14.452: Problem Set 2

Due date: November 10, 2017.

Please only hand in Question 3, which will be graded. The rest will be reviewed in the recitation but should not be handed in.

**Question 1:** Consider a modified version of the continuous-time Solow growth model where the aggregate production function is

\[ F(K, L, Z) = L^\beta K^\alpha Z^{1-\alpha-\beta}, \]

where \( Z \) is land, available in fixed inelastic supply. Assume that \( \alpha + \beta < 1 \), capital depreciates at the rate \( \delta \), and there is an exogenous saving rate of \( s \).

1. First suppose that there is no population growth. Find the steady-state capital-labor ratio and the steady-state output level. Prove that the steady state is unique and globally stable.

2. Show that, in the steady-state equilibrium, there is a monotonic relationship between the interest rate and the saving rate of the economy. Using this result, show that there exists a saving rate \( s^* \) such that above this, the interest rate is negative. Show that when the interest rate is negative, starting from the steady-state equilibrium, it is possible to reallocate resources so that consumption increases at all points in time. Explain what this means and why such a possibility is present in this model.

3. Now suppose that there is population growth at the rate \( n \), that is, \( \dot{L}/L = n \). Does a steady-state equilibrium exists? What happens to the capital-labor ratio and output level as \( t \to \infty \)? What happens to returns to land and the wage rate as \( t \to \infty \)?

4. Would you expect the population growth rate \( n \) or the saving rate \( s \) to change over time in this economy? If so, how? What other adjustments might you expect in this economy as \( t \to \infty \)?

**Question 2:** Consider the discrete-time Solow growth model with constant population growth at the rate \( n \), no technological change and depreciation rate of capital equal to \( \delta \). Assume that the saving rate is a function of the capital-labor ratio, thus given by \( s(k) \).

1. Suppose that \( f(k) = Ak \) and \( s(k) = s_0 k^{-1} - 1 \). Show that if \( A + \delta - n = 2 \), then for any \( k(0) \in (0, As_0/(1+n)) \), the economy immediately settles into an asymptotic cycle and continuously fluctuates between \( k(0) \) and \( As_0/(1+n) - k(0) \). [Suppose that \( k(0) \) and the parameters are given such that \( s(k) \in (0, 1) \) for both \( k = k(0) \) and \( k = As_0/(1+n) - k(0) \)].
2. Now consider more general continuous production function $f(k)$ and saving function $s(k)$, such that there exist $k_1, k_2 \in \mathbb{R}_+$ with $k_1 \neq k_2$ and

\[
\begin{align*}
  k_2 &= \frac{s(k_1) f(k_1) + (1 - \delta) k_1}{1 + n} \\
  k_1 &= \frac{s(k_2) f(k_2) + (1 - \delta) k_2}{1 + n}.
\end{align*}
\]

Show that when such $(k_1, k_2)$ exist, there may also exist a stable steady state.

3. Prove that such cycles are not possible in the continuous-time Solow growth model for any (possibly non-neoclassical) continuous production function $f(k)$ and continuous $s(k)$.

4. What does the result in parts 1-3 imply for the approximations of discrete time by continuous time in the Solow model (suggested in Section 2.4 of the textbook)? What does this imply for the cycles in parts 1 and 2?

5. Show that if $f(k)$ is nondecreasing in $k$ and $s(k) = k$, cycles as in parts 1 and 2 are not possible in discrete-time either.

**Question 3:** Consider the Solow growth model with constant saving rate $s$ and depreciation rate of capital equal to $\delta$. Assume that population is constant and the aggregate output is given by the CES production function

\[
F(A_K(t) K(t), A_L(t) L) = \left[ \gamma (A_K(t) K(t))^{\sigma - 1} + (1 - \gamma) (A_L(t) L)^{\sigma - 1} \right]^{\frac{1}{\sigma - 1}},
\]

where $\dot{A}_L(t)/A_L(t) = g_L > 0$ and $\dot{A}_K(t)/A_K(t) = g_K > 0$. Suppose the elasticity of substitution between capital and labor is less than one, $\sigma < 1$, and capital-augmenting technological progress is faster than labor-augmenting progress, i.e., $g_K \geq g_L$. Show that as $t \to \infty$, the economy converges to a BGP where the share of labor in national income is equal to 1, and capital, output and consumption all grow at the rate $g_L$. In light of this result, discuss the often-made claim that capital-augmenting technological change is inconsistent with balanced growth.

**Question 4:** Consider the basic Solow model in continuous time and suppose that $A(t) = A$, so that there is no technological progress of the usual kind. However, assume that the relationship between investment and capital accumulation is modified to

\[
\dot{K}(t) = q(t) I(t) - \delta K(t),
\]

where $\{q(t)\}_{t=0}^\infty$ is an exogenously given time-varying process. Intuitively, when $q(t)$ is high, the same investment expenditure translates into a greater increase in the capital stock. Therefore, we can think of $q(t)$ as the inverse of the relative prices of machinery to output. When $q(t)$ is high, machinery is relatively cheaper, and thus suppose that $\dot{q}(t) > 0$.

1. Suppose that $\dot{q}(t)/q(t) = \gamma_K > 0$. Show that for a general production function, $F(K, L)$, there exists no steady-state equilibrium.
2. Now suppose that the production function is Cobb-Douglas, \( F(K, L) = K^\alpha L^{1-\alpha} \), and characterize the unique steady-state equilibrium.

3. Show that this steady-state equilibrium does not satisfy the Kaldor fact of constant \( K/Y \). Is this a problem? [Hint: how is “\( K \)” measured in practice? How is it measured in this model?]