Lecture Note: A Ricardian Model of Task-Biased Technological Change

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14.662 Graduate Labor Economics, Spring 2018

February 20, 2018

1 Introduction

This model works to capture three forces that I believe are central to understanding the relationships among skills, technologies and wages:

1. The supply and demand for worker skills (as in Tinbergen, Katz-Murphy, etc.).

2. Self-selection of workers of given *skill* across different *tasks* according to comparative advantage.

3. Technology as a substitute for specific tasks and a complement to others.

A key starting point is the distinction between skills and tasks. A skill is a worker's *endowment* of capabilities for performing various tasks. This endowment is a stock, and it may be exogenously assigned or, more generally, acquired through schooling or other training. A task is a unit of a specific work activity that produces output. Workers apply their skill endowments to tasks in exchange for wages. Thus, skills are applied to tasks to produce output; skills do not directly produce output.

It may be convenient to assume a one-to-one mapping between skills and tasks. Indeed, one can interpret the Tinbergen/Katz-Murphy models as adopting this convention. More generally, the distinction between skills and tasks is irrelevant when the assignment of skills to tasks is fixed (or, if not fixed, unchanging during the period of study). The distinction between skills and tasks becomes relevant, however, when workers of a given skill level can potentially perform a variety of tasks, so the equilibrium assignment of skills to tasks is likely to be endogenous to demand conditions, technologies, and factor supplies. This endogenous assignment of skills to tasks lies at the heart of the Acemoglu-Autor model in the *Handbook of Labor Economics Volume 4* chapter (in progress). this model builds on Acemoglu and Zilibotti (2001) and is also related to Costinot and Vogel (2010). The relationship between the framework here and these models is discussed further below.

2 Environment

Output is a Cobb-Douglas aggregate of a continuum of tasks

$$Y = \exp\left[\int_0^1 \ln y(i)di\right], \tag{1}$$

and is produced competitively. Thus, all tasks are both q-complements and p-substitutes, with elasticity of substitution equal to one. The Cobb-Douglas function also implies that the expenditure share on each task is the same.

There are three factors of production, high, medium and low skilled workers. In addition, we will introduce capital (or technology embedded in capital) below. We first assume that there is a fixed supply of the three types of workers, L, M and H, supplied inelastically.

Each task i has the following production function

$$y(i) = A_L \cdot [\alpha_L(i)\, l(i)] + A_M \cdot [\alpha_M(i)\, m(i)] + A_H \cdot [\alpha_H(i)\, h(i)] + A_K \cdot [\alpha_K(i)\, k(i)], \qquad (2)$$

where A terms represent factor augmenting technology, $\alpha_L(i)$ is the productivity of low skill workers in task i and $l(i)$ is the number of low skill workers allocated to the task. The remaining terms are defined analogously. Note that the labor of each skill group is perfectly substitutable in production of each task. But skill groups differ in their efficiency at each task (e.g., $A_L a_L(i), A_M a_M(i), A_H a_H(i)$). Factor market clearing requires

$$\int_0^1 l(i)di \leq L, \int_0^1 h(i)di \leq H, \text{ and } \int_0^1 m(i)di \leq M. \qquad (3)$$

When we introduce capital, we will assume that it is available at some constant price r.

We impose the following assumption throughout:

Assumption 1 $\alpha_L(i)/\alpha_M(i)$ and $\alpha_M(i)/\alpha_H(i)$ are continuously differentiable and strictly decreasing.

This assumption implies that higher indices correspond to "more complex" tasks in which high skill workers are better than medium skill workers and medium skill workers are better than low skill workers. This is the standard *comparative advantage* assumption.

2.1 Equilibrium

An equilibrium is defined in the usual manner as an allocation in which (final good) producers maximize profits and labor markets clear. For now there is no labor supply decision on the part of the workers.

Let us first ignore capital (equivalently, $\alpha_K(i) \equiv 0 \forall i$). You might intuit that the optimal allocation of workers to tasks involves some I_L and I_H such that all tasks $i < I_L$ will be performed by low skill workers, and all tasks $j > I_H$ will be performed by high skill workers. Intermediate tasks will be performed by medium skilled workers. We can think of these as routine production, monitoring, clerical and administrative tasks. This intuition would be correct.

More formally, we have:

Lemma 1 *In any equilibrium there exist* I_L *and* I_H *such that* $0 < I_L < I_H < 1$ *and for any* $i < I_L$, $m(i) = h(i) = 0$, *for any* $i \in (I_L, I_H)$, $l(i) = h(i) = 0$, *and for any* $i > I_H$, $l(i) = m(i) = 0$.

The proof of this lemma follows a similar argument to that in Acemoglu and Zilibotti (2001), extended to environment in which there are three types of workers.

