Abstract

This paper establishes some asymptotic results such as central limit theorems and consistency of variance estimation in factor models. We consider a setting common to modern macroeconomic and financial models where many counties/regions/macro-variables/assets are observed for many time periods, and when estimation of a global parameter includes aggregation of a cross-section of heterogeneous micro-parameters estimated separately for each entity. We establish a central limit theorem for quantities involving both cross-sectional and time series aggregation, as well as for quadratic forms in time-aggregated errors. We also study sufficient conditions when one can consistently estimate the asymptotic variance. These results are useful for making inferences in two-step estimation procedures related to factor models. We avoid structural modeling of cross-sectional dependence but impose time-series independence.

Keywords: factor models, two-step procedure, dimension asymptotics, central limit theorem.

JEL classification codes: C13, C33, C38, C55.
1 Introduction

Data that have a factor structure are increasingly used in empirical macroeconomics and finance. Often this data consists of a long time series of observations on a large cross-section of many assets, portfolios, regions or industries. Quite a few new estimation strategies have appeared in the empirical literature that use both cross-sectional and time series variation in order to estimate global structural parameters. Often the structural parameter of interest arises from aggregation or estimation using cross-sectional variation of individual parameters for each entity.

Examples of such an approach are linear factor pricing models in asset pricing (Fama & MacBeth (1973) and Shanken (1992)), where we usually have a long time series of excess returns on a large cross-section of portfolios or assets priced by a small number of risk factors. Each portfolio or stock may have its own (heterogeneous) exposure to risk, often referred to as betas, which can be estimated separately from time series observations for each portfolio. The parameter of interest, a risk premium, is defined as the coefficient of proportionality in the cross-sectional relation between the average excess return on a portfolio and its individual beta.

A vast majority of macroeconomic shocks are only weakly identified via structural VARs that use only time series observations on leading macro variables. A new approach to the estimation of causal effects of different macro shocks on the economy is to use cross-sectional variation in data on regions, countries or industries. For example, Serrato & Wingender (2016) use cross-sectional variation in federal spending programs due to a Census shock to identify the causal impact of government spending on the economy. Cross-sectional variation among counties in government spending and in the accuracy of census-based estimates of population provides a better justified treatment effect framework, allows for the estimation of local fiscal multipliers (allowing for heterogeneous government spending effects), and finally gives a better global estimate of the fiscal multiplier via aggregation of local multipliers. Hagedorn, Manovskii & Mitman (2015) estimate the aggregate effect of unemployment-benefit duration on employment and labor force participation using cross-sectional differences across US states. Sarto (2018) discusses how, in a very large class of macroeconomic models with heterogeneous regions, sensitivities of regions to aggregate policy variables, micro-global elasticities, can be used to recover macro elasticities of interest such as, for example, a fiscal multiplier.

A shared feature of the above-mentioned examples is the use of time-series observations on multiple entities (stocks, portfolios, counties, states or industries), while those
entities typically cannot be considered to be independent and identically distributed. Moreover, variables for different entities will often display strong co-movements to the extent that the data have a factor structure, and estimation of this co-movement is a main goal. Indeed, the realization of a risk factor in the economy moves returns on all portfolios simultaneously, while a federal fiscal shock moves spending in all US counties, though in both cases heterogeneously so. A valid estimation procedure must explicitly model and account for the data’s factor structure to the extent that the error terms (or residuals) can be considered idiosyncratic (see Kleibergen & Zhan (2015) and Anatolyev & Mikusheva (2018) for how a factor structure that is unaccounted for can lead to misleading results). However, idiosyncrasy of the errors usually implies only that the correlation among errors for different entities is relatively small and does not introduce first-order bias to the estimation procedure. Usually, it is not reasonable to assume that errors for different entities are completely independent; indeed, stocks in the same industry are likely to co-move even after global-economy risks are removed, while errors for neighboring counties are more likely to be correlated even after one accounts for federal shocks. At the same time, we typically want to remain agnostic about the correlation structure of shocks and avoid their structural modeling as long as this does not introduce biases.

The second common feature of the above-mentioned examples is the two-step nature of the estimation procedure, where in the first step we estimate entity-specific coefficients (risk exposures/betas, local fiscal multipliers, micro-global elasticities) by running a time-series regression separately for each entity. In the second step, we estimate the global coefficient of interest by either aggregating entity-specific coefficients (Serrato & Wingerder (2016) and Hagedorn, Manovskii & Mitman (2015)), or by running an OLS regression on the cross-section of entity-specific coefficients (Fama & MacBeth (1973) and Sarto (2018)), or by running an IV regression on the cross-section of entity-specific coefficients (Anatolyev & Mikusheva (2018)).

The goal of this paper is to establish asymptotic normality of the estimate obtained in the two-step estimation procedure and to come up with a consistent estimator for the asymptotic variance, while being flexible in modeling the cross-sectional dependence of errors. The main difficulty here is that even though the second step cross-sectional regression has nearly uncorrelated errors (which is usually sufficient to obtain consistency of the two-step estimator), this condition is usually insufficient for the central limit theorem, which typically requires that stronger discipline be imposed on the dependence (such as independence, or a martingale-difference structure, or mixing). Our solution to this problem is to discipline the time series behavior while staying agnostic about the
cross-sectional dependence. We assume time-series independence of idiosyncratic errors, which is quite consistent with market efficiency (for factor asset pricing models) and the non-predictability of macro shocks (in macroeconomic settings). The estimation noise in the two-stage procedure will involve aggregation both over time (from the first step) and over entities (from the second step). We show that under certain conditions it will be sufficient to have a central limit theorem over just one of these directions, and we will use time-series direction for establishing asymptotic gaussianity.

The second difficulty of our task is that for the cases when OLS or IV is used in the second step, we will need to establish a central limit theorem for quadratic forms, since both the second-stage-dependent variable and the second stage regressor/instrument contain first-stage estimation noise. A similar issue was resolved by Chao, Swanson, Hausman, Newey & Woutersen (2012) where asymptotic gaussianity for the jackknife IV estimator in a many-weak-instruments setting was established. There, the estimator comes from an IV regression at the second stage of a two-step procedure. However, Chao, Swanson, Hausman, Newey & Woutersen (2012) assume full independence of all errors and are not concerned with the possibility of weak cross-sectional dependence. Technically, we use an approach close to that of Chao, Swanson, Hausman, Newey & Woutersen (2012), which relies on the CLT by de Jong (1987).

Our paper also contributes to the literature on structural estimation in panel data models. Related papers include Pesaran (2006), Ando & Bai (2015) and Bai (2009). The results obtained here are used in Anatolyev & Mikusheva (2018), which introduces a new version of a factor asset pricing estimator that is robust to weak observed factors and strongly cross-sectionally correlated pricing errors.

The paper proceeds as follows. Section 2 explains problems with establishing asymptotic gaussianity of the two-step estimator and shows how discipline in the time series direction can help. Section 3 introduces assumptions on idiosyncratic errors, states central limit theorems for two cases, and discusses the relevance of those cases to empirical practice. Section 4 discusses estimation of asymptotic variances. All proofs appear in the Appendix.

2 Setup and Notation

2.1 Estimation environment

Assume that we use data that contain observations on many entities indexed by \( i = 1, \ldots, N \) observed for multiple time periods \( t = 1, \ldots, T \) to estimate some structural pa-
rameter $\lambda$. We will assume that both $N$ and $T$ increase to infinity without restrictions on their rates. Assume that we have a well justified two-step procedure, where in the first step we estimate parameters $\beta_i$ separately for each entity using time-series observations for that entity. Assume that the estimator looks like $\hat{\beta}_i = \beta_i + \epsilon_i$ with estimation noise $\epsilon_i = \frac{1}{T} \sum_{t=1}^{T} v_t e_{it}(1 + \omega(1))$, where $e_{it}$ is an idiosyncratic error, and $v_t$ is some variable common to all entities; the term $\omega(1)$ is uniformly small over entities. This holds, for example, if the $\hat{\beta}_i$’s are OLS or IV estimates with common variables as regressors/instruments. In the case of factor pricing, $v_t$ includes a constant and risk factors; in Sarto (2018), $v_t$ contains aggregate policy variables. Note that, this setting can also accommodate entity-specific regressors, say $v_{it}$, that have factor structure themselves. Indeed, assume that $v_{it} = a_i u_t + u_{it}$ where $u_t$ is a common co-movement in the regressors and $u_{it}$ is idiosyncratic. Then

$$\epsilon_i = \frac{1}{T} \sum_{t=1}^{T} v_t e_{it} = \frac{1}{T} \sum_{t=1}^{T} u_t (a_i e_{it}) + \frac{1}{T} \sum_{t=1}^{T} u_{it} e_{it} = \frac{1}{T} \sum_{t=1}^{T} v_t e_{it},$$

where $v_t = (u_t, 1)'$ and $e_{it}^* = (a_i e_{it}, u_{it} e_{it})$. Thus, by idiosyncratic error we mean the factor-removed part of entity-specific variables.

In the second step, the parameter of interest $\lambda$ is estimated from the cross-section of entities either by some weighted averaging $\hat{\lambda} = \frac{1}{N} \sum_{i=1}^{N} \gamma_i \hat{\beta}_i$, or by an OLS regression of some components of the first-step estimates $\beta_i$ on some other components of $\beta_i$: $\hat{\lambda} = \left( \frac{1}{N} \sum_{i=1}^{N} \hat{\beta}_i^{(1)} \hat{\beta}_i^{(1)} \right)^{-1} \frac{1}{N} \sum_{i=1}^{N} \hat{\beta}_i^{(1)} \hat{\beta}_i^{(2)}$, or by IV in a regression with some components serving as instruments: $\hat{\lambda} = \left( \frac{1}{N} \sum_{i=1}^{N} \hat{\beta}_i^{(1)} \hat{\beta}_i^{(3)} \right)^{-1} \frac{1}{N} \sum_{i=1}^{N} \hat{\beta}_i^{(1)} \hat{\beta}_i^{(2)}$.

The consistency and the rate of convergence of $\hat{\lambda}$ usually come from convergence in probability to zero of properly normalized sums $\sum_{i=1}^{N} \gamma_i \epsilon_i$ or $\sum_{i=1}^{N} \epsilon_i^{(1)} \epsilon_i^{(2)}$. Results such as these mainly rely on Chebyshev’s inequality and for the most part restrict the covariance structure of the variables involved. The goal of this paper is to establish a foundation for statistical inferences of $\hat{\lambda}$, namely, to obtain asymptotic theorems that can be used to establish asymptotic gaussianity of $\hat{\lambda}$ and to obtain a consistent estimate of the asymptotic variance. That is, we want to find sufficient conditions for asymptotic normality of the properly normalized sums $\sum_{i=1}^{N} \gamma_i \epsilon_i$ and $\sum_{i=1}^{N} \epsilon_i^{(1)} \epsilon_i^{(2)}$ and to estimate their asymptotic variances. For an application of this paper to the estimation of a factor pricing model, see Anatolyev & Mikusheva (2018, Section 6).

The set of idiosyncratic errors $\{e_{it}, i = 1, \ldots, N, t = 1, \ldots, T\}$, which are typically errors in first-step regressions, in most cases cannot be assumed to be independent and/or identically distributed. In most applications, in order to guarantee the consistency of $\hat{\lambda}$,
these idiosyncratic errors are assumed to be weakly cross-sectionally correlated to such an extent that the asymptotic correlation between the $\varepsilon_i$’s with different $i$’s is negligible. See Andrews (2005) on how cross-sectional correlation can create biases. In particular, a well-formulated first step must not have errors with a factor structure, see Kleibergen & Zhan (2015) and Anatolyev & Mikusheva (2018) for insight into how a factor structure which is unaccounted for can lead to misleading results. In most realistic applications, we are usually willing to assume that $\{e_{it}, i = 1, ..., N, t = 1, ..., T\}$ do not have a factor structure, but still allow for some correlation between different entities, which would not affect the consistency of $\hat{\lambda}$. For example, it is reasonable to think that stocks of firms in the same industry or of the same size may react to some local shocks and be correlated, though when averaged over all stocks (and all industries), this co-movement of returns would have no first-order impact on an estimation.

The paradigm in this paper is to be agnostic with regard to possible cross-correlation among errors for different entities and to avoid explicit modeling of its structure whenever possible. However, vanishing correlation among the $\varepsilon_i$’s with different $i$’s is insufficient for a central limit result, as the majority of central limit theorems impose independence, or a martingale difference structure, or stationarity and some mixing condition on the dependence. None of these conditions seems appealing in applications where we do not have a reasonable ordering of entities or a clear structure of cross-sectional dependence.

2.2 Idea of solution

In many applications of interest, it is more credible to impose strong dependence assumptions in a time-series direction rather than in a cross-sectional direction. For example, the efficient market hypothesis implies mean non-predictability of excess returns given past history, which is equivalent to a martingale difference property for the errors. The definition of shocks in macroeconomics similarly assumes their time-series independence. In this paper, we assume time-series independence, which in some cases may be weakened to the martingale-difference property or stationarity with some proper mixing condition, but we will not ascribe this generalization here.

We show that the assumption about time-series independence allows one to obtain asymptotic gaussianity for normalized sums of interest. Assume that $E(e_{it}) = 0$ and is uncorrelated with the first-step regressor $v_t$. For the case when the second step involves
the averaging of the first step estimates, we have

\[ \sqrt{\frac{T}{N}} \sum_{i=1}^{N} \gamma_i \varepsilon_i = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} v_t \left( \sum_{i=1}^{N} \gamma_i e_{it} \right). \]

Here, we have changed the order of summation. Clearly, it will be enough to obtain some CLT along a time-series dimension that will eventually place some restrictions on the cross-sectional dependence of \( e_{it} \) by restricting the moments of cross-sectional sums.

