Machine learning for set-identified linear models

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Abstract

This paper provides estimation and inference methods for set-identified linear models with high-dimensional covariates where the model selection is based on machine learning tools. I characterize the boundary (i.e, support function) of the identified set using a semiparametric moment condition. Combining Neyman-orthogonality and sample splitting ideas, I construct a root-N consistent, uniformly asymptotically Gaussian estimator of the support function. I propose a weighted bootstrap procedure to conduct inference about the identified set. I provide a general method to construct a Neyman-orthogonal moment condition for the support function. I apply this result to estimate sharp bounds on the average treatment effect in Lee (2008)'s model of endogenous selection and tighten the bounds on this parameter in Angrist et al. (2006)'s empirical setting. Other applications include sharp identified sets for a new parameter, called a partially linear predictor, and the average partial derivative when the outcome variable is recorded in intervals.

1 Introduction

Economists are often interested in bounds on parameters when parameters themselves are not point-identified (e.g., Manski (2010)). In practice, however, bounds are often wide. For example, the upper and lower bounds on average treatment effect frequently have opposite signs and cannot determine whether the treatment helps or hurts. As discussed in Lee (2008) and Manski and Pepper (2011), covariates can help tighten the bounds. However, economists rarely know which covariates have the strongest tightening ability. As a result, the reported bounds may not be as tight as possible.

The covariate selection problem has gained recent attention in the context of high-dimensional data sets that contain hundreds of covariates per observation. On the one hand, ex-ante covariate selection delivers valid inference but leads to wide bounds since important covariates may be dropped in this approach. On the other hand, ex-post covariate selection is prone to overfitting. To perform data-driven model selection and obtain valid inference at the same time, economists have used modern machine learning tools to control for omitted variable bias (Belloni et al. (2016), Chernozhukov et al. (2017a), Chernozhukov et al. (2017b)) and to model treatment effect heterogeneity (Wager and Athey (2016)) in point-identified settings. However, exploiting the predictive power of machine learning tools to tighten the bounds in set-identified models is a novel idea.

The main contribution of this paper is to provide estimation and inference methods for identified sets where the selection among high-dimensional covariates is based on machine learning tools. I develop a general set-identified linear model with high-dimensional covariates that covers a broad variety of set-identified models: for example, those considered in Beresteanu and Molinari (2008), Bontemps et al. (2012), Chandrasekhar et al. (2011), and Kaido (2017). I propose a root-N consistent, uniformly asymptotically Gaussian estimator of the identified set's boundary (i.e., support function) and conduct uniform inference about the boundary. As one example, this paper provides estimation and inference methods for sharp (that is, tightest possible) nonparametric bounds on the average treatment effect (ATE) (i.e., Lee (2008) bounds) in the presence of endogenous sample selection.

This paper focuses on identified sets whose boundaries can be characterized by a semiparametric moment equation. In this equation, the parametric component gives the description of the

boundary (i.e., support function) and the nonparametric component is a nuisance parameter, for example, a conditional mean function. A naive approach would be to plug-in a machine learning estimate of the nuisance parameter into the moment equation and solve the moment equation for the boundary. However, modern regularized methods (machine learning techniques) have bias converging slower than the parametric rate. As a result, plugging such estimates into the moment equation produces a biased, low-quality estimate of the identified set's boundary.

The major challenge of this paper is to overcome the transmission of the biased estimation of the first-stage nuisance parameter into the second stage. A basic idea, previously proposed in the point-identified case, is to make the moment equation insensitive, or, formally, Neyman-orthogonal, to the biased estimation of the first-stage parameter (Neyman (1959)). Combining Neyman-orthogonality and sample splitting, (Chernozhukov et al. (2017a), Chernozhukov et al. (2017b)) derive a root-N consistent and asymptotically normal estimator of the low-dimensional parameter identified by a semiparametric moment equation. However, extending this idea to a set-identified case presents additional challenges.

The main distinction between the point- and set-identified cases is that the target parameter is no longer a finite-dimensional vector, but a boundary that consists of continuum points. Therefore, in addition to pointwise inference, economists are interested in uniform statistical properties over the identified set's boundary. Second, because the moment condition for the boundary depends on the nuisance parameter in a non-smooth way, establishing Neyman-orthogonality is a non-trivial exercise. I develop high-level sufficient conditions for Neyman-orthogonality and derive a uniformly root-N consistent, uniformly asymptotically Gaussian estimator of the identified set's boundary.

