Deterministic Dynamic Choice

- Today 2 periods, \( t = 0, 1 \).

- Use fairly general representations to clarify some conceptual points; the next lecture will look at more structured representations (e.g. additively separable) for choice over multiple time periods.

- Warning: the model covered today has a lot of notation we don’t usually both with; its purpose is to let make assumptions explicit and not hidden when they are implicit in the notation.

- Spaces \( Z_0, Z_1 \) of alternatives.

- In period 0 the agent may face a choice of period-1 menus, and also choose a period-0 alternative, and these two may be linked. So \( \) chooses an element of \( X_0 := Z_0 \times M(Z_1) \). (recall that \( M(Z) = 2^Z \setminus \{\emptyset\} \).)
• In period 1 the agent chooses an element of $X_1 := Z_1$.

• Example: agent has initial wealth $w_0$. Can spend some of it but not borrow; can save at real rate $r$. So period-0 menu is

$$\left\{(c_0, M(c_0)) : c_0 \in [0, w_0], M(c_0) = \{c_1 \in [0, (1 + r)(w_0 - c_0)]\} \right\}.$$  

Note that not all second period menus are consistent with all first-period choices: if the agent consumes all her wealth in period 0 she can’t consume in period 1.

• In general, period-0 choices are described by a choice correspondence

$$c_0 : M(X_0) \to M(X_0) \text{ s.t. } c_0(A_0) \subseteq A_0 \text{ for all } A_0 \in M(X_0). \quad (0)$$  

Here $c_0(A_0)$ is a finite collection of pairs $\{(z'_0, A'_i), (z''_0, A''_i), \ldots\}$. 
• Period 1 choices might depend on period-0 consumption.

• And (for now) let’s allow the possibility they also depend on the choice problem the agent faced in period 0.

• So define the period-0 histories $h_0$ to be pairs $(A_0, x_0)$, where $A_0$ is the menu the agent faced, and $x_0$ is the choice she made.

• The set of all possible period-0 histories is then

\[ H_0 := \{(A_0, x_0) \in M(X_0) \times X_0 : x_0 \in c_0(A_0)\}. \]

• Note the restriction to “intended choices” $x_0 \in c_0(A_0)$: can’t observe choice in period 1 when the agent didn’t get their intended period-0 outcome.

• Would learn more about preferences if agents were forced to “tremble” as in Frick, Iijima, and Strzalecki [2017]; won’t cover that here.
For each period-1 menu $A_1$, let $H_0(A_1)$ be the period-0 histories consistent with it:

$$H_0(A_1) := \{(A_0, x_0) \in H_0 : x_0 = (z_0, A_1) \text{ for some } z_0 \in Z_0\}.$$

The domain of period 1 choice correspondence - the set it’s defined on - is the current menu $A_1$ and the history that preceded it. Denote this as

$$\mathcal{D}_1 := \{(A_1, A_0, x_0) \in M(X_1) \times H_0 : (A_0, x_0) \in H_0(A_1)\}.$$

- The period 1 choice correspondence is a

$$c_1 : \mathcal{D}_1 :\rightarrow M(X_1) \text{ with } c_1(A_1, A_0, x_0) \subseteq A_1. \quad (1)$$

- A dynamic choice correspondence is a pair $(c_0, c_1)$ that satisfies (0) and (1).
• **Definition:** A dynamic choice correspondence is *consequentialist* if

$$c_1(A_1, A_0, x_0) = c_1(A_1, B_0, x_0)$$
for all $A_0, B_0$ s.t. for all $(A_0, x_0), (B_0, x_0) \in H_0(A_1)$
s.t. $x_0 = (z_0, A_1)$ for some $z_0 \in Z_0$.

-Note that this *does* let period-1 choice vary with the period-0 choice $x_0$.