If the second step involves OLS or IV estimation, the situation becomes more involved. Denote \( \mathbb{E} (e_{it}^2) = \sigma_i^2 \), then we have

\[ \frac{T}{\sqrt{N}} \sum_{i=1}^{N} \left( \varepsilon_i^{(1)} \varepsilon_i^{(2)} - \sigma_i^2 \right) = \frac{T}{\sqrt{N}} \sum_{i=1}^{N} \left[ \left( \frac{1}{T} \sum_{t=1}^{T} v_{it}^{(1)} e_{it} \right) \left( \frac{1}{T} \sum_{s=1}^{T} v_{is}^{(2)} e_{is} \right) - \sigma_i^2 \right] \]

\[ = \frac{1}{T} \sum_{s=1}^{T} \sum_{t<s} (v_{s}^{(1)} v_{t}^{(2)} + v_{t}^{(1)} v_{s}^{(2)}) \left( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} e_{it} e_{is} \right) \]

\[ + \frac{1}{T} \sum_{s=1}^{T} (v_{s}^{(1)} v_{s}^{(2)}) \left( \sum_{i=1}^{N} (e_{is}^2 - \sigma_i^2) \right). \]

The second term can be handled as in the previous ‘averages’ case: indeed, we can define a new idiosyncratic error \( e_{it}^* = e_{it}^2 - \sigma_i^2 \) and a new common variable \( v_t^* = v_t^{(1)} v_t^{(2)} \); then some assumptions imposed on the new variables will lead to a CLT. The first term, however, is much more complicated, as it involves a double summation over time and represents itself as a quadratic form in errors. Such quadratic forms are quite common for asymptotics of two-step procedures; see, for example, Chao, Swanson, Hausman, Newey & Woutersen (2012). In our case, it is complicated by the presence of cross-sectional dependence. In the proof of the CLT for quadratic forms we will follow de Jong (1987).

Given these observations, the eventual goal of this paper is to find the conditions under which the following statement will hold:

\[ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \xi_i \Rightarrow \mathcal{N}(0, \Sigma), \quad (1) \]

where

\[ \xi_i = \left( \frac{1}{\sqrt{T}} \sum_{s=1}^{T} v_{s} \gamma_i e_{is} \right). \]

Here, the \( e_{it} \) are idiosyncratic errors, and \( v_s \) and \( w_{st} \) are common variables. We want to explore the trade-off for cross-sectional dependence if we are willing to discipline the time-series behavior.
As a next step, we wish to learn the circumstances when we can also consistently estimate the asymptotic covariance – that is, the sufficient conditions for a statement like

$$\frac{1}{N} \sum_{i=1}^{N} \xi_i \xi'_i \rightarrow \Sigma_\xi.$$  \hspace{1cm} (2)

3 Central Limit Theorem

In this paper we consider asymptotics as both cross-sectional and time-series sample sizes, $N$ and $T$, increase to infinity. We allow the data-generating process for all variables to vary with $N$ and $T$. Define $\mathcal{F}$ to be a sigma-algebra containing $v_s$ and $w_{st}$ for all $s$ and $t$, as well as, potentially, other variables common to all entities – for example, factors, macroeconomic shocks, macroeconomic state variables, and so on. We treat $\gamma_i$ as non-random $k_\gamma \times 1$ vectors. In order to simplify the notation, in what follows we will denote $C$ to be a positive generic constant, which may be different in different equations. We will use the following notation: for a square matrix $A$ we denote by $\text{tr}(A)$ its trace, by $\max\text{ev}(A)$ – its maximal eigenvalue, and by $\text{dg}(A)$ a diagonal matrix of the same size with the elements from the diagonal of $A$.

**Assumption C** (Assumptions about common variables): The random vectors $v_s$ and $w_{st}$ are measurable with respect to the sigma-algebra $\mathcal{F}$ for all $s$, $t$, and

(i) $\frac{1}{T} \sum_{s=1}^{T} \mathbb{E} (v_s v'_s) \rightarrow \Omega_v$ and $\frac{1}{T^2} \sum_{s=1}^{T} \sum_{t<s} \mathbb{E} (w_{st} w'_{st}) \rightarrow \Omega_w$, where $\Omega_v$ and $\Omega_w$ are full rank matrices;

(ii) $\max_{1 \leq s \leq T} \mathbb{E} [\|v_s\|^4] < C$ and $\max_{1 \leq t, s \leq T} \mathbb{E} [\|w_{st}\|^4] < C$;

(iii) $\mathbb{E} \left[ \left\| \frac{1}{T} \sum_{s=1}^{T} \sum_{t<s} (w_{st} w'_{st} - \mathbb{E} w_{st} w'_{st}) \right\|^2 \right] \rightarrow 0$;

(iv) $\mathbb{E} \left[ \left\| \frac{1}{T} \sum_{s=1}^{T} (v_s v'_s - \mathbb{E} v_s v'_s) \right\|^2 \right] \rightarrow 0$.

**Assumptions L** (Assumption about loadings): $\max_{1 \leq i \leq N} \|\gamma_i\| < C$.

**Assumption E** (Assumptions about idiosyncratic errors that are needed in all cases):

(i) Conditional on $\mathcal{F}$, the random vectors $e_t = (e_{1t}, ..., e_{Nt})'$ are serially independent, and $\mathbb{E}(e_t|\mathcal{F}) = 0$ for all $t$;
(ii) \( \max_{1 \leq i \leq N, 1 \leq t \leq T} \mathbb{E}(e_{it}^4) < C. \)

Assumption C imposes very mild restrictions on the time-series behavior of the common (non-entity specific) variables. For example, the part related to \( v_t \) is trivially satisfied if \( f_t \equiv v_t v_t' \) is weakly stationary with summable auto-covariances. Assumption L restricts the influence of any one entity in the cross-sectional average and will eventually contribute to asymptotic negligibility of the cross-sectional summands needed for the CLT. Assumption E(i) is a restrictive assumption which imposes discipline on the time-series structure, and the restriction \( \mathbb{E}(e_t | \mathcal{F}) = 0 \) is a form of strict exogeneity in the first step regression. Uniform moment boundedness in Assumption E(ii) is traditional.

Apparently, Assumptions C, L and E are insufficient to establish a central limit theorem, and we need to put some restrictions on the cross-sectional dependence and dependence between idiosyncratic errors and common variables. Indeed, we will use a change of summation ordering:

\[
\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \xi_i = \left( \frac{1}{\sqrt{T}} \sum_{s=1}^{T} v_s \left( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \gamma_i e_{is} \right) \right),
\]

and will establish asymptotic convergence in the time-series direction. For that, however, we need some sort of asymptotic negligibility of summands with different time indexes, in particular, terms like \( v_s \left( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \gamma_i e_{is} \right) \) and \( w_{st} \left( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} e_{it} e_{is} \right) \) should behave well. Apparently, there is a trade-off in how much dependence of idiosyncratic errors across entities and how much dependence between idiosyncratic errors and common variables are allowed. Below we consider two particular cases. In the first case, full independence between the \( e_{it} \)'s and \( \mathcal{F} \) is assumed; as a result, we can be agnostic about the structure of cross-sectional dependence, so any assumptions about it are very mild. In the second case, we allow for conditional heteroskedasticity in \( e_{it} \), and it is driven by common variables from \( \mathcal{F} \) producing dependence in higher-order conditional moments, but will impose some structure, though also pretty mild, on the cross-sectional behavior of \( e_{it} \).

### 3.1 Independence from common variables

**Assumptions I** (Model with independence between errors and common variables)

(i) The errors \( e_t = (e_{1t}, \ldots, e_{Nt})', t = 1, \ldots, T \) are independent from the sigma-algebra \( \mathcal{F} \) and identically distributed for different \( t \);
(ii) For the $N \times N$ covariance matrix $\mathcal{E}_{N,T} = \mathbb{E}[e_t e_t']$, $\limsup_{N,T \to \infty} \max \text{ ev} (\mathcal{E}_{N,T}) < \infty$, and $\frac{1}{N} \text{tr}(\mathcal{E}_{N,T}^2) \to a$;

(iii) $\frac{1}{N} \gamma' \mathcal{E}_{N,T} \gamma \to \Gamma_{\sigma}$, where $\Gamma_{\sigma}$ is full rank;

(iv) $\frac{1}{N^2} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \sum_{i_3=1}^{N} \sum_{i_4=1}^{N} \left| \mathbb{E} (e_{i_1 t} e_{i_2 t} e_{i_3 t} e_{i_4 t}) \right| < C$.

**Theorem 3.1** Under Assumptions C, L, E and I, the central limit theorem stated in equation (1) holds.

Numerous papers that establish inferences in factor models commonly assume that the set of factors is independent from the set of idiosyncratic errors, as in Assumption I(i), though cross-sectional dependence of errors is allowed; see, for example, Assumption D in Bai & Ng (2006). We intended for the first part of Assumption I(ii) to impose weak cross-sectional dependence as expressed by the covariance matrix; in particular, it means that no strong factor structure is left in the errors; similar assumptions appear in Onatski (2012) and Bai & Ng (2006). The convergence of the trace in Assumptions I(ii) and I(iii) is needed for the asymptotic covariance matrix to be properly defined. Assumption I(iv) is another way to restrict pervasive dependence in multiple variables, in particular, precluding outliers to realize in too many error terms simultaneously.

One of the important steps in the proof of Theorem 3.1 verifies asymptotic negligibility of time-series summands by checking boundedness of the fourth moments of the cross-sectional sums $\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \gamma_i e_{is}$ and $\frac{1}{\sqrt{N}} \sum_{i=1}^{N} e_{it} e_{is}$; that imposes the main way we restrict cross-sectional dependence.

### 3.2 Conditional heteroskedasticity

Assumption I(i) of independence is much stronger than Assumption E(i) about exogeneity: it does not allow higher conditional moments of $e_{it}$ to co-move with the common variables; in particular, it imposes conditional homoskedasticity. It may be especially problematic in financial applications where time-varying volatility is of strong empirical relevance, and returns on many stocks display patterns of changing volatility driven by some common variables. The assumptions below allow for conditional heteroskedasticity.

**Assumption H** (Heteroskedastic model with weak factor structure) Assume that the errors $e_{it}$ have the following weak (unobserved) factor structure:

$$e_{it} = \pi_i f_t + \eta_{it},$$
where the following assumptions hold.

(i) The $k_f \times 1$ process $f_t$ is serially independent, conditionally on $\mathcal{F}$, with $\mathbb{E}(f_t|\mathcal{F}) = 0$, $\mathbb{E}(f_tf'_t) = I_{k_f}$, $\max_{1 \leq t,s \leq T} T \mathbb{E}[(\|v_s\|^4 + 1)\|f_t\|^4] < C$, and $\max_{1 \leq s,t,t' \leq T} T \mathbb{E}[\|w_{st}\|^4\|f_t\|^8] < C$;

(ii) $\max_{v t} \left(\sum_{i=1}^{N} \pi_i \pi'_i\right) < C$ and $\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \pi_i \gamma'_i \to \Gamma \pi \gamma$;

(iii) The random variables $\eta_{it}$ are independent both cross-sectionally and across time, independent from both $f_s$’s and $\mathcal{F}$, have mean zero and variances $\mathbb{V}(\eta_{it}) = \omega^2_i$ that are bounded from above and such that $\frac{1}{N} \sum_{i=1}^{N} \omega^4_i \to \omega^4$, $\frac{1}{N} \sum_{i=1}^{N} \omega^2_i \gamma_i \gamma'_i \to \Gamma \omega$, where $\Gamma \omega$ has full rank, and $\max_{1 \leq i \leq N, 1 \leq t \leq T} T \mathbb{E}(\eta^4_{it}) < C$;

(iv) Additionally, if $\Gamma \pi \gamma \neq 0$, then there exists a matrix $\Sigma_{fv}$ such that

$$
\mathbb{E}\left[\left\|\frac{1}{T} \sum_{s=1}^{T} (f_{st}f'_s) \otimes (v_{s}v'_s) - \Sigma_{fv}\right\|^2\right] \to 0.
$$

**Theorem 3.2** Under Assumptions C, L, E and H, the statement of the central limit theorem stated in equation (1) holds.

An interesting feature of this example is that it allows the errors to be weakly cross-sectionally dependent to the extent that they may possess a weak (latent) factor structure. The condition $\mathbb{E}(f_tf'_t) = I_{k_f}$ is a normalization and involves no loss of generality. Assumption H(ii) forces the factors to be weak to such an extent that the factor structure cannot be consistently detected; it implies that the covariance matrix of idiosyncratic errors would satisfy the first half of Assumption I(ii). Moreover, this factor structure may be closely related to the common variables in $\mathcal{F}$, which causes the cross-sectional dependence among the errors $e_{it}$ to change with the common variables and allows a very flexible form of conditional heteroskedasticity. Indeed, the conditional cross-sectional covariance is

$$
\mathbb{E}(e_{it}e_{jt}|\mathcal{F}) = \pi'_i \mathbb{E}(f_tf'_t|\mathcal{F}) \pi_j + \mathbb{I}_{(i=j)} \omega^2_i.
$$

Since we do not restrict $\mathbb{E}(f_tf'_t|\mathcal{F})$ beyond proper moment conditions, the strength of any cross-sectional dependence as well as error variances may change stochastically depending on realizations of the common variables.

The moment conditions in Assumption H(i) help to establish asymptotic negligibility of the time-series summands. Assumption H(iii) about $\Gamma \omega$ and Assumption H(iv) allow us to define properly the asymptotic covariance matrix.
4 Estimation of Covariance Matrix

Statistical inferences such as confidence set construction and hypotheses testing about the structural parameter typically require consistent estimation of asymptotic variances of all important quantities that are asymptotically gaussian. The easiest to implement and thus the most appealing from an applied perspective are those that use the same variables and have a structure similar to the original averages, such as the statement in equation (2).

Notice that equation (2) contains the cross-sectional summation outside, and hence it treats the cross-section as nearly uncorrelated observations, or at least it ignores the cross-sectional correlation. A relevant analog is the difference between the long-run covariance and instantaneous covariance in a classical time series. However, implementing an analog of long-run covariance estimation here would be a challenge since we do not have any cross-sectional stationarity or a measure of distance between cross-sectional entities. Rather, we explore under which conditions equation (2) holds.

Theorem 4.1 below obtains a statement for the case when the common variables are independent from the idiosyncratic errors, while Theorem 4.2 establishes a similar statement for the conditionally heteroskedastic case.

Theorem 4.1 If in addition to Assumptions C, L, E, I we also have that

$$\|E_{N,T} - \text{dg}(E_{N,T})\| \to 0 \quad \text{as} \quad N, T \to \infty;$$

(3)

then consistency statement (2) holds.

Theorem 4.2 If in addition to Assumptions C, L, E, I we also have that $\Gamma_{\pi\gamma} = 0$, then consistency statement (2) holds.

