right)\right) \\
&\quad + \sum_{j \in N} \left(\frac{\pi_{ij}}{1 - \delta_i \left(1 - \sum_{k \in N} \pi_{ik}\right)} - \frac{\pi'_{ij}}{1 - \delta_i \left(1 - \sum_{k \in N} \pi'_{ik}\right)}\right) \max\left(f_{ij} - \delta_j v'_j, \delta_i v'_i\right) \Bigg| \\
&\leq \sum_{j \in N} \frac{\pi_{ij}}{1 - \delta_i \left(1 - \sum_{k \in N} \pi_{ik}\right)} \left|\max\left(f_{ij} - \delta_j v_j, \delta_i v_i\right) - \max\left(f_{ij} - \delta_j v'_j, \delta_i v'_i\right)\right| \\
&\quad + \sum_{j \in N} \left|\frac{\pi_{ij}}{1 - \delta_i \left(1 - \sum_{k \in N} \pi_{ik}\right)} - \frac{\pi'_{ij}}{1 - \delta_i \left(1 - \sum_{k \in N} \pi'_{ik}\right)}\right| \max\left(f_{ij} - \delta_j v'_j, \delta_i v'_i\right) \\
&\leq \sum_{j \in N} \frac{\pi_{ij}}{\sum_{k \in N} \pi_{ik}} \max_{k \in N} \delta_k \|v - v'\|_\infty + \frac{1 - \max_{k \in N} \delta_k}{\max_{k,l \in N} f_{kl}} \varepsilon \max_{k,l \in N} f_{kl} \\
&= \max_{k \in N} \delta_k \|v - v'\|_\infty + (1 - \max_{k \in N} \delta_k)\varepsilon,
\end{aligned}$$

where the first bound utilizes the triangle inequality, while the second relies on

$$\begin{aligned}
|\max(f_{ij} - \delta_j v_j, \delta_i v_i) - \max(f_{ij} - \delta_j v'_j, \delta_i v'_i)| &\leq \max(\delta_j |v_j - v'_j|, \delta_i |v_i - v'_i|) \leq \\
&\max_{k \in N} \delta_k \max(|v_j - v'_j|, |v_i - v'_i|) \leq \max_{k \in N} \delta_k \|v - v'\|_\infty
\end{aligned}$$

(see Lemma 5 in Manea, 2011), $1 - \delta_i \left(1 - \sum_{k \in N} \pi_{ik}\right) \geq \sum_{k \in N} \pi_{ik}$, $\max\left(f_{ij} - \delta_j v'_j, \delta_i v'_i\right) \leq \max_{k,l \in N} f_{kl}$, and assumption (16).

It follows that

$$\|v - v'\|_\infty = \max_{i \in N} |v_i - v'_i| \leq \max_{k \in N} \delta_k \|v - v'\|_\infty + (1 - \max_{k \in N} \delta_k)\varepsilon,$$

which leads to $(1 - \max_{k \in N} \delta_k)(\|v - v'\|_\infty - \varepsilon) \leq 0$. Since $1 - \max_{k \in N} \delta_k > 0$, it must be that $\|v - v'\|_\infty \leq \varepsilon$.

Condition (16) trivially holds for $\varepsilon = 0$ and $\mu' = \mu$. The arguments above then establish that the function $g : [0, \infty)^N \to [0, \infty)^N$ defined by

$$g_i(v) = \sum_{j \in N} \frac{\pi_{ij}(\mu)}{1 - \delta_i \left(1 - \sum_{k \in N} \pi_{ik}(\mu)\right)} \max\left(f_{ij} - \delta_j v_j, \delta_i v_i\right), \forall i \in N$$

is a contraction of modulus $\max_{k \in N} \delta_k$ with respect to the norm $\| \cdot \|_\infty$. The contraction mapping theorem implies that g has exactly one fixed point $v(\mu)$. The vector $v(\mu)$ constitutes the unique solution to the system of payoff equations (3).

Now fix $\mu \in \mathcal{M}$ and $\varepsilon > 0$. Since π is continuous on $\mathcal{M}$, there exists a neighborhood $\mathcal{N}$ of μ such that (16) is satisfied for all $\mu' \in \mathcal{N}$. The inequalities above imply that $\|v(\mu) - v(\mu')\|_\infty \leq \varepsilon$ for all $\mu' \in \mathcal{N}$. This demonstrates the continuity of $v(\cdot)$.

Step 1.2. The definition of κ.