To make the orthogonal approach useful, I provide a general recipe to construct a Neyman-orthogonal moment equation starting from a non-orthogonal one, extending the previous work on orthogonal estimation (Hardle and Stoker (1989), Newey and Stoker (1993), Newey (1994), Ichimura and Newey (2017), Chernozhukov et al. (2017b)) from a point- to a set-identified case. I also provide a weighted bootstrap algorithm to conduct inference about the identified set's boundary. The procedure simplifies the weighted bootstrap algorithm from Chandrasekhar et al. (2011): instead of re-estimating the first-stage parameter in each bootstrap repetition, I estimate the first-stage parameter once on an auxiliary sample. My algorithm is faster to compute because only the

second stage is repeated in the simulation. I show that the simpler weighted bootstrap procedure is valid when the moment equation is Neyman-orthogonal.

I demonstrate my method's utility with three applications. In the first application, I estimate sharp bounds on the average treatment effect in the presence of endogenous sample selection and non-compliance. Reporting nonparametric bounds on the average treatment effect in addition to the point estimates derived under stronger identification assumptions is a common robustness check in labor and education studies (Angrist et al. (2006), Lee (2008), Engberg et al. (2014), Huber et al. (2017), Abdulkadiroglu et al. (2018), Sieg and Wang (2018)). In some cases, such as Engberg et al. (2014), the bounds have opposite signs and are therefore uninformative.[1] To tighten these bounds, Lee (2008) suggests splitting the observations into several categories, performing the analysis within each category, and then averaging the lower and the upper bounds across categories. I show how to tighten the bounds even further by conditioning on high-dimensional covariates.

In the second application, I study the partially linear model from Robinson (1988) in the presence of high-dimensional covariates when the outcome variable is recorded in intervals. I characterize the identified set for the causal parameter in this model and provide estimation and inference methods for the identified set's boundary. I provide primitive conditions on the problem design that allow to incorporate machine learning tools to conduct uniform inference about the boundary. Because Robinson (1988)'s model may be misspecified in practice, I introduce a new parameter, called a partially linear predictor, to measure the predictive effect of an endogenous variable on an outcome variable in the presence of high-dimensional controls. I show that the identified set for the causal parameter in Robinson (1988) is the sharp identified set for the partially linear predictor.

In the third application, I study the average partial derivative (Hardle and Stoker (1989), Newey and Stoker (1993)) in the presence of high-dimensional controls when the outcome variable is recorded in intervals. Kaido (2017) characterized the identified set's boundary. He also derived an orthogonal moment equation for the boundary and proposed the estimator for the boundary when the number of control variables is small. I extend his result, allowing the number of covariates to exceed the sample size. I also provide primitive sufficient conditions on the problem design that

[1] For example, Engberg et al. (2014) reports the effect of attending a magnet program on the Mathematics test score lies between $-24.22(148.06)$ and $87.09(57.62)$. The results are taken from Table 8 of Engberg et al. (2014), which reports the ATE of attending a magnet program in a mid-sized urban school district on the high school achievement in Mathematics, as measured by a standardized achievement test score. The standard errors are indicated in parentheses.

allow to incorporate machine learning tools to conduct uniform inference about the boundary.

As an empirical application, I revisit the bounds analysis of Lee (2008) using the data in Angrist et al. (2006) and substantially tighten the bounds suggested by Lee (2008)'s method. In the original study, Angrist et al. (2006) examined the effect of a private school-subsidizing voucher on test scores. To derive the bounds, Angrist et al. (2006) assumed that the voucher can neither deter the test participation nor harm the test score. Following the approach in Lee (2008), I use only the first assumption and estimate sharp bounds using all available covariates. For both Mathematics and Language, the estimated bounds are substantially tighter than the original bounds reported in Angrist et al. (2006) and are both positive. For Language, the estimated bounds are both positive and significant.

The paper is organized as follows. Section 2 provides motivating examples and constructs an estimator for the support function in a one-dimensional case of the partially linear predictor. Section 3 introduces a general set-identified linear model with high-dimensional covariates and establishes theoretical properties of the support function estimator. Section 4 describes the applications of the proposed framework to bounds analysis and to models where an outcome variable is recorded in intervals. Section 5 revisits the empirical application in Angrist et al. (2006) and sharpens the bounds on the treatment effect. Section 6 states my conclusions.