Additional assumption (3) strengthens conditions on weakness of the cross-sectional correlation; in particular, it requires that the covariance matrix converges to one that is diagonal. The additional assumption in Theorem 4.2 requires that the weights with which cross-sectional entities are averaged are orthogonal to the loadings on the latent factor structure, which precludes the latent factor structure (that represents the cross-sectional dependence) from being amplified.
5 Appendix: Proofs

5.1 Preliminary results

We use the following central limit theorem for vector-valued martingale-difference sequence:

**Lemma 5.1** Let the sequence \((Z_{t;T}, \mathcal{F}_{t;T}), t = 1, \ldots, T\), be a martingale difference sequence of \(r \times 1\) random vectors with \(\Sigma_T = \mathbb{V} \left( \sum_{t=1}^{T} Z_{t;T} \right)\). If the following two conditions hold as \(T \to \infty\),

1. \((\min \text{ev}(\Sigma_T))^{-2} \sum_{t=1}^{T} \mathbb{E} \left[ \|Z_{t;T}\|^4 \right] \to 0,
2. \((\min \text{ev}(\Sigma_T))^{-2} \mathbb{E} \left[ \left\| \sum_{t=1}^{T} Z_{t;T} Z'_{t;T} - \Sigma_T \right\|^2 \right] \to 0,

then, as \(T \to \infty\),

\[
\Sigma_T^{-1/2} \sum_{t=1}^{T} Z_{t;T} \Rightarrow \mathcal{N}(0, I_r).
\]

**Proof of Lemma 5.1** Indeed, the statement of Lemma 5.1 holds if for any non-random \(r \times 1\) vector \(\lambda\), we have \((\lambda' \Sigma_T \lambda)^{-1/2} \sum_{t=1}^{T} \lambda' Z_{t;T} \Rightarrow \mathcal{N}(0, 1)\). Let us define a scalar martingale difference sequence \(z_t = \lambda' Z_{t;T}\) with variance \(\sigma_t^2 = \mathbb{V} \left( \sum_{t=1}^{T} \lambda' Z_{t;T} \right) = \lambda' \Sigma_T \lambda\). Let us check that all conditions of the central limit theorem by Heyde & Brown (1970) are satisfied for \(\delta = 1\). Indeed,

\[
\frac{1}{\sigma_T^2} \sum_{t=1}^{T} \mathbb{E} \left[ |z_t|^4 \right] = \frac{1}{(\lambda' \Sigma_T \lambda)^2} \sum_{t=1}^{T} \mathbb{E} \left[ |\lambda' Z_{t;T}|^4 \right] \leq \frac{1}{(\|\lambda\|^2 \min \text{ev}(\Sigma_T))^2} \sum_{t=1}^{T} ||\lambda||^4 \mathbb{E} \left[ ||Z_{t;T}||^4 \right] \to 0,
\]

and

\[
\mathbb{E} \left[ \frac{\sum_{t=1}^{T} z_t^2}{\sigma_T^2} - 1 \right]^2 = \mathbb{E} \left[ \frac{\sum_{t=1}^{T} (\lambda' Z_{t;T})^2}{\lambda' \Sigma_T \lambda} - 1 \right]^2 \leq \frac{1}{(\|\lambda\|^2 \min \text{ev}(\Sigma_T))^2} \mathbb{E} \left[ \left\| \lambda' \left( \sum_{t=1}^{T} Z_{t;T} Z'_{t;T} - \Sigma_T \right) \lambda \right\|^2 \right]
\]

\[
\leq \frac{1}{(\|\lambda\|^2 \min \text{ev}(\Sigma_T))^2} \|\lambda\|^4 \mathbb{E} \left[ \left\| \sum_{t=1}^{T} Z_{t;T} Z'_{t;T} - \Sigma_T \right\|^2 \right] \to 0.
\]

These two conditions imply that \(\sigma_T^{-1} \sum_{t=1}^{T} z_t \Rightarrow \mathcal{N}(0, 1)\). This finishes the proof. \(\square\)
As a preliminary result, we establish a central limit theorem for quadratic forms. The idea of this result comes from the CLT for quadratic forms by de Jong (1987). All random variables are implicitly indexed by the sample sizes $T$ (or $N, T$ in the further application to factor models), which are omitted to reduce clutter; for example, $W_{st}$ in full notation is indexed as $W_{st,T}$ or $W_{st,N,T}$.

**Lemma 5.2** Let $W_{st} = W_{st}(X_{st}, e_s, e_t)$ be a set of random vectors defined for all $s > t$, where $s, t \in \{1, \ldots, T\}$, such that $X_{st}$ is a random vector measurable with respect to the $\sigma$-algebra $\mathcal{F}$, and all $e_t$ are independent from each other, conditionally on $\mathcal{F}$. Assume that

$$E(W_{st}|\mathcal{F}, e_t) = 0 \quad \text{and} \quad E(W_{st}|\mathcal{F}, e_s) = 0.$$ \hspace{1cm} (4)

Define $W(T) = \sum_{s=1}^{T} \sum_{t < s} W_{st}$ and $\Sigma_{W,T} = \mathbb{V}(W(T))$. Assume the following statements hold as $T \to \infty$:

(i) $\Sigma_{W,T} \to \Sigma_W$, where $\Sigma_W$ is a full rank matrix;

(ii) $T^4 \max_{1 \leq s, t \leq T} \mathbb{E}[\|W_{st}\|^4] < C$;

(iii) $\mathbb{E} \left[ \left\| \sum_{s=1}^{T} \sum_{t < s} W_{st}W_{st}' - \Sigma_{W,T} \right\|^2 \right] \to 0$;

(iv) $T^4 \max_{s_1 \neq s_2, t_1 \neq t_2; t_1 < s_1, t_2 < s_2} \mathbb{E} \left( W_{s_1t_2}W_{s_2t_1}W_{s_2t_2}W_{s_1t_1} \right) \to 0$.

Then, as $T \to \infty$,

$$W(T) \Rightarrow \mathcal{N}(0, \Sigma_W).$$

**Lemma 5.3** Let $W_{st} = W_{st}(X_{st}, e_s, e_t)$ satisfy all conditions of Lemma 5.2. Let $V_s = V_s(X_s, e_s)$ be a random vector defined for all $s \in \{1, \ldots, T\}$ such that $X_s$ is a random vector measurable with respect to the $\sigma$-algebra $\mathcal{F}$, and $E(V_s|\mathcal{F}) = 0$. Define $V(T) = \sum_{s=1}^{T} V_s$ and $\Sigma_{V,T} = \mathbb{V}(V(T))$. Assume the following statements hold as $T \to \infty$:

(a) $\Sigma_{V,T} \to \Sigma_V$, where $\Sigma_V$ is a full rank matrix;

(b) $T^3 \max_{1 \leq s \leq T} \mathbb{E}[\|V_s\|^4] \to 0$;

(c) $\mathbb{E} \left[ \left\| \sum_{s=1}^{T} V_sV_s' - \Sigma_{V,T} \right\|^2 \right] \to 0$;

(d) $T^3 \max_{1 \leq t \leq \min\{s_1, s_2\} \leq T} \mathbb{E} \left( W_{s_1t}V_{s_1}V_{s_2}W_{s_2t} \right) \to 0$.

Then, as $T \to \infty$,

$$\begin{pmatrix} V(T) \\ W(T) \end{pmatrix} \Rightarrow \mathcal{N} \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma_V & 0 \\ 0 & \Sigma_W \end{pmatrix} \right).$$
Lemma 5.1 holds.

When at least one index from the set \{s_1, t_1, \ldots, s_k, t_k\} has a value that occurs only once. The functional form of \(W_{st}\) and the condition stated in (4) guarantee that in our case \(W_{st}\) is clean. Indeed, if, for example, the index \(s_1\) occurs only once, then

\[
\mathbb{E} (W_{s_1t_1} \otimes W_{s_2t_2} \otimes \ldots \otimes W_{s_kt_k}) = 0
\]

We call \(W_{st}\) clean if

\[
\mathbb{E} (W_{s_1t_1} \otimes W_{s_2t_2} \otimes \ldots \otimes W_{s_kt_k}) = \mathbb{E} [\mathbb{E}(W_{s_1t_1} \otimes W_{s_2t_2} \otimes \ldots \otimes W_{s_kt_k} \mid \mathcal{F}, e_{t_1}, e_{s_2}, e_{t_2}, \ldots, e_{t_k})]
\]

\[
= \mathbb{E} [\mathbb{E}(W_{s_1t_1} \mid \mathcal{F}, e_{t_1}, e_{s_2}, e_{t_2}, \ldots, e_{t_k}) \otimes W_{s_2t_2} \otimes \ldots \otimes W_{s_kt_k}]
\]

\[
= \mathbb{E} [\mathbb{E}(W_{s_1t_1} \mid \mathcal{F}, e_{t_1}) \otimes W_{s_2t_2} \otimes \ldots \otimes W_{s_kt_k}]
\]

Now, \(W(T) = \sum_{s=1}^{T} \sum_{t<s} W_{st} = \sum_{s=1}^{T} Z_{s,T}\), where \(Z_{s,T} = \sum_{t<s} W_{st}\). We denote by \(\mathcal{F}_s\) the \(\sigma\)-algebra generated by \(\mathcal{F}\) and \(e_t\) for all \(t < s\). Then, \((Z_{s,T}, \mathcal{F}_s)\) is a martingale difference sequence. Below we check that all conditions of Lemma 5.1 are satisfied.

Condition (i) implies that \(\min \text{ev}(\Sigma_{W,T}) \to C > 0\). Now let us check condition (1) of Lemma 5.1:

\[
\mathbb{E} [\|Z_{s,T}\|^4] = \mathbb{E} \left[ \left\| \sum_{t<s} W_{st} \right\|^4 \right]
\]

\[
= \mathbb{E} \left[ \left( \sum_{t_1<s} W_{st_1} \right)' \left( \sum_{t_2<s} W_{st_2} \right) \left( \sum_{t_3<s} W_{st_3} \right)' \left( \sum_{t_4<s} W_{st_4} \right) \right]
\]

\[
\leq \sum_{t<s} \mathbb{E} [\|W_{st}\|^4] + C \sum_{t_1<s} \sum_{t_2<s} \sum_{t_1<s, t_2<s, \neq t_1} \mathbb{E} [\|W_{st_1}\|^2 \|W_{st_2}\|^2]
\]

The last statement follows from the fact that \(W_{st}\) is clean, and non-zero summands are only those where either \(t_1 = t_2 = t_3 = t_4\) or the set \(\{t_1, t_2, t_3, t_4\}\) consists of two distinct elements each occurring twice. We also notice that \(\mathbb{E} [\|W_{st_1}\|^2 \|W_{st_2}\|^2] \leq \frac{1}{2} (\mathbb{E} [\|W_{st_1}\|^4] + \mathbb{E} [\|W_{st_2}\|^4]) \leq \max_{1 \leq t, s \leq T} \mathbb{E} [\|W_{st}\|^4] < CT^{-4}\) due to condition (ii). Hence, \(\mathbb{E} [\|Z_{s,T}\|^4] \leq CT^{-2}\). Thus, \(\sum_{s=1}^{T} \mathbb{E} [\|Z_{s,T}\|^4] \leq CT^{-1}\), implying that condition (1) of Lemma 5.1 holds.

Now let us turn to condition (2). First, notice that

\[
\Sigma_{W,T} = \mathbb{V}(W(T)) = \mathbb{V} \left( \sum_{s=1}^{T} \sum_{t<s} W_{st} \right) = \sum_{s=1}^{T} \sum_{t<s} \mathbb{V}(W_{st})
\]
the last equality holding because \( W_{st} \) is clean. Next,

\[
\mathbb{E} \left[ \left\| \sum_{s=1}^{T} Z_{s,T} Z'_{s,T} - \Sigma_{W,T} \right\|_F^2 \right] \\
= \mathbb{E} \left[ \left\| \sum_{s=1}^{T} \left( \sum_{t_1 < s} W_{st_1} \right) \left( \sum_{t_2 < s} W_{st_2} \right)' - \Sigma_{W,T} \right\|_F^2 \right] \\
= \mathbb{E} \left[ \left\| \sum_{s=1}^{T} \sum_{t < s} (W_{st} W'_{st} - \mathbb{E}[W_{st} W'_{st}]) + \sum_{s=1}^{T} \sum_{t_1 \neq t_2} W_{st_1} W_{st_2}' \right\|_F^2 \right] \\
= \mathbb{E} \left[ \sum_{s=1}^{T} \sum_{t < s} (W_{st} W'_{st} - \mathbb{E}[W_{st} W'_{st}]) \right]^2 + \mathbb{E} \left[ \sum_{s=1}^{T} \sum_{t_1 \neq t_2} W_{st_1} W_{st_2}' \right]^2. \\
\tag{5}
\]

The last equality holds because of the clean form, as the expectation of the Frobenius norm is equal to the trace of the sums of various products of four terms, and any such product that contains two of the same indexes \( t \) and two different indexes \( t_1 \neq t_2 \), has a zero expectation. Now, the first summand in equation (5) converges to zero due to condition (iii) of the Lemma. Now consider the second term in (5):

\[
\mathbb{E} \left[ \sum_{s=1}^{T} \sum_{t_1 \neq t_2 < s} W_{st_1} W'_{st_2} \right]^2 = \sum_{s=1}^{T} \sum_{t_1 \neq t_2 < s} \sum_{s=1}^{T} \sum_{t_3 \neq t_4} \mathbb{E} \left[ \text{tr} \left( W_{s_1 t_1} W'_{s_1 t_2} W_{s_2 t_3} W'_{s_2 t_4} \right) \right] \\
= C \sum_{s=1}^{T} \sum_{s=1}^{T} \mathbb{E} \left[ \text{tr} \left( W_{s_1 t_1} W'_{s_1 t_2} W_{s_2 t_1} W'_{s_2 t_2} \right) \right],
\]

the last equality holding because \( W_{st} \) is clean. The last summation can be divided into a category when \( s_1 \neq s_2 \), the corresponding sum being asymptotically \( o(1) \) due to condition (iv), and a category when \( s_1 = s_2 \), there being at most \( CT^3 \) of such summands, each smaller than \( C \max_{1 \leq t, s \leq T} \mathbb{E} \left[ \| W_{st} \|^4 \right] < CT^{-4} \). Thus,

\[
\mathbb{E} \left[ \left\| \sum_{s=1}^{T} W_{st_1} W_{st_2}' \right\|_F^2 \right] \to 0. \\
\tag{6}
\]

Putting statements (5) and (6) together we obtain that condition (2) of Lemma 5.1 is satisfied. Thus, the conclusion of Lemma 5.2 holds. \( \square \)

**Proof of Lemma 5.3.** Let us define \( Z_s = (V_s', \sum_{t < s} W_{st}')' \), and let \( \mathcal{F}_s \) be defined as in the proof of Lemma 5.2. We will show that all conditions of Lemma 5.1 are satisfied.
Notice that
\[ \mathbb{E}[V_s W'_s] = \mathbb{E}\left( \mathbb{E}[V_s W'_s | \mathcal{F}, e_s] \right) = \mathbb{E}(V_s \mathbb{E}[W'_s | \mathcal{F}, e_s]) = 0. \]