The activity condition (1) implies that $\max_{i \in N} v_i(\mu) > 0$ for any $\mu \in \mathcal{M}$. Indeed, (1) guarantees that for every $\mu \in \mathcal{M}$ there exist $i, j \in N$ such that $\pi_{ij}(\mu) f_{ij} > 0$. If $\pi_{ij}(\mu) f_{ij} > 0$ and $v_j(\mu) = 0$, then the incentive constraint $v_i(\mu) \geq \pi_{ij}(\mu)(f_{ij} - \delta_j v_j(\mu)) + (1 - \pi_{ij}(\mu))\delta_i v_i(\mu)$ leads to

$$v_i(\mu) \geq \frac{\pi_{ij}(\mu) f_{ij}}{1 - (1 - \pi_{ij}(\mu))\delta_i} > 0.$$

Let $\underline{\lambda} = \min_{i \in N} \lambda_i$; $\underline{\lambda}$ is positive, as $\lambda_i > 0$ for all $i \in N$. Note that

$$\kappa' = \min_{\mu \in \mathcal{M}, \|\mu\| \in [\underline{\lambda}/2, 2\underline{\lambda}]} \max_{i \in N} v_i(\mu)$$

is well-defined and positive since the objective function $\max_{i \in N} v_i(\cdot)$ is continuous and positive on the compact domain $\{\mu \in \mathcal{M} \mid \|\mu\| \in [\underline{\lambda}/2, 2\underline{\lambda}]\}$.

Let $\underline{f} = \min\{f_{ij} \mid i, j \in N, f_{ij} > 0\}$ and define

$$\kappa = \min\left(\kappa', \frac{\underline{f}}{6 - 4\min_{i \in N} \delta_i}\right).$$

Clearly, $\kappa > 0$.

Step 1.3. Show that for any $\mu \in \mathcal{M}$ such that $\|\mu\| \geq \underline{\lambda}/2$, there exists $\tilde{\mu} \in S^\rho(\mu)$ with $\|\tilde{\mu}\| > \underline{\lambda}/2$.

Consider first $\mu \in \mathcal{M}$ with $\|\mu\| \in [\underline{\lambda}/2, 2\underline{\lambda}]$. In this case, $\|\tilde{\mu}\| > \underline{\lambda}/2$ for all $\tilde{\mu} \in S^\rho(\mu)$. Indeed, the definition of κ' implies the existence of $i \in N$ such that $v_i(\mu) \geq \kappa'$. Then $v_i(\mu) \geq \kappa' \geq \kappa > c_i$, and hence $\beta_i \in Z(v_i(\mu) - c_i)$ only if $\beta_i = 1$. Therefore, $\tilde{\mu}_i \geq \lambda_i > \underline{\lambda}/2$ and $\|\tilde{\mu}\| > \underline{\lambda}/2$ for all $\tilde{\mu} \in S^\rho(\mu)$.

Now fix $\mu \in \mathcal{M}$ with $\|\mu\| > 2\underline{\lambda}$. Choose $\alpha_{ij} \in Z(f_{ij} - \delta_i v_i(\mu) - \delta_j v_j(\mu))$ for $i, j \in N$ such that $\alpha_{ij} = 0$ whenever $f_{ij} = 0$ (if $f_{ij} = 0$ then $f_{ij} - \delta_i v_i(\mu) - \delta_j v_j(\mu) \leq 0$, so $0 \in Z(f_{ij} - \delta_i v_i(\mu) - \delta_j v_j(\mu))$). Also select $\beta_i \in Z(v_i(\mu) - c_i)$ for $i \in N$, and define

$$\tilde{\mu}_i = \rho\Big(\mu_i - \sum_{j \in N}(\alpha_{ij}\pi_{ij}(\mu)\mu_i + \alpha_{ji}\pi_{ji}(\mu)\mu_j)\Big) + \beta_i \lambda_i.$$

By construction, $\tilde{\mu} \in S^\rho(\mu)$. We set out to show that $\|\tilde{\mu}\| > \underline{\lambda}/2$.

If $v_i(\mu) \geq \kappa$ for some $i \in N$, then arguments similar to those for $\|\mu\| \in [\underline{\lambda}/2, 2\underline{\lambda}]$ show that $\beta_i = 1$ and $\tilde{\mu}_i \geq \lambda_i$, so $\|\tilde{\mu}\| > \underline{\lambda}/2$.

