Online Appendix for “Minimizing Justified Envy in School Choice: The Design of New Orleans’ OneApp”

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S.1 Proof of Proposition 2

The example below shows that TTC-Counters, TTC-Clinch and Trade, and Equitable-TTC do not minimize justified envy in the class of Pareto efficient and strategy-proof mechanisms, when schools have multiple seats.

Consider TTC-Counters (as will become clear, the argument is similar for TTC-Clinch and Trade or Equitable-TTC). We build a Pareto efficient and strategy-proof mechanism ϕ that allows for strictly less justified envy than TTC-Counters. Suppose ϕ selects the same matching as TTC-Counters except for the following economy: there are four individuals i1, i2, i3, and i4 and three schools s1, s2, s3, where q_{s1} = 1 and q_{s2} = q_{s3} = 3. In this economy, ϕ selects a matching that is free of justified envy and Pareto efficient, which can be computed by student-proposing DA.

ϕ characterizes priorities under which, for any individual preference, there exists a matching that is Pareto efficient and justified envy-free. Priorities must be such that there is no Ergin-cycle. A profile ≻ has an Ergin-cycle if there are three individuals i1, i2, and i3 and two schools s1, s2 such that the two conditions are satisfied:

1. Cycle condition. i1 ≻_{s1} i2 ≻_{s1} i3 and i3 ≻_{s2} i1,
2. Scarcity condition. There are (possibly empty) disjoint sets N_{s1} and N_{s2} ⊆ I\{i1, i2, i3\} s.t. N_{s1} ⊆ U_{s1}(i2) and N_{s2} ⊆ U_{s2}(i1) and |N_{s1}| = q_{s1} - 1 and |N_{s2}| = q_{s2} - 1 where U_{s1}(i2) and U_{s2}(i1) are the strict upper contour set of i2 and i1, respectively (i.e., U_{s1}(i2) := \{ℓ : ℓ ≻_{s1} i2\} and U_{s2}(i1) := \{ℓ : ℓ ≻_{s2} i1\}).

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In the economy described above, the scarcity condition in the definition of an Ergin-cycle can never be satisfied. To see this, observe that for a school \( s \in \{s_2, s_3\} \),

\[ |N_s| = 3 - 1 = 2, \]

while \( N_s \subseteq I\backslash\{i_1, i_2, i_3\} \) implies that

\[ |N_s| \leq 1, \]

since \( |I| = 4 \). Therefore, sets satisfying the scarcity condition do not exist. Hence, any profile of priority relations is Ergin-acyclic.

Finally, it is enough for our purpose to build some \((P, \succ)\) where the set of blocking pairs of \( \varphi \) is a proper subset of the set of blocking pairs of TTC-Counters. Since \( \varphi \) eliminates justified envy, we only need to show that there is \((P, \succ)\) under which TTC-Counters does not eliminate justified envy.

Consider the following profile of preferences and priority relations:

<table>
<thead>
<tr>
<th>( P_{i_1} )</th>
<th>( P_{i_2} )</th>
<th>( P_{i_3} )</th>
<th>( P_{i_4} )</th>
<th>( \succ_{s_1} )</th>
<th>( \succ_{s_2} )</th>
<th>( \succ_{s_3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_3 )</td>
<td>( s_1 )</td>
<td>( s_1 )</td>
<td>( s_2 )</td>
<td>( i_1 )</td>
<td>( i_1 )</td>
<td>( i_3 )</td>
</tr>
<tr>
<td>( s_3 )</td>
<td>( i_4 )</td>
<td>( i_3 )</td>
<td>( i_4 )</td>
<td></td>
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<tr>
<td>( i_2 )</td>
<td>( i_2 )</td>
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</tr>
<tr>
<td>( i_3 )</td>
<td>( i_4 )</td>
<td>( i_1 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TTC-Counters produces:

\( \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ s_3 & s_3 & s_1 & s_2 \end{pmatrix} \),

where \((i_2, s_1)\) is a blocking pair.

This completes the argument for TTC-Counters. It is easy to check that TTC-Counters, TTC-Clinch and Trade, and Equitable TTC coincide to produce the same assignment for the above profile of preferences and priority relations. Hence, the same argument can be used for TTC-Clinch and Trade and Equitable TTC.