-It requires that the agent makes the same choice from $A_1$ regardless of the period-0 menu $A_0$ or $B_0$. The “s.t.” part says to only look at situations where the period-0 menu allows picking $x_0$. 
Suppose that in period 0 you eat lunch and pick a restaurant for dinner, in period 1 you order dinner.

• Consequentialism allows what you order at dinner (period 1) to depend on the $z_0$ you had for lunch. But it requires that what you order at dinner in an Italian restaurant doesn’t depend on whether the alternative restaurant you considered was Greek or Spanish.

• Same idea as the “consequentialism” I defined for static choice under risk, which said that if first Nature decides whether to implement lottery $r$ or give you a choice between $p$ and $q$, your choice doesn’t depend on what $r$ was.

• In both settings, consequentialism is a form of “no regret” condition.

• It allows dependence on things that have happened in the past, but not on things that “might have happened.”
• From here on assume consequentialism and write $c_1(A_1, x_0)$.

• Note that w/o consequentialism period-1 choice at $A_1$ can be different for each $A_0$ that leads to it- so hard to have non-vacuous consistency conditions on period-1 choice.

• The next condition is not needed to define “rational” dynamic choice, and in some applications such as habit formation it is relaxed. But it is commonly assumed to simplify:

**Definition:** Choice is *history-independent* if $c_1(A_1, z_0) = c_1(A_1, z'_0)$ for all $z_0, z'_0 \in Z, A_1 \in M(X_1)$.

• Will assume history independence (aka *time separability*) for the rest of this lecture and write period-1 choice as $c_1(A_1)$. 
• In fact will now go further and suppose there is no period-0 consumption choice, and set $X_0 = M(Z_1)$.

• Assume that both $c_0$ and $c_1$ satisfy WARP.

• So they correspond to maximizing complete transitive preferences $\succeq_0$ and $\succeq_1$, and (because $Z_1$ is finite) to maximizing utility functions $u_0$ and $u_1$.

• Note: $u_0$ and $\succeq_0$ are defined on $X_0 = M(Z_1)$.

• They induce a utility function and preference on $Z_1$ by looking at singleton menus $\{z_1\}$.

• But the domain of singleton menus is too small to determine period-0 preferences without additional conditions.
**Definition:** Preferences $(\succeq_0, \succeq_1)$ have a **recursively consistent representation** if they can be represented by $u_0, u_1$ s.t. $u_0(A_1) = \max_{z_1 \in A_1} u_1(z_1)$.

(Note: Strzalecki calls this a “dynamically consistent representation” but then calls another condition “dynamic consistency of preferences.”)

- Economists usually (but not always!) use recursively consistent representations.

- Let’s try to understand them better by seeing when they apply.
**Definition**: Period-0 preference $\succeq_0$ is *strategically rational* if
\[ A_1 \succeq_0 B_1 \implies A_1 \sim_0 A_1 \cup B_1. \]

- Note that this needn’t be true if the period-1 choice isn’t made to maximize period-0 preferences.

- For example $B_1$ might be a “temptation”: Let $B_1$ be $\{\text{Scotch}\}$ and $A_1 = \{\text{Pellegrino}\}$.

- If Drew thinks that if he buys Scotch now (period 0) he will drink more of it tonight than he ought to, then we could have $A_1 \succ_0 B_1$ and $A_1 \succ_0 A_1 \cup B_1$.

- OTOH if period-1 preference is stochastic, then it might be that $A_1 \succeq_0 B_1$ and $A_1 \prec_0 A_1 \cup B_1$; the larger menu adds an “option value.”
Lemma (Kreps Ema [1979]): Period-0 preference $\succsim_0$ is *strategically rational* iff the function $u_0(A_i) = \max_{z_i \in A_i} u_0(\{z_i\})$ represents $\succsim_0$.

*Proof sketch:* For each $A_i$ list its elements in decreasing preference order, $\{z_1^1\} \succsim_0 \{z_1^2\} \succsim_0 \cdots \succsim_0 \{z_1^{\#A_i}\}$.

