GENERAL EQUILIBRIUM AND WELFARE THEOREMS FOR INATTENTIVE ECONOMIES

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Abstract. We develop a flexible framework for studying the general equilibrium and the welfare properties of economies featuring a generalized form of rational inattention, or cognitive frictions. Our framework departs from the canonical Arrow-Debreu framework only in the introduction of such frictions. But it also nests several specifications of such frictions found in the literature, including entropy costs, sparsity, bounded recall, and local thinking. This helps illuminate the robust properties of the class of economies we are interested in.

We first use an example to illustrate how the equilibrium prices in such economies may fail to reveal demand and supply to the outside observer, leading to potentially erroneous inferences about consumer surplus and complicating welfare analysis. We next show more generally how the Second Welfare Theorem in such economies can be understood as the problem of a mediator whose messages—shadow prices—are observed by the agents with endogenous idiosyncratic noise and can be interpreted by them only at the expense of cognitive effort. This means that there can be welfare gains from sending "simpler" messages, or perhaps implementing less volatile market prices, relative to the frictionless benchmark.

We finally identify general restrictions on the cognitive structure that suffice for these welfare gains to be taken care by the "invisible hand" of the market mechanism, that is, for appropriate extensions of the two Welfare Theorems to hold despite the aforementioned complications. We also explore few policy implications.

Motivation

- **Challenges** with rational inattention/cognitive frictions in GE
  - How are budgets satisfied?
  - How do prices clear markets?

> If both sides of the market react to market prices with rational inattention, then neither side is reacting precisely and immediately. Prices therefore cannot play their usual market-clearing role.
>
> Sims (2010)

- **Welfare properties?**
  - Efficiency of both allocation and attention
  - Policy: should governments intervene to simplify prices?

This Paper

- **General framework** for GE with RI/cognitive constraints
  - extends Arrow-Debreu
  - maps RI to **endogenous measurability constraints**
- Nests
  - RI with entropy costs, as in Sims (2003)
  - more general costs or signals, as in Myatt and Wallace (2012)
  - costly contemplation as in Tirole (2015)
  - sparsity as in Gabaix (2014)
  - local thinking as in Lian (2018)
- Clarifies role of assumptions in existing applications
  - **Reveals what is robust**
- Makes available "high-level toolkit" of GE theory

Results

- Show, with example, how to bypass aforementioned challenges
- Identify a new challenge:
  - Demand may not be informative about preferences
  - “Harberger triangles” may have no welfare interpretation
  - Suggests difficulty in doing normative analysis, and yet. . .
- Extend **Welfare Theorems**
  - both for allocations and for attention choice
- Requires certain restriction of inattention/cognition structures
  - rules out *“pecuniary externalities working through cognition”*
  - satisfied for mutual information cost

Related Literature

- **Consumer Theory with Inattention**: Sims (2003, 2010), Gabaix (2014, 2017), Farhi and Gabaix (2017)

  Macro Settings with Inattention: Woodford (2003), Mackowiak and Wiederholt (2009)

  New: More flexible environment, bypassing of challenges with budget constraints and GE, proof of Welfare Theorems

- **Information Choice Games**: Tirole (2015)

  New: Clarification about the role of externalities in creating "cognitive trap" failure of FWT

- **Welfare Theorems with Island Economies**: Prescott and Rios-Rull (1992), Prescott and Townsend (1984)