### S.2 Another Justified Envy-Minimal Mechanism

When each school has a single seat, we build a mechanism different from TTC that is strategy-proof, Pareto-efficient, and justified envy-minimal. The mechanism is identical to TTC except at the following instance of priorities: we have three students \( i_1, i_2 \) and \( i_3 \) and two schools \( s_1 \) and \( s_2 \) each with a single seat. Priorities are given by

\[
\begin{array}{c|c}
\succ_{s_1} & \succ_{s_2} \\
\hline
i_2 & i_1 \\
\hline
i_3 & i_3 \\
\hline
i_1 & i_2 \\
\end{array}
\]
In essence, the mechanism will rank $i_3$ on top of each school’s ranking and run standard TTC on these modified priorities (except for some preference profiles where there is a unique efficient and stable allocation where the original priorities will still be used to run TTC). This mechanism will be denoted TTC*. Let us describe it precisely. For the instance of priorities described above and for each profile of preferences $P$, TTC* selects a matching as follows.

Case A.
If under $P$ there is an individual who ranks all schools as unacceptable then run TTC.

Case B.1.
If under $P$ both $i_1$ and $i_2$ rank $s_1$ first and $i_1$ finds $s_2$ unacceptable then run TTC

Case B.2.
If under $P$ both $i_1$ and $i_2$ rank $s_2$ first and $i_2$ finds $s_1$ unacceptable then run TTC

Case C.
If none of the above cases apply, move $i_3$ to the top of each school’s ranking. Run TTC on the modified priorities.

Clearly, TTC* is Pareto efficient. We prove below that it is strategy-proof.

**Proposition S1.** TTC* is strategy-proof.

**Proof.** Fix $P$ falling into case A. If some student $i$ deviates to $P'_i$, this cannot be profitable if we remain into case A or fall into Case B.1 or B.2 (since TTC is strategy-proof). Therefore, consider the case where we fall into Case C after $i$’s deviation. After the deviation, all individuals rank at least one school acceptable (since we are not in Case A anymore). Since at least one individual must rank all schools unacceptable before the deviation and since we are looking at a single deviation by individual $i$, we conclude that $P_i$ ranks all schools unacceptable. Hence, under TTC*, $i$ is unmatched under $P$ and since $P_i$ ranks all schools unacceptable, there cannot be any profitable deviation.

Fix $P$ falling into case B.1 (and not in case A). If some student $i$ deviates to $P'_i$, this cannot be profitable if we remain into case B.1 or fall into Case A or B.2 (since TTC is strategy-proof). Therefore, consider the case where we fall into Case C after $i$’s deviation. This must mean that before deviation both $i_1$ and $i_2$ rank $s_1$ first and $i_1$ finds $s_2$ unacceptable, though this is not the case anymore after deviation. Note that this must mean that $i$ is either $i_1$ or $i_2$. Further, since $P$ falls into Case B.1, TTC* runs standard TTC. Hence, $i_2$ gets matched to her top choice $s_1$ and so $i_2$ has no incentive to deviate (recall that each individual finds at least one school acceptable since we are not in Case A). Hence, let us consider $i = i_1$. The only way to reach (by a deviation of $i_1$) Case C is for $i_1$ to claim that $s_2$ is acceptable (while $s_2$ is not acceptable to $i_1$ under the original preferences $P_i$). Now, to complete the argument, we distinguish two cases. First, assume that $i_3$ ranks $s_2$ first. Then, since we fall into Case C after deviation, TTC*($P'_i, P_{-i}$) is
given by
\[
\begin{pmatrix}
i_1 & i_2 & i_3 \\
i_1 & s_1 & s_2
\end{pmatrix}.
\]
In particular, \(i_1\) cannot get \(s_1\) (the only acceptable school under \(P_i\)) so the deviation to \(P'_i\) cannot be profitable. Similarly, in the other case where \(i_3\) ranks \(s_1\) first, \(\text{TTC}^*(P'_i,P_{-i})\) is given by
\[
\begin{pmatrix}
i_1 & i_2 & i_3 \\
s_2 & i_2 & s_1
\end{pmatrix}.
\]
Here again, \(i_1\) fails to obtain \(s_1\) and so the deviation is not profitable. The case under which \(P\) falls into Case B.2 can be treated in the same way.