This lemma shows that the set of tasks will be partitioned into three (convex) sets, one performed by low skill workers, one performed by medium skill workers and one performed by high skill workers. Crucially, the boundaries of these sets, I_L and I_H, are endogenous and will respond to changes in skill supplies and technology. This introduces the first type of substitution that will play an important role in our model: *the substitution of skills across tasks*. Given the types of skills supplied in the market, firms decide (or, as an alternative interpretation, workers decide based on market wages) which tasks will be performed by which skill groups.

In the sections that follow, we will impose a series of constraints on the model to derive the equilibrium effects of changes to the supply or productivity of certain groups.

2.2 Allocation of labor among tasks within a skill group

Let $p(i)$ denote the price of task i. Throughout, we normalize the price of a final good to 1 (remember the ideal price index from first-year macro). Mathematically,

$$\exp\left[\int_0^1 \ln p(i) di\right] = 1.$$

This price index is important to what follows because it provides an appropriate numeraire for studying the real wages paid to each task.

2.3 Deriving labor allocated to each task

First, we will derive the amount of labor allocated to each task. In any equilibrium all tasks employing low-skill workers must pay them the same wage, w_L, all tasks employing medium skill workers must pay w_M, and all tasks employing high skill workers must also pay them the same wage, w_H. This is called the "Law of One Price for Skill". In view of (2) and Lemma 1, this implies:

$$w_L = p(i) A_L \alpha_L(i) \text{ for any } i < I_L.$$

$$w_M = p(i) A_M \alpha_M(i) \text{ for any } I_L < i < I_H.$$

$$w_H = p(i) A_H \alpha_H(i) \text{ for any } i > I_H.$$

Thus, workers of each skill level are indifferent among tasks that are exclusively performed by their own skill group. This has a convenient implication. For any $i, i' < I_L$, we must have

$$p(i)\alpha_L(i) = p(i')\alpha_L(i') \equiv P_L, \tag{4}$$

for any $I_H > i, i' > I_L$,

$$p(i)\alpha_M(i) = p(i')\alpha_M(i') \equiv P_M, \tag{5}$$

and for any $i, i' > I_H$,

$$p(i)\alpha_H(i) = p(i')\alpha_H(i') \equiv P_H. \tag{6}$$

Second, note that the Cobb-Douglas technology featuring an identical exponent for each input seen in equation (1) implies that the share of output paid to each task is identical:

$$\begin{aligned} p(i)y(i) &= p(i')y(i') \\ p(i)A_L a_L(i)\, l(i) &= p(i')A_L a_L(i')\, l(i'). \end{aligned}$$

Therefore, for any $i, i' < I_L$, these two conditions imply that $l(i) = l(i')$. This further implies that,

$$l(i) = \frac{L}{I_L} \text{ for any } i < I_L, \tag{7}$$

and with a similar argument,

$$m(i) = \frac{M}{I_H - I_L} \text{ for any } I_H > i > I_L. \tag{8}$$

$$h(i) = \frac{H}{1 - I_H} \text{ for any } i > I_H. \tag{9}$$

This establishes that any two tasks performed exclusively by workers of one skill group use identical amounts of labor, equal to the group's total labor supply divided by the fraction of the task continuum performed by the group. [In trying to map this model to the real world, we could always consider tasks to be finite connected sets along the continuum of i and alter the size of these sets to fit the Cobb-Douglas assumption. For this reason, these insights do not depend on the Cobb-Douglas form.]

2.4 No arbitrage condition

To derive equilibrium wage conditions, we will need to establish a no-arbitrage condition. Recall that the threshold task I_H must be such that it can be profitably produced using either high skilled or medium skilled workers. This is equivalent to task I_H having the same supply either when produced only with skilled or unskilled workers, that is, it implies the no-arbitrage condition

$$\frac{A_M \alpha_M (I_H) M}{I_H - I_L} = \frac{A_H \alpha_H (I_H) H}{1 - I_H}. \tag{10}$$

(Formally, this equation can be defined by taking the left and the right limits towards I_H). With an analogous argument, we also have

$$\frac{A_L \alpha_L (I_L) L}{I_L} = \frac{A_M \alpha_M (I_L) M}{I_H - I_L}. \tag{11}$$

Given that each skill group can perform each task, efficiency requires that there is no incentive in equilibrium for firms to alter the assignment of skills to tasks. This implies that from any two tasks produced with high *and* medium skill workers,

$$\frac{P_M A_M M}{I_H - I_L} = \frac{P_H A_H H}{1 - I_H},$$

Rearranging,

$$\frac{P_H}{P_M} = \left(\frac{A_H H}{1 - I_H}\right)^{-1} \left(\frac{A_M M}{I_H - I_L}\right). \tag{12}$$

Similarly,

$$\frac{P_M}{PL} = \left(\frac{A_M M}{I_H - I_L}\right)^{-1} \left(\frac{A_L L}{I_L}\right). \tag{13}$$

We'll be using these equations below.