By strategic rationality $\{z_1^1\} \sim_0 \{z_1^1, z_1^2\}$, and by induction $\{z_1^1\} \sim_0 A_i$ so $u_0(A_i) = \max_{z_i \in A_i} u_0(\{z_i\})$.

Conversely if $u_0(A_i) = \max_{z_i \in A_i} u_0(\{z_i\})$,

then $u_0(A_i \cup B_i) = u(A_i)$ so $A_i \sim A_i \cup B_i$. ■

- Strategic rationality is a condition only on period-0 choice, so it can’t link period-0 and period-1 choices. (Although it can suggest possibilities that are consistent with the period-0 choice and a given way of linking the two periods together.)
**Definition:** Preferences \((\succeq_0, \succeq_1)\) are *dynamically stable* if
\[
\{z_1\} \succeq_0 \{w_1\} \iff z_1 \succeq_1 w_1 .
\]

**Theorem:** Preferences \((\succeq_0, \succeq_1)\) have a *recursively consistent representation* iff they are strategically rational and dynamically stable.

**Proof:** easy HW.

Recursive consistency rules out stochastic period-1 preferences. To allow stochastic preference, Kreps ([Ema 1979]) provides a representation theorem for the representation

\[
u_0(A_1) = \sum_{s \in S} p(s) \max_{a_1 \in A_1} u_1(a_1, s) \] for some \(S, p, \) and \(u_1.\)  \hspace{1cm} (3)

Here \(u_1\) can depend on \(s\) but neither it nor \(p\) can depend on \(A_1.\) *(Note this reduces to recursive consistency when \(S\) is a singleton.)*
With representation (3), if \( \{a,b\} \sim_0 \{a\} \) then the agent can never strictly prefer to add \( b \) to any menu that contains \( a \): if adding \( b \) helps menu \( C \) it means sometimes \( b \) is better than \( a \) so we would have \( \{a,b\} \succ_0 \{a\} \).

More generally, representation (3) satisfies *modularity*:

If \( A_1 \sim_0 A_1 \cup B_1 \) then \( A_1 \cup C_1 \sim_0 A_1 \cup C_1 \cup B_1 \) for all \( C_1 \).

Kreps shows that period-0 menu preferences have representation (3) iff they satisfy modularity and

*“Preference for Flexibility”:* If \( A_1 \supseteq B_1 \) then \( A_1 \succ_0 B_1 \). (*)

*Aside:* I prefer to call (*) *monotonicity* as it can have other interpretations.

Note that this result is about the representation of choice in period 0, and doesn’t say anything about period-1 choice; see Ahn and Sarver *Ema* [2013].
Temptation and Self Control

A strategically rational agent never strictly prefers a smaller menu, and wouldn’t sign up for a monitoring program that has only fines and no positive payments.

Many experiments where a non-trivial (though sometimes small) fraction of participants prefer a smaller menu:
Ashraf, Karlan and Yin QJE [2006] 28% of subjects choose a commitment savings account; Gine, Karlan and Zimmerman AER [2010] 11% of smokers agree to be fined if they don’t quit; Kaur, Kremer and Mullainathan JPE [2015] commitment contracts chosen 35% of the time (averaging over days and workers); Houser et al [2010] 36% of subjects use commitment device to keep from web surfing during a lab experiment; Augenblick, Nierderle, and Sprenger QJE [2015] 58% of subjects prefer costless commitment in a work now/work later task (though only 9% will pay more than $0.25 for it.)
Period-0 preference \( \vartriangleleft_0 \) has a \textit{Strotz representation} (Strotz \textit{REStud} [1955]) if there are \( u_0, u_1 \) s.t.

\[
c_1(A_1) = \arg \max_{z_1 \in A_1} u_1(z_1) \quad \text{and} \quad c_0(A_0) = \arg \max_{A_i \in A_0, z_i \in c(A_i)} u_0(z_1).
\]

- Here the agent is “sophisticated”: she knows not only that her period-1 preferences will be different but what they will be, and picks a menu accordingly- it’s as if the observed choices are the equilibria of a game between these two selves.