1.1 Literature Review

This paper is related to two lines of research: estimation and inference in set-identified models and Neyman-orthogonal semiparametric estimation. This paper contributes to the literature by introducing Neyman-orthogonal semiparametric estimation to the set-identified literature.

Set-identification is a vast area of research (Manski (1989), Manski and Tamer (2002), Beresteanu and Molinari (2008), Bontemps et al. (2012), Beresteanu et al. (2011), Ciliberto and Tamer (2009), Chen et al. (2011), Kaido and White (2014), Kaido and Santos (2014), Chandrasekhar et al. (2011), Kaido (2016), Kaido (2017)), see e.g. Tamer (2010) or Molinari and Molchanov (2018) for a review. There are two approaches to estimate and conduct inference on identified sets: the moment inequalities approach (Chernozhukov et al. (2007), Kaido and White (2014)) and the support function approach (Beresteanu and Molinari (2008), Bontemps et al. (2012)), which applies only to

convex and compact identified sets. A framework to unify these approaches was proposed by Kaido (2016). In this paper, I extend the support function approach, allowing the moment equation for the identified set's boundary to depend on a nuisance parameter that can be high-dimensional and is estimated by machine learning methods. In Semenova (2018), I introduce the same dependence in moment inequalities.

Within the first line of research, my empirical applications are most connected to work that derives nonparametric bounds on the average treatment effect in the presence of endogenous sample selection and non-compliance. This literature (Angrist et al. (2002), Angrist et al. (2006), Engberg et al. (2014), Huber et al. (2017), Abdulkadiroglu et al. (2018), Sieg and Wang (2018)) derives nonparametric bounds on the average treatment effect. Specifically, I build on Lee (2008), who derived sharp bounds on the average treatment effect and highlighted the role of covariates in achieving sharpness. However, Lee (2008)'s estimator only applies to a small number of discrete covariates. In this paper, I permit a large number of both discrete and continuous covariates and leverage the predictive power of machine learning tools to identify sharp bounds.

The second line of research obtains a $\sqrt{N}$-consistent and asymptotically normal estimator of a low-dimensional target parameter θ in the presence of a high-dimensional nuisance parameter η (Neyman (1959), Neyman (1979), Hardle and Stoker (1989), Newey and Stoker (1993), Newey (1994), Robins and Rotnitzky (1995), van der Vaart (1998), Robinson (1988), Chernozhukov et al. (2017a), Chernozhukov et al. (2017b)). It is common to estimate the target parameter in two stages, where a first-stage estimator of the nuisance $\hat{\eta}$ is plugged into a sample analog of a mathematical relation that identifies the target, such as a moment condition, a likelihood function, etc. A statistical procedure is called Neyman-orthogonal (Neyman (1959), Neyman (1979)) if it is locally insensitive with respect to the estimation error of the first-stage nuisance parameter. In a point-identified problem, the orthogonality condition is defined at the true value of the target θ_0. Since the notion of unique true value θ_0 no longer exists in a set-identified framework, I extend the orthogonality condition to hold on a slight expansion of the boundary of the identified set.

2 Setup and Motivation

2.1 General Framework

I focus on identified sets that can be represented as weighted averages of an outcome variable that is known to lie within an interval. Let Y be an outcome and Y_L, Y_U be random variables such that

$$Y_L \leq Y \leq Y_U \text{ a.s.} \tag{2.1}$$

Consider an identified set of the following form

$$\mathcal{B} = \{\beta = \Sigma^{-1}\mathbb{E}VY, \quad Y_L \leq Y \leq Y_U\}, \tag{2.2}$$

where $V \in \mathcal{R}^d$ is a d-vector of weights and $\Sigma \in \mathcal{R}^{d\times d}$ is a full-rank normalizing matrix. Σ can be either known or unknown, covering a variety of cases. For example, $\Sigma = V = 1$ corresponds to the expectation of an outcome Y. For another example, $\Sigma = (\mathbb{E}VV^\top)^{-1}$ corresponds to the set-valued best linear predictor of the outcome Y when V is used as a predictive covariate. I have adopted this structure because it allows me to cover a wide class of set-identified models that are usually studied separately.