Suppose next that $v_i(\mu) < \kappa$ for all $i \in N$. The payoff equations (3) can be rewritten as follows:

$$(1 - \delta_i)v_i(\mu) = \sum_{j \in N} \pi_{ij}(\mu) \max(f_{ij} - \delta_i v_i(\mu) - \delta_j v_j(\mu), 0). \tag{17}$$

We obtain the sequence of inequalities

$$(1-\delta_i)\kappa > (1-\delta_i)v_i(\mu) = \sum_{j\in N}\pi_{ij}(\mu)\max(f_{ij}-\delta_i v_i(\mu)-\delta_j v_j(\mu),0)$$
$$\geq \sum_{j\in N}\pi_{ij}(\mu)\max\left(f_{ij}-2\kappa,0\right) \geq (\underline{f}-2\kappa)\sum_{j\in N, f_{ij}>0}\pi_{ij}(\mu).$$

Note that the last summation is restricted to i's partners j with $f_{ij}>0$, for which it must be the case that $f_{ij}\geq \underline{f}$. As $\underline{f}>2\kappa$, it follows that for all $i\in N$,

$$\sum_{j\in N, f_{ij}>0}\pi_{ij}(\mu) < \frac{(1-\delta_i)\kappa}{\underline{f}-2\kappa}.$$

But the definition of κ implies that $\kappa \leq \underline{f}/(6-4\delta_i)$ for any $i\in N$, which is equivalent to

$$\frac{(1-\delta_i)\kappa}{\underline{f}-2\kappa} \leq \frac{1}{4}.$$

Therefore,

$$\sum_{j\in N, f_{ij}>0}\pi_{ij}(\mu) < \frac{1}{4}, \forall i\in N. \tag{18}$$

It follows that

$$\sum_{i\in N}\Big(\mu_i - \sum_{j\in N}(\alpha_{ij}\pi_{ij}(\mu)\mu_i + \alpha_{ji}\pi_{ji}(\mu)\mu_j)\Big) = \|\mu\| - 2\sum_{i\in N}\sum_{j\in N}\alpha_{ij}\pi_{ij}(\mu)\mu_i$$
$$= \|\mu\| - 2\sum_{i\in N}\mu_i\sum_{j\in N, f_{ij}>0}\alpha_{ij}\pi_{ij}(\mu) \geq \|\mu\| - 2\sum_{i\in N}\mu_i\sum_{j\in N, f_{ij}>0}\pi_{ij}(\mu)$$
$$\geq \|\mu\| - \frac{1}{2}\sum_{i\in N}\mu_i = \|\mu\|/2 > \underline{\lambda},$$

where the first equality collects the two $\alpha_{ij}\pi_{ij}(\mu)\mu_i$ terms in the expression, the second equality relies on our deliberate selection of $\alpha_{ij}=0$ for $f_{ij}=0$, the first inequality is a consequence of $\alpha_{ij}\leq 1$, the second inequality follows from (18), and the last inequality reflects the assumption that $\|\mu\|>2\underline{\lambda}$.

Since $\rho\geq 1/2$ and $\beta_i\lambda_i\geq 0$ for all $i\in N$, we have

$$\|\tilde{\mu}\| \geq \rho\sum_{i\in N}\Big(\mu_i - \sum_{j\in N}(\alpha_{ij}\pi_{ij}(\mu)\mu_i + \alpha_{ji}\pi_{ji}(\mu)\mu_j)\Big) > \underline{\lambda}/2,$$

as desired.

Step 1.4. The restriction of S^ρ to $\mathcal{C}$.

Aiming to apply Kakutani's theorem to show that S^ρ has a fixed point, we restrict the domain and range of S^ρ to a common compact set $\mathcal{C}$ defined by

$$\mathcal{C} = \left\{\mu\in\mathcal{M} \mid \|\mu\|\geq \underline{\lambda}/2 \;\&\; \mu_i\leq M_i, \forall i\in N\right\},$$

where

$$M_i = \lambda_i \left(1 + \frac{\max_{j\in N} f_{ij}}{c_i(1-\delta_i)}\right).$$

The restriction of S^ρ to $\mathcal{C}$, $\bar{S}^\rho : \mathcal{C} \rightrightarrows \mathcal{C}$, is given by $\bar{S}^\rho(\mu) = S^\rho(\mu) \cap \mathcal{C}, \forall \mu \in \mathcal{C}$.

Step 1.5. Show that $\bar{S}^\rho$ is non-empty valued.

