Online Appendix for “Adverse Selection with Ex-Post Signals”

Daniel Clark∗
March 18, 2020

OA.1 Omitted Calculations from Section 3

OA.1.1 Subsection 3.1

Consider a mechanism $e(\cdot)$, $t(\cdot)$ that is both individually rational and incentive compatible for fixed $p \in [0, 1]$. The incentive compatibility constraint in the problem given by Equation 1, along with the integral envelope theorem (see Milgrom and Segal [2002]), applied for fixed $p \in [0, 1]$, implies that

$$t(c, p) - \frac{c}{2} e(c, p)^2 = t(2, p) - e(2, p)^2 + \int_c^2 \frac{1}{2} e(\tilde{c}, p)^2 d\tilde{c}$$

holds for all $(c, p) \in [1, 2] \times [0, 1]$. Rearranging this, we find that

$$t(c, p) = t(2, p) - e(2, p)^2 + \int_c^2 \frac{1}{2} e(\tilde{c}, p)^2 d\tilde{c} + \frac{c}{2} e(c, p)^2$$

(OA 1)

holds for all $(c, p) \in [1, 2] \times [0, 1]$. From this, we obtain

$$\mathbb{E}_{(c, p)}[(y + bp)e(c, p) - t(c, p)] = \int_0^1 \int_1^2 \left[ (y + bp)e(c, p) - t(2, p) + e(2, p)^2 \right.
\left. - \int_c^2 \frac{1}{2} e(\tilde{c}, p)^2 d\tilde{c} - \frac{c}{2} e(c, p)^2 \right] dcdp.$$
From integration by parts, it follows that
\[ \int_{1}^{2} \int_{c}^{2} \frac{1}{2} e(\bar{c}, p)^2 d\bar{c} dc = \int_{1}^{2} \frac{1}{2} (c - 1) e(c, p)^2 dc. \]

Combining these two equations, we find that
\[ E_{(c, p)}[(y+bp)e(c, p) - t(c, p)] = \int_{0}^{1} \int_{1}^{2} \left[ -(t(2, p) - e(2, p)^2) + (y+bp)e(c, p) - \left( \frac{2c - 1}{2} \right) e(c, p)^2 \right] dcdp. \]

Performing point-wise maximization on the integrand in the above equation with respect to \( e(c, p) \), we obtain
\[ e^*(c, p) = \frac{y + bp}{2c - 1} \] (OA 2)
as a candidate for the optimal allocation rule. Note that the individual rationality constraint for the agent with type \((c, p) = (2, p)\) translates into \( t(2, p) - e(2, p)^2 \geq 0 \). The fact that this constraint must bind for the optimal mechanism, along with Equations OA 1 and OA 2, leads to
\[ t^*(c, p) = \frac{c}{2} \left( \frac{y + bp}{2c - 1} \right)^2 + \frac{(y + bp)^2}{4} \left( \frac{1}{2c - 1} - \frac{1}{3} \right) \] (OA 3)
as a candidate for the optimal transfer rule. Note that the mechanism given by the allocation rule in Equation OA 2 and the transfer rule in Equation OA 3 has the corresponding positive and convex information rent function
\[ V^*(c, p) = \frac{(y + bp)^2}{4} \left( \frac{1}{2c - 1} - \frac{1}{3} \right), \]
which satisfies \(-1/2)e^*(c, p)^2 = (\partial V^*/\partial c)(c, p)\) for all \((c, p) \in [1, 2] \times [0, 1]\). By Corollary 1 of Clark [2020], we conclude that it is individually rational and incentive compatible for fixed \( p \in [0, 1] \). We therefore conclude that the mechanism given in Equation 2 is indeed the optimal mechanism that is both individually rational and incentive compatible for fixed \( p \in [0, 1] \).
OA.1.2 Subsection 3.2

We verify that
\[ V^*(c, p) = \frac{(y + bp)^2}{4} \left( \frac{1}{2c - 1} - \frac{1}{3} \right) \]
is not a convex function of \((c, p) \in [1, 2] \times [0, 1]\). This follows because
\[
\frac{\partial^2 V^*}{\partial c^2}(c, p) \frac{\partial^2 V^*}{\partial p^2}(c, p) - \left( \frac{\partial^2 V^*}{\partial c \partial p}(c, p) \right)^2 = -\frac{b^2(y + bp)^2}{3(2c - 1)^3}
\]
for all \((c, p) \in [1, 2] \times [0, 1]\).