A key innovation of my framework is that the bounds Y_L, Y_U and the weighting variable V can depend on an identified nuisance parameter that I allow to be high-dimensional. To fix ideas, let W be a vector of observed data and P_W denote its distribution. Then, I allow each coordinate of the weighting vector V and the bounds Y_L, Y_U to depend on an identified parameter of the data distribution P_W. The examples below demonstrate the importance of this innovation.

2.2 Motivating Examples

Example 1. Endogenous Sample Selection. In this example I revisit the model of endogenous sample selection from Lee (2008). I use the following notation for the potential outcomes. Let $D \in \{1,0\}$ denote an indicator for whether an unemployed subject has won a lottery to participate in a job training program. Let $S_0 = 1$ be a dummy for whether the subject would have been employed after losing the lottery, and $S_1 = 1$ be a dummy for whether the subject would have been

employed after winning the lottery. Similarly, let $\{Y_d, d \in \{1,0\}\}$ represent the potential wages in case of winning and losing the lottery, respectively. The object of interest is the average effect on wages $\beta = \mathbb{E}[Y_1 - Y_0 | S_1 = 1, S_0 = 1]$ for the group of people who would have been employed regardless of lottery's outcome, or, briefly, the always-employed.

The data consist of the admission outcome D, the observed employment status $S = DS_1 + (1 - D)S_0$, and the baseline covariates X (e.g., age, gender, race). In addition, the data contain wages for employed subjects $S \cdot Y = S \cdot (DY_1 + (1-D)Y_0)$. As discussed in Lee (2008) (see Lemma 14 for details) , the average wage in case of non-admission, $\mathbb{E}[Y_0 | S_1 = 1, S_0 = 1]$, is point-identified, but the quantity $\mathbb{E}[Y_1 | S_1 = 1, S_0 = 1]$ is not. The sharp bounds on $\mathbb{E}[Y_1 | S_1 = 1, S_0 = 1]$ are given by

$$[\mathbb{E}Y_L(X), \mathbb{E}Y_U(X)], \tag{2.3}$$

where the lower bound $Y_L(X)$ is equal to

$$Y_L = Y_L(X) := \frac{D \cdot S \cdot Y \cdot 1_{\{Y \leq y_{\{p_0(X),X\}}\}} \Pr(D=0)}{\Pr(D=0, S=1) \Pr(D=1)} \tag{2.4}$$

and the upper bound is equal to

$$Y_U = Y_U(X) := \frac{D \cdot S \cdot Y \cdot 1_{\{Y \geq y_{\{1-p_0(X),X\}}\}} \Pr(D=0)}{\Pr(D=0, S=1) \Pr(D=1)}, \tag{2.5}$$

where $s(D,X), p_0(X), y_{\{p_0(X),X\}}, y_{\{1-p_0(X),X\}}$ are functions of X defined as follows

$$\begin{aligned} s(D,X) = \mathbb{E}[S=1|D,X], p_0(X) = \frac{s(0,X)}{s(1,X)}, \\ y_{\{u,X\}}: \quad \Pr(Y \leq y_{\{u,X\}} | X, D=1, S=1) = u, \quad u \in [0,1]. \end{aligned} \tag{2.6}$$

Specifically, $s(D,X)$ is the probability of employment given X, $p_0(X)$ is the ratio of conditional probabilities, and $y_{\{u,X\}}$ is the quantile function of employed and admitted individuals given X. As a result, the sharp bounds on the program effect depend on the first-stage parameter $\eta_0(X) = \{s(0,X), s(1,X), y_{u,X}\}$. Therefore, the identified set (2.3) is a special case of model (2.1)-(2.2) with $V = \Sigma = \Pr(D=0, S=1)$ and the first-stage parameter $\eta_0(X)$.

Table 1: Lee (2008)'s bounds on Voucher Effect on Test Scores using the data in Angrist et al. (2006)

Covariates	None (1)	{ Age, gender } (2)	**My result, all 7 covs** (3)
A. *Mathematics*			
Estimate	[-1.304, 2.073]	[-1.100, 1.827]	**[0.160, 0.904]**
95% CR	(-2.131, 2.886)	(-1.875, 2.599)	**(-0.168, 1.570)**
B. *Language*			
Estimate	[-1.192, 2.640]	[-0.946, 2.341]	**[0.473, 1.112]**
95% CR	(-2.086, 3.542)	(-1.8007, 3.211)	**(0.144, 1.847)**

Table 1 reports estimated bounds for the voucher effect (Estimates) and a 95% confidence region (95% CR) for the identified set for the voucher effect for test scores in Mathematics (Panel A) and Language (Panel B). I report the results for 3 specifications: without covariates (Column 1), with age and gender covariates (Column 2), and my result based on all 7 covariates (Column 3).