Fix a market distribution $\mu \in \mathcal{C}$. Since $\|\mu\| \geq \underline{\lambda}/2$, Step 1.3 implies the existence of some $\tilde{\mu} \in S^\rho(\mu)$ with $\|\tilde{\mu}\| > \underline{\lambda}/2$. We seek to establish that $\tilde{\mu} \in \mathcal{C}$, which delivers the conclusion that $\tilde{\mu} \in \bar{S}^\rho(\mu)$. As $\|\tilde{\mu}\| > \underline{\lambda}/2$, we only have to show that $\tilde{\mu}_i \leq M_i$ for all $i \in N$. Assume by contradiction that $\tilde{\mu}_i > M_i$ for a given $i \in N$. By definition, $\tilde{\mu} \in S^\rho(\mu)$ entails that

$$\tilde{\mu}_i = \rho\Big(\mu_i - \sum_{j\in N}(\alpha_{ij}\pi_{ij}(\mu)\mu_i + \alpha_{ji}\pi_{ji}(\mu)\mu_j)\Big) + \beta_i\lambda_i$$

for some $\alpha_{ij} \in Z(f_{ij} - \delta_i v_i(\mu) - \delta_j v_j(\mu))$ for $j \in N$ and $\beta_i \in Z(v_i(\mu) - c_i)$.

Then $\tilde{\mu}_i > M_i$ implies that

$$\rho\Big(\mu_i - \sum_{j\in N}\big(\alpha_{ij}\pi_{ij}(\mu)\mu_i + \alpha_{ji}\pi_{ji}(\mu)\mu_j\big)\Big) + \beta_i\lambda_i > M_i. \tag{19}$$

As $\rho < 1$, $\mu_i \leq M_i, \lambda_i > 0$, and the summation is nonnegative, this is possible only if $\beta_i > 0$. The condition $\beta_i \in Z(v_i(\mu) - c_i)$ leads to $v_i(\mu) \geq c_i$. Along with $c_i > 0$, the last inequality implies that $\max_{j\in N} f_{ij} > 0$.

Combining (19) with the conditions $\rho < 1$ and $\beta_i\lambda_i \leq \lambda_i$, we find that

$$\mu_i - \sum_{j\in N}\big(\alpha_{ij}\pi_{ij}(\mu)\mu_i + \alpha_{ji}\pi_{ji}(\mu)\mu_j\big) + \lambda_i > M_i.$$

This inequality, along with $\mu_i \leq M_i$, entails that $\sum_{j\in N}\alpha_{ij}\pi_{ij}(\mu)\mu_i < \lambda_i$. The same inequality also leads to $\mu_i > M_i - \lambda_i$. Since $\max_{j\in N} f_{ij} > 0$, the definition of M_i implies that $M_i > \lambda_i$. We obtain

$$\sum_{j\in N}\alpha_{ij}\pi_{ij}(\mu) < \frac{\lambda_i}{\mu_i} < \frac{\lambda_i}{M_i - \lambda_i}. \tag{20}$$

As $\alpha_{ij} \in Z\big(f_{ij} - \delta_i v_i(\mu) - \delta_j v_j(\mu)\big)$, we have that $\max(f_{ij} - \delta_i v_i(\mu) - \delta_j v_j(\mu), 0) \neq 0$ only if $\alpha_{ij} = 1$. Hence the payoff equation (17) is equivalent to

$$(1-\delta_i)v_i(\mu) = \sum_{j\in N}\alpha_{ij}\pi_{ij}(\mu)\max(f_{ij} - \delta_i v_i(\mu) - \delta_j v_j(\mu), 0).$$

Since $\max(f_{ij} - \delta_i v_i(\mu) - \delta_j v_j(\mu), 0) \leq f_{ij}$, it must be that

$$(1-\delta_i)v_i(\mu) \leq \sum_{j\in N}\alpha_{ij}\pi_{ij}(\mu)f_{ij} \leq \Big(\max_{j\in N} f_{ij}\Big)\sum_{j\in N}\alpha_{ij}\pi_{ij}(\mu).$$

Then (20), along with $v_i(\mu) \geq c_i$ and $\max_{j\in N} f_{ij} > 0$, leads to

$$(1-\delta_i)c_i \leq (1-\delta_i)v_i(\mu) \leq \Big(\max_{j\in N} f_{ij}\Big)\sum_{j\in N}\alpha_{ij}\pi_{ij}(\mu) < \Big(\max_{j\in N} f_{ij}\Big)\frac{\lambda_i}{M_i - \lambda_i}.$$

It follows that

$$M_i < \lambda_i \left(1 + \frac{\max_{j \in N} f_{ij}}{c_i(1-\delta_i)}\right),$$

which contradicts the definition of M_i. The contradiction demonstrates that $\tilde{\mu} \in \bar{S}^\rho(\mu)$ and thus $\bar{S}^\rho(\mu) \neq \emptyset$.