2.5 Wage levels and wage ratios

Now we can use the definitions of the $P's$ above to obtain expressions for wages. We know that wages are simply the value marginal products of different types of skills. For low skill workers, for example, this is

$$w_L = P_L A_L. \tag{14}$$

More useful than the level of wages are the ratios, which inform us about the wage structure and inequality. For example, between high and medium skills, we have

$$\frac{w_H}{w_M} = \frac{P_H A_H}{P_M A_M}.$$

A more convenient way of expressing these is to use (12) and write the relative wage is simply in terms of relative supplies and the threshold sectors, I_L and I_H. That is,

$$\frac{w_H}{w_M} = \left(\frac{1-I_H}{I_H-I_L}\right)\left(\frac{H}{M}\right)^{-1}. \tag{15}$$

Similarly, for the wages medium relative to low skill workers, we have

$$\frac{w_M}{w_L} = \left(\frac{I_H-I_L}{I_L}\right)\left(\frac{M}{L}\right)^{-1}. \tag{16}$$

2.6 Equilibrium assignment of skills to tasks

The final step is to determine the equilibrium assignment of skills to tasks, given by I_L and I_H. The equations for skill prices P_L, P_m, and P_H together with the choice of the numeraire (as the ideal price index), $\int_0^1 \ln p(i)di = 0$, determine the equilibrium. In particular, using (4)-(6), we can write the last equilibrium condition as:

$$\int_0^{I_L} (\ln P_L - \ln \alpha_L(i))\, di + \int_{I_L}^{I_H} (\ln P_M - \ln \alpha_M(i))\, di + \int_{I_H}^{1} (\ln P_H - \ln \alpha_H(i))\, di = 0. \tag{17}$$

Proposition 2 *There exists a unique equilibrium summarized by* $(I_L, I_H, P_L, P_M, P_H, w_L, w_M, w_H)$ *given by equations (12), (13), (10), (11), (14), (15), (16) and (17).*

The only part of this proposition that requires proof is the uniqueness. This follows because in fact the equilibrium is considerably easier to characterize than it first appears because it has a block recursive structure. In particular, we can first use (10) and (11) to determine I_L and I_H. Given these we can then compute relative wages from (15) and (16). Finally, to compute wage and price levels, we can use (12), (13), (14) and (17). Figure 22 of the Handbook chapter shows a diagrammatic representation of the equilibrium, in which the curves corresponding to (10) and (11) determine I_L and I_H. Both curves are upward sloping in the (I_L, I_H) space, but the first one is everywhere steeper than (19)—see below for a proof. This establishes the existence of a unique intersection between the two curves in Figure 22, and thus there exists unique equilibrium values of I_L and I_H. Given these values, P_L, P_M, P_H, w_L, w_M and w_H are uniquely determined from (12), (13), (14), (15), (16) and (17).

2.7 Special Cases

Some special cases help clarify the workings of the model. Suppose first that there are no medium skill workers. Then we are back to a two factor world as in the canonical model. In addition, we could assume that instead of a continuum of tasks, there are only two tasks, one in which high skill workers have a strong comparative advantage and the other one in which low skill workers have a strong comparative advantage. This would be identical to the canonical model, except with Cobb-Douglas production function (elasticity of substitution between high and low skill workers equal to 1).

Another special case is the model studied by Acemoglu and Zilibotti. They also assume that there are only two types of workers, high and low skill, and impose the following functional form on the comparative advantage schedules.

$$a_L(i) = 1 - i \text{ and } a_H(i) = i.$$

Then an equivalent of (10) implies that all tasks below I will be performed by low skill workers and those above I will be performed by high skill workers. Moreover, I is given by

$$\frac{1-I}{I} = \left(\frac{A_H H}{A_L L}\right)^{1/2},$$

and thus

$$\frac{P_H}{P_L} = \left(\frac{A_H H}{A_L L}\right)^{-1/2},$$

and

$$\frac{w_H}{w_L} = \left(\frac{A_H}{A_L}\right)^{1/2}\left(\frac{H}{L}\right)^{-1/2}.$$

Thus, in this case the model is isomorphic to the canonical model with an elasticity of substitution equal to 2. This also shows that by choosing different forms for the comparative advantage schedules, one could obtain any elasticity of substitution or any constant returns to scale production function is a special case of the model shown here.

3 Basic Comparative Statics

Let us now return to obtain basic comparative statics from the general model. To do this, let us take logs in equations (10) and (11) to obtain

$$\ln A_M - \ln A_H + \beta_H(I_H) + \ln M - \ln H - \ln(I_H - I_L) + \ln(1 - I_H) = 0, \qquad (18)$$

and

$$\ln A_L - \ln A_M + \beta_L(I_L) + \ln L - \ln M + \ln(I_H - I_L) - \ln(I_L) = 0, \quad (19)$$

where $\beta_H(I) \equiv \ln \alpha_M(I) - \ln \alpha_H(I)$ and $\beta_L(I) \equiv \ln \alpha_L(I) - \ln \alpha_M(I)$ are both strictly decreasing in view of Assumption 1.[1] It can be easily verified that both of these curves are upward sloping in the (I_L, I_H) space, but (18) is everywhere steeper than (19) as claimed above. Therefore, there is a unique intersection between the two curves shown in Handbook Figure 22.

