Intrapersonal Level-$k$ Thinking

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MIT Theory Lunch

November 22, 2016
Motivation

- Intertemporal consumption-Euler equation has strong implications for the path of consumption and its sensitivity to income and interest rate.

- Empirical violations of intertemporal Euler equation are prevalent:
  - excess smoothness
  - excess sensitivity
  - drop in consumption at retirement
  - forward guidance
  - …
This Paper

- a behavioral approach to dynamic decision problems
- The decision-maker acts as if he is playing a game with his future selves and believes his future selves to be boundedly rational.
- level-\(k\) thinking as the model of bounded rationality
- application to a standard consumption-saving problem
Review of Level-\(k\) Thinking

- beauty contest game
  - \(n\) players
  - Each player picks a number between 0 and 9
  - The player wins who picks the number closest to \(\frac{2}{3} \times \text{average}\).

- level-0
  - follow a "default" strategy
    - What would the player do without thinking?
    - Pick a number uniformly at random!

- level-1
  - act as if everyone else is level-0
    - choose 3

- level-2
  - act as if everyone else is level-1
    - choose 2

- ...
an agent lives for $T + 1$ periods

utility of period $t$ “self”

$$u(c_t) + u(c_{t+1}) + \cdots + u(c_{T+1})$$

gets one unit every day except for the last day when he gets nothing
Intrapersonal Level-$k$ Thinking: An Example

- **level-0 time-$t$ self:**
  - follow some default strategy
  - consume everything you have: $c_t = a_t + y_t$

- **level-1 time-$t$ self:**
  - act as if all future selves are level-0
  - consume everything you have if $t < T$: $c_t = a_t + y_t$
  - consume $\frac{1}{2}$ of everything you have if $t = T$: $c_T = \frac{1}{2}(a_T + y_T + y_{T+1})$

<table>
<thead>
<tr>
<th>$t$</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>$T - 1$</th>
<th>$T$</th>
<th>$T + 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_t^{(0)}$</td>
<td>1</td>
<td>1</td>
<td>...</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$c_t^{(1)}$</td>
<td>1</td>
<td>1</td>
<td>...</td>
<td>(\frac{1}{2})</td>
<td>(\frac{1}{2})</td>
<td>(\frac{1}{2})</td>
</tr>
<tr>
<td>$c_t^{(2)}$</td>
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<td>1</td>
<td>...</td>
<td>(\frac{2}{3})</td>
<td>(\frac{2}{3})</td>
<td>(\frac{2}{3})</td>
</tr>
<tr>
<td>$c_t^*$</td>
<td>$T$</td>
<td>$T$</td>
<td>...</td>
<td>$T$</td>
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- Time-1 consumption of a level-$k$ agent is *higher* and *less sensitive* to $y_{T+1}$ than for the rational agent.
Consumption-Saving with Intrapersonal Level-\(k\) Thinkers

- utility of the period-\(t\) self

\[
\sum_{s=0}^{\infty} \beta^s u(c_{t+s}).
\]

- isoelastic utility

\[
u(c) = \begin{cases} 
    c^{1-\frac{1}{\sigma}} - 1 & \text{if } \sigma \neq 1, \\
    \log(c) & \text{if } \sigma = 1
\end{cases}
\]

- asset \(a_t\) with gross rate of return \(R_{t+1}\)

- forward-looking consumption function for period-\(t\) self

\[
C_t (a_t, \{y_s\}_{s \geq t}, \{R_{s+1}\}_{s \geq t}).
\]

- time profile of consumption function \(C = \{C_t\}_{t \geq 0}\)

- \(C\): space of all forward-looking consumption function profiles
the continuation payoff according to a level- \( k + 1 \) thinker

\[
V_t^{(k)} (a_t, \{y_s\}_{s \geq t}, \{R_{s+1}\}_{s \geq t}) = u \left( C_t^{(k)} (a_t, \{y_s\}_{s \geq t}, \{R_{s+1}\}_{s \geq t}) \right)
+ \beta V_{t+1}^{(k)} \left( R_t + y_t - C_t^{(k)} (a_t, \{y_s\}_{s \geq t}, \{R_{s+1}\}_{s \geq t}), \{y_s\}_{s \geq t+1}, \{R_{s+1}\}_{s \geq t+1} \right)
\]

A level- \( k + 1 \) period- \( t \) self of the consumer solves

\[
C_t^{(k+1)} (a_t, \{y_s\}_{s \geq t}, \{R_{s+1}\}_{s \geq t}) = \arg \max_c u(c)
+ \beta V_{t+1}^{(k)} \left( R_t + y_t - c, \{y_s\}_{s \geq t+1}, \{R_{s+1}\}_{s \geq t+1} \right)
\]