Step 1.6. Show that $\bar{S}^\rho$ has a fixed point $\mu^\rho \in \mathcal{C}$.

Clearly, $\mathcal{C}$ is a non-empty, compact, and convex subset of $\mathbb{R}^N$. By Step 1.5, $\bar{S}^\rho$ is non-empty valued. One can easily check that $\bar{S}^\rho$ is convex-valued. The continuity of $v(\cdot)$ and $\pi(\cdot)$, along with the upper hemicontinuity of Z, implies that $\bar{S}^\rho$ has a closed graph. By Kakutani's theorem, the correspondence $\bar{S}^\rho$ has a fixed point $\mu^\rho \in \mathcal{C}$.

Step 1.7. Prove that the sequence $(\mu^{1-1/n})_{n\geq 2}$ has a limit point that constitutes a steady state.

For $\rho \in [1/2, 1)$, $\mu^\rho \in \bar{S}^\rho(\mu^\rho)$ implies the existence of selections $\alpha_{ij}^\rho \in Z(f_{ij} - \delta_i v_i(\mu^\rho) - \delta_j v_j(\mu^\rho))$ and $\beta_i^\rho \in Z(v_i(\mu^\rho) - c_i)$ for $i, j \in N$ such that

$$\mu_i^\rho = \rho\Big(\mu_i^\rho - \sum_{j \in N}\Big(\alpha_{ij}^\rho \pi_{ij}(\mu^\rho)\mu_i^\rho + \alpha_{ji}^\rho \pi_{ji}(\mu^\rho)\mu_j^\rho\Big)\Big) + \beta_i^\rho \lambda_i, \forall i \in N.$$

The sequence $(\mu^{1-1/n}, \alpha^{1-1/n}, \beta^{1-1/n})_{n\geq 2}$ is contained in a sequentially compact space, so it must admit a convergent subsequence. Hence there is a subsequence $(\rho_k)_{k\geq 0}$ of $(1-1/n)_{n\geq 2}$ such that $(\mu^{\rho_k}, \alpha^{\rho_k}, \beta^{\rho_k})_{k\geq 0}$ converges to some profile (μ, α, β) as $k \to \infty$. The continuity of $v(\cdot)$ and $\pi(\cdot)$ and the upper hemicontinuity of Z, along with $\lim_{k\to\infty} \rho_k = 1$, imply that for all $i, j \in N$,

$$\begin{aligned} &\alpha_{ij} \in Z\left(f_{ij} - \delta_i v_i(\mu) - \delta_j v_j(\mu)\right) \\ &\beta_i \in Z\left(v_i(\mu) - c_i\right) \\ &\mu_i = \mu_i - \sum_{j \in N}\left(\alpha_{ij}\pi_{ij}(\mu)\mu_i + \alpha_{ji}\pi_{ji}(\mu)\mu_j\right) + \beta_i\lambda_i. \end{aligned} \tag{21}$$

Suppose now that $v_i(\mu) < c_i$ for some $i \in N$. Then the continuity of $v_i(\cdot)$ and $\lim_{k\to\infty} \mu^{\rho_k} = \mu$ imply the existence of $\underline{k}$ such that $v_i(\mu^{\rho_k}) < c_i$ for all $k \geq \underline{k}$. It follows that $Z\left(v_i(\mu^{\rho_k}) - c_i\right) = \{0\}$ and $\beta_i^{\rho_k} = 0$, and thus

$$\mu_i^{\rho_k} = \rho_k\Big(\mu_i^{\rho_k} - \sum_{j \in N}\Big(\alpha_{ij}^{\rho_k}\pi_{ij}(\mu^{\rho_k})\mu_i^{\rho_k} + \alpha_{ji}^{\rho_k}\pi_{ji}(\mu^{\rho_k})\mu_j^{\rho_k}\Big)\Big)$$

for $k \geq \underline{k}$. As $\rho_k < 1$ and the sum in the formula above is non-negative, we need $\mu_i^{\rho_k} = 0$ for all $k \geq \underline{k}$. Hence, $\mu_i = \lim_{k\to\infty} \mu_i^{\rho_k} = 0$.