OA.1.3 Verification that Relaxed Incentive Compatibility Constraints are Satisfied

Here we show that the mechanism given by the effort rule \(e^*(c, p)\) and the transfer rules, \(t^*_H(c, p)\) and \(t^*_L(c, p)\), from Equation 7 satisfy the constraints in Equation 6. By construction, \(pt^*_H(c, p) + (1 - p)t^*_L(c, p) = t^*(c, p)\) for all \((c, p) \in [1, 2] \times [0, 1]\). Since
\[
c \in \arg \max_{c \in [1, 2]} t^*(\hat{c}, p) - \frac{c}{2} e^*(\hat{c}, p)^2 \text{ for all } (c, p) \in [1, 2] \times [0, 1],
\]
it follows that the condition ensuring the agent truthfully reports \(c\) when they report \(p\) truthfully is satisfied. Now we show that the condition ensuring the agent truthfully reports \(p\) when they report \(c\) truthfully is also satisfied. Note that
\[
pt^*_H(c, \hat{p}) + (1 - p)t^*_L(c, \hat{p}) - \frac{c}{2} e^*(c, \hat{p})^2 = \left( \frac{(y + bp)(y + b\hat{p})}{2} - \frac{(y + b\hat{p})^2}{4} \right) \left( \frac{1}{2c - 1} - \frac{1}{3} \right)
\]
for all \(c \in [1, 2]\) and \(p, \hat{p} \in [0, 1]\). Straightforward calculations show that
\[
p \in \arg \max_{\hat{p} \in [0, 1]} \frac{(y + bp)(y + b\hat{p})}{2} - \frac{(y + b\hat{p})^2}{4} \text{ for all } p \in [0, 1].
\]
OA.2 The No Information Benchmark

We consider a benchmark adverse selection problem in which neither the agent nor the principal know the value of $\phi$.

A mechanism in this problem is a collection of two functions, $x : \Theta \rightarrow X$ and $t : \Theta \rightarrow \mathbb{R}$, where $x(\theta)$ is the allocation when the agent reports $\theta$ and $t(\theta)$ is the transfer from the principal to the agent when the agent reports $\theta$. The principal’s optimal mechanism solves the problem given by

$$
\max_{x(\cdot), t(\cdot)} \mathbb{E}_{(\theta, \phi)} [u(\theta, \phi, x(\theta)) - t(\theta)] \\
\text{subject to} \\
\text{IR: } \theta x(\theta) + t(\theta) \geq 0 \text{ for all } \theta \in \Theta, \\
\text{IC: } \theta \in \arg \max_{\theta \in \Theta} \theta x(\theta) + t(\theta) \text{ for all } \theta \in \Theta.
$$

(1)

**Definition 1.** The no information benchmark is the problem given by Equation 1.

Just as the public information benchmark provides an upper bound on the payoff the principal can achieve, the no information benchmark provides a corresponding lower bound. The reason for this is that the principal can design a mechanism that has the same allocation rule as that of the solution to the no information benchmark and such that the $n$ transfer rules all equal the transfer rule of the solution to the no information benchmark.

Moreover, when the probability distribution on $\Theta \times \Phi$ is atomless and there is no informative ex-post signal, not only does the payoff the principal obtains in the no information benchmark provide a lower bound on the principal’s payoff, but these payoffs are in fact equal.

**Claim 1.** Suppose $F$ is an atomless distribution on $\Theta \times \Phi$.\(^1\) Then the principal’s payoff

\(^1\)The assumption that $F$ is an atomless distribution on $\Theta \times \Phi$ is essential for the conclusion of
in the no information benchmark equals the principal’s payoff in the full problem given by Equation 9.

The intuition behind Claim 1 is that, when there is no informative ex-post signal, the agent’s true value of \( \phi \) does not impact their utility. Hence, the principal must rely on the agent voluntarily revealing \( \phi \) without being able to incentivize the agent to report \( \phi \) truthfully. One consequence of this is that all agents with the same value of \( \theta \) must receive the same payoff from the mechanism. For this to be the case, the mechanism can result in a different allocation or transfer for two agent types with the same value of \( \theta \) only for values of \( \theta \) belonging to a countable subset of \( \Theta \). Therefore, when the probability distribution on \( \Theta \times \Phi \) is atomless, to every mechanism that satisfies the IR and IC constraints, there is a corresponding mechanism with the same payoff to the principal that depends only on the reported value of \( \theta \) and also satisfies the IR and IC constraints.