Fix \(P\) falling into case C. If some student \(i\) deviates to \(P'_i\), this cannot be profitable if we remain into case C (since \(\text{TTC}\) is strategy-proof). Therefore, consider the case where we fall into Case A. This means that \(P'_i\) ranks all schools unacceptable. Since \(\text{TTC}^*\) is individually rational, \(i\)'s deviation cannot be profitable. So consider the case where after deviation we fall into Case B.1 (and not into Case A). This means that before deviation, either \(i_1\) or \(i_2\) does not rank \(s_1\) first, or \(i_1\) finds \(s_2\) acceptable while after deviation both \(i_1\) and \(i_2\) rank \(s_1\) first and \(i_1\) finds \(s_2\) unacceptable. This means that \(i\) is either \(i_1\) or \(i_2\). If \(i = i_1\), this means that, \(i_2\) ranks \(s_1\) first (recall that before and after deviation, each individual has at least one acceptable school since we do not fall into Case A before and after deviation). So, in particular, after deviation, \(i_1\) cannot get \(s_1\) (\(s_1\) will be allocated to \(i_2\)). Since after deviation, \(i_1\) finds \(s_2\) unacceptable, \(i_1\) will not get \(s_2\) either, and so she will remain unmatched. So the deviation cannot be profitable to \(i_1\).

Now, consider the other case where deviator \(i = i_2\). This means that \(i_1\) ranks \(s_1\) first and ranks \(s_2\) unacceptable. This also means that before deviation, \(i_2\) ranks \(s_2\) first while after deviation \(i_2\) ranks \(s_1\) first. To complete the proof, we distinguish two cases. First, assume that \(i_3\) ranks \(s_2\) first. Then, at \(P\), \(i_2\) gets \(s_1\) if she finds \(s_1\) acceptable or remains unmatched. After deviation, \(i_2\) ranks \(s_1\) on top and \(i_2\) must be getting \(s_1\) after deviation, so this cannot be profitable. In the other case where \(i_3\) ranks \(s_1\) first, before deviation, \(i_2\) is getting \(s_2\), which is her top choice. So the deviation cannot improve on this. The same reasoning holds if the deviation falls into Case B.2.

We fix any Pareto efficient and strategy-proof mechanism \(\varphi\) with less justified envy than \(\text{TTC}^*\). We claim that \(\varphi = \text{TTC}^*\).

**Proposition S2.** Fix any \(P\) that falls into Case A, B.1 or B.2. \(\varphi(P) = \text{TTC}^*(P)\).

**Proof.** Fix any \(P\) falling into Case A. Some individual must rank all schools as unacceptable. It is easy to check that, in such a case, there is a unique efficient and stable allocation that is selected by TTC (with only two students, priorities are trivially Ergin-acyclic). Hence, because
\( \varphi \) has less justified envy than TTC*, \( \varphi \) must also select the unique efficient and stable allocation, and we obtain \( \varphi(P) = \text{TTC}^*(P) \).

Now, fix any \( P \) falling into Case B.1. Both \( i_1 \) and \( i_2 \) rank \( s_1 \) first, and \( i_1 \) finds \( s_2 \) unacceptable. Here again, one can check that TTC selects the unique efficient and stable allocation, and we obtain \( \varphi(P) = \text{TTC}^*(P) \). A similar reasoning holds for any \( P \) falling into Case B.2.

**Proposition S3.** Fix any \( P \) that falls into Case C. \( \varphi(P) = \text{TTC}^*(P) \).

**Proof.** We assume that \( P \) falls into Case C and prove the above proposition in the four following claims.

*Claim 1.* Assume that \( s_1 P_i s_2 \) and \( s_2 P_i s_1 \).

\[
\varphi(P) = \text{TTC}^*(P).
\]

**Proof.** Clearly, under TTC*, \( i_3 \) is never part of any blocking pair. Hence, because \( \varphi \) has less justified envy than TTC*, we must have that \( i_3 \) is never part of any blocking pair under \( \varphi \) as well. Assume wlog that \( s_1 \) is \( i_3 \)'s top choice (recall that because \( P \) falls into Case C, each individual finds at least one school acceptable).