Table 1 shows the bounds on the voucher's effect on the test scores using the data in Angrist et al. (2006). The empirical details are discussed in Section 5. Without any covariates, Lee (2008)'s bounds have opposite signs and cannot determine the direction of the effect (Column 1). Including age and gender covariates, selected by Angrist et al. (2006), does not help determine the direction of the effect (Column 2). However, conditioning on all covariates (Column 3) that better predict test participation results in bounds that are both positive and substantially tighter. The bounds for Language are also significant.

Example 2. Average Partial Derivative. An important parameter in economics is the average partial derivative. This parameter shows the average effect of a small change in an endogenous variable D on the outcome Y conditional on the covariates X. To describe this change, define the conditional expectation function of an outcome Y given the endogenous variable D and exogenous variable X as $\mu(D,X) := \mathbb{E}[Y|D,X]$ and its partial derivative with respect to D as $\partial_D \mu(D,X) := \partial_d \mu(d,X)|_{d=D}$. Then, the average partial derivative is defined as $\beta = \mathbb{E}\partial_D \mu(D,X)$. For example, when Y is the logarithm of consumption, D is the logarithm of price, and X is the vector of other demand attributes, the average partial derivative stands for the average price elasticity.

Assume that the endogenous variable D has bounded support $\mathcal{D} \subset \mathcal{R}^d$ and has positive density on this support. Hardle and Stoker (1989) have shown that the average partial derivative can be

represented as $\beta = \mathbb{E}VY$, where $V = -\partial_D \log f(D|X) = -\frac{\partial_D f(D|X)}{f(D|X)}$ is the negative partial derivative of the logarithm of the density $f(D|X)$.

Suppose the outcome Y is interval-censored. As discussed in Kaido (2017), the sharp identified set $\mathcal{B}$ for the average partial derivative can be represented as

$$\mathcal{B} = \{\mathbb{E}VY, \quad Y_L \leq Y \leq Y_U\}, \tag{2.7}$$

which is a special case of model (2.1)-(2.2) with $\Sigma = 1$, $V = -\frac{\partial_D f(D|X)}{f(D|X)}$, and the nuisance parameter $\eta_0(X) = \frac{\partial_D f(D|X)}{f(D|X)}$. In contrast to Kaido (2017), I allow the vector of covariates X to be high-dimensional.

Example 3. Partially Linear Predictor. A widely used approach to measure the causal effect of an endogenous variable D on an outcome variable Y is to adopt the partially linear model (Robinson (1988)):

$$Y = D\beta_0 + f_0(X) + U, \quad \mathbb{E}[U|D,X] = 0,$$

where X is a vector of covariates. However, when the model is misspecified, the parameter β_0 has no interpretation. An alternative parameter, which is robust to misspecification of the partially linear model, is a partially linear predictor. This parameter is defined as

$$\beta = \arg \min_{b \in \mathcal{R}^d, f \in \mathcal{M}} \mathbb{E}(Y - D^\top b - f(X))^2,$$

or the linear component of the projection of Y on a partially linear combination of the endogenous variable D and the covariates X, where $\mathcal{M}$ is a set of functions of X. Equivalently, the parameter β can be represented as the best linear predictor of variable Y in terms of the first-stage residual V (see Lemma 15) as

$$\beta = \arg \min_{b \in \mathcal{R}^d} \mathbb{E}(Y - V^\top b)^2,$$

where the first-stage residual V is defined as $V = D - \mathbb{E}[D|X]$.