  New: Variant for economies with RI/cognitive constraints

A Toy Example

Warm-up Example, with Complete Information

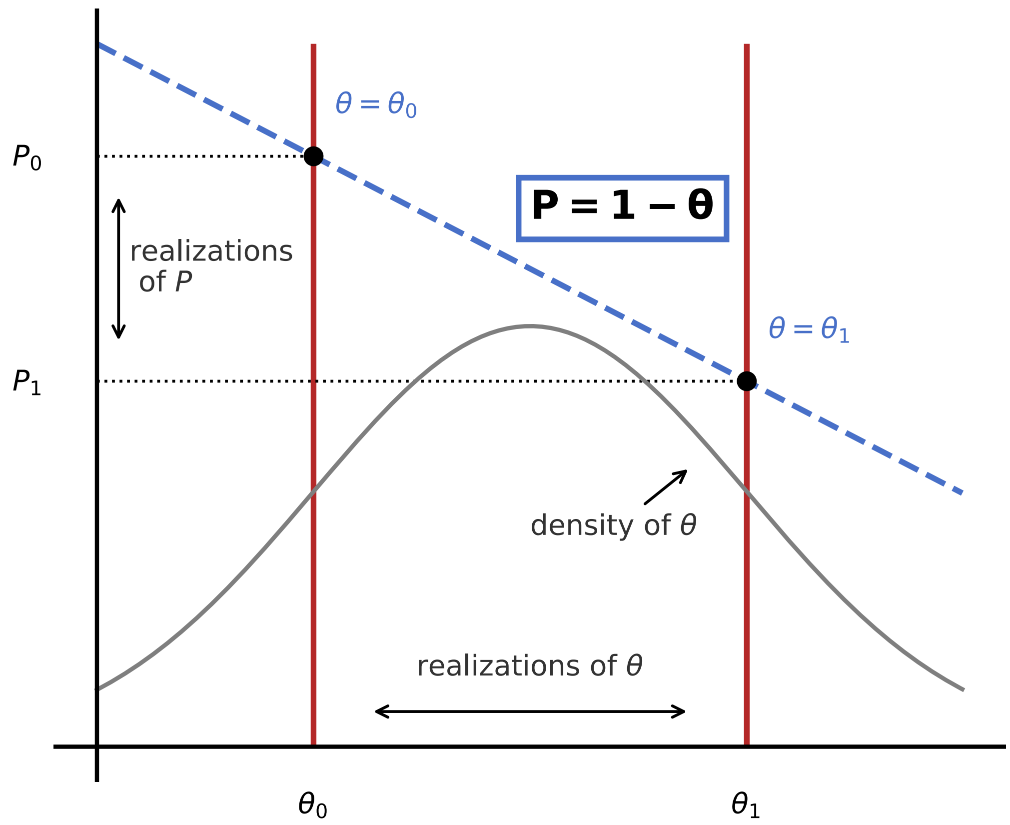
- **Endowment economy**
  - Continuum of agents
  - Linear-quadratic preferencs
- Household problem

$$\max_{X,Y} X - \frac{X^2}{2} + Y$$
$$\text{s.t. } PX + Y \leq P\theta + 1$$

- $\theta \sim N(\mu, \sigma^2)$, endowment of X for each agent
- Demand, from first order conditions, is

$$X^*(\theta, P) = 1 - P$$
$$Y^*(\theta, P) = 1 + P\theta - P(1 - P)$$

General Equilibrium, with Complete Information



Adding Inattention

- Inattention as a **measurability constraint** on action (X, Y)
  - *Interpretation*: difficulty in tracking (θ, P) or (X^*, Y^*)
- How is the **budget constraint** met?
  - One good adjusts with state of nature

$$X = X(\omega)$$
$$Y = Y(\omega, \theta, P)$$

    where ω is some signal
  - More flexible: $X = X(\omega_1), Y = Y(\omega_2)$
- **Rational inattention**: choose signal structure given costs
  - Joint distribution $\phi(\omega, \theta, P)$

Inattentive General Equilibrium

LLN across $\omega \rightarrow$ *aggregate demand* function of $(\theta, P) \rightarrow P = P(\theta)$

Definition

A **rational expectations equilibrium** *is a goods allocation $(X(\cdot), Y(\cdot))$, attention allocation $\phi(\omega, \theta)$, and a price function $P(\theta)$ such that*

(i) *Agents choose demand $(X(\omega), Y(\omega, \theta, P))$ and and attention $\phi(\omega, \theta)$ to maximize expected utility*

(ii) *Price functional $P = P(\theta)$ is a fixed point*

(iii) *Markets clear for all realizations of θ*

Remark: existence under standard assumptions of continuity

Equilibrium with Gaussian Signals about θ

- Let signal have the form $\omega = \theta + \xi_i$, with $\xi_i \sim N(0, \tau^2)$
- Agents play (given signal-to-noise ratio δ):

$$\begin{aligned} X_i = 1 - \mathbb{E}_i[P] &= 1 - \overbrace{(a - b\mathbb{E}_i[\theta])}^{\mathbb{E}_i[\text{Conjectured } P(\theta)]} \\ &= 1 - a + b \underbrace{(\delta\omega + (1-\delta)\mu)}_{\text{Gaussian conditional}} \end{aligned}$$

- After applying market clearing $\int X_i \mathrm{d}i = \theta$, we recover

$$X := \int X_1 \mathrm{d}i = \delta + (1-\delta)\mu - \underbrace{\delta P}_{\text{Attenuated price response}}$$