We have established that $\mu_i = 0$ whenever $v_i(\mu) < c_i$, which along with (21) implies that μ is a fixed point of S. Therefore, $(\mu, v(\mu), \alpha, \beta)$ constitutes a steady state of the bargaining game. □

Proof of Theorem 2. Fix a matching model (N, f, r, π, ω). We shall argue that the system of payoff equations (12) has a unique solution $v(\mu)$ for any $\mu \in \mathcal{M}$. Recall the definition (15) of the correspondence S^ρ derived from the function $v(\cdot)$ in Section 3. Note that μ constitutes a steady state unmatched distribution if and only if there exists $\rho > 0$ such that μ is a fixed point of S^ρ. We aim to apply Kakutani's fixed-point theorem to prove that S^ρ has a fixed point for a certain choice of $\rho > 0$. We need to restrict the domain and range of S^ρ to a common compact and convex set $\mathcal{C} \subset \mathcal{M}$. The proof is divided into a series of steps. Step 2.1 uses a contraction argument similar to Step 1.1 in the proof of Theorem 1 to show the uniqueness of the solution for the system of payoff equations (12) for every $\mu \in \mathcal{M}$ and the continuity of the solution in μ. Steps 2.2 and 2.3 define the appropriate $\mathcal{C}$ and ρ, respectively. In Step 2.4, we establish that $S^\rho(\mathcal{C}) \subseteq \mathcal{C}$. Step 2.5 completes the fixed-point argument.

Step 2.1. Uniqueness and continuity of the solution for the payoff equations.

Suppose that $\mu, \mu' \in \mathcal{M}$ and $\varepsilon \geq 0$ satisfy

$$\sum_{j\in N}\left|\frac{\pi_{ij}(\mu)}{r_i+\omega+\sum_{k\in N}\pi_{ik}(\mu)}-\frac{\pi_{ij}(\mu')}{r_i+\omega+\sum_{k\in N}\pi_{ik}(\mu')}\right| \leq \frac{\varepsilon}{\max_{k,l\in N} f_{kl}}\min_{k\in N}\frac{r_k+\omega}{r_k+\omega+\sum_{l\in N}\pi_{kl}(\mu)}$$

for all $i \in N$.[10] As in Step 1.1, we use the shorthand π_{ij} and π'_{ij} for $\pi_{ij}(\mu)$ and $\pi_{ij}(\mu')$, respectively. Assume that the vectors v and v' solve the payoff equations (12) corresponding to the unmatched market distributions μ and μ', which are equivalent to

$$v_i=\sum_{j\in N}\frac{\pi_{ij}}{r_i+\omega+\sum_{k\in N}\pi_{ik}}\max\left(f_{ij}-v_j, v_i\right)$$

$$v'_i=\sum_{j\in N}\frac{\pi'_{ij}}{r_i+\omega+\sum_{k\in N}\pi'_{ik}}\max\left(f_{ij}-v'_j, v'_i\right)$$

for all $i \in N$. Analogously to Step 1.1, we obtain

$$\begin{aligned}
|v_i-v'_i| &= \Bigg|\sum_{j\in N}\frac{\pi_{ij}}{r_i+\omega+\sum_{k\in N}\pi_{ik}}\left(\max\left(f_{ij}-v_j,v_i\right)-\max\left(f_{ij}-v'_j,v'_i\right)\right)\\
&\quad+\sum_{j\in N}\left(\frac{\pi_{ij}}{r_i+\omega+\sum_{k\in N}\pi_{ik}}-\frac{\pi'_{ij}}{r_i+\omega+\sum_{k\in N}\pi'_{ik}}\right)\max\left(f_{ij}-v'_j,v'_i\right)\Bigg|\\
&\leq \sum_{j\in N}\frac{\pi_{ij}}{r_i+\omega+\sum_{k\in N}\pi_{ik}}\left|\max\left(f_{ij}-v_j,v_i\right)-\max\left(f_{ij}-v'_j,v'_i\right)\right|\\
&\quad+\sum_{j\in N}\left|\frac{\pi_{ij}}{r_i+\omega+\sum_{k\in N}\pi_{ik}}-\frac{\pi'_{ij}}{r_i+\omega+\sum_{k\in N}\pi'_{ik}}\right|\max\left(f_{ij}-v'_j,v'_i\right)\\
&\leq \sum_{j\in N}\frac{\pi_{ij}}{r_i+\omega+\sum_{k\in N}\pi_{ik}}\|v-v'\|_\infty\\
&\quad+\frac{\varepsilon}{\max_{k,l\in N} f_{kl}}\left(\min_{k\in N}\frac{r_k+\omega}{r_k+\omega+\sum_{l\in N}\pi_{kl}}\right)\max_{k,l\in N} f_{kl}\\
&= \frac{\sum_{j\in N}\pi_{ij}}{r_i+\omega+\sum_{j\in N}\pi_{ij}}\|v-v'\|_\infty+\varepsilon\min_{k\in N}\frac{r_k+\omega}{r_k+\omega+\sum_{l\in N}\pi_{kl}}.
\end{aligned}$$