- Strotz preferences aren’t concerned by items in the menu that won’t be chosen. This rules out a preference to avoid temptations that would be resisted.
**Definition** Period-0 preference \( \succeq_0 \) satisfies **no compromise** if for all \( A_1, B_1 \in M(Z_1) \) either \( A_1 \sim_0 A_1 \cup B_1 \) or \( B_1 \sim_0 A_1 \cup B_1 \).

**Theorem** (Gul and Pesendorfer *RESTud* [2005]): Period-0 preference \( \succeq_0 \) satisfies no compromise iff it has a Strotz representation.

*Not-very-revealing proof outline*: define a revealed preference on subsets, and a separate revealed preference on singletons, and then relate them.

- The Strotz representation is closely related to quasi-hyperbolic discounting, where preferences represented by \( U_0 = u_0 + \beta \delta u_1 + \beta \delta^2 u_2 \) and \( U_1 = u_1 + \beta \delta u_2 \).

- Here the period-1 utility function \( U_1 \) differs from period-0 utility \( U_0 \) in its tradeoffs between immediate rewards in period 1 and rewards that will arrive later, say in a period 2 where the agent has no decision to make. The idea is that the period-1 agent will sacrifice period 2 consumption for consumption in period 1 at worse terms than the period-0 agent would like.
• O’Donoghue and Rabin *AER* [1999] define “partially naïve” Strotz models: agent realizes that his future self will be present biased but mis-forecasts how large that bias will be.


• Gul and Pesendorfer *Ema* [2001]: the value of a menu is

\[
u_0(A_1) = \max_{z_1 \in A_1} u_1(z_1) - \left( \max_{x_1 \in A_1} v_1(x_1) - v_1(z_1) \right).
\]

• Here \( v_1 \) is an arbitrary “temptation function,” and the control cost of choosing \( z_1 \) from menu \( A_1 \) is the resisted temptation \( \left( \max_{z_1 \in A_1} v_1(z_1) - v_1(z_1) \right) \).

• Important: only the biggest temptation in \( A_1 \) matters- fits with perfectly foreseen preferences.
• This comes from their assumption of set betweenness:
  \[ A_1 \sim B_1 \rightarrow A_1 \sim A_1 \cup B_1 \sim B_1. \]

• Control cost is linear in foregone utility from a version of the independence axiom, on a different and more complex space. But there is some evidence for convex costs.

• To model period-1 choice GP specify dynamic consistency, so that
  \[
  c_1(A_1) = \arg\max_{z_1} \left( u_1(z_1) - \left( \max_{x_1 \in A_1} v_1(x_1) - v_1(z_1) \right) \right),
  \]
  \[
  = \arg\max_{z_1} u_0(z_1) + v_1(z_1)
  \]

  *Note that the “temptation” \[ \max_{x_1 \in A_1} v_1(x_1) \] doesn’t alter period-1 choice.*
Fudenberg and Levine AER [2006] show that a similar representation describes the equilibrium of a game between a “long run self” and a sequence of completely myopic short-run selves who have the same per-period utilities (and so differ from the LR only in their discount factors).

Dual selves: a long-run “planner” can exert effort to change the preferences of a myopic “doer.

As in GP this control cost depends on foregone utility.

The subgame-perfect equilibrium is as if LR self maximizes

\[
(1 - \delta) \sum_{t=0}^{\infty} \delta^t \left( u_t(z_t) - \gamma(\Delta_t) \right)
\]

where \( \Delta_t = \max_{z_t' \in A_t} u(z_t') - u(z_t) \) is the difference between the maximum feasible utility in the current period and the utility actually received.
- More restrictive than GP because same utility function for temptation and choice instead of the pair \((u, v)\): the two selves differ only in their discount factors, as in quasi-hyperbolic preferences.