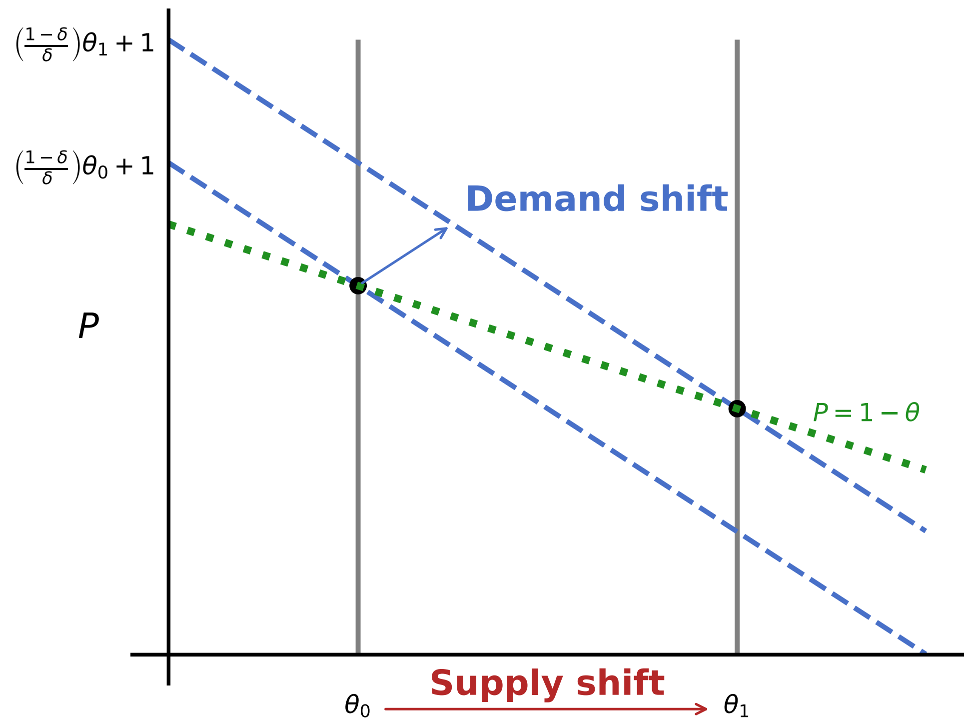
- "Aggregate demand," with price which does clear markets

New Challenge: Welfare and Surplus

- **Harberger triangle analysis**
  - Say econometrician traces out demand by plotting P versus θ
  - Can we use this to compute welfare costs of a tax?
- **Result**: no
- Some intuition
  - Measured slope $-\delta \in [-1, 0)$ depends on noise
  - True slope -1 if tax is certain

An Illustration: Shifts of Supply and Demand

Example: mean of θ distribution shifts from θ_0 to θ_1



General Environment and Welfare Theorems

Arrow-Debreu Architecture

- State of nature $\theta \in \Theta \subseteq \mathbb{R}^{|\Theta|}$ with common prior $\pi(\theta)$
  - Encodes preferences and endowments
- Continuum of consumers of $J < \infty$ "types," mass ξ^j each
  - Today, **endowment economy**
  - Production in paper's version
- N distinct goods per state
  - Quantities $\boldsymbol{x} := [x(\theta)]_{\theta \in \Theta} \in \mathbb{R}^{N|\Theta|}$
  - Prices $\boldsymbol{p} := [p(\theta)]_{\theta \in \Theta} \in \mathbb{R}^{N|\Theta|}$
- Preference ordering $\succsim^j$ over goods

Inattentive Twist

- Measurability constraint $x_i = x_i(\omega_i)$
  - Let $x(\omega) = [x_i(\omega_i)]_{i=1}^N$ stack these functions $\omega_i \mapsto x_i$
- Joint distribution $\phi(\omega, Z) = \phi([\omega_i]_{i=1}^N, Z)$ is choice variable
- **Tracking prices versus tracking states**
  - Most natural to set $Z = p(\theta)$
  - Will start with $Z = \theta$, an easier case
- Cost of information: disutilty $-C[\phi(\omega, \theta)]$
  - More abstract: any preference ordering $\succsim^j$ over $(x(\cdot), \phi(\cdot, \cdot))$