It follows that

[10] The proof assumes that $\max_{k,l\in N} f_{kl} > 0$. The result is trivial if no pair of players produces output.

$$\begin{aligned}\|v - v'\|_\infty &= \max_{i \in N} |v_i - v'_i| \\ &\leq \max_{i \in N} \frac{\sum_{j \in N} \pi_{ij}}{r_i + \omega + \sum_{j \in N} \pi_{ij}} \|v - v'\|_\infty + \varepsilon \min_{i \in N} \frac{r_i + \omega}{r_i + \omega + \sum_{j \in N} \pi_{ij}}.\end{aligned}$$

Then

$$1 - \max_{i \in N} \frac{\sum_{j \in N} \pi_{ij}}{r_i + \omega + \sum_{j \in N} \pi_{ij}} = \min_{i \in N} \frac{r_i + \omega}{r_i + \omega + \sum_{j \in N} \pi_{ij}} > 0$$

leads to

$$\min_{i \in N} \frac{r_i + \omega}{r_i + \omega + \sum_{j \in N} \pi_{ij}} \left(\|v - v'\|_\infty - \varepsilon\right) \leq 0.$$

The inequalities above hold only if $\|v - v'\|_\infty \leq \varepsilon$. The same arguments from Step 1.1 then imply that the system of payoff equations (12) has a unique solution $v(\mu)$ for every $\mu \in \mathcal{M}$, which is continuous in μ.

Step 2.2. Definition of $\mathcal{C}$.

The regularity condition (9) implies that

$$\lim_{\mu \to \mathbf{0}} \left(\omega(\theta_i - \mu_i) - \sum_{j \in N} \left(\pi_{ij}(\mu)\mu_i + \pi_{ji}(\mu)\mu_j\right)\right) = \omega\theta_i > 0.$$

Thus there exists $m > 0$ such that

$$\|\mu\| < 2m \Rightarrow \omega(\theta_i - \mu_i) - \sum_{j \in N} \left(\pi_{ij}(\mu)\mu_i + \pi_{ji}(\mu)\mu_j\right) > 0, \forall i \in N. \quad (22)$$

Pick such an m, which satisfies the additional constraint $m \leq \|\theta\|$. Define

$$\mathcal{C} = \{\mu \in \mathcal{M} \mid \|\mu\| \geq m\}.$$

The condition $m \leq \|\theta\|$ guarantees that $\mathcal{C} \neq \emptyset$.

Step 2.3. Definition of ρ.

Note that the value

$$z = \max_{\mu \in \mathcal{C}} \sum_{i \in N} \sum_{j \in N} \left(\pi_{ij}(\mu)\mu_i + \pi_{ji}(\mu)\mu_j\right)$$

is well-defined and finite since the objective function is continuous in μ on the compact domain $\mathcal{C}$. Set

$$\rho = \min\left(\frac{m}{z}, \frac{1}{\omega}\right).$$

Step 2.4. Show that $S^\rho(\mathcal{C}) \subseteq \mathcal{C}$.

Fix $\mu \in \mathcal{C}$ and $\tilde{\mu} \in S^\rho(\mu)$. Then there exist $\alpha_{ij} \in Z\left(f_{ij} - v_i(\mu) - v_j(\mu)\right)$ for $i, j \in N$ such that

$$\tilde{\mu}_i = \mu_i + \rho\left(\omega(\theta_i - \mu_i) - \sum_{j \in N} \left(\alpha_{ij}\pi_{ij}(\mu)\mu_i + \alpha_{ji}\pi_{ji}(\mu)\mu_j\right)\right), \forall i \in N.$$

We need to show that $\tilde{\mu} \in \mathcal{C}$.