In the sequel, we claim that \( i_3 \) is assigned its top choice \( s_1 \) under matching \( \varphi(P) \). If \( i_3 \) is not assigned its top choice \( s_1 \) under \( \varphi \), then in order to ensure that \((i_3, s_1)\) does not block \( \varphi(P) \), we must have that \( i_2 \) is matched to \( s_1 \) under \( \varphi(P) \). Now, consider two cases. First, \( i_3 \) is matched to \( s_2 \) under \( \varphi(P) \). In that case, \( i_2 \) and \( i_3 \) would be better off switching their assignments, a contradiction with Pareto efficiency of \( \varphi \). In the other case, \( i_3 \) must be unmatched under \( \varphi(P) \). If \( i_1 \) gets matched to \( s_2 \) under \( \varphi(P) \), allowing \( i_2 \) and \( i_1 \) to switch their assignments would be beneficial to both of them, again a contradiction with Pareto efficiency of \( \varphi \). If \( i_1 \) is not matched to \( s_2 \) under \( \varphi(P) \) then \( s_2 \) is unmatched, and by assigning it to \( i_2 \) we Pareto-improve on \( \varphi(P) \), a contradiction.

Thus, we proved that \( \varphi(P)(i_3) = s_1 = \text{TTC}^*(P)(i_3) \). Now, let us complete the argument and show that \( \varphi(P) = \text{TTC}^*(P) \). First, consider the case where \( i_1 \) finds \( s_2 \) acceptable. TTC* yields the following matching

\[
\begin{pmatrix}
i_1 & i_2 & i_3 \\
s_2 & i_2 & s_1
\end{pmatrix}.
\]

If under \( \varphi(P) \), \( i_1 \) remains unmatched, then \((i_1, s_2)\) would block \( \varphi(P) \) while it does not block TTC*(\( P \)), a contradiction with our assumption that \( \varphi \) has less justified envy than TTC*. Thus, \( \varphi(P)(i_1) = s_2 = \text{TTC}^*(P)(i_1) \) and so we conclude that \( \varphi(P) = \text{TTC}^*(P) \). Now, consider the second case where \( i_1 \) finds \( s_2 \) unacceptable. Recall that \( s_2 \) must be acceptable to \( i_2 \), and so TTC* yields the following matching

\[
\begin{pmatrix}
i_1 & i_2 & i_3 \\
i_1 & s_2 & s_1
\end{pmatrix}.
\]
Clearly, since we showed that \( i_3 \) gets matched to \( s_1 \) under \( \varphi(P) \), \( i_1 \) remains unmatched under \( \varphi(P) \). So by Pareto efficiency of \( \varphi \), it must be that \( \varphi(P)(i_2) = s_2 = \text{TTC}^*(P)(i_2) \). We conclude that \( \varphi(P) = \text{TTC}^*(P) \).

Claim 2. Assume that \( s_1 P_{i_1} s_2 \) and \( s_1 P_{i_2} s_2 \).

\[ \varphi(P) = \text{TTC}^*(P). \]

Proof. There are two cases.

Case 1. \( s_1 P_{i_3} s_2 \). Because \( P \) falls into Case C, each individual ranks at least one school acceptable (since \( P \) does not fall into Case A) and \( i_1 \) finds \( s_2 \) acceptable (since \( P \) does not fall into Case B.1). Thus, \( \text{TTC}^* \) yields the following matching

\[
\begin{pmatrix}
  i_1 & i_2 & i_3 \\
  s_2 & i_2 & s_1
\end{pmatrix}.
\]