Basic comparative statics of the allocation of tasks across different skill groups can now be obtained from this figure. For example, an increase in H or A_H shifts (18) in, so both I_L and I_H decrease. This is intuitive, if either there are more high skill workers or high skill workers become uniformly more productive (because of "skill-biased technological change"), then they should perform more of the tasks. Thus the allocation of tasks endogenously shifts away from medium skill workers to high skill workers. However, if I_L remained constant this would imply that, from (8) there would be "excess" supply of medium skill workers in the remaining tasks. Therefore, the indirect effect of the increase in H or A_H would be to also reduce I_L, thus shifting some of tasks previously performed by low skill workers to medium skill workers. Similarly, we can analyze the implications of changes in the supply or the technologies complementing medium skill or low skill workers and changes in the comparative advantages schedules using this figure. The only comparative static that is slightly more involved is when M or A_M increases. In this case, the curve corresponding to both (18) and (19) shift up as shown in Handbook Figure 25. From the figure, the impact on I_L looks ambiguous, but it can be shown that I_L decreases and I_H increases, so that both tasks previously performed by low and high skill workers would now be performed by medium skilled workers. These comparative statics illustrate the substitution of skills across tasks.

It is harder to visually represent the changes in the wage structure resulting from changes in technology or supplies, because it depends on how I_L changes relative to I_H. Nevertheless, obtaining these comparative static results is also straightforward. To do this, let us consider a change in A_H and let us totally differentiate (18) and (19). We thus obtain:

$$\begin{pmatrix} \beta_H'(I_H) - \frac{1}{I_H - I_L} - \frac{1}{1 - I_H} & \frac{1}{I_H - I_L} \\ \frac{1}{I_H - I_L} & \beta_L'(I_L) - \frac{1}{I_H - I_L} - \frac{1}{I_L} \end{pmatrix} \begin{pmatrix} dI_H \\ dI_L \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} d\ln A_H.$$

It can be easily verified that all of the terms in the diagonals of the matrix on the left hand

[1] We make use of the fact that $P_M/P_H = p(I_H)\,\alpha_M(I_H)/p(I_H)\,\alpha_H(I_H)$ for the marginal task performed by H and M. Thus, $\ln(P_M/P_H) = \ln(\alpha_M(I_H)) - \ln(\alpha_H(I_H))$.

side are negative (again Assumption 1), and moreover, its determinant is positive, given by

$$\Delta = \left(\beta_H'(I_H) - \frac{1}{1-I_H}\right)\left(\beta_L'(I_L) - \frac{1}{I_L}\right).$$

Therefore,

$$\frac{dI_H}{d\ln A_H} = \frac{\beta_L'(I_L) - \frac{1}{I_H - I_L} - \frac{1}{I_L}}{\Delta} < 0 \text{ and } \frac{dI_L}{d\ln A_H} = \frac{-\frac{1}{I_H - I_L}}{\Delta} < 0,$$

as we obtain from the diagrammatic analysis. But in addition, we can also now see that

$$\frac{d(I_H - I_L)}{d\ln A_H} = \frac{\beta_L'(I_L) - \frac{1}{I_L}}{\Delta} < 0.$$

Using these expressions, we can obtain comparative statics for how various different relative wages change when there is high skill biased technological change. A similar exercise can be performed for low skill biased technological change and medium skill biased technological change. The next proposition summarizes the main results.

Proposition 3 *The following comparative static results apply:*

1. *(The response of task location to technology and skill supplies):*

$$\begin{aligned}
\frac{dI_H}{d\ln A_H} &= \frac{dI_H}{d\ln H} < 0,\ \frac{dI_L}{d\ln A_H} = \frac{dI_L}{d\ln H} < 0 \textit{ and } \frac{d(I_H - I_L)}{d\ln A_H} = \frac{d(I_H - I_L)}{d\ln H} < 0;\\
\frac{dI_H}{d\ln A_L} &= \frac{dI_H}{d\ln L} > 0,\ \frac{dI_L}{d\ln A_L} = \frac{dI_L}{d\ln L} > 0 \textit{ and } \frac{d(I_H - I_L)}{d\ln A_L} = \frac{d(I_H - I_L)}{d\ln L} < 0;\\
\frac{dI_H}{d\ln A_M} &= \frac{dI_H}{d\ln M} > 0,\ \frac{dI_L}{d\ln A_M} = \frac{dI_L}{d\ln M} < 0 \textit{ and } \frac{d(I_H - I_L)}{d\ln A_M} = \frac{d(I_H - I_L)}{d\ln M} > 0
\end{aligned}$$

2. *(The response of relative wages to skill supplies):*

$$\begin{aligned}
\frac{d\ln(w_H/w_L)}{d\ln H} &< 0,\ \frac{d\ln(w_H/w_M)}{d\ln M} > 0,\ \frac{d\ln(w_H/w_L)}{d\ln L} > 0,\\
\frac{d\ln(w_M/w_L)}{d\ln L} &> 0,\ \frac{d\ln(w_M/w_L)}{d\ln M} < 0.
\end{aligned}$$