The above equations define a mapping \( f : C \rightarrow C \).
The behavior of intrapersonal level-$k$ thinkers is determined by two objects:

1. default consumption function profile $C^{(0)} = \{ C_{t}^{(0)} \}_{t \geq 0} \in C$
2. mapping $f : C \rightarrow C$

These define the consumption function profiles of level-$k$ thinkers as:

$$
\begin{align*}
C^{(0)} \xrightarrow{f} C^{(1)} \xrightarrow{f} C^{(2)} \xrightarrow{f} \cdots
\end{align*}
$$
Aside: The Default Strategy

- How to pick the default strategy?
- Let $S$ be the set of state variables.
- Consider a baseline environment with state variables belonging to $\hat{S} \subseteq S$.
- It's reasonable to demand that
  \[ C_t^{(0)}(s) = C_t^*(s) \quad \forall s \in \hat{S}. \]
- Experience may lead the agent to learn the optimal strategy over $\hat{S}$.
Given $C^{(0)}$, the level-$k$ consumption function profile $C^{(k)}$ is what is obtained by $k$ rounds of policy function iteration starting from initial guess $C^{(0)}$.

As $k$ goes to infinity, $C^{(k)}$ converges to the optimal consumption function profile $C^*$. 
Linear Consumption Functions

restrict attention to the following class of consumption functions:

\[ \mathcal{L} = \left\{ C \in \mathcal{C} : C_t(a_t, \{y_s\}_{s \geq t}, \{R_{s+1}\}_{s \geq t}) = \gamma_t(\{R_{s+1}\}_{s \geq t}) \left( a_t + \sum_{s=0}^{\infty} \frac{y_{t+s}}{R_{t+s}} \right) \right\} \]

Observation
Suppose that the consumer has CRRA preferences. Then \( C^* \in \mathcal{L} \).

Proposition
Suppose that the consumer has CRRA preferences. Then \( f(\mathcal{L}) \subseteq \mathcal{L} \) for all \( R \).
Generalized Euler Equation

- the FOC of a level-$k+1$ thinker:

\[
u'(C_t^{(k+1)}) = \beta R_t^{t+1} \gamma_{t+1}^{(k)} u' \left( \gamma_{t+1}^{(k)} R_t^{t+1} (w_t - C_t^{(k+1)}) \right)
+ \beta^2 R_t^{t+2} \gamma_{t+2}^{(k)} (1 - \gamma_t^{(k)}) u' \left( \gamma_{t+2}^{(k)} (1 - \gamma_{t+1}^{(k)}) R_t^{t+2} (w_t - C_t^{(k+1)}) \right)
+ \ldots
\]

- the FOC given CRRA utility:

\[
\gamma_t^{(k+1)} = \frac{\left[ \sum_{s=1}^{\infty} \beta^s \left( R_t^{t+s} \gamma_t^{(k)} \prod_{\tau=1}^{s-1} \left( 1 - \gamma_t^{(k)} \right) \right)^{1 - \frac{1}{\sigma}} \right]^{-\sigma}}{1 + \left[ \sum_{s=1}^{\infty} \beta^s \left( R_t^{t+s} \gamma_t^{(k)} \prod_{\tau=1}^{s-1} \left( 1 - \gamma_t^{(k)} \right) \right)^{1 - \frac{1}{\sigma}} \right]^{-\sigma}}.
\]

\[
\{ \gamma_t^{(k)} \}_{t \geq 0} \mapsto \{ \gamma_t^{(k+1)} \}_{t \geq 0}
\]
Main Result

**Proposition**

Suppose that the consumer's utility function is CRRA with IES $\sigma$.

1. If $C^{(0)} = C^*$ or $\sigma = 1$, then $C^{(k)} = C^*$ for all $k \geq 1$.
2. If $C^{(0)} \neq C^*$ and $\sigma > 1$, then $C_t^{(k)}(\cdot) > C_t^*(\cdot)$ for all $k \geq 1$ and all $t$.
3. If $C^{(0)} \neq C^*$ and $\sigma < 1$, then $C_t^{(k)}(\cdot) < C_t^*(\cdot)$ for all $k \geq 1$ and all $t$. 
Future selves are expected not to consume their inheritance optimally. This lowers the effective rate of return of saving for the future. This has *income* and *substitution* effects. IES determines which effect is dominant.
Marginal Propensity to Consume out of Permanent Income