We first claim that under \( \varphi(P) \), \( i_2 \) must remain unmatched. Indeed, if \( i_2 \) is matched under \( \varphi(P) \), then consider the new preference profile where \( s_2 P_{i_2} i_1 P_{i_2} i_2 \). Note that \( (P'_{i_2}, P_{-\{i_2\}}) \) falls into the cases considered in Claim 1. Hence, by Claim 1, we know that \( \varphi(P'_{i_2}, P_{-\{i_2\}})(i_3) = \text{TTC}^*(P'_{i_2}, P_{-\{i_2\}})(i_3) = s_1 \) and \( \varphi(P'_{i_2}, P_{-\{i_2\}})(i_2) = \text{TTC}^*(P'_{i_2}, P_{-\{i_2\}})(i_2) = i_2 \). Thus, from profile \( (P'_{i_2}, P_{-\{i_2\}}) \), \( i_2 \) can misreport her preference profile as \( P_{i_2} \). In turn, she gets matched and is strictly better-off, which contradicts the strategy-proofness of \( \varphi \). Hence, under \( \varphi(P) \), \( i_2 \) must be unmatched. Next, we claim that \( i_3 \) is assigned \( s_1 \) under \( \varphi(P) \). Indeed, if \( i_3 \) is not assigned \( s_1 \) under \( \varphi(P) \), then \( i_1 \) must be assigned \( s_1 \) since it is acceptable to her (and we already know that \( i_2 \) must be unmatched). But then \( (i_3, s_1) \) would block \( \varphi(P) \) but does not block \( \text{TTC}^*(P) \), which contradicts our assumption that \( \varphi \) has less justified envy than \( \text{TTC}^* \). To conclude, under \( \varphi(P) \), \( i_3 \) gets \( s_1 \), \( i_2 \) is unmatched, and so, since \( s_2 \) is acceptable to \( i_1 \), \( i_1 \) gets matched to \( s_2 \). Thus, \( \varphi(P) = \text{TTC}^*(P) \).

Case 2. \( s_2 P_{i_3} s_1 \). \( \text{TTC}^* \) yields the following matching

\[
\begin{pmatrix}
  i_1 & i_2 & i_3 \\
  i_1 & s_1 & s_2
\end{pmatrix}.
\]

We first claim that \( \varphi(P) \) matches \( i_3 \) to her top choice \( s_2 \). Indeed, if \( i_3 \) is not matched to \( s_2 \) under \( \varphi(P) \) then in order for \( (i_3, s_2) \) not to block \( \varphi(P) \), \( i_1 \) must match to \( s_2 \). But then, in order for \( (i_2, s_1) \) not to block \( \varphi(P) \), which is necessary since it does not block \( \text{TTC}^*(P) \), \( i_2 \) must also match \( s_1 \). So if \( i_3 \) is not matched to \( s_2 \) under \( \varphi(P) \) the only candidate for \( \varphi(P) \) is

\[
\begin{pmatrix}
  i_1 & i_2 & i_3 \\
  s_2 & s_1 & i_3
\end{pmatrix}.
\]

Now, let us assume that \( i_3 \) ranks \( s_1 \) acceptable under \( P_{i_3} \). Next, consider the preference profile \( (P'_{i_3}, P_{-\{i_3\}}) \) where \( s_1 P_{i_3} s_2 P_{i_3} i_3 \). Note that \( (P'_{i_3}, P_{-\{i_3\}}) \) falls into Case 1 considered just
above. Hence, we know that \( \varphi(P^\prime_{i_3}, P_{-\{i_3\}}) = \text{TTC}^*(P^\prime_{i_3}, P_{-\{i_3\}}) \) and so \( i_3 \) is matched to \( s_1 \) under \( \varphi(P^\prime_{i_3}, P_{-\{i_3\}}) \). Since \( i_3 \) is unmatched under \( \varphi(P) \), because we assumed that \( i_3 \) ranks \( s_1 \) acceptable under \( P_{i_3} \), we found a profitable deviation for \( i_3 \), a contradiction with the strategy-proofness of \( \varphi \). Thus, provided that \( P_{i_3} \) ranks \( s_1 \) as acceptable, we obtained \( \varphi(P_{i_3}, P_{-\{i_3\}})(i_3) = \text{TTC}^*(P_{i_3}, P_{-\{i_3\}})(i_3) = s_2 \).