3. *(The response of wages to factor augmenting technologies):*

$$\begin{aligned}
\frac{d\ln(w_H/w_L)}{d\ln A_H} &> 0,\ \frac{d\ln(w_H/w_M)}{d\ln A_H} > 0,\ \frac{d\ln(w_M/w_L)}{d\ln A_H} < 0; \\
\frac{d\ln(w_H/w_L)}{d\ln A_L} &< 0,\ \frac{d\ln(w_H/w_M)}{d\ln A_L} > 0,\ \frac{d\ln(w_M/w_L)}{d\ln A_L} < 0; \\
\frac{d\ln(w_H/w_M)}{d\ln A_M} &< 0,\ \frac{d\ln(w_M/w_L)}{d\ln A_M} > 0, \text{ and} \\
\frac{d\ln(w_H/w_L)}{d\ln A_M} &\lesseqqgtr 0 \text{ if and only if } \left|\beta_L'(I_L)\, I_L\right| \gtreqqless \left|\beta_H'(I_H)\,(1-I_H)\right|.
\end{aligned}$$

Part 1 of this proposition follow by straightforward differentiation and manipulation of the expressions in (18) and (19) for I_L and I_H. Parts 2 and 3 then follow readily from the expressions for relative wages in (15) and (16) using the behavior of these thresholds. Here we therefore just give the intuition for the main results (proof available upon request).

First, an increase in A_H or H expands the set of tasks performed by high skill workers and contracts the set of tasks performed by low and medium skill workers. This is equivalent to both I_L and I_H and decreasing. An increase in A_M or M similarly expands the set of tasks performed by medium skill workers and contracts those allocated to low and high skill workers. Mathematically, this corresponds to a decline in I_L and an increase in I_H. The implications of an increase in A_L or L are analogous, and raise both I_L and I_H, expanding the set of tasks performed by low skill workers. Finally, the implications of a change in A_M or M on $I_H - I_L$ are ambiguous for reasons we will encounter again below. When supply or productivity of medium skill workers increase, they will perform more tasks, taking over some of the tasks previously performed both by high and low skill workers. The impact on $I_H - I_L$ captures whether there will be greater expansion of the upper or lower margin. When $I_H - I_L$ increases, this implies that medium skill workers will be displacing high skilled workers more. When $I_H - I_L$ decreases, they will be displacing low skill workers more. In this light, the condition which determines whether the effect on $I_H - I_L$ is positive or negative is intuitive. When the absolute value of $\beta_L'(I_L)$ is high (relative to $\beta_H'(I_H)$), this implies that there is a strong comparative advantage for low skill workers for tasks below I_L. Consequently, medium skill workers will not be displacing low skill workers much. The converse happens when the absolute value of $\beta_L'(I_L)$ is low relative to the absolute value of $\beta_H'(I_H)$.

Second, the fact that relative demand curves are downward sloping for all factors, as claimed in Part 2, parallels the results in the canonical model (or in fact the more general results in Acemoglu, 2009, for any model with constant or diminishing returns at the aggregate level).

Third, an increase in A_H corresponds to "high skill biased technological change" and increases both w_H/w_L and w_H/w_M increase (high skill workers have higher relative wages both compared to medium skill and low skill workers). Perhaps more interestingly, an increase in A_H also reduces w_M/w_L despite the fact that it is contracting the set of tasks performed both by medium skill and low skill workers. Intuitively, the direct effect of an increase in A_H is on medium skill workers. The impact on low skill workers is indirect, resulting from the fact that medium skill workers become cheaper and this makes firms expand the set of tasks that these workers perform. This indirect effect never dominates the direct effect, and thus the wages of medium skill workers decreases relative to those of low skill workers when there is a high skill biased technological change. The implications of low skill biased technological change are analogous. This intuition also highlights why the implications of medium skill biased technological change are different. In this case there is a direct effect both on high skill and low skill workers. Therefore, the behavior of w_H/w_L is ambiguous. In particular, it depends on the exact form of the comparative advantage schedules. The condition in the proposition highlights that if $\beta_L'(I_L)$ is larger in absolute value relative to $\beta_H'(I_H)$, w_H/w_L is more likely to decline. Intuitively this again corresponds to the case in which low skill workers have strong comparative advantage for tasks below I_L (relative to the comparative advantage of high skill workers for tasks above I_H) and thus medium skill workers will expand more into high skill occupations than low skill occupations. In addition, I_L and $1 - I_H$ also matter, since the higher is I_L, the smaller will be the impact of displacement of low skill workers from some of the tasks they were previously performing (and vice versa for $1 - I_H$).

One attractive feature of the model, highlighted by the characterization results and the comparative statics in Proposition 3, is that all equilibrium objects depend on the set of tasks performed by the three different groups of workers. Depending on which set of tasks expands (contracts) more, wages of the relevant group increases (decreases). This is both useful intuition for understanding the workings of the model and also provides us with a tight connection between the model and the data.