Thus,
\[ \Sigma_T = \mathbb{V} \left( \sum_{s=1}^{T} Z_s \right) = \begin{pmatrix} \Sigma_{V,T} & 0 \\ 0 & \Sigma_{W,T} \end{pmatrix} \to \begin{pmatrix} \Sigma_V & 0 \\ 0 & \Sigma_W \end{pmatrix}. \]

The right-hand-side is a full rank matrix by condition (i) of Lemma 5.2 and condition (a) of Lemma 5.3. Thus, the minimal eigenvalue of \( \Sigma_T \) is separated away from zero for large \( T \). Now,
\[ \sum_{s=1}^{T} \mathbb{E} \left[ \| Z_s \|_4^4 \right] \leq C \sum_{s=1}^{T} \mathbb{E} \left[ || V_s ||_4^4 \right] + C \sum_{s=1}^{T} \mathbb{E} \left[ \sum_{t<s} W_{st} \right]_F^4. \]

The first term here is bounded by \( T \max_{1 \leq s \leq T} \mathbb{E} \left[ || V_s ||_4^4 \right] \) which goes to zero by condition (b) of Lemma 5.3, while convergence to zero of the second sum has been already shown during the proof of Lemma 5.2. Thus, condition (1) of Lemma 5.1 holds. Next,
\[ \mathbb{E} \left[ \left\| \sum_{s=1}^{T} Z_s Z'_s - \Sigma_T \right\|_F^2 \right] \leq \mathbb{E} \left[ \left\| \sum_{s=1}^{T} Z_s Z'_s - \Sigma_T \right\|_F^2 \right] \]
\[ = \mathbb{E} \left[ \left\| \sum_{s=1}^{T} V_s V'_s - \Sigma_{V,T} \right\|_F^2 \right] + 2 \mathbb{E} \left[ \left\| \sum_{s=1}^{T} \left( \sum_{t<s} W_{st} \right) V'_s \right\|_F^2 \right] \]
\[ + \mathbb{E} \left[ \left\| \sum_{s=1}^{T} \left( \sum_{t<s} W_{st} \right) \left( \sum_{t<s} W_{st} \right)' - \Sigma_{W,T} \right\|_F^2 \right]. \]

Here we use that the Frobenius norm of a matrix equals to the sum of squares of all elements and can be decomposed into sums over four blocks of the matrix. Condition (c) guarantees that
\[ \mathbb{E} \left[ \left\| \sum_{s=1}^{T} V_s V'_s - \Sigma_{V,T} \right\|_F^2 \right] \leq C \mathbb{E} \left[ \left\| \sum_{s=1}^{T} V_s V'_s - \Sigma_{V,T} \right\|_F^2 \right] \to 0. \]

During the proof of Lemma 5.2 we show that
\[ \mathbb{E} \left[ \left\| \sum_{s=1}^{T} \left( \sum_{t<s} W_{st} \right) \left( \sum_{t<s} W_{st} \right)' - \Sigma_{W,T} \right\|_F^2 \right] \to 0. \]
Finally,
\[
E \left[ \left\| \sum_{s=1}^{T} \left( \sum_{t<s} W_{st} \right) V_s \right\|_2^2 \right] = \sum_{s_1=1}^{T} \sum_{t_1<s_1}^{T} \sum_{s_2=1}^{T} \sum_{t_2<s_2}^{T} \text{tr} \left( E \left( W_{s_1 t_1} V'_{s_1} V_s W'_{s_2 t_2} \right) \right)
\]
\[
= \sum_{s_1=1}^{T} \sum_{s_2=1}^{T} \sum_{t<s \in \{s_1, s_2\}} \text{tr} \left( E \left( W_{s_1 t} V'_{s_1} V_s W'_{s_2 t} \right) \right)
\]
\[
\leq CT^3 \max_{1 \leq s_1, s_2, t \leq T} \| E \left( W_{s_1 t} V'_{s_1} V_s W'_{s_2 t} \right) \| \to 0.
\]

Here we used that \( E \left( W_{s_1 t_1} V'_{s_1} V_s W'_{s_2 t_2} \right) = 0 \) if \( t_1 \neq t_2 \) and condition (d) of the Lemma.

To conclude, condition (2) of Lemma 5.1 also holds. \( \square \)

**Lemma 5.4** For an \( N \times N \) symmetric matrix \( A = (a_{ij}) \) denote \( \otimes \) to be the Hadamard product. Then \( \| A \otimes A \| \leq \sqrt{N} \| A \|^2 \).

**Proof.** Using the equivalence of norms, we have

\[
\| A \otimes A \| \leq \| A \otimes A \|_F = \sqrt{\sum_{1 \leq i,j \leq N} a_{ij}^4} \leq \sqrt{\max_{1 \leq i,j \leq N} a_{ij}^2} \sqrt{\sum_{1 \leq i,j \leq N} a_{ij}^2} \leq \| A \| \| A \|_F \leq \sqrt{N} \| A \|^2.
\]

\( \square \)

### 5.2 Proofs for Independent case

**Proof of Theorem 3.1.** We will check that all conditions of Lemma 5.3 are satisfied for

\[
W_{st} = \frac{1}{T \sqrt{N}} w_{st} \sum_{i=1}^{N} e_{it} e_{is} = \frac{1}{T} w_{st} \frac{e_i^t e_s}{\sqrt{N}}
\]

and

\[
V_s = \frac{1}{\sqrt{TN}} \sum_{i=1}^{N} \gamma_i e_{is} \otimes v_s = \frac{1}{\sqrt{T \sqrt{N}}} \gamma' e_s \otimes v_s.
\]

(i) First notice that due to Assumption I(ii)

\[
E \left[ \left( \frac{e_i^t e_s}{\sqrt{N}} \right)^2 \right] = \frac{\text{tr} (\mathcal{E}_{N,T}^2)}{N} \to a.
\]

Due to the independence between the common variables and \( e_{it} \) and because \( W_{st} \) is clean, we have:

\[
\Sigma_{W,T} = \mathcal{V} \left( \sum_{s=1}^{T} \sum_{t<s} W_{st} \right) = \frac{1}{T^2} \sum_{s=1}^{T} \sum_{t<s} E(w_{st} w'_{st}) \mathcal{V} \left[ \left( \frac{e_i^t e_s}{\sqrt{N}} \right)^2 \right] \to a \Omega_w,
\]

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and the limit is a positive definite matrix.

(ii) By Assumption I(i) and the i.i.d. nature of \( e_t \), we have:

\[
T^4 \mathbb{E} \left[ \| W_{st} \|^4 \right] = \mathbb{E} \left[ \| w_{st} \|^4 \right] \mathbb{E} \left[ \left( \frac{e'_s e_t}{\sqrt{N}} \right)^4 \right] \leq \frac{C}{N^2} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \sum_{i_3=1}^{N} \sum_{i_4=1}^{N} \mathbb{E} (e_{i_1} e_{i_2} e_{i_3} e_{i_4})^2 < C.
\]

here we used that \( |\mathbb{E} (e_{i_1} e_{i_2} e_{i_3} e_{i_4})| \leq \max_{1 \leq i \leq N, 1 \leq t \leq T} \mathbb{E} (e_t^4) < C \) and Assumption I(iv).

(iii) Next,

\[
\sum_{s=1}^{T} \sum_{t<s} W_{st} W'_{st} - \mathbb{E}_{W,T} = \frac{1}{T^2} \sum_{s=1}^{T} \sum_{t<s} w_{st} w'_{st} \left[ \left( \frac{e'_s e_t}{\sqrt{N}} \right)^2 - \frac{1}{N} \text{tr} (\mathcal{E}_{N,T}^2) \right]
+ \frac{1}{N} \text{tr} (\mathcal{E}_{N,T}^2) \left[ \frac{1}{T^2} \sum_{s=1}^{T} \sum_{t<s} (w_{st} w'_{st} - \mathbb{E} (w_{st} w'_{st})) \right]
= A_1 + A_2,
\]

hence it is enough to prove that \( \mathbb{E} \| A_1 \|^2 \to 0 \) and \( \mathbb{E} \| A_2 \|^2 \to 0 \). The latter is postulated by Assumption C(iii). Notice that all summands in \( A_1 \) are uncorrelated with each other due to Assumptions E(i) and I(i). Thus,

\[
\mathbb{E} [ \text{tr} (A_1 A'_1) ] = \frac{1}{T^2} \sum_{s=1}^{T} \sum_{t<s} \mathbb{E} \left[ \| w_{st} \|^4 \right] \mathbb{E} \left[ \left( \frac{e'_s e_t}{\sqrt{N}} \right)^4 \right] \mathbb{E} \left[ \left( \frac{e'_s e_t}{\sqrt{N}} - \frac{\text{tr} (\mathcal{E}_{N,T}^2)}{N} \right)^2 \right]
\leq \frac{1}{T^2} \sum_{s=1}^{T} \sum_{t<s} \mathbb{E} \left[ \| w_{st} \|^4 \right] \mathbb{E} \left[ \left( \frac{e'_s e_t}{\sqrt{N}} \right)^4 \right] < \frac{C}{T^2}.
\]

In the last inequality, we use the proof of statement (ii) above. This implies that condition (iii) of Lemma 5.2 holds.

(iv) If the set \( \{ s_1, s_2, t_1, t_2 \} \) contains four distinct indexes, then

\[
T^4 \mathbb{E} \left[ W_{s_1 t_2} W_{s_2 t_1} W'_{s_1 t_1} W'_{s_2 t_2} \right] \leq \mathbb{E} \left[ \| w_{st} \|^4 \right] \text{tr} \left( E \left( e_{s_1} e'_{s_1} e_{t_1} e'_{t_1} e_{s_2} e'_{s_2} e_{t_2} e'_{t_2} \right) \right) \frac{1}{N^2} \text{tr} (\mathcal{E}_{N,T}^4) \leq \frac{C}{N^2} \text{tr} (\mathcal{E}_{N,T}^4) \leq \frac{C}{N^2} \max \text{ev}(\mathcal{E}_{N,T}^4) \leq \frac{C}{N} \to 0.
\]

We now move to conditions (a)-(d) of Lemma 5.3.

(a) By Assumptions I(iii) and C(i) we have

\[
\Sigma_{V,T} = \left( \frac{1}{N} \gamma' \mathcal{E}_{N,T} \gamma \right) \otimes \left( \frac{1}{T} \sum_{s=1}^{T} \mathbb{E} (v_s v'_s) \right) \to \Gamma \otimes \Omega_v,
\]

and the limit is a full rank matrix.
(b) Next,

\[ T \mathbb{E} [||V_s||^4] = \frac{1}{T} \mathbb{E} \left[ \left( \frac{1}{\sqrt{N}} \gamma' e_s \right)^4 \right] \mathbb{E} [||v_s||^4] , \]

where \( \mathbb{E} [||v_s||^4] \leq C \) due to Assumption C(ii). Assumptions L and I(iv) imply that

\[ \mathbb{E} \left[ \left( \frac{1}{\sqrt{N}} \gamma' e_s \right)^4 \right] = \frac{1}{N^2} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \sum_{i_3=1}^{N} \sum_{i_4=1}^{N} \mathbb{E} (e_{i_1t} e_{i_2t} e_{i_3t} e_{i_4t}) \gamma' e_{i_1} \gamma' e_{i_2} \gamma' e_{i_3} \gamma' e_{i_4} \]

\[ \leq \max_{1 \leq t \leq N} ||\gamma||^4 \frac{1}{N^2} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \sum_{i_3=1}^{N} \sum_{i_4=1}^{N} |\mathbb{E} (e_{i_1t} e_{i_2t} e_{i_3t} e_{i_4t})| < C. \]

(c) Next,

\[ \sum_{s=1}^{T} V_s V_s' - \Sigma_{V,T} = \left( \frac{1}{N} \gamma' \mathbb{E}_{N,T} \gamma \right) \otimes \left( \frac{1}{T} \sum_{s=1}^{T} (v_s v_s' - \mathbb{E} (v_s v_s')) \right) \]

\[ + \frac{1}{T} \sum_{s=1}^{T} \left( \frac{\gamma' e_s e_s' \gamma}{N} - \frac{\gamma' \mathbb{E}_{N,T} \gamma}{N} \right) \otimes (v_s v_s') \]

\[ = A_1 + A_2. \]

Notice that \( A_1 \) and \( A_2 \) are uncorrelated, hence

\[ \mathbb{E} \left[ \left\| \sum_{s=1}^{T} V_s V_s' - \Sigma_{V,T} \right\|^2_F \right] = \text{tr} (\mathbb{E}(A_1' A_1)) + \text{tr} (\mathbb{E}(A_2' A_2)). \]

Assumption C(iv) guarantees the convergence of the first term. Notice that the summands in \( A_2 \) are uncorrelated due to time independence of errors, hence

\[ \text{tr} (\mathbb{E}(A_2' A_2)) = \frac{1}{T^2} \sum_{s=1}^{T} \mathbb{E} \left[ \left( \frac{\gamma' e_s e_s' \gamma}{N} - \frac{1}{N} \gamma' \mathbb{E}_{N,T} \gamma \right) \right] \mathbb{E} [||v_s||^4] \]

\[ \leq C \frac{1}{T} \mathbb{E} \left[ \left( \frac{\gamma' e_s e_s' \gamma}{N} - \frac{1}{N} \gamma' \mathbb{E}_{N,T} \gamma \right) \right] . \]

Given the bounds on the fourth moment of \( N^{-1/2} \gamma' e_s \) derived in the proof of part (b) we get that condition (c) holds.