Let us now assume that \( i_3 \) ranks \( s_1 \) unacceptable under \( P_{i_3} \). Consider a deviation of \( i_3 \) to \( P^\prime_{i_3} \) satisfying \( s_2 P^\prime_{i_3} s_1 P^\prime_{i_3} i_3 \), i.e., where \( s_1 \) is ranked as acceptable. We just saw that, in such a case, \( \varphi(P^\prime_{i_3}, P_{-\{i_3\}})(i_3) = \text{TTC}^*(P^\prime_{i_3}, P_{-\{i_3\}})(i_3) = s_2 \) and so \( i_3 \) gets matched to \( s_2 \) under \( \varphi(P^\prime_{i_3}, P_{-\{i_3\}}) \). Here again, we find a profitable deviation for \( i_3 \), which contradicts the strategy-proofness of \( \varphi \).

We conclude that \( \varphi(P) \) matches \( i_3 \) to her top choice \( s_2 \). Now, \( \text{TTC}^*(P) \) matches \( i_2 \) with \( s_1 \), and, in order not have the blocking pair \((i_2, s_1)\) under \( \varphi(P) \), \( i_2 \) and \( s_1 \) must also be matched together under \( \varphi(P) \). We conclude that \( \varphi(P) = \text{TTC}^*(P) \). \( \square \)

**Claim 3.** Assume that \( s_2 P_{i_1} s_1 \) and \( s_2 P_{i_2} s_1 \).

\[ \varphi(P) = \text{TTC}^*(P). \]

**Proof.** The proof is similar to that of Claim 2. \( \square \)

**Claim 4.** Assume that \( s_2 P_{i_1} s_1 \) and \( s_1 P_{i_2} s_2 \).

\[ \varphi(P) = \text{TTC}^*(P). \]

**Proof.** Without loss of generality, assume that \( s_2 P_{i_1} s_1 \) (the same argument applies when \( s_1 P_{i_2} s_2 \)). \( \text{TTC}^* \) yields the following matching

\[
\begin{pmatrix}
i_1 & i_2 & i_3 \\
i_1 & s_1 & s_2
\end{pmatrix}.
\]

If, under \( \varphi(P), i_3 \) is not matched to her top choice \( s_2 \), then in order for \((i_3, s_2)\) not to block \( \varphi(P) \) (which is necessary, since it does not block \( \text{TTC}^*(P) \)), \( i_1 \) must be matched to \( s_2 \) under \( \varphi(P) \). But then for \((i_2, s_1)\) not to block \( \varphi(P) \) (which is necessary since it does not block \( \text{TTC}^*(P) \)), \( i_2 \) must be matched to \( s_1 \). Thus, if \( i_3 \) is not matched to her top choice \( s_2 \), the only candidate for \( \varphi(P) \) is

\[
\begin{pmatrix}
i_1 & i_2 & i_3 \\s_2 & s_1 & i_3
\end{pmatrix}.
\]

Now, consider \( P^\prime_{i_1} \) such that \( s_1 P^\prime_{i_1} s_2 P^\prime_{i_1} i_1 \). Since \( i_1 \) ranks \( s_2 \) acceptable under \( P^\prime_{i_1} \), \((P^\prime_{i_1}, P_{-\{i_3\}})\) falls in to the profile of preferences considered in Claim 2. Hence, \( \varphi(P^\prime_{i_1}, P_{-\{i_3\}}) \) and \( \text{TTC}^*(P^\prime_{i_1}, P_{-\{i_3\}}) \) both yield the same matching given by

\[
\begin{pmatrix}
i_1 & i_2 & i_3 \\
i_1 & s_1 & s_2
\end{pmatrix}.
\]
Now, if the true preference profile is \((P'_{i_1}, P_{-\{i_1\}})\) and \(i_1\) misreports to \(P_{i_1}\), then \(i_1\) gets matched to \(s_2\) under \(\varphi(P_{i_1}, P_{-\{i_1\}})\). Hence, the misreport \(P_{i_1}\) is profitable to \(i_1\), which contradicts the strategy-proofness of \(\varphi\).

We conclude that under \(\varphi(P)\), \(i_3\) must be matched to her top choice \(s_2\). But now, if \(i_2\) is not matched to \(s_1\) under \(\varphi(P)\) then \((i_2, s_1)\) blocks \(\varphi(P)\) but does not block \(\text{TTC}^*(P)\), which is a contradiction. Hence, \(i_2\) must be matched to \(s_1\), and we conclude that \(\varphi(P) = \text{TTC}^*(P)\). □

These four claims together establish the proposition. □