Suppose the outcome Y is interval-censored. Then, the sharp identified set $\mathcal{B}$ for the partially

linear predictor is

$$\mathcal{B} = \{(\mathbb{E}VV^\top)^{-1}\mathbb{E}VY, \quad Y_L \leq Y \leq Y_U\}, \tag{2.8}$$

which is a special case of model (2.1)-(2.2) with $V = D - \mathbb{E}[D|X]$, $\Sigma = \mathbb{E}VV^\top$, and the nuisance parameter $\eta_0(X) = \mathbb{E}[D|X]$. Moreover, the identified set $\mathcal{B}$ is a non-sharp identified set for the causal parameter β_0 when the partially linear model is correctly specified.

2.3 Main ideas

This section is a demonstration of two well-known ideas, Neyman-orthogonality (Neyman (1959), Chernozhukov et al. (2017a), Chernozhukov et al. (2017b)) and sample splitting, for the partially linear predictor (Example 3) when the endogenous variable D is one-dimensional. Combining these two ideas, I obtain a root-N consistent, asymptotically Gaussian estimator of the lower and the upper bound on the partially linear predictor.

2.3.1 Neyman-orthogonality

When the endogenous variable D is one-dimensional, the identified set (2.8) is a closed interval $[\beta_L, \beta_U]$. The moment function for the upper bound β_U takes the form

$$m(\text{data}, \beta_U, \eta_0) := (Y^{\text{UBG}}(\eta_0) - (D - \eta_0(X))\beta_U)(D - \eta_0(X)), \tag{2.9}$$

where the *upper bound generator* $Y^{\text{UBG}}(\eta)$ is defined as $Y^{\text{UBG}}(\eta) = \begin{cases} Y_L, & D - \eta(X) \leq 0 \\ Y_U, & D - \eta(X) \geq 0. \end{cases}$. A naive estimator $\hat{\beta}_U^{\text{NAIVE}}$, based on plugging in a regularized estimator $\hat{\eta}$ into the moment condition (8.2), converges at a rate slower than root-N and cannot be used to conduct inference about β_U using standard asymptotic theory. Because $\hat{\eta}$ is a regularized estimator, its own bias converges at a rate slower than root-N and carries over into the second stage. The second-stage bias is non-zero $\partial_{\eta_0}\mathbb{E}m(W, \beta_U, \eta_0)(\hat{\eta} - \eta_0) = -\mathbb{E}[Y^{\text{UBG}}(\eta_0)(\hat{\eta}(X) - \eta_0(X))] \neq 0$.

To overcome the transmission of the bias, I replace the moment equation (8.2) by another moment equation that is less sensitive to the biased estimation of its first-stage parameters. Specif-

ically, I replace the upper bound generator $Y^{\text{UBG}}(\eta)$ by its residual after projection on the set of covariates $Y^{\text{UBG}}(\eta) - \mathbb{E}[Y^{\text{UBG}}(\eta)|X]$, which makes the derivative above equal to zero. This replacement corresponds to a new moment function

$$g(\text{data}, \beta_U, \xi_0) = (Y^{\text{UBG}}(\eta_0) - \mathbb{E}[Y^{\text{UBG}}(\eta_0)|X] - (D - \eta_0(X))\beta_U)(D - \eta_0(X)),$$

which is orthogonal with respect to its first-stage parameter $\xi_0 = \{\eta_0, \mathbb{E}[Y^{\text{UBG}}(\eta_0)|X]\}$ (see Theorem 5). By orthogonality, the first-stage bias of $\hat{\xi}$ does not carry over into the second-stage.

2.3.2 Sample splitting

I can use machine learning methods in the first stage because of sample splitting. In the absence of sample splitting, the estimation error of the first-stage machine learning estimator may be correlated with the true values of the first and second-stage residuals. This correlation leads to bias, referred to as overfitting bias. While sample splitting helps overcome overfitting bias, it cuts the sample used for the estimation in half. This problem can lead to the loss of efficiency in small samples. To overcome this problem, I use the cross-fitting technique from Chernozhukov et al. (2017a) defined in Section 3. Specifically, I partition the sample into two halves. To estimate the residuals for each half, I use the other half to estimate the first-stage nuisance parameter. Then, the upper bound is estimated using the whole sample. As a result, each observation is used both in the first and second stages, improving efficiency in small samples.

3 Main Results

In this section, I introduce a general set-identified linear model with a high-dimensional nuisance parameter. I describe the boundary of the identified set (support function) by a semiparametric moment equation. I introduce a new sufficient condition for a moment equation - uniform near-orthogonality - and establish uniform asymptotic theory for the support function estimator and bootstrap support function process under that condition. Finally, I provide a general recipe to construct a uniformly near-orthogonal moment equation starting from a non-orthogonal one.