Inattentive Consumer's Problem

$$\max_{x(\cdot),\phi(\cdot,\cdot)} \sum_{\omega,\theta} u^j(x(\omega),\theta)\phi(\omega,\theta) - \overbrace{C^j[\phi(\omega,\theta)]}^{\text{Rational inattention}}$$

$$\text{s.t.} \sum_{\omega,\theta} \overbrace{(p(\theta)\cdot x(\omega))}^{\text{Dot product over } N \text{ goods}})\phi(\omega,\theta) \leq \sum_{\theta} p(\theta)\cdot e(\theta) \quad \text{Budget}$$

$$x_i(\omega) = x_i(\omega') \text{ if } \omega_i = \omega_i' \qquad \text{“Local thinking”}$$

Equilibrium Concept

Define **aggregate demand** as $\boldsymbol{x}(\theta)$ and **aggregate supply** as $\boldsymbol{e}(\theta)$

Recall from example: Law of Large Numbers allows for this

Definition

An Arrow-Debreu competitive equilibrium with inattention is a goods allocation $[x^j(\omega)]_{j=1}^J$, an attention allocation $[\phi^j]_{j=1}^J$, and a price function $p(\theta)$ such that

(i) *Consumers optimize in their budget set*

(ii) *Markets clear, meaning that $\boldsymbol{x}(\theta) = \boldsymbol{e}(\theta)$ for all $\theta \in \Theta$*

The Planner's Problem

Definition

An efficient allocation (social planner's solution) $[x^j(\omega), \phi^j(\omega,\theta)]_{j=1}^J$*, for Pareto weights* $[\chi^j]_{j=1}^J$*, solves*

$$\max_{[x^j(\cdot),\phi^j(\cdot,\cdot)]_{j=1}^J} \sum_{j=1}^N \chi^j \xi^j \left[\sum_{\omega,\theta} u(x^j(\omega),\theta)\phi^j(\omega,\theta) - C^j\left[\phi^j(\omega,\theta)\right]\right]$$

$$s.t. \ \sum_j \xi^j \sum_\omega x^j(\omega)\phi^j(\omega \mid \theta) \leq \sum_j \xi^j e^j(\theta), \forall\theta \in \Theta$$

Example of **"team problem"**

Intuition: Same FOC

- Lagrange multiplier $\lambda(\theta)\pi(\theta)$
- **Planner's FOC** for $x_i^j(\omega)$

$$\mathbb{E}\left[\frac{\partial u(x^j(\omega),\theta)}{\partial x_i} \mid \omega\right] = \mathbb{E}\left[(\chi^j)^{-1}\lambda_i(\theta) \mid \omega\right]$$

- Looks like **Agent's FOC**

$$\mathbb{E}\left[\frac{\partial u(x^j(\omega),\theta)}{\partial x_i} \mid \omega\right] = \mathbb{E}\left[\mu^j p_i(\theta) \mid \omega\right]$$

- Given this, can show attention choice also equivalent
- To generalize: use familiar GE Theory tools

First Welfare Theorem: Statement

Assumption (Costly to track θ)

Consumers maximize $\mathbb{E}[u^j(x(\omega), \theta)] - C^j[\phi(\omega, \theta)]$

Assumption (Local non-satiation for goods)

For every $j \in \{1, \ldots, J\}$, $x(\omega)$, ϕ, *and* $\epsilon > 0$, *there exists some* $x'(\omega) \in \mathcal{B}_\epsilon(x(\omega)) := \{x'(\omega) : \|[x'(\omega) - x(\omega)]_{\omega \in \Omega}\| < \epsilon\}$ *such that* $\mathbb{E}_\phi[u^j(x'(\omega), \theta)] > \mathbb{E}_\phi[u^j(x(\omega), \theta)]$.

Theorem (First Welfare Theorem)

Assume a competitive equilibrium allocation exists with prices $p(\theta) > 0$ *for all* $\theta \in \Theta$. *Then, given local non-satiation, there does not exist another feasibile allocation that Pareto dominates the competitive equilibrium.*

Second Welfare Theorem: Statement

Assumption (Costly to track θ)

Consumers maximize $\mathbb{E}[u^j(x(\omega), \theta)] - C^j[\phi(\omega, \theta)]$

Assumption (Convexity)

There are costless lotteries for consumers or firms, or convexity of the (additive form) cost functionals plus convexity of the "standard" preferences and production sets

Theorem (Second Welfare Theorem)

Consider any Pareto optimal allocation of goods, $[x^j(\omega)]_{j=1}^J$, and attention, $[\phi^j]_{j=1}^J$. This allocation can be supported as an Arrow-Debreu equilibrium with inattention with transfers.