(d) By Assumption I(i) we have that

\[ T^3 \left\| \mathbb{E} (W_{s_1t} V_{s_1} V_{s_2} W_{s_2t}) \right\| = \left\| \mathbb{E} (w_{s_1t} v_{s_1} v_{s_2} v_{s_2t}) \mathbb{E} \left( \frac{e_{s_1} \gamma' e_{s_2} e_{s_1} e_{s_2} e_t}{\sqrt{N}} \right) \right\|. \]
Using that scalars can be reshuffled to make two $e_t$ with the same index stand back to back and employing time series independence of errors, we obtain that
\[
\left| \mathbb{E} \left( \frac{e_1' \gamma' e_2' e_1' e_1' e_t} {\sqrt{N}} \right) \right| = \frac{1}{N^2} \left| \text{tr}(\gamma' \mathbf{\mathbb{E}}(e_{s_2} e_{s_2}') \mathbf{\mathbb{E}}(e_t e_t') \mathbf{\mathbb{E}}(e_{s_1} e_{s_1}')) \right|
\leq \frac{1}{N^2} \text{tr}(\gamma' \gamma') \max \text{ev}(\mathbf{\mathbb{E}}_{N,T}^3) \leq \frac{C}{N}.
\]

Here we use the assumption L to get $N^{-1} \text{tr}(\gamma' \gamma') < C$ and Assumption I(ii). Given Assumption C(ii) we obtain that
\[
T^3 \max_{1 \leq t < \min(s_1, s_2) \leq T} \| \mathbb{E} \left( W_{s_1 t} V_{s_1} V_{s_2} V_{s_2} \right) \| \leq \frac{C}{N} \rightarrow 0.
\]
Thus, condition (d) of Lemma 5.3 is satisfied. This concludes the proof of Theorem 3.1.

\[\Box\]

Proof of Theorem 4.1. We will prove the following three statements for
\[
\xi_{V,i} = \frac{1}{\sqrt{T}} \sum_{s=1}^{T} \gamma_i e_{is} \otimes v_s
\]
and
\[
\xi_{W,i} = \frac{1}{T} \sum_{s=1}^{T} \sum_{t<s} w_{st} e_{st} e_{is}:
\]

(i) $N^{-1} \sum_{i=1}^{N} \xi_{V,i} \xi_{V,i}' \overset{P}{\rightarrow} \Sigma_V$;

(ii) $N^{-1} \sum_{i=1}^{N} \xi_{W,i} \xi_{W,i}' \overset{P}{\rightarrow} \Sigma_W$;

(iii) $N^{-1} \sum_{i=1}^{N} \xi_{V,i} \xi_{W,i}' \overset{P}{\rightarrow} 0$.

Let us start with statement (i). Denote by $\sigma^2$ the diagonal and by $\sigma_{ij}$ the off-diagonal elements of matrix $\mathbf{\mathbb{E}}_{N,T}$. Notice that the additional assumption of Theorem 4.1 implies that
\[
\Gamma_{\sigma} = \lim \frac{\gamma' \mathbf{\mathbb{E}}_{N,T} \gamma}{N} = \lim \frac{1}{N} \sum_{i=1}^{N} \gamma_i \gamma^2_i \sigma^2_i.
\]
Let us define $\tilde{\Sigma}_{V,T} = \left( N^{-1} \sum_{i=1}^{N} \gamma_i \gamma_i' \sigma_i^2 \right) \left( T^{-1} \sum_{s=1}^{T} \mathbb{E} (v_s v_s') \right)$, and notice that $\tilde{\Sigma}_{V,T} \rightarrow \Sigma_V$. Thus,

$$\frac{1}{N} \sum_{i=1}^{N} \xi_i \xi_i' - \tilde{\Sigma}_{V,T} = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} \left( (\gamma_i \gamma_i' e_{is} e_{it}) \otimes (v_i v_i') - I \{ s = t \} \sigma_i^2 \left( \gamma_i \gamma_i' \right) \otimes \mathbb{E} (v_i v_i') \right)$$

$$= \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left( e_{it}^2 - \sigma_i^2 \right) \left( \gamma_i \gamma_i' \right) \otimes (v_i v_i')$$

$$+ \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s \neq t} \left( \gamma_i \gamma_i' e_{is} e_{it} \right) \otimes (v_s v_i')$$

$$+ \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left( \gamma_i \gamma_i' \sigma_i^2 \right) \otimes (v_i v_i' - \mathbb{E} (v_i v_i'))$$

$$= A_1 + A_2 + A_3.$$ 

Notice that the three terms are uncorrelated, so it is enough to prove that $\text{tr} \left( \mathbb{E} (A_j A'_j) \right) \rightarrow 0$ for $j = 1, 2, 3$. First,

$$\text{tr} \left( \mathbb{E} (A_1 A'_1) \right) = \text{tr} \left[ \mathbb{E} \left( \frac{1}{N^2 T^2} \sum_{i,j=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} \left( \gamma_i \gamma_i' \gamma_j \gamma_j' (e_{it}^2 - \sigma_i^2) (e_{js}^2 - \sigma_j^2) \right) \otimes (v_i v_i' v_s v_s') \right) \right]$$

$$= \frac{1}{N^2 T^2} \sum_{i,j=1}^{N} \sum_{t=1}^{T} \text{tr} \left( \gamma_i \gamma_i' \gamma_j \gamma_j' \text{cov}(e_{it}^2, e_{jt}^2) \right) \text{tr} \left( \mathbb{E} (v_i v_i' v_t v_t') \right)$$

$$\leq \frac{1}{T^2} \sum_{t=1}^{T} \max_{1 \leq i \leq N} \| \gamma_i \|^4 \max_{1 \leq i \leq N} \mathbb{E} \left[ (e_{it}^2 - \sigma_i^2)^2 \right] \mathbb{E} \left[ \| v_t \|^4 \right] \leq \frac{C}{T}.$$ 

Here we used that $e_{it}$’s are independent from each other for different $t$ by Assumption E(i), which forces $s = t$. The last inequality uses Assumptions C(ii), E(ii) and L.

Consider the term $A_2$ and notice that any two summands in the two-directional sum (over $t$ and over $s$) are uncorrelated due to time series independence of $e_i$’s and all summands are mean zero. Thus,

$$\text{tr} \left( \mathbb{E} (A_2 A'_2) \right) = \frac{1}{N^2 T^2} \sum_{t=1}^{T} \sum_{s \neq t} \sum_{i,j=1}^{N} \text{tr} \left( \mathbb{E} (\gamma_i \gamma_i' \gamma_j \gamma_j' e_{it} e_{is} e_{jt} e_{js}) \otimes \mathbb{E} (v_s v_i' v_t v_s') \right)$$

$$= \frac{1}{N^2} \sum_{i,j=1}^{N} \text{tr} \left( \gamma_i \gamma_i' \gamma_j \gamma_j' \sigma_i^2 \right) \frac{1}{T^2} \sum_{t=1}^{T} \sum_{s \neq t} \text{tr} \left( \mathbb{E} (v_s v_i' v_t v_s') \right)$$

We notice that $T^{-2} \sum_{t=1}^{T} \sum_{s \neq t} \text{tr} \left( \mathbb{E} (v_s v_i' v_t v_s') \right) \leq \mathbb{E} \left[ \| v_t \|^4 \right] < C$ due to Assumption C(ii). Denote $r, r^*$ to be indexes that go over $1, ..., \dim(\gamma_i)$. For any fixed value of $r, r^*$ denote
\[ B^{(r, r^*)} = ((\gamma_i \gamma_i)_{r, r^*})_{i=1}^N, \text{ an } N \times 1 \text{ vector. Then,} \]
\[
\frac{1}{N^2} \sum_{i,j=1}^N \text{tr}(\gamma_i \gamma_j \gamma_j' \sigma_{ij}^2) = \frac{1}{N^2} \sum_{i,j=1}^N \sum_{r,r^*} (\gamma_i \gamma_i' (\gamma_j \gamma_j')_{r, r^*} \sigma_{ij}^2)
\]
\[
= \sum_{r,r^*} \frac{1}{N^2} \sum_{i=1}^N (\gamma_i \gamma_i' (\gamma_i \gamma_i')_{r, r^*} \sigma_i^4)
\]
\[
+ \sum_{r,r^*} \frac{1}{N^2} B^{(r, r^*)} [(\mathcal{E}_{N,T} - \text{dg}(\mathcal{E}_{N,T})) \odot (\mathcal{E}_{N,T} - \text{dg}(\mathcal{E}_{N,T}))] B^{(r, r^*)}
\]
\[
\leq (\dim(\gamma_i))^2 \max_{1 \leq i \leq N} \|\gamma_i\|^4 \left( \frac{1}{N^2} \sum_{i=1}^N \sigma_i^4 + \sqrt{N} \|\mathcal{E}_{N,T} - \text{dg}(\mathcal{E}_{N,T})\| \right)
\]
\[
\leq \frac{C}{\sqrt{N}} \to 0,
\]
where in the second to last inequality we used Lemma 5.4 and the last inequality is due to Assumptions L, I(ii) and the additional assumption stated in Theorem 4.1. This shows that \(\text{tr} (\mathbb{E}(A_2 A_2')) \to 0.\)

Finally, \(\text{tr} (\mathbb{E}(A_3 A_3')) \to 0\) due to Assumption C(iv). This ends the proof of statement (i).

Let us turn to statement (ii):
\[
\frac{1}{N} \sum_{i=1}^N \xi_{W,i} \xi_{W,i}' - \Sigma_{W,T} = \frac{1}{T^2 N} \sum_{i=1}^N \sum_{s=1}^T \sum_{t<s} \left( e_{it}^2 e_{is}^2 - \sigma_i^4 \right) w_{st} w_{st}'
\]
\[
+ \frac{1}{T^2 N} \sum_{i=1}^N \sum_{s=1}^T \sum_{t<s} \sum_{t'<s} \sum_{t_1=t', \{s_1, t_1\} \neq \{s, t\}} w_{st_1} w_{st_1}' e_{it_1} e_{is_1} e_{st_1} e_{is_1}
\]
\[
+ \frac{1}{N} \sum_{i=1}^N \sigma_i^4 \frac{1}{T^2} \sum_{s=1}^T \sum_{t<s} \left( w_{st} w_{st}' - \mathbb{E}(w_{st} w_{st}') \right)
\]
\[
= A_1 + A_2 + A_3.
\]

Again, \(A_1, A_2\) and \(A_3\) are uncorrelated with each other. Thus, we can deal with each one of them separately.

Let us start with
\[
\text{tr} (\mathbb{E}(A_1 A_1')) = \frac{1}{T^3 N^2} \sum_{i,j=1}^N \sum_{s,s'}=1 \sum_{t<s,t'<s'} \text{tr} (\mathbb{E}(w_{st} w_{s't'} w_{s't'})) \mathbb{E}(b_{i,t,s} b_{j,t',s'}),
\]
where
\[
b_{i,t,s} = e_{it}^2 e_{is}^2 - \sigma_i^4 = (e_{it}^2 - \sigma_i^2) (e_{is}^2 - \sigma_i^2) + \sigma_i^2 (e_{is}^2 - \sigma_i^2) + \sigma_i^2 (e_{it}^2 - \sigma_i^2).
\]

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Notice that $\mathbb{E}(b_{ij,t}s_j,t^*,s^*) \neq 0$ only if at least one of the indexes from the set $\{t, t^*, s, s^*\}$ appears twice. Thus, the summation over time index is three-dimensional and there are at most $CT^3N^2$ non-zero summands in $\text{tr}(\mathbb{E}(A_1A'_1))$. Let us bound every summand from above. Notice that since $t < s$ and $t^* < s^*$, all indexes in the set $\{t, t^*, s, s^*\}$ can appear at most twice; also errors with different time indexes are independent from each other, so the largest moment of the error term we will have is the fourth. To sum up, each non-zero summand is bounded above by $T^{-4}N^{-2}C \max_{1 \leq i, s \leq T} \mathbb{E}[[w_{st}]^4] \max_{1 \leq i \leq N, 1 \leq t \leq T} \mathbb{E}(e_{it}^4)^2$, thus $\text{tr}(\mathbb{E}(A_1A'_1)) \leq C/T \rightarrow 0$.

The term $\text{tr}(\mathbb{E}(A_2A'_2))$ includes summation over eight time indexes but most of the summands are zeros. The non-zero terms place at least four restrictions on the time indexes. We note that the non-trivial part of the sum in $\text{tr}(\mathbb{E}(A_2A'_2))$ includes summation over $i, j = 1, ..., N$ and over time indexes $\{s_1, s_1^*, t_1, t_1^*, s_2, s_2^*, t_2, t_2^*\}$, where in the last set any distinct index appears at least twice. The summands are

$$
\frac{1}{T^4N^2} \mathbb{E} \left( w_{s_1t_1}w_{s_1^*t_1^*}w_{s_2t_2}w_{s_2^*t_2^*} \right) \mathbb{E} \left( e_{it_1}e_{is_1}e_{it_1^*}e_{is_1^*}e_{jt_2}e_{jt_2^*}e_{j_t}e_{jt} \right).
$$

Notice also that due to restrictions that $t$’s are strictly smaller than their corresponding $s$’s, each time index can appear at most four times, hence we get at most fourth power of each error term.

First, consider the case when the set $\{s_1, s_1^*, t_1, t_1^*, s_2, s_2^*, t_2, t_2^*\}$ contains at most three distinct indexes (this makes the summation over time three-dimensional). We can show that each summand is bounded by $T^{-4}N^{-2} \max_{1 \leq i, s \leq T} \mathbb{E}[[w_{st}]^4] \max_{1 \leq i \leq N, 1 \leq t \leq T} \mathbb{E}(e_{it}^4)^2 \leq C/(T^4N^2)$ in absolute value, and as there are at most $N^2T^3$ of them (two-dimensional cross-sectional and three-dimensional over time summations), the sum of such terms will go to zero.

Finally, we consider the case when the set $\{s_1, s_1^*, t_1, t_1^*, s_2, s_2^*, t_2, t_2^*\}$ contains four distinct indexes. Then each summand of this type is bounded in absolute value by

$$
C \left| \sigma_{ij} \right|^{a}(\sigma_i^2)^{b}(\sigma_j^2)^{c} \max_{1 \leq i, s \leq T} \mathbb{E}[[w_{st}]^4],
$$

where $a + b + c = 4$, and the values of $a, b$ and $c$ depend on which indexes coincide with which; however, due to the conditions $\{s_1, t_1\} \neq \{s_2, t_2\}$ and $t_1 < s_1, t_2 < s_2$, we know that the set $\{s_1, s_2, t_1, t_2\}$ contains at least three distinct indexes. Thus, $c$ and $b$ are either 0 or 1 each, and $a \geq 2$. Hence, due to Assumption C(ii), the corresponding sum is bounded.
above by
\[
\frac{C}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij}^a (\sigma_i^2)^b (\sigma_j^2)^c \leq \frac{C}{N^2} \max_{1 \leq i \leq N} \sigma_i^4 \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij}^2
\]
\[
= \frac{C}{N^2} \sum_{i=1}^{N} \sigma_i^4 + \frac{C}{N^2} \sum_{i=1}^{N} \sum_{j \neq i}^{N} \sigma_{ij}^2
\]
(7)
\[
\leq \frac{C}{N} + \frac{C}{N^2} \| \epsilon_{N,T} \|_F^2 \leq \frac{C}{N}.
\]

In the first inequality we use \( |\sigma_{ij}| \leq \sigma_i \sigma_j \). In the last line we use that for any symmetric matrix \( A \), we have \( \| A \|_F^2 \leq N \| A \|_2^2 \) and assumption stated in Theorem 4.1. Thus, \( \text{tr} \left( \mathbb{E}(A_2A'_2) \right) \to 0 \).