Corollary

1. If $\sigma > 1$ and the default strategy is not optimal, then

$$\frac{dC_t^{(k)}}{dy_{t+s}} > \frac{dC_t^*}{dy_{t+s}}$$

2. If $\sigma < 1$ and the default strategy is not optimal, then

$$\frac{dC_t^{(k)}}{dy_{t+s}} < \frac{dC_t^*}{dy_{t+s}}$$
Proof of the Main Result, Part 1

- when $\sigma = 1$

$$C_t^{(k+1)}(a_t, \{y_s\}_{s \geq t}, \{R_{s+1}\}_{s \geq t}) = (1 - \beta) \left( a_t + \sum_{s=0}^{\infty} \frac{y_{t+s}}{R_t^{t+s}} \right)$$

$$= C_t^* (a_t, \{y_s\}_{s \geq t}, \{R_{s+1}\}_{s \geq t}).$$

- when $C^{(0)} = C^*$, by principle of optimality

$$C^{(1)} = f(C^{(0)}) = f(C^*) = C^*$$

- so by induction

$$C^{(k+1)} = f(C^{(k)}) = C^*$$
Proof of the Main Result, Part 2

▶ define

\[
\chi \left( \{ \gamma_{s+1}^{(k)} \}_{s \geq t}, \{ R_{s+1} \}_{s \geq t} \right) = \sum_{s=1}^{\infty} \beta^s \left( R_t^{s+1} \gamma_{t+s}^{(k)} \prod_{\tau=1}^{s-1} \left( 1 - \gamma_{t+\tau}^{(k)} \right) \right)^{1 - \frac{1}{\sigma}}
\]

▶ then the level-\(k+1\) MPC is given by

\[
\gamma_t^{(k+1)} = \frac{\chi \left( \{ \gamma_{s+1}^{(k)} \}_{s \geq t}, \{ R_{s+1} \}_{s \geq t} \right)^{-\sigma}}{1 + \chi \left( \{ \gamma_{s+1}^{(k)} \}_{s \geq t}, \{ R_{s+1} \}_{s \geq t} \right)^{-\sigma}}
\]

▶ need to show

\[
\chi \left( \{ \gamma_{s+1} \}_{s \geq t}, \{ R_{s+1} \}_{s \geq t} \right) \geq \chi \left( \{ \gamma_{s+1}^* \}_{s \geq t}, \{ R_{s+1} \}_{s \geq t} \right)
\]

▶ with inequality strict whenever \(\{ \gamma_t \}_{t \geq 0} \neq \{ \gamma_t^* \}_{t \geq 0}\)
Proof of the Main Result, Part 2

- $w_t$ permanent income
- $V (w_t, \{\gamma_{s+1}\}_{s \geq t}, \{R_{s+1}\}_{s \geq t})$ life-time utility

$$
\chi \left( \{\gamma_{s+1}\}_{s \geq t}, \{R_{s+1}\}_{s \geq t} \right) = \beta \left( \frac{R_{t+1}}{w_{t+1}} \right)^{1-\frac{1}{\sigma}} \left[ \left( 1 - \frac{1}{\sigma} \right) V (w_t, \{\gamma_{s+1}\}_{s \geq t}, \{R_{s+1}\}_{s \geq t}) + \frac{1}{1 - \beta} \right]
$$

- $\{\gamma_{s+1}\}_{s \geq t}$ is the unique maximizer of $V (w_t, \cdot, \{R_{s+1}\}_{s \geq t})$
Suppose the level-0 MPC is independent of the interest rates:

\[
\frac{\partial \gamma_t^{(0)}}{\partial R_{t+s}} = 0
\]

What is the level-1 interest rate elasticity of consumption?

\[
\epsilon_{0,t}^{(1)} \equiv \frac{\partial C_0^{(1)}}{\partial R_t} \frac{R_t}{C_0^{(1)}} \bigg|_{R_s=R=\beta^{-1}, y_s=y, a_0=0, \gamma_t^{(0)}=\gamma}
\]
Proposition

Suppose that the consumer’s utility function is isoelastic with IES $\sigma$.

1. If $\gamma = \gamma^*$ or $\sigma = 1$, then $\epsilon_{0,t}^{(1)} = \epsilon_{0,t}^*$.
2. If $\sigma > 1$ and $\gamma < \gamma^*$, then $\epsilon_{0,t}^{(1)} < \epsilon_{0,t}^*$.
3. If $\sigma < 1$ and $\gamma > \gamma^*$, then $\epsilon_{0,t}^{(1)} < \epsilon_{0,t}^*$. 
thank you!