- Less restrictive than GP because \(\gamma\) can be strictly convex.

- When \(\gamma\) strictly convex, it’s more than twice as hard/costly to resist twice the temptation.

- Convex control costs allow certain (but not all!) violations of WARP, such as the “compromise effect”: pick fruit from \{fruit, small desert\} but small desert from \{fruit, small desert, large desert\}.

- This can’t happen with GP’s linear cost function because there the action chosen maximizes \(w(z_1) := u_0(z_1) + v_1(z_1)\) - so choice satisfies WARP.
• Convex costs explain why more self-indulgent in one domain (e.g. diet or exercise) when exerting more self control in another (e.g. hours of work).

• Can also be used to explain the effect of cognitive load (Shiv and Fedorikhin J. Cons. Res. [1999]):

• Subjects were asked to memorize either a two- or a seven-digit number, and then walk to a table with a choice of two deserts, chocolate cake and fruit salad.

• Subjects would then pick a ticket for a desert and report the number and their choice in a second room. Longer number to memorize: % choosing cake increases from 41% to 63%.


• Dual-self explanation: control cost depends on the sum of cognitive load and foregone utility.
Toussaert [2016]: Subjects paid for doing a tedious task, face temptation to forego earnings to hear a story.

- Elicit preference ordering over \{story\}, \{no story\}, \{choose later\}, when the preferred choice is implemented *stochastically*: highest ranked menu is most likely but not certain. *Here “choose later” means choose the subsequent menu \{story, no story\}*

- “self-control types” can rank \{no story\} \succ \{choose later\} \succ \{story\}.

- Strotz preferences can’t do this, as they aren’t concerned by items in the menu that won’t be chosen.

- Implement (stochastic) distribution over menus: now can observe 2\textsuperscript{nd} period choices of subjects who preferred commitment. *(w/o this stochastic element, the 2\textsuperscript{nd} period choice of someone who chose not to have a 2\textsuperscript{nd} period choice is hypothetical/counterfactual.)*
• To distinguish indifference from strict preference, offered subjects the chance to pay to get their 1st choice if lottery says get 2nd- either in $ or in extra time to work (using a price list mechanism as in the BDM procedure.)

• Also asked subjects’ beliefs about what they’d do w/o commitment. (and used predictions of other subject’s second-period choice as an instrument...
Findings:

- 36% of subjects report \{no story} \rightarrow \{choose later} \rightarrow \{story\}.

- 58% of this group (so 25% of overall pool) willing to pay for a commitment.

- Only 2.5% of subjects consistent with Strotz preferences.

- Almost all self-control types predicted that they would resist the temptation to learn the story in the absence of commitment- no support for “random temptation” or “random Strotz” models.

- Perceived self-control almost exactly matches observed self control when making the choice: 18%.

Now back to “standard” preferences and work up to more than 2 periods...
Additively separable discounting: \( U(z_0, z_1, \ldots) = \sum_t \delta^t u(z_t) \).

- Used by (most?) models of choice over time.

- Here the same utility function is used in every period, and the discount factor \( \delta \) is constant.

- This rules out e.g. exogenously changing tastes, habit formation as in Becker-Murphy *JPE* [1988], and a preference for consumption streams that increase over time. Can pick these up with a state variable that tracks the payoff-relevant aspects of past consumption, won’t do that here.

- To understand the restrictions that the discounting representation imposes on choice, need to first understand when preferences are *additively separable*, e.g. when can we decompose \( u(\text{apples, oranges}) \) into \( u_a(\text{apples}) + u_o(\text{oranges}) \)?
Separable Preferences

• Let $I$ be a finite (for now) set of indices (e.g. time periods, fruits, states). (We will see a representation theorem for countably many time periods, it needs more assumptions. And the expected utility representations extend to uncountable state spaces, this also needs more structure.)

• For each $i \in I$ there is a set $X_i$, let $X := \times_{i \in I} X_i$.