Next, Assumption C(iii) implies the convergence of \( A_3 \). This finishes the proof of (ii).

Finally, we need to prove statement (iii) that
\[
\frac{1}{NT^{3/2}} \sum_{i=1}^{N} \sum_{s=1}^{T} \sum_{t<s}^{T} \sum_{s'=1}^{T} (\gamma_i \otimes v_{si}) w_{is} \epsilon_{it} \epsilon_{is} \epsilon_{is} \to^p 0.
\]
As before, we look at the expectation of the square of the sum above, which involves six-dimensional summation over time indexes and two-dimensional summation over cross-section (over \( i, j \)) and is normalized by \( N^{-2}T^{-3} \). Due to time-series independence of \( \epsilon_{it} \), the six-dimensional summation over time indexes has mostly zeros and can be reduced to three-dimensional summation over time indexes as the set \( \{ s_1, t_1, s_1^*, s_2, t_2, s_2^* \} \) should have any distinct index to appear at least twice.

First, consider only those terms for which the set \( \{ s_1, t_1, s_1^*, s_2, t_2, s_2^* \} \) contains at most two distinct indexes; there are at most \( N^2T^2 \) of such terms. Since \( t_1 < s_1 \) and \( t_2 < s_2 \), each time index can appear at most four times; thus, the highest power of each individual shock can be the fourth. As a result, each summand is bounded above by \( N^{-2}T^{-3} \max_{1 \leq i \leq N} \| \gamma_i \|_2 \max_{1 \leq t \leq T} \mathbb{E} [ \| v_t \|_2^2 ] \max_{1 \leq t, s \leq T} \mathbb{E} [ \| w_{st} \| ] \max_{1 \leq t \leq N, 1 \leq t \leq T} \mathbb{E} [ \| \epsilon_i \|_4^3 ]^{1/2} \).

Given Assumptions C(ii) and E(ii), the sum of these terms is bounded above by \( C/T \).

Finally, consider only those terms for which the set \( \{ s_1, t_1, s_1^*, s_2, t_2, s_2^* \} \) contains exactly three distinct indexes. The summation over these indexes is equal to
\[
\text{tr} \left( \frac{1}{N^2} \sum_{i,j=1}^{N} \gamma_{ij} (C_1 \sigma_{ij} \sigma_i^2 \sigma_j^2 + C_2 \sigma_{ij}^3) \right).
\]
The term \( \sigma_{ij}^3 \) appears when \( \{s_1, t_1, s_1^*\} = \{s_2, t_2, s_2^*\} \), while \( \sigma_{ij}^2 \) arises when the sets \( \{s_1, t_1, s_1^*\} \) and \( \{s_2, t_2, s_2^*\} \) have two coinciding indexes each. Therefore,

\[
\text{tr} \left( \frac{1}{N^2} \sum_{i,j=1}^{N} \gamma_i \gamma_j' \sigma_{ij} \sigma_i^2 \sigma_j^2 \right) = \text{tr} \left( \frac{1}{N^2} \sum_{i=1}^{N} \gamma_i \gamma_i' \sigma_i^6 \right) + \text{tr} \left( \frac{1}{N^2} \sum_{i\neq j} (\gamma_i \sigma_i^2)(\gamma_j' \sigma_j^2) \sigma_{ij} \right) \\
\leq \frac{C}{N} + \frac{1}{N} \sum_{i=1}^{N} \|\gamma_i\|^2 \sigma_i^4 \|E_{N,T} - \text{dg}(E_{N,T})\| \to 0.
\]

Also,

\[
\text{tr} \left( \frac{1}{N^2} \sum_{i,j=1}^{N} \gamma_i \gamma_j' \sigma_{ij}^3 \right) = \frac{1}{N^2} \sum_{i=1}^{N} \|\gamma_i\|^2 \sigma_i^6 + \frac{1}{N} \sum_{i\neq j} \text{tr}(\gamma_i \gamma_j') \sigma_{ij}^3 \\
\leq \frac{C}{N} + \max_{1 \leq i \leq N} \|\gamma_i\|^2 \max_{1 \leq i \leq N} \sigma_i^2 \frac{1}{N^2} \sum_{i,j=1}^{N} \sigma_{ij}^2 \to 0.
\]

Here we used the statement \( N^{-2} \sum_{i,j=1}^{N} \sigma_{ij}^2 \to 0 \), which is proved in equation (7). This ends the proof of Theorem 4.1. □

### 5.3 Proofs for Conditional Heteroskedasticity case

**Proof of Theorem 3.2.** In order to apply Lemma 5.3 we check conditions (i)-(iv) of Lemma 5.2 and conditions (a)-(d) of Lemma 5.3 for

\[
W_{st} = \frac{1}{T} w_{st} e_t' e_s \sqrt{N}
\]

and

\[
V_s = \frac{1}{\sqrt{T}} \gamma' e_s \otimes v_s.
\]

(i) Due to serial independence of \( e_{st} \) conditionally on \( \mathcal{F} \), we have

\[
\Sigma_{W,T} = \frac{1}{T^2} \sum_{s=1}^{T} \sum_{t<s} \mathbb{E} \left[ w_{st} w_{st}' \mathbb{E} \left( \left( \frac{e_t' e_s}{\sqrt{N}} \right)^2 | \mathcal{F} \right) \right].
\]

Notice that \( (e_t' e_s)^2 = \text{tr} \left( (e_t' e_t) (e_t' e_s) \right) = \text{tr} \left( (e_t e_t') (e_s e_s') \right) \), and hence, given the conditional independence assumption,

\[
\mathbb{E} \left[ \left( \frac{e_t' e_s}{\sqrt{N}} \right)^2 | \mathcal{F} \right] = \frac{1}{N} \text{tr} \left( \mathbb{E}(e_t e_t'| \mathcal{F}) \mathbb{E}(e_s e_s'| \mathcal{F}) \right).
\]
Recall that $e_t = \pi f_t + \eta_t$. We will use the notation $\Omega = E(\eta_t \eta_t') = dg(\omega_i^2)_{i=1}^N$. Then,

$$E \left[ \left( \frac{c_t e_s}{\sqrt{N}} \right)^2 | \mathcal{F} \right] = \frac{1}{N} \text{tr} \left( \pi E(f_t f'_t|\mathcal{F}) \pi' + \Omega \right) = \frac{1}{N} \sum_{i=1}^N \omega_i^4 + \Delta_{N,T},$$

where

$$\Delta_{N,T} \leq \frac{C}{N} E \left[ (\|f_t\|^2 + 1)(\|f_s\|^2 + 1)|\mathcal{F} \right].$$

Indeed, $\Delta_{N,T}$ has three terms each of which is easy to bound. For example,

$$\frac{1}{N} \text{tr} \left( \Omega_{t} \pi E(f_s f'_s|\mathcal{F}) \pi' \right) \leq \frac{1}{N} \max_{1 \leq i \leq N} \omega_i^2 \cdot \text{tr} \left( E(f_s f'_s|\mathcal{F}) \pi' \pi \right) \leq \frac{1}{N} \max_{1 \leq i \leq N} \omega_i^2 \cdot \max ev(\pi' \pi) \cdot E \left[ ||f_s||^2 | \mathcal{F} \right].$$

Since we assumed that $\max_{1 \leq i \leq N} \omega_i^2 < C$ and from Assumption H(ii), it follows that

$$\left\| \frac{1}{T^2} \sum_{s=1}^{T} \sum_{t<s} E \left[ w_{st} w'_{st} \Delta_{N,T} \right] \right\| \leq \frac{C}{NT^2} \sum_{s=1}^{T} \sum_{t<s} E \left[ ||w_{st}||^2 (||f_t||^2 + 1)(||f_s||^2 + 1) \right] \leq \frac{C}{N} \to 0,$$

where the last inequality is due to Assumption H(i). So, we obtain that

$$\Sigma_{W,T}(T, N) = \lim \frac{1}{T^2} \sum_{s=1}^{T} \sum_{t<s} E \left[ w_{st} w'_{st} \right] \frac{1}{N} \sum_{i=1}^{N} \omega_i^4 = \omega^4 \Omega_w = \Sigma_W.$$ (ii) Notice that

$$\frac{c_t e_s}{\sqrt{N}} = f'_t \pi' \pi f_s \frac{\eta_t}{\sqrt{N}} + f'_t \pi' \eta_s \frac{\eta_t}{\sqrt{N}} + f'_s \pi' \eta_t \frac{\eta_s}{\sqrt{N}}.$$

Using the Marcinkiewicz–Zygmund inequality for a second power applied twice we notice that in order to bound $E \left[ \left( \frac{c_t e_s}{\sqrt{N}} \right)^4 | \mathcal{F} \right]$ from above it is enough to bound the fourth moment of each summand. Using serial and cross-sectional conditional independence of $\eta$’s as well as their conditional independence from $f$’s, we obtain

$$E \left[ \left( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \eta_{it} \eta_{is} \right)^4 \right] = \frac{1}{N^2} \sum_{i=1}^{N} E \left[ (\eta_{it} \eta_{is})^4 \right] + C \frac{1}{N^2} \sum_{i_1 \neq i_2} E(\eta_{it_1}^2 \eta_{i_1}^2 \eta_{is_1}^2 \eta_{is_2}^2) \leq C,$$

$$E \left[ \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \pi_i \eta_{is} \right\|^4 \right] \leq \frac{1}{N^2} \sum_{i=1}^{N} E \left[ ||\pi_i \eta_{is}||^4 \right] + C \frac{1}{N^2} \sum_{i_1 \neq i_2} ||\pi_{i_1}||^2 ||\pi_{i_2}||^2 E \left[ \eta_{i_1}^2 \eta_{i_2}^2 \right] \leq \frac{C}{N^2},$$

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where we use Assumption H(ii,iii), and that \( \sum \| \pi_i \|^4 \leq (\sum \| \pi_i \|^2)^2 \leq C \). Hence,
\[
\mathbb{E} \left[ \left( \frac{e'e_s}{\sqrt{N}} \right)^4 | \mathcal{F} \right] \leq \frac{C}{N^2} \mathbb{E} \left[ \| f_t \|^4 \| f_s \|^4 | \mathcal{F} \right] + \frac{C}{N^2} \left( \mathbb{E} \left[ \| f_t \|^4 | \mathcal{F} \right] + \mathbb{E} \left[ \| f_s \|^4 | \mathcal{F} \right] \right) + C.
\]

Finally, due to Assumption H(i),
\[
T^4 \mathbb{E} \left[ \| W_{st} \|^4 \right] \leq \mathbb{E} \left[ \left\| w_{st} \frac{e'e_s}{\sqrt{N}} \right\|^4 \right] \leq C \mathbb{E} \left[ \| w_{st} \|^4 (\| f_t \|^4 + 1)(\| f_s \|^4 + 1) \right] < C.
\]

Thus, condition (ii) of Lemma 5.2 holds.

(iii) Let us define a sigma-algebra \( \mathcal{A} = \mathcal{F} \cup \{ f_t, t = 1, \ldots, T \} \). Let us now denote
\[
\vartheta_{st} = \mathbb{E} \left[ \left( \frac{e's_t}{\sqrt{N}} \right)^2 | \mathcal{A} \right] = \mathbb{E} \left[ \left( (\pi f_s + \eta_s)'(\pi f_t + \eta_t) \right)^2 | \mathcal{A} \right] = \frac{1}{N} \left( \left( f_s \pi' f_t \right)^2 + f_s \pi \pi' f_t + f_t \pi \pi' f_s + \sum_{i=1}^N \omega_i^4 \right).
\]

We have:
\[
\sum_{s=1}^T \sum_{t<s} W_{st}W_{st}' - \Sigma_W = \frac{1}{T^2} \sum_{s=1}^T \sum_{t<s} w_{st}w_{st}' \left[ \left( \frac{e's_t}{\sqrt{N}} \right)^2 - \vartheta_{st} \right] + \frac{1}{T^2} \sum_{s=1}^T \sum_{t<s} (w_{st}w_{st}' \vartheta_{st} - \mathbb{E} \left[ w_{st}w_{st}' \vartheta_{st} \right]) = A_1 + A_2,
\]
so, it is enough to prove convergence of each term separately. Now, \( \mathbb{E} \left[ \text{tr}(A_1A_1') \right] \) is equal to
\[
\frac{1}{T^4} \sum_{s_1,s_2=1}^T \sum_{t_1,t_2} \mathbb{E} \left[ \text{tr}(w_{s_1t_1}w_{s_1t_1}'w_{s_2t_2}w_{s_2t_2}') \left( \left( \frac{e's_t}{\sqrt{N}} \right)^2 - \vartheta_{s_1t_1} \right) \left( \left( \frac{e's_t}{\sqrt{N}} \right)^2 - \vartheta_{s_2t_2} \right) \right].
\]
Notice that in order for a summand from the last sum to be non-zero we need that some indexes in the set \( \{ s_1, s_2, t_1, t_2 \} \) coincide, and we obtain at most \( CT^3 \) non-zero summands. Each non-zero summand is bounded above by a constant due to the moment assumptions formulated in Assumption H(i,iii). Thus, \( \mathbb{E} \left[ \text{tr}(A_1A_1') \right] \to 0 \).

Notice that due to Assumption H, and similar to the argument above,
\[
\left| \vartheta_{st} - \frac{1}{N} \sum_{i=1}^N \omega_i^4 \right| \leq \frac{C}{N}(\| f_s \| + \| f_t \| + 1)^4.
\]
Thus,
\[
A_2 = \left( \frac{1}{N} \sum_{i=1}^{N} \omega_i^4 \right) \frac{1}{T^2} \sum_{s=1}^{T} \sum_{t<s} \left( w_{st} w_{st}' - \mathbb{E} (w_{st} w_{st}') \right) + \frac{1}{T^2} \sum_{s=1}^{T} \sum_{t<s} \left( w_{st} w_{st}' (\vartheta_{st} - \frac{1}{N} \sum_{i=1}^{N} \omega_i^4) - \mathbb{E} \left[ w_{st} w_{st}' (\vartheta_{st} \frac{1}{N} - \sum_{i=1}^{N} \omega_i^4) \right] \right),
\]
where the first sum converges to zero due to Assumption C(iii), while expectation of the second moment of the second term is bounded by
\[
\frac{1}{T^4} \sum_{s_1,s_2,t_1,t_2} C \frac{N}{N^2} \mathbb{E} \left[ \|f_{s_1}|| + \|f_{t_1}|| + 1 \right]^4 \left( \|f_{s_2}|| + \|f_{t_2}|| + 1 \right)^4 \|w_{s_1 t_1}||^2 \|w_{s_2 t_2}||^2 \leq \frac{C}{N^2},
\]
due to inequality (8) and Assumption H(i). Thus, condition (iii) of Lemma 5.2 holds.