4 Machines Replacing Tasks

An important advantage of this framework is that it can be used to investigate the implications of machines replacing some of the tasks previously performed by one of the set of workers. In general, we expect machines to replace tasks done by all three types of workers. However, given the salient patterns shown in the data, the most relevant set of tasks replaced by machines appear to be those previously performed by medium skill workers. For this reason, let us suppose that there now exists a range of tasks $[I', I''] \subset [I_L, I_H]$ for which $\alpha_K(i)$ increases sufficiently so that they will now be performed by machines. For all the remaining tasks, i.e., for all $i \notin [I', I'']$, we continue to assume that $\alpha_K(i) = 0$. What are the implications of this type of technological change for the supply of different types of tasks and for wages?

Our analysis directly applies to this case and implies that they will now be a new equilibrium characterized by thresholds $\hat{I}_L$ and $\hat{I}_H$. Moreover, we have the following proposition Lemma 1 and Proposition 2:

Proposition 4 *Suppose we start with an equilibrium characterized by thresholds* $[I_L, I_H]$ *and technological change implies that the tasks in the range* $[I', I''] \subset [I_L, I_H]$ *are now performed by machines. Then there exists new unique equilibrium characterized by new thresholds* $\hat{I}_L < I_L$ *and* $\hat{I}_H > I_H$ *such that* $0 < \hat{I}_L < I' < I'' < \hat{I}_H < 1$ *and for any* $i < \hat{I}_L$, $m(i) = h(i) = 0$ *and* $l(i) = L/\hat{I}_L$*; for any* $i \in (\hat{I}_L, I') \cup \left(I'', \hat{I}_H\right)$, $l(i) = h(i) = 0$ *and* $m(i) = M/\left(\hat{I}_H - I'' + I' - \hat{I}_L\right)$*; for any* $i \in (I', I'')$, $l(i) =$, $m(i) = h(i) = 0$*; and for any* $i > \hat{I}_H$, $l(i) = m(i) = 0$ *and* $h(i) = H/\left(1 - \hat{I}_H\right)$.

This proposition ishows that,as a consequence of machines replacing tasks previously performed by medium skill workers, there will be reallocation of tasks in the economy. In particular, medium skill workers will now start performing some of the tasks previously allocated to low skill workers, thus increasing the supply of these tasks (the same will happen at the top with an expansion of some of the high skill tasks). This proposition therefore gives us a way of thinking about how new technologies replacing routine, semi-skilled tasks will directly lead to the expansion of low skill tasks (corresponding to service occupations).

We next investigate the wage inequality implications of the introduction of these machines. For simplicity, we focus on the case where we start with $[I', I''] = \varnothing$ and then the set of tasks expands to an interval of size ε', where ε' is small. This is simply for expositional reasons and enables us to use differential calculus is we have done so far. None of the results depend on the set of tasks performed by machines being small.

Under the assumptions outlined here, and using the results in Proposition 4, we can write the equivalents of (18) and (19) as

$$\ln A_M - \ln A_H + \beta_H(I_H) + \ln M - \ln H - \ln(I_H - I_L - \varepsilon) + \ln(1 - I_H) = 0, \tag{20}$$

and

$$\ln A_L - \ln A_M + \beta_L(I_L) + \ln L - \ln M + \ln(I_H - I_L - \varepsilon) - \ln(I_L) = 0. \tag{21}$$

When $\varepsilon = 0$, these equations give the equilibrium before the introduction of machines replacing medium skill tasks, and when $\varepsilon = \varepsilon' > 0$, they describe the new equilibrium. The convenient feature here is that we can obtain the relevant comparative statics by using these two equations. In particular, the implications of the introduction of these new machines on the allocation of tasks is obtained from the following system:

$$\begin{pmatrix} \beta_H'(I_H) - \frac{1}{I_H-I_L} - \frac{1}{1-I_H} & \frac{1}{I_H-I_L} \\ \frac{1}{I_H-I_L} & \beta_L'(I_L) - \frac{1}{I_H-I_L} - \frac{1}{I_L} \end{pmatrix} \begin{pmatrix} dI_H \\ dI_L \end{pmatrix} = \begin{pmatrix} -\frac{1}{I_H-I_L} \\ \frac{1}{I_H-I_L} \end{pmatrix} d\varepsilon.$$

One can then to verify that

$$\begin{aligned} \frac{dI_H}{d\varepsilon} &= \frac{1}{I_H-I_L}\frac{-\beta_L'(I_L)+\frac{1}{I_L}}{\Delta} > 0, \\ \frac{dI_L}{d\varepsilon} &= \frac{1}{I_H-I_L}\frac{\beta_H'(I_H)+\frac{1}{1-I_H}}{\Delta} < 0, \\ \frac{d(I_H-I_L)}{d\varepsilon} &= \frac{1}{I_H-I_L}\frac{-\beta_L'(I_L)-\beta_H'(I_H)+\frac{1}{1-I_H}+\frac{1}{I_L}}{\Delta} > 0, \end{aligned}$$

where recall that Δ is the determinants of the matrix on the left hand side. These results confirm the statements in Proposition 4 concerning the set of tasks performed by low and high skill workers expanding.