• Analyst observes complete transitive preference $\succeq$ on $X$.

• **Definition:** $\succeq$ has an *additively separable representation* if there are $u_i : X_i \rightarrow \mathbb{R}$ s.t. $U(x_1,\ldots,x_n) = u_1(x_1) + \ldots + u_n(x_n)$ represents $\succeq$.

• In an additively separable representation, the tradeoff between any $x_i$ and $x_j$ is independent of the other components, i.e. of $X_{-i,j}$.
• For any \( E \subseteq I \) and any \( x, y \in X \) define \( x_E y \in X := \begin{cases} x_i & i \in E \\ y_i & i \notin E \end{cases} \).

• **Definition:** \( \succcurlyeq \) is *singleton separable* if for all \( i \in I \) and all \( x, y, z, z' \in X \),
\[
  x_i z \succcurlyeq y_i z \iff x_i z' \succcurlyeq y_i z'.
\]
(this should remind you of the independence axiom of expected utility!)

• Singleton separability implies that for each index \( i \) we have a complete transitive preference \( \succcurlyeq_i \) on \( X_i \) that is independent of the other components:
\[
x_i \succcurlyeq_i y_i \text{ if } x_i z \succcurlyeq y_i z \text{ for some } z.
\]

• Already restrictive, but not sufficient, because it doesn’t yet imply that the tradeoff between any \( x_i \) and \( x_j \) is independent of the other components.
• For example if \( X_1 = \{1, 2, 3\} \), \( X_2 = \{1, 3, 5\} \), the preference induced by 
\[ u(x_1, x_2) = x_1 x_2 + x_1^x_2 \] is strictly increasing in \( x_1 \) for each \( x_2 \) and vice versa. But it 
does not have an additive representation (HW).

• In the discounting application, we need an additive representation if we want 
the tradeoff between consumption in periods \( t \) and \( s \) to be independent of 
consumption in other periods.

• So we use a stronger condition:

\[ \succsim \text{ has jointly separable indices} \] if for any \( E \subseteq I \) and all \( x, y, z, z' \in X \), 
\[ x_E z \succsim y_E z \iff x_E z' \succsim y_E z' \]. (Strzalecki calls this “separable.”).

• With 3 or more indices this say the tradeoffs between \( x_i \) and \( x_j \) don’t depend 
on the level of some 3rd index \( k \).
• For this to have any bite we need 3 indices that “matter.”

• **Reason:** with only 2 indices, jointly separable indices reduces to singleton separability, and as we saw that doesn’t imply an additive representation. And having 3 with one that doesn’t matter is like having 2.

• **Definition:** An index \( i \) is **null** if for all \( x, y, z \in X \), \( x_i z \sim y_i z \).

• The next theorem will ask that every index is non-null, and also that each \( X_i \) is connected; together these two conditions mean that each coordinate has a continuum of elements.

• Since additively separable representations have a utility function, we expect to need a continuity condition when \( X \) isn’t finite.
• “Technical condition”: Assume each $X_i$ is a connected subset of $\mathbb{R}^k$ (or more generally a connected topological space) and that the preference $\succeq$ on $X := \times_{i \in I} X_i$ is continuous w.r.t. the product topology. (result extends to more general topological spaces)

**Theorem** (Debreu [1960], generalized by Wakker *J Math Psychol*ogy [1988]): Suppose complete transitive preference $\succeq$ satisfies the technical condition and has at least three non-null indices. Then it has jointly separable indices iff it has an additively separable representation by continuous utility functions $u_i : X_i \to \mathbb{R}$ s.t. $u_i$ is constant whenever $i$ is null. Moreover, if $v_1, \ldots, v_n$ also represent $\succeq$, then there are $\alpha > 0$ and $\beta_i$ s.t. $v_i = \alpha u_i + \beta_i$.

**Proof**: omitted.

• **Reading for next time**: Strzalecki Ch 3.1-3.3, Ch 4.