Let us check condition (iv):
\[
T^4 \mathbb{E} \left( W_{s_1 t_2} W_{s_2 t_2} W_{s_1 t_1} \right) = \frac{1}{N^2} \mathbb{E} \left[ w_{s_1 t_2} w_{s_2 t_2} w_{s_1 t_1} \mathbb{E} \left( e_{s_1} e_{t_1} e_{s_2} e_{t_2} e_{s_1} \mathbb{E} (f_{s_1 t_1} | \mathcal{F}) \right) \right],
\]
where we used that the scalar products $e_t e_s = e'_t e'_s$ are scalars and they can be reshuffled to make two $e_t$ with the same index stand back to back. Let us bound the $N \times N$ matrix \( \mathbb{E}(e_t e'_t | \mathcal{F}) = \pi \mathbb{E}(f_{t} f'_{t} | \mathcal{F}) \pi' + \Omega_{\pi} \):
\[
\max \text{ev} (\mathbb{E}(e_t e'_t | \mathcal{F})) \leq \max \text{ev} (\pi^t \mathbb{E}(f_{t} f'_{t} | \mathcal{F}) \pi) + \max \text{ev} (\Omega_{\pi})
\leq \text{tr} (\pi^t \mathbb{E}(f_{t} f'_{t} | \mathcal{F}) \pi) + \max_{1 \leq i \leq N} \omega_i^2
\leq \max \text{ev} (\pi^t \pi) \mathbb{E} (\|f_t||^2 | \mathcal{F}) + C
\leq C \mathbb{E} (\|f_t||^2 + 1 | \mathcal{F}). \tag{9}
\]
As a result,
\[
\left| \mathbb{E}(e_{s_1} e_{t_1} e'_{t_1} e_{s_2} e'_{s_2} e_{t_2} e'_{t_2} e_{s_1} | \mathcal{F}) \right| = \left| \text{tr} (\mathbb{E}(e_{t_1} e'_{t_1} | \mathcal{F}) \mathbb{E}(e_{s_2} e'_{s_2} | \mathcal{F}) \mathbb{E}(e_{t_2} e'_{t_2} | \mathcal{F}) \mathbb{E}(e_{s_1} e'_{s_1} | \mathcal{F})) \right|
\leq N \max \text{ev} \left( \prod_{t \in \{s_1,s_2,t_1,t_2\}} \mathbb{E}(e_t e'_t | \mathcal{F}) \right)
\leq N \max \text{ev} \left( \prod_{t \in \{s_1,s_2,t_1,t_2\}} \mathbb{E}(e_t e'_t | \mathcal{F}) \right)
\leq N C \max \text{ev} \left( \prod_{t \in \{s_1,s_2,t_1,t_2\}} \mathbb{E}(\|f_t||^2 + 1 | \mathcal{F}) \right).
\]
Also using Assumption H(i) we obtain that
\[
T^4 \mathbb{E} \left( W_{s_1 t_2} W_{s_2 t_2} W_{s_1 t_1} \right) \leq \frac{C}{N^2} \max_{1 \leq s,t,t' \leq T} \mathbb{E} \left[ \|w_{st}\|^4 \|f_{t'}||^8 \right] \rightarrow 0.
\]

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Thus, condition (iv) holds as well.

Now we will check assumptions (a)-(d) of Lemma 5.3. First, we find the limit of the covariance matrix $\Sigma_{V,T}$.

$$
\mathbb{E} \left[ \left( \frac{\gamma' e_s}{\sqrt{N}} \right) \left( \frac{\gamma' e_s}{\sqrt{N}} \right) ^\prime | \mathcal{F} \right] = \left( \frac{\gamma' \pi}{\sqrt{N}} \right) \mathbb{E} [f_s f_s' | \mathcal{F}] \left( \frac{\pi' \gamma}{\sqrt{N}} \right) + \frac{1}{N} \sum_{i=1}^{N} \omega_i^2 \gamma_i \gamma_i' \\
\rightarrow \Gamma_{\pi \gamma} \mathbb{E} (f_s f_s' | \mathcal{F}) \Gamma_{\pi \gamma} + \Gamma_\omega.
$$

Here we used the Assumptions H(ii,iii). Therefore,

$$
\Sigma_{V,T} = V \left( \sum_{s=1}^{T} V_s \right) = \mathbb{E} \left[ \frac{1}{T} \sum_{s=1}^{T} \mathbb{E} \left[ \left( \frac{\gamma' e_s}{\sqrt{N}} \right) \left( \frac{\gamma' e_s}{\sqrt{N}} \right) ^\prime | \mathcal{F} \right] \otimes (v_s v_s') \right] \\
= \frac{1}{T} \sum_{s=1}^{T} \mathbb{E} \left[ (\Gamma_{\pi \gamma} f_s f_s' \Gamma_{\pi \gamma} + \Gamma_\omega) \otimes (v_s v_s') \right] \\
\rightarrow (\Gamma_{\pi \gamma} \otimes I_{k_v}) \Sigma_{f_0} (\Gamma_{\pi \gamma} \otimes I_{k_v}) + \Gamma_\omega \otimes \Omega_v.
$$

The limit matrix is positive definite since both $\Gamma_\omega$ and $\Omega_v$ are positive-definite due to Assumptions C(i) and H(iii).

Now note that due to Assumption H(ii)

$$
\mathbb{E} \left[ \left\| \frac{\gamma' e_t}{\sqrt{N}} \right\|^4 | \mathcal{F} = \frac{1}{N^2} \mathbb{E} \left[ \left\| \gamma' f_t + \gamma' \eta_t \right\|^4 | \mathcal{F} \right] \leq C \mathbb{E} \left[ \left\| f_t \right\|^4 | \mathcal{F} \right] + C' N^2 \mathbb{E} \left[ \left\| \eta_t \right\|^4 \right] ,
$$

$$
\mathbb{E} \left[ \left\| \gamma' \eta_t \right\|^4 \right] = \mathbb{E} \left[ \left\| \sum_{i=1}^{N} \gamma_i \eta_{t_i} \right\|^4 \left\| \right\| \right] \leq \sum_{i=1}^{N} \left\| \gamma_i \right\|^4 \mathbb{E} \eta_{t_i}^4 + C \sum_{i=1}^{N} \left\| \gamma_i \right\|^2 \left\| \gamma_{i_2} \right\|^2 \omega_i^2 \omega_{i_2}^2.
$$

Due to Assumptions L and H(iii) we have that $\mathbb{E} \left[ \left\| \gamma' \eta_t \right\|^4 \right] \leq C N^2$, and thus

$$
\mathbb{E} \left[ \left\| \frac{\gamma' e_t}{\sqrt{N}} \right\|^4 | \mathcal{F} \right] \leq C \mathbb{E} \left[ \left\| f_t \right\|^4 + 1 | \mathcal{F} \right].
$$

Collecting the pieces,

$$
T \mathbb{E} \left\| V_s \right\|^4 \leq C T \mathbb{E} \left[ \frac{1}{T^2} \mathbb{E} \left[ \left\| \frac{\gamma' e_s}{\sqrt{N}} \right\|^4 | \mathcal{F} \right] \otimes \left\| v_s \right\|^4 \right] \leq T C \frac{1}{T^2} \mathbb{E} \left[ \left( \left\| f_s \right\|^4 + 1 \right) \left\| v_s \right\|^4 \right] \rightarrow 0.
$$

This gives us the validity of condition (b) of Lemma 5.3.
(c) Denote $\Gamma_{\omega,N} = N^{-1} \sum_{i=1}^{N} \omega_i^2 \gamma_i^2 \rightarrow \Gamma_{\omega}$. Then,

$$
\sum_{t=1}^{T} V_t V_t' - \Sigma_{VT} = \frac{1}{T} \sum_{t=1}^{T} \left( \frac{\gamma' e_t e'_t}{\sqrt{N}} \right) \otimes (v_t v_t') - \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \left( \frac{\gamma' \pi f_t \pi' f_t}{\sqrt{N}} + \Gamma_{\omega,N} \right) \otimes (v_t v_t') \right]
$$

$$
= \frac{1}{T} \sum_{t=1}^{T} \left( \frac{\gamma' e_t e'_t}{\sqrt{N}} \frac{\pi' \gamma}{\sqrt{N}} - \frac{\gamma' \pi f_t \pi' f_t}{\sqrt{N}} - \Gamma_{\omega,N} \right) \otimes (v_t v_t')
$$

$$
+ \frac{1}{T} \sum_{t=1}^{T} \left[ \frac{\gamma' \pi f_t \pi' \gamma}{\sqrt{N}} + \Gamma_{\omega,N} - \mathbb{E} \left[ \frac{\gamma' \pi f_t \pi' \gamma}{\sqrt{N}} + \Gamma_{\omega,N} \right] \right] \otimes (v_t v_t')
$$

$$
= A_1 + A_2.
$$

Notice that given the conditional independence of $\eta_t$’s, the two terms in the last expression, $A_1$ and $A_2$ are uncorrelated, so in order to check condition (c) of Lemma 5.3 we can prove convergence to zero of $\mathbb{E} \| A_1 \|^2$ and $\mathbb{E} \| A_2 \|^2$ separately. First,

$$
\mathbb{E} \left[ \| A_1 \|^2 \right] = \mathbb{E} \left[ \left\| \frac{1}{T} \sum_{t=1}^{T} \left( \frac{\gamma' \pi f_t \eta_t \gamma}{\sqrt{N}} + \gamma' \eta_t f_t \pi' \gamma + \left( \frac{\gamma' \eta_t \eta_t^{'} \gamma}{\sqrt{N}} - \Gamma_{\omega,N} \right) \right) \otimes (v_t v_t') \right\|^2 \right]
$$

$$
\leq \frac{1}{T^2} \sum_{t=1}^{T} \mathbb{E} \left[ \left\| \frac{\gamma' \pi f_t \eta_t \gamma}{\sqrt{N}} + \gamma' \eta_t f_t \pi' \gamma + \left( \frac{\gamma' \eta_t \eta_t^{'} \gamma}{\sqrt{N}} - \Gamma_{\omega,N} \right) \right\|^2 \| v_t \|^4 \right]
$$

$$
\leq \frac{1}{T^2} C \mathbb{E} \left[ (\| f_t \|^2 + 1) \| v_t \|^4 \right] \rightarrow 0.
$$

The former inequality is due to $\eta_t$’s being conditionally serially uncorrelated, and thus the summation over $t$ can be taken outside the expectation of the square; the latter inequality uses bounds on the moments of $\eta_t^{'} \gamma / \sqrt{N}$ we derived before. Second, the convergence of term $A_2$ is due to Assumptions H(iv) and C(iv). Putting all terms together, we obtain that condition (c) is satisfied.

Finally, we check the condition (d):

$$
T^2 \| \mathbb{E} \left( W_{s_1 t} V_{s_1} V_{s_2} W_{s_2 t} \right) \| = \| \mathbb{E} \left[ w_{s_1 t} v_{s_1} v_{s_2} w_{s_2 t} \mathbb{E} \left( \frac{e_{s_1}^{'} \gamma \gamma' e_{s_1}^{'} e_{s_1}^{'} e_{s_2}^{'} e_{s_2}^{'} e_{t}^{'} e_{t}^{'} | \mathcal{F} \right) \right] \right| .
$$

Using that scalars could be reshuffled to make two $e_t$ with the same index stand back to
back and employing conditional independence we obtain:

\[
\left| \mathbb{E} \left( \frac{e_s}{\sqrt{N}} \frac{\gamma e_{s_2} e'_{s_2} e_t e'_t}{\sqrt{N}} \big| \mathcal{F} \right) \right| = \frac{1}{N^2} \left| \text{tr}(\gamma' \mathbb{E}(e_{s_2} e'_{s_2} | \mathcal{F}) \mathbb{E}(e_t e'_t | \mathcal{F}) \mathbb{E}(e_{s_1} e'_{s_1} | \mathcal{F})) \right| \\
\leq \frac{1}{N^2} \text{tr}(\gamma' \prod_{s \in \{s_1, s_2, t\}} \max \text{ev}(\mathbb{E}(e_s e'_s | \mathcal{F}))) \\
\leq \frac{C}{N} \mathbb{E} \left[ \prod_{s \in \{s_1, s_2, t\}} (\|f_s\|^2 + 1) | \mathcal{F} \right].
\]

We use Assumption L that $N^{-1} \text{tr}(\gamma') < C$ and the bound (9) we derived before. In the last equality, we also exploit that $f_t$'s are conditionally independent of each other. Thus, Assumption H (i) implies that

\[ T^3 \max_{1 \leq t < \min\{s_1, s_2\} \leq T} \| \mathbb{E} \left( W_{s_1} V_{s_1} V_{s_2} W'_{s_2} \right) \| \leq \frac{C}{N} \to 0. \]

Thus, condition (d) of Lemma 5.3 is satisfied. This concludes the proof of Theorem 3.2. \qed

**Proof of Theorem 4.2.** We will prove three statements:

(i) $N^{-1} \sum_{i=1}^{N} \xi_{v_i} \xi'_{v_i} \to \Sigma_V$;

(ii) $N^{-1} \sum_{i=1}^{N} \xi_{w_i} \xi'_{w_i} \to \Sigma_W$;

(iii) $N^{-1} \sum_{i=1}^{N} \xi_{v_i} \xi'_{w_i} \to 0$.