Some Remarks

- Relatively “standard” proofs
- What happened to the choice of ϕ?
    - Affects other agents through prices
    - Hence **pecuniary externalities** through budget
        - Handled by **complete markets**
    - No *direct effect* through cognitive constraints
- Contrast with literature
    - Tirole (2015) on cognitive traps: *interaction with other externalities*, not non-convexity of $C[\cdot]$
    - Gabaix (2014) on welfare properties: wrong efficiency concept, not sparsity

Tracking Prices

Why is this Tricky?

- Consider this version of the linear-quadratic example:

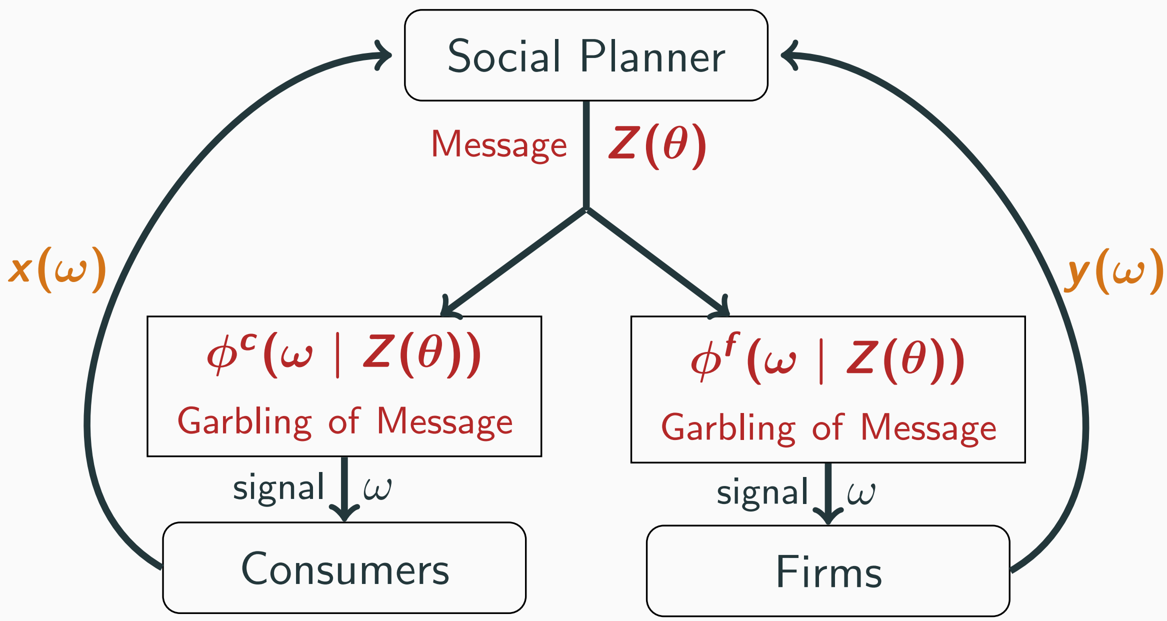
$$\max_{X(\cdot),Y(\cdot),\phi(\cdot,\cdot)} \mathbb{E}_\phi\left[X(\omega) - \frac{X(\omega)^2}{2} + Y(\omega,\theta,P)\right] - \mathrm{Cov}_\phi[\omega, P(\theta)]$$
$$\text{s.t. } P(\theta)X(\omega) + Y(\omega,\theta,P) \leq P(\theta)\theta + 1$$

  Consumers will miss the externality

- Pecuniary externalities in budget defeated by complete markets, but what about **pecuniary externalities working through cognitive constraint**
- **Our result: not necessarily**

The Planner as a Communicator

Choose **information structure** and **actions**



When does the equilibrium (Walrasian) mechanism lead to the same allocation as this one?

Intuition from Planner's Problem

- Previously restricted to $Z(\theta) = \theta$
- Recall the FOC

$$\mathbb{E}\left[\frac{\partial u(x^j(\omega), \theta^j)}{\partial x_i} \mid \omega\right] = \mathbb{E}\left[(\chi^j)^{-1}\lambda(\theta) \mid \omega\right]$$

- Say, for simplicity, agent knows θ^j
- $\lambda(\theta)$ a **sufficient statistic** for others' types
- **Sensible assumption**: cost of tracking $\lambda(\theta)$ is weakly lower than cost of tracking θ
    - Sending $Z(\theta) = \lambda(\theta)$ is weakly better than sending $Z(\theta) = \theta$
    - Proving its optimality requires some more assumptions

Informational Equivalence for the Cost Function

Assumption (Informational Equivalence about ω)

Let random variables X and Z be such that $\omega \perp X \mid Z$ and $\omega \perp Z \mid X$. Then $C[\omega, Z] = C[\omega, X]$.