**Proof.** Consider an individually rational, incentive compatible mechanism \( x : \Theta \times \Phi \rightarrow X, t : \Theta \times \Phi \rightarrow \mathbb{R} \). We will show that there exists an individually rational, incentive compatible mechanism \( \tilde{x} : \Theta \times \Phi \rightarrow X, \tilde{t} : \Theta \times \Phi \rightarrow \mathbb{R} \) such that \( \tilde{x}(\theta, \phi_1) = \tilde{x}(\theta, \phi_2) \) and \( \tilde{t}(\theta, \phi_1) = \tilde{t}(\theta, \phi_2) \) for all \( \theta \in \Theta \) and \( \phi_1, \phi_2 \in \Phi \), and \( E_{(\theta,\phi)}[u(\theta, \phi, x(\theta, \phi)) - t(\theta, \phi)] = E_{(\theta,\phi)}[u(\theta, \phi, \tilde{x}(\theta, \phi)) - \tilde{t}(\theta, \phi)] \).

Incentive compatibility of the mechanism \( x(\cdot), t(\cdot) \) implies that

\[
(\theta, \phi) \in \arg \max_{(\hat{\theta}, \hat{\phi}) \in \Theta \times \Phi} \theta x(\hat{\theta}, \hat{\phi}) + t(\hat{\theta}, \hat{\phi})
\]

for all \( (\theta, \phi) \in \Theta \times \Phi \). Because there is increasing differences between \( \theta \) and \( x \), this implies that for all \( \theta_1, \theta_2 \in \Theta \) such that \( \theta_1 > \theta_2 \), \( x(\theta_1, \phi_1) \geq x(\theta_2, \phi_2) \) for all \( \phi_1, \phi_2 \in \Phi \). Therefore, there exists a subset \( \Theta' \subset \Theta \) containing all but a countable subset of \( \Theta \)

Claim 1. There are counterexamples with discrete distributions on \( \Theta \times \Phi \) in which the principal can utilize the reported value of \( \phi \) to attain a strictly higher payoff than in the no information benchmark.
such that, for all $\theta \in \Theta$, $x(\theta, \phi_1) = x(\theta, \phi_2)$ for all $\phi_1, \phi_2 \in \Phi$. Incentive compatibility then implies that, for all $\theta \in \Theta$, $t(\theta, \phi_1) = t(\theta, \phi_2)$ for all $\phi_1, \phi_2 \in \Phi$.

Fix $\phi_3 \in \Phi$. Consider the mechanism, $\tilde{x} : \Theta \times \Phi \to X$, $\tilde{t} : \Theta \times \Phi \to \mathbb{R}$ such that $\tilde{x}(\theta, \phi) = x(\theta, \phi_3)$ and $\tilde{t}(\theta, \phi) = t(\theta, \phi_3)$ for all $(\theta, \phi) \in \Theta \times \Phi$. This mechanism is individually rational and incentive compatible. Moreover, $\tilde{x}(\theta, \phi) = x(\theta, \phi)$ and $\tilde{t}(\theta, \phi) = t(\theta, \phi)$ for all $(\theta, \phi) \in \Theta \times \Phi$. Since $F$ is atomless, $\mathbb{E}_{(\theta, \phi)}[u(\theta, \phi, x(\theta, \phi)) - t(\theta, \phi)] = \mathbb{E}_{(\theta, \phi)}[u(\theta, \phi, \tilde{x}(\theta, \phi)) - \tilde{t}(\theta, \phi)]$. □

OA.3 Omitted Calculations from Subsection 4.2

We verify that

$$\tilde{V}(c, p_1, p_2) = \frac{(y + bp_1)^2}{4} \left( \frac{1}{2c - 1} - \frac{1}{3} \right) + Ap_1^2 - Ap_2$$

is a convex function of $(c, p_1, p_2) \in [1, 2] \times [0, 1] \times [0, 1]$ for all $A \geq b^2/12$. Since $-Ap_2$ is a convex function of $p_2 \in [0, 1]$, we need only show that

$$W(c, p_1) = \frac{(y + bp_1)^2}{4} \left( \frac{1}{2c - 1} - \frac{1}{3} \right) + Ap_1^2$$

is a convex function of $(c, p_1) \in [1, 2] \times [0, 1]$. This follows because

$$\frac{\partial^2 W}{\partial c^2} (c, p_1) \frac{\partial^2 W}{\partial p_1^2} (c, p_1) - \left( \frac{\partial^2 W}{\partial c \partial p_1} (c, p_1) \right)^2 = \frac{4(y + bp_1)^2}{(2c - 1)^3} \left( A - \frac{b^2}{12} \right) \geq 0$$

for all $(c, p_1) \in [1, 2] \times [0, 1]$ whenever $A \geq b^2/12$.

References