Given these results on the allocation of tasks, we can also characterize the impact on relative wages. These are stated in the next proposition (these results are stated for general case rather than the case in which the range of tasks performed by machines is infinitesimal, since they can be generalized to this case in a straightforward manner; proof omitted).

Proposition 5 *Suppose we start with an equilibrium characterized by thresholds* $[I_L, I_H]$ *and technological change implies that the tasks in the range* $[I', I''] \subset [I_L, I_H]$ *are now performed by machines. Then:*

1. w_H/w_M *increases;*

2. w_M/w_L *decreases;*

3. w_H/w_L *increases if* $|\beta_L'(I_L)\, I_L| < |\beta_H'(I_H)(1-I_H)|$ *and* w_H/w_L *decreases if* $|\beta_L'(I_L)\, I_L| > |\beta_H'(I_H)(1-I_H)|$.

The first two parts of the proposition are intuitive. Because new machines replace the tasks previously performed by medium skill workers, their relative wages both compared to high skill and low skill workers decline. In terms of the data this corresponds to the wages of workers in the middle of the income distribution, previously performing relatively routine tasks, falling compared to those at the top and the bottom of the wage distribution. The impact of this type of technological change for the wage of high skill workers relative to low skill workers is ambiguous. It depends on whether medium skill workers displaced by machines are better substitutes for low or high skill workers. The condition $|\beta_L'(I_L) I_L| > |\beta_H'(I_H)(1 - I_H)|$ is the same as the condition we encountered in Proposition 4, and the intuition is similar. This condition implies that medium skill workers are better substitute for high skill workers, in the sense that around I_L, there is a stronger comparative advantage of low skill workers for the tasks that they are performing compared to the comparative advantage of high skill workers around I_H (these explain the terms $\beta_L'(I_L)$ and $\beta_H'(I_H)$; the terms I_L and $(1 - I_H)$ also have a similar intuition; if previously there are many tasks performed by high skill workers relative to low skill workers, a few of those going to medium skill workers will have less of an effect on there wages than those of low skill workers). The extent that in practice we think that medium skill (semi-skilled workers previously performing the routine tasks) are better substitutes for low skill workers employed in service occupations, Part 3 of this proposition implies that we should also see an increase in w_H/w_L. However, if sufficiently many of these workers displaced by machines go into high skill occupations, w_H/w_L may also increase. This latter case would correspond to one in which, in relative terms, low skill workers are the main beneficiaries of the introduction of new machines into the production process.

5 Endogenous Supply of Skills [This Section is for Optional Self-Study]

We have so far focused on one type of substitution, which we referred to as substitution of skills across tasks. A complementary force is *substitution of workers across different types of skills*, meaning that in response to changes in technology or factor supplies, workers change the types of skills they supply to the market. To allow for this type of substitution and also to generate richer types of wage inequality patterns, we now assume that each worker j is endowed with some amount of "low skill," some amount of "medium skill" and some amount of "high skill," respectively l^j, m^j and h^j. We can think of this as the worker having one unit of time and the "skill transformation" constraint being

$$t_l^j + t_m^j + t_h^j \leq 1.$$

The worker's income is

$$w_L t_l^j l^j + w_M t_m^j m^j + w_H t_h^j h^j,$$

which captures the fact that the worker with skill vector (l^j, m^j, h^j) will have to allocate his time between jobs requiring different types of skills. Generically, we will see that each worker will prefer to allocate his or her time entirely to one type of task.

The production side of the economy is identical to the framework we have analyzed so far. Our analysis then applies once we know the aggregate amount of skills of different types. Let us denote these by

$$L = \int_{j \in E_l} l^j dj,\ M = \int_{j \in E_m} m^j dj, \text{ and } H = \int_{j \in E_h} h^j dj,$$

where E_l, E_m and E_h are the sets of workers choosing to supply their low, medium and high skills respectively.

Clearly, the worker will choose to be in the set E_h only if

$$\frac{l^j}{h^j} \leq \frac{w_H}{w_L} \text{ and } \frac{m^j}{h^j} \leq \frac{w_H}{w_M}.$$

There are similar inequality is determining when a worker will be in the sets E_m and E_l. To keep the model tractable, we now impose a type of *single-crossing* assumption. We order workers over the interval $(0, 1)$ (with the total mass of workers normalized 1 without loss of any generality). Then:

Assumption 2 h^j/m^j and m^j/l^j are both strictly decreasing in j and $\lim_{j \to 0} h^j/m^j = \infty$ and $\lim_{j \to 1} m^j/l^j = 1$.