Interpretation: say ω is a signal about $Z(\theta)$, and contains no other information about θ. This has the same cost relative to θ as it does relative to $Z(\theta)$.

Remarks

- Satisfied by mutual information
- Rules out scale effects (e.g., $\mathrm{Cov}[\omega, P]$)

Extended Second Welfare Theorem

Theorem (Welfare Theorems for Price-tracking Economies)

Assume informational equivalence in the cost functionals $C^j[\cdot,\cdot]$ Assume that θ^j is known to each type j. The previous Welfare Theorems (given previous assumptions) extend to price-tracking economies.

Some intuition for FWT: Assume, as a contradiction, there exists a simpler message that is better than θ. Because of informational equivalence, it is identical to send θ as the message. Because of the sufficiency of $\lambda(\theta)$, it is identical to send $\lambda(\theta)$ as well.

Discussion

Implementation with Prices

Corollary (Prices are Optimal Messages)

Consider some Pareto optimal allocation with signal $Z(\theta)$. This allocation can be implemented as a competitive equilibrium (with transfers) with prices $p(\theta)$. Moreover, the planner could have implemented the same goods and attention allocation by originally sending $p(\theta)$.

Mechanism design interpretation (?)

Macro/PF: no need to use taxes to make price system simpler

Conclusion

- Set up general environment for inattentive GE to
  - clarify role of assumptions
  - explore robust properties
- Inattentive economies can be (constrained) efficient because
  - Complete markets wash out pecuniary externalities
  - Prices are "sufficient statistic" for state of nature
- **Prices play dual role**
  - Clear markets and aggregate information (others' types)
  - Compare with the prior belief that they might do neither!
- Possible extensions
  - Concrete example of "surprisingly efficient" price system
    - Macro: when are sticky prices optimal?
  - **Optimal taxation**: principles should be same
    - Extended Diamond-Mirlees or Atkinson-Stiglitz?

Convexity for Second Welfare Theorem SWT Statement

- Two paths forward to make economy "convex enough" for the Second Welfare Theorem
  (i) Costless lotteries
    - **Deterministic split** of each type's continuum into "subtypes," which have their own ϕ and their own consumption bundle
    - Alternatively, zero cognitive cost to introducing a **stochastic lottery** (conditioned on extrinsic variable) in ϕ

  (ii) Convexity of $C[\cdot]$: in shorthand notation,
  $C[a\phi + (1-a)\phi'] < aC[\phi] + (1-a)C[\phi']$
- Is either option consistent wit the narrative of inattention?
  - Point: lottery further "noises up" consumer demand
  - Counterpoint: executing lottery requires cognitive overhead and high coordination
- After verifying conditions, can apply very general Theorem of Debreu (1954)

Preliminary Definition: $Z(\theta)$ Equilibrium

Let $Z(\theta) : \Theta \to \mathcal{Z}$ map θ to some arbitrary space. Consider the inattentive consumer problem, but the dependence on $\phi(\omega, \theta)$ is replaced by dependence on some cost functional $C^j[\omega, Z(\theta)]$.

Definition ($Z(\theta)$ Equilbrium, Approximately)

An $Z(\theta)$ competitive equilibrium with inattention is a goods allocation $[x^j(\omega)]_{j=1}^J$, an attention allocation $[\phi^j]_{j=1}^J$, and a price system $p(\theta)$ such that all agents solve their $Z(\theta)$ problems and markets clear. Denote the set of equilibria as $\mathcal{E}[Z(\theta)]$.

Price-tracking Equilibrium as a Fixed Point `Planning Picture`

Important, but technical: θ^j is privately known. Alternate approach, which requires tweaked definition: "charge" each agent for tracking their own θ^j, in addition to the price.

Definition (Price-Tracking Equilbrium)

The set of price-tracking equilibria are all allocations $([x^j(\omega), \phi^j]_{j=1}^J, p(\theta))$ *such that*

$$([x^j(\omega), \phi^j]_{j=1}^J, \underbrace{p(\theta)}_{\text{Equilibrium price}}) \in \mathcal{E}[\overbrace{p(\theta)}^{\text{Conjecture}}]$$

Remark

The price system, unnaturally, is part of the "allocation."