First, note that for all $i \in N$, the inequalities $\mu_i \le \theta_i$ and $\rho \le 1/\omega$ lead to $\tilde{\mu}_i \le \mu_i + \rho\omega(\theta_i - \mu_i) = \theta_i - (1-\rho\omega)(\theta_i - \mu_i) \le \theta_i$. Moreover, $\alpha_{ij} \in [0,1]$ for $i, j \in N$ implies that

$$\tilde{\mu}_i \ge \mu_i + \rho\Big(\omega(\theta_i - \mu_i) - \sum_{j\in N}\big(\pi_{ij}(\mu)\mu_i + \pi_{ji}(\mu)\mu_j\big)\Big), \forall i \in N. \tag{23}$$

We now show that $\|\tilde{\mu}\| \ge m$ by considering two cases. Suppose first that $\|\mu\| < 2m$. Then (22) and (23) lead to $\tilde{\mu}_i > \mu_i$ for all $i \in N$. Hence, in this case, $\|\tilde{\mu}\| > \|\mu\| \ge m$.

Assume instead that $\|\mu\| \ge 2m$. By (23), we have $\tilde{\mu}_i \ge \mu_i - \rho\sum_{j\in N}\big(\pi_{ij}(\mu)\mu_i + \pi_{ji}(\mu)\mu_j\big)$ for all $i \in N$. Then the definition of z, along with the assumption that $\|\mu\| \ge 2m$ and the inequality $\rho \le m/z$, leads to

$$\|\tilde{\mu}\| \ge \|\mu\| - \rho\sum_{i\in N}\sum_{j\in N}\big(\pi_{ij}(\mu)\mu_i + \pi_{ji}(\mu)\mu_j\big) \ge 2m - \rho z \ge m.$$

We have established that $\tilde{\mu}_i \le \theta_i$ for all $i \in N$ and $\|\tilde{\mu}\| \ge m$, so $\tilde{\mu} \in \mathcal{C}$. Since $\mu \in \mathcal{C}$ and $\tilde{\mu} \in S^\rho(\mu)$ were chosen arbitrarily, it follows that $S^\rho(\mathcal{C}) \subseteq \mathcal{C}$.

Step 2.5. Show that S^ρ has a fixed point, which constitutes a steady state.

Note that $\mathcal{C}$ is a non-empty, compact, and convex subset of $\mathbb{R}^N$. Moreover, $S^\rho(\mu)$ is a non-empty convex set for every $\mu \in \mathcal{C}$. By Step 2.4, $S^\rho(\mathcal{C}) \subseteq \mathcal{C}$. The continuity of $v(\cdot)$ established in Step 2.1, along with the continuity of π and the upper hemicontinuity of Z, implies that the restriction of S^ρ to $\mathcal{C}$ has a closed graph. By Kakutani's theorem, S^ρ has a fixed point in $\mathcal{C}$. The fixed point constitutes a steady state unmatched distribution for the matching model, as argued in the preamble of the proof. □

References

Atakan, A.E., 2006. Assortative matching with explicit search costs. Econometrica 74, 667–680.

Diamond, P.A., Maskin, E., 1979. An equilibrium analysis of search and breach of contract I: steady states. Bell J. Econ. 10, 282–316.

Duffie, D., Garleanu, N., Pedersen, L.H., 2005. Over-the-counter markets. Econometrica 73, 1815–1847.

Gale, D., 1987. Limit theorems for markets with sequential bargaining. J. Econ. Theory 43, 20–54.

Lauermann, S., 2013. Dynamic matching and bargaining games: a general approach. Am. Econ. Rev. 103, 663–689.

Lauermann, S., Noldeke, G., 2015. Existence of steady-state equilibria in matching models with search frictions. Econ. Lett. 131, 1–4.

Manea, M., 2011. Bargaining in stationary networks. Am. Econ. Rev. 101, 2042–2080.

Manea, M., 2013. Bargaining in dynamic markets. Mimeo.

Noldeke, G., Troger, T., 2009. Matching heterogeneous agents with a linear search technology. Mimeo.

Rubinstein, A., Wolinsky, A., 1985. Equilibrium in a market with sequential bargaining. Econometrica 63, 295–328.

Satterthwaite, M., Shneyerov, A., 2007. Dynamic matching, two-sided incomplete information, and participation costs: existence and convergence to perfect competition. Econometrica 75, 155–200.

Shimer, R., Smith, L., 2000. Assortative matching and search. Econometrica 68, 343–369.

Sun, Y., 2006. The exact law of large numbers via Fubini extension and characterization of insurable risks. J. Econ. Theory 126, 31–69.