This assumption implies that lower index workers have a comparative advantage in high skill tasks and higher index workers have a comparative advantage in low skill tasks. Moreover, at the extremes these comparative advantages are strong enough that there will always be some workers choosing to supply high and low skills. An immediate implication is the following lemma:

Lemma 6 *For any ratios of wages w_H/w_M and w_M/w_L, there exist $J^*(w_H/w_M)$ and $J^{**}(w_M/w_L)$ such that $t_h^j = 1$ for all $j < J^*(w_H/w_M)$, $t_m^j = 1$ for all $j \in (J^*(w_H/w_M), J^{**}(w_M/w_L))$ and $t_l^j = 1$ for all $j > J^{**}(w_M/w_L)$. $J^*(w_H/w_M)$ and $J^{**}(w_M/w_L)$ are both strictly increasing in their arguments.*

Clearly, $J^*(w_H/w_M)$ and $J^{**}(w_M/w_L)$ are defined such that

$$\frac{m^{J^*(w_H/w_M)}}{h^{J^*(w_H/w_M)}} = \frac{w_H}{w_M} \text{ and } \frac{l^{J^{**}(w_M/w_L)}}{m^{J^{**}(w_M/w_L)}} = \frac{w_M}{w_L}. \tag{22}$$

In light of this lemma, we can write

$$H = \int_0^{J^*(w_H/w_M)} h^j dj, \; M = \int_{J^*(w_H/w_M)}^{J^{**}(w_M/w_L)} m^j dj \text{ and } L = \int_{J^{**}(w_M/w_L)}^{1} l^j dj. \tag{23}$$

We can next prove:

Proposition 7 *In the model with endogenous supplies, there exists a unique equilibrium summarized by*
$(I_L, I_H, P_L, P_M, P_H, w_L, w_M, w_H, J^*(w_H/w_M), J^{**}(w_M/w_L)\, L, M, H)$ *given by equations (12), (13), (10), (11), (14), (15), (16), (17), (22) and (23).*

The main difference from the analysis so far is that when there is factor augmenting technological change or introduction of new machines, the induced changes in wages will also affect supplies. Consequently, there will be substitution of workers across different types of skills. In particular, when new machines replace medium skill workers in a set of tasks, this will induce some of the workers that were previously supplying medium skills to now supply low or high skills. In particular, if the more elastic margin is the one between medium and low skills, we would expect a significant fraction of the workers previously supplying medium skills, and working in middle tasks to now supply low skills and start performing relatively low-ranked tasks. This type of substitution therefore complements the substitution of skills across tasks. Finally, with the distribution of effective supplies across workers, this model also generates a richer distribution of earnings inequality (instead of the three point distribution generated by the baseline model).

6 Interpretation

We can try to interpret the changes in the US wage and employment structures over the last several decades through the lenses of this framework. Throughout, we take the comparative advantage schedules as given, and consider what combinations of factor augmenting technological changes, introduction of new machines replacing tasks previously performed by different types of workers, and supply changes would be necessary to explain the patterns we observe. As we have seen, during the 1980s the US labor market experienced declining wages at the bottom of the distribution together with a relative contraction in employment in low-wage occupations (in particular service occupations), and also increasing wages at the top together with a relative expansion of employment in the highest-paid occupations. In terms of our model, this would be a consequence of an increase in A_H/A_M and A_M/A_L, which is the analog of skill-biased technological change in this three factor model. However, starting in the 1990s, the pattern is different. We see them more U-shaped behavior both in employment shares and in wage percentiles. In terms of our model, this would be an implication of the spread of computers and robotics to more sectors and occupations in the economy, replacing middling tasks, particularly those with relatively heavy routine components. As we have seen, this will depress both the wages medium skill workers and employment in tasks that were previously performed by these medium skill workers. There is an additional contrast between the 1990s and 2000s, which is that there is a larger relative increase in employment and low wage service-type occupations during the 2000s. In terms of our model, this could be an implication of more displacement of medium skilled workers, thus again substitution of skills across tasks. In addition, this pattern would be strengthened if there is also substitution of workers across different types of skills, so that some of the workers previously supplying routine medium skills, now switch to supplying low skill and manual labor. This would complement the process via which firms would be substituting given bundle of skills to different tasks.

7 Generalization

The framework presented here can be further generalized by considering more types of skills. In the limit, as well as a continuum of tasks, we could have a continuum of skills. The resulting model would be similar to the assignment models that have been studied in different forms in the labor literature over the past 50 years. A general version of such a model is considered by Costinot and Vogel (2010). Under a comparative advantage (log supermodularity) assumption, which generalizes our comparative advantage assumption here, Costinot and Vogel characterize the equilibrium in terms of two ordinary differential equations, one determining the match between skills and tasks and the other one determining the wage as a function of assignment. They show that a variety of changes in the patterns of comparative advantage will lead to unambiguous comparative static results. The framework of Costinot and Vogel can thus also be used to study similar issues.However, like other assignment models, to get sharp predictions one would need to impose additional structure on the pattern of comparative advantage.