(i) Given assumption $\Gamma_{\pi \eta} = 0$, we have $\Sigma_V = \Gamma_{\omega} \otimes \Omega_v$. Then,

\[
\frac{1}{N} \sum_{i=1}^{N} \xi_{v_i} \xi'_{v_i} = \frac{1}{NT} \sum_{t=1}^{T} \sum_{s=1}^{T} \left( \sum_{i=1}^{N} \gamma_i \gamma'_i (\pi'_i f_s + \eta_i)(\pi'_i f_t + \eta_i) \right) \otimes (v_i v'_i). \tag{10}
\]

After we open up the brackets there will be three different types of terms. We will show that

\[
\frac{1}{NT} \sum_{t=1}^{T} \sum_{s=1}^{T} \sum_{i=1}^{N} (\gamma_i \gamma'_i \eta_i \eta_i) \otimes (v_i v'_i) \overset{p}{\to} \Sigma_V, \tag{11}
\]

while terms that involve $\pi'_i w_s \pi'_i w_t$ or $\eta_i \pi'_i f_s$ converge to zero in probability. Indeed,

\[
\frac{1}{NT} \sum_{t=1}^{T} \sum_{s=1}^{T} \sum_{i=1}^{N} (\gamma_i \gamma'_i \eta_i \eta_i) \otimes (v_s v'_s) - \Sigma_{V,T} = \frac{1}{NT} \sum_{t=1}^{T} \sum_{s \neq t}^{T} \sum_{i=1}^{N} (\gamma_i \gamma'_i \eta_i \eta_i) \otimes (v_s v'_s) \\
+ \frac{1}{NT} \sum_{t=1}^{T} \sum_{i=1}^{N} (\gamma_i \gamma'_i) \otimes (\eta_i^2 v_tv'_t - \omega^2 \mathbb{E}(v_t v'_t)).
\]
We check that the first sum in the last expression is negligible:

\[
\mathbb{E} \left[ \text{tr} \left( \left( \frac{1}{NT} \sum_{t=1}^{T} \sum_{s \neq t} \sum_{i=1}^{N} (\gamma_{it}^i \eta_{st}^i \otimes (v_s v_t^i))^2 \right) \right) \right] \leq \frac{1}{N^2 T^2} \sum_{t=1}^{T} \sum_{s \neq t} \sum_{i=1}^{N} ||\gamma_i||^4 \omega_i^4 \mathbb{E} ||v_t||^2 ||v_s||^2 \\
\leq \frac{C}{N^2} \sum_{i=1}^{N} ||\gamma_i||^4 \omega_i^4 \to 0.
\]

Here we use the conditional cross-sectional and temporal independence of \( \eta_{ht} \); that is, for \( s \neq t \) we have \( \mathbb{E}(\eta_{ht} \eta_{st}^i | \mathcal{F}) = \omega_i^4 \) if \( i = j \) and \( \{t, s\} = \{t^*, s^*\} \), and zero otherwise.

We also use Assumptions L and H(iii). As for the second sum, we notice that all summands in the expression below are uncorrelated with each other, hence

\[
\text{tr} \left( \mathbb{E} \left[ \left( \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (\gamma_{it}^i \eta_{st}^i \otimes (v_t v_t^i) - \Sigma_v)^2 \right) \right] \right) \\
= \frac{1}{N^2 T^2} \sum_{i=1}^{N} \sum_{t=1}^{T} \text{tr} \left( \mathbb{E} \left[ (\gamma_{it}^i \eta_{st}^i \otimes (v_t v_t^i) - \mathbb{E} [\gamma_{it}^i \eta_{st}^i \otimes (v_t v_t^i)])^2 \right] \right) \\
\leq \frac{C}{N^2 T^2} \sum_{i=1}^{N} \sum_{t=1}^{T} ||\gamma_i||^4 \mathbb{E} [||v_t||^4] \to 0.
\]

Thus, we showed the convergence (11).

Now consider terms in (10) that involve \( \pi_i' f_s \pi_i' f_i \):

\[
\frac{1}{NT} \sum_{t=1}^{T} \sum_{s=1}^{T} \left( \sum_{i=1}^{N} \gamma_{it}^i \pi_i' f_s \pi_i' f_t \right) \otimes (v_s v_t^i) \\
= \left( \frac{1}{N} \sum_{i=1}^{N} (\gamma_i \gamma_i^i \otimes (\pi_i' \otimes \pi_i') \otimes I_{k_v}) \right) \left( \frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{T} I_{k_v} \otimes \text{vec}(f_s f_t') \otimes (v_s v_t^i) \right).
\]

Using Assumption L and H(ii) we can show that

\[
\left\| \frac{1}{N} \sum_{i=1}^{N} (\gamma_i \gamma_i^i \otimes (\pi_i' \otimes \pi_i')) \right\| \leq \frac{1}{N} \sum_{i=1}^{N} \|\gamma_i\|^2 \|\pi_i\|^2 \leq \frac{C}{N} \sum_{i=1}^{N} \|\pi_i\|^2 \to 0.
\]
Now observe that

\[
\mathbb{E} \left[ \left\| \frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{T} \text{vec}(f_s f_t') \otimes (v_s v_t') \right\|_F^2 \right] = \text{tr} \left( \frac{1}{T^2} \mathbb{E} \left[ \sum_{t=1}^{T} \sum_{s=1}^{T} \sum_{t'=1}^{T} \sum_{s'=1}^{T} (\text{vec}(f_s f_t') \text{vec}(f_{s'} f_{t'}')) \otimes (v_s v_{t'} v_{s'} v_{t'}) \right] \right) \leq C \frac{1}{T^2} \mathbb{E} \left[ \sum_{t=1}^{T} \sum_{s=1}^{T} \|f_t\|^2 \|f_s\|^2 \|v_s\|^2 \|v_t\|^2 \right] < C.
\]

Here the equality is due to \( f_t \)'s being serially independent and mean zero conditionally on \( \mathcal{F} \) by Assumption H(i) and \( v_t \in \mathcal{F} \); hence, among the four summation indexes at most two may be distinct. The last inequality is due to Assumption H(i). Thus, we showed that

\[
\frac{1}{NT} \sum_{t=1}^{T} \sum_{s=1}^{T} \left( \sum_{i=1}^{N} \gamma_i^t \gamma_i^s f_i f_i' \right) \otimes (v_s v_t) \overset{p}{\to} 0.
\]

And finally, we show that

\[
\frac{1}{NT} \sum_{t=1}^{T} \sum_{s=1}^{T} \left( \sum_{i=1}^{N} \gamma_i^t \gamma_i^s f_i \eta_{st} \right) \otimes (v_s v_t) \overset{p}{\to} 0.
\]

This holds because \( \eta_{t,s} \)'s are mean zero, cross-sectionally independent and independent from \( f_t \) conditionally on \( \mathcal{F} \). This implies that the mean of the sum above is zero, and all summands are uncorrelated with each other. The second moment of the sum is bounded above by

\[
\frac{C}{N^2 T^2} \sum_{t=1}^{T} \sum_{s=1}^{T} \sum_{i=1}^{N} \|\gamma_i^t\|^2 \|\gamma_i^s\|^2 \omega_i^2 \mathbb{E} \left[ \|f_t\|^2 \|v_t\|^2 \|v_s\|^2 \right] \to 0.
\]

Thus, we proved statement (i).

Let us turn to statement (ii):

\[
\frac{1}{N} \sum_{t=1}^{T} \sum_{s=1}^{T} w_{st} \xi_{s,t} - \Sigma_{w,t} = \frac{1}{T^2 N} \sum_{t=1}^{N} \sum_{s=1}^{T} \sum_{t<s} w_{st} w_{st}' (e_{it}^2 e_{is}^2 - \omega_i^2)
\]

\[
+ \frac{1}{T^2 N} \sum_{t=1}^{N} \sum_{s=1}^{T} \sum_{t<s} \sum_{t'<s} \sum_{s'=1}^{T} \sum_{s,s* \neq s,t} w_{st} w_{st}' e_{it}^2 e_{is}^2 e_{is'}^2 e_{is*}^2
\]

\[
+ \frac{1}{N} \sum_{t=1}^{N} \sum_{s=1}^{T} \sum_{t<s} \left( w_{st} w_{st}' - \mathbb{E}(w_{st} w_{st}') \right)
\]

\[
= A_1 + A_2 + A_3.
\]
As for $A_1$, we can notice that all summands with indexes $\{s, t\} \neq \{s^*, t^*\}$ are uncorrelated with each other, so the correlation for summands with different indexes $i$ can come only from the $\pi_i f_i$ part. Thus,

$$\mathbb{E}[\|A_1\|^2_F] = \frac{1}{T^4N^2} \sum_{s=1}^T \sum_{t<s}^T \mathbb{E} \left[ \left\| \sum_{i=1}^N w_{st} w_{st}' \left( e_i^2 e_i^2 - \omega_i^2 \right) \right\|_F^2 \right] \leq \frac{C}{T^4N^2} \sum_{s=1}^T \sum_{t<s}^T \sum_{i=1}^N \left( \mathbb{E}[\|w_{st}\|^4] \max_{1 \leq i \leq N} \mathbb{E}[(\eta_{st}^4)^2] + \sum_{j=1}^N \|\pi_i\|^4 \|\pi_j\|^4 \mathbb{E}[\|w_{st}\|^4\|f_i\|^4\|f_s\|^4] \right) \to 0.$$  

In the last convergence we used that due to Assumption H,

$$\frac{C}{N^2} \sum_{i=1}^N \sum_{j=1}^N \|\pi_i\|^4 \|\pi_j\|^4 \leq \frac{C}{N^2} \max_{1 \leq i \leq N} \|\pi_i\|^4 \left( \sum_{i=1}^N \|\pi_i\|^2 \right)^2 \to 0, \quad (12)$$

and hence the term $A_1$ converges to zero.

Term $\mathbb{E}[\|A_2\|^2_F]$ equals the following expression:

$$\frac{1}{T^4N^2} \sum_{s_1, s_1', s_2, s_2' \in \{s_m, t_m\} \neq \{s^*, t^*\}} \sum_{i,j=1}^N \sum_{s_1, s_1', s_2, s_2' \in \{s_m, t_m\} \neq \{s^*, t^*\}} \mathbb{E} \left[ \text{tr}(w_{s_1 t_1} w_{s_1' t_1} w_{s_2 t_2} w_{s_2' t_2} e_{i t_1} e_{i s_1} e_{i t_1'} e_{i s_1'} e_{j t_2} e_{j s_2} e_{j t_2'} e_{j s_2'}) \right]. \quad (13)$$

Notice that if $s_m < t_m$, $s_m^* < t_m^*$ and $\{s_m, t_m\} \neq \{s^*, t^*\}$ for $m = 1, 2$, the only ways when the expectation

$$\mathbb{E}(e_{i t_1} e_{i s_1} e_{i t_1'} e_{i s_1'} e_{j t_2} e_{j s_2} e_{j t_2'} e_{j s_2'}) | \mathcal{F} \neq 0 \quad (14)$$

can be non-zero is when we place at least four restrictions on the time indexes. Indeed, if $\{s_1, s_1', t_1, t_1'\}$ are all distinct, then to get a non-zero expectation we need indexes to coincide as sets: $\{s_1, s_1', t_1, t_1'\} = \{s_2, s_2', t_2, t_2'\}$. If the set $\{s_1, s_1', t_1, t_1'\}$ contains three distinct indexes, for example, $s_1 = s_2^*$ (this is one restriction), then the set $\{s_2, s_2', t_2, t_2'\}$ should contain $\{t_1, t_1'\}$ (these are two restrictions), and the remaining indexes should be either equal to each other (one restriction) or equal to the ones previously mentioned (two restrictions). Thus, instead of eight-dimensional summation over time indexes in equation (13) we have a four-dimensional summation.

Let us consider those terms in (13) when the summation index $j$ is equal to $i$. Notice that since each $t$ index is strictly smaller than the corresponding $s$ index, then any
distinct time index can appear in the set \( \{s_1, s_1^*, t_1, t_1^*, s_2, s_2^*, t_2, t_2^*\} \) at most four times, thus any individual error term \( e_{it} \) may appear in at most power four. Thus, all non-zero terms are bounded above by \( \max_{1 \leq i \leq N, 1 \leq s,t,t' \leq T} \mathbb{E} \left[ \|w_{st}\|^4 (\mathbb{E} (\eta_{it}^4) + C\|f_{it}\|^4)^2 \right] < C \) due to Assumption H(i,iii). There are at most \( CT^4N \) of such terms while the normalization is \( N^{-2}T^{-4} \), hence that sum converges to zero.

Now consider those terms in \( (13) \) when \( i \neq j \). Since \( e_{it} = \pi'_if_i + \eta_{it} \), with \( \eta_{it} \)'s independent of each other both cross-sectionally and temporally, \( i \neq j \) and \( \{s_m, t_m\} \neq \{s_1, t_1^*\} \), we have that all terms including \( \eta_{it} \) are zero, and only a non-trivial part of the term in \( (14) \) is the one including \( \pi'_if_i \) in place of \( e_{it} \). So, every non-zeros term in the sum \( (13) \) is bounded above in the same manner as stated in equation \( (12) \). Thus, we showed that \( A_2 \overset{p}{\to} 0 \). The convergence \( A_3 \overset{p}{\to} 0 \) comes from Assumption C(iii). This finishes the proof of (ii).

Finally, let us prove statement (iii):

\[
\frac{1}{NT^{3/2}} \sum_{i=1}^{N} \sum_{s=1}^{T} \sum_{t<s} \sum_{s'=1}^{T} (\gamma_i \otimes v_{s^*}) w_{st} e_{is} e_{is} \to 0.
\]

As before, we look at the expectation of the square of the sum above, which involves six-dimensional summation over time indexes and two-dimensional cross-sectional summation (over \( i, j \)) and is normalized by \( N^{-2}T^{-3} \). Due to time-series independence of \( e_{it} \), the six-dimensional summation over time indexes has mostly zeros and can be reduced to three-dimensional summation over time indexes as the set \( \{s_1, t_1, s_1^*, s_2, t_2, s_2^*\} \) should have any distinct index to appear at least twice. If we consider the cases when \( i = j \), then all terms are bounded above by a constant and the number of non-zero terms is \( NT^3 \); given the normalization, this sum converges to zero. When we sum over \( i \neq j \), the only part of \( e_{it} \) that yields a non-trivial effect is \( \pi'_if_i \); hence this sum is bounded by

\[
\frac{1}{N^2} \sum_{i,j=1}^{T} \|\gamma_i\| \|\gamma_j\| \|\pi_i\|^3 \|\pi_j\|^3 \max_{1 \leq s,t,s' \leq T} \mathbb{E} \left[ \|v_{s^*}\|^2 \|w_{st}\|^2 \|f_{s^*}\|^2 \|f_s\|^2 \|f_{t}\|^2 \right]
\leq C \left( \frac{1}{N} \sum_{i=1}^{N} \|\gamma_i\|^2 \|\pi_i\|^3 \right)^2 \leq \frac{1}{N^2} \max_{1 \leq i \leq N} \|\gamma_i\|^2 \max_{1 \leq i \leq N} \|\pi_i\|^4 \sum_{i=1}^{N} \|\pi_i\|^2 \to 0.
\]

This ends the proof of Theorem 4.2. \( \square \)
References