Notation. I use the following standard notation. Let $\mathcal{S}^{d-1} = \{q \in \mathcal{R}^d, \|q\| = 1\}$ be the d-dimensional unit sphere and $q \in \mathcal{S}^{d-1}$ be a generic vector on the unit sphere. I use the following notation

$$\Gamma(t_1, t_2 - t_1, t_3) := t_1 + (t_2 - t_1) 1_{\{t_3 \geq 0\}}, \tag{3.1}$$

where $1_{\{t_3 \geq 0\}} = 1$ if t_3 is non-negative and $1_{\{t_3 \geq 0\}} = 0$ otherwise. I use standard notation for numeric and stochastic dominance. For two numeric sequences $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$, let $a_n \lesssim b_n$ stand for $a_n = O(b_n)$. For two sequences of random variables $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$, let $a_n \lesssim_P b_n$ stand for $a_n = O_P(b_n)$. For a random variable ξ, $(\xi)^0 := \xi - \mathbb{E}[\xi]$. Let $L^\infty(\mathcal{S}^{d-1})$ be the space of absolutely surely bounded functions defined on the unit sphere $\mathcal{S}^{d-1}$. Define an $L_{P,c}$ norm of a vector-valued random variable W as: $\|W\|_{L_{P,c}} := \left(\int_{w \in \mathcal{W}} \|W\|^c\right)^{1/c}$. Let $\mathcal{W}$ be the support of the data vector W of the distribution P_W. Let $(W_i)_{i=1}^N$ be an i.i.d sample from the distribution P_W. Denote the sample average of a function $f(\cdot)$ as $\mathbb{E}_N[f(W_i)] := \frac{1}{N}\sum_{i=1}^N f(W_i)$ and the centered, root-N scaled sample average as $\mathbb{G}_N[f(W_i)] := \frac{1}{\sqrt{N}}\sum_{i=1}^N [f(W_i) - \mathbb{E}f(W_i)]$.

3.1 High-Level Assumptions

One of this paper's key innovations is to allow the identified set $\mathcal{B}$, given in (2.2), to depend on an identified parameter of data distribution P_W. Definition 1 formalizes this dependence.

Definition 1 (Constructed Random Variable)**.** *Let W be the vector of the observed data, P_W its distribution, and $\mathcal{W}$ its support. I refer to V as a constructed random variable if there exists an identified parameter $\eta_0, \eta_0 \in \mathcal{T}$ from a linear and convex set $\mathcal{T}$ and a known measurable map $H(W, \eta) : \mathcal{W} \bigtimes \mathcal{T} \to \mathcal{R}$ such that $V = H(W, \eta_0)$ a.s. I refer to η as a nuisance parameter and η_0 as the true value of η.*

ASSUMPTION 1 (Constructed Random Variables)**.** *Each coordinate of the random vector (V, Y_L, Y_U) in the identified set (2.2) is either an observed or constructed random variable.*

To complete the model, I need to identify matrix Σ in (2.2) when Σ is unknown. If this is the case, I assume that Σ is identified by a semiparametric moment condition (Assumption 2).

ASSUMPTION 2 (Identification of Σ). *1. There exists an identified parameter η of the distribution P_W and a known measurable map $A(W,\eta) : \mathcal{W} \bigtimes \mathcal{T} \to \mathcal{R}$ such that*

$$\Sigma = \mathbb{E}A(W,\eta_0). \tag{3.2}$$

2. There exist constants $\lambda_{\min} > 0$ and $\lambda_{\max} < \infty$ that bound the eigenvalues of Σ from above and below $0 < \lambda_{\min} \leq \min eig(\Sigma) \leq \max eig(\Sigma) < \lambda_{\max}$.

According to Beresteanu and Molinari (2008) and Bontemps et al. (2012), the identified set $\mathcal{B}$ is a compact and convex set. Therefore, it can be equivalently represented by its support function. Fix a direction q on a unit sphere $\mathcal{S}^{d-1} := \{q \in \mathcal{R}^d, \quad \|q\| = 1\}$. Define the support function

$$\sigma(q,\mathcal{B}) := \sup_{b \in \mathcal{B}} q^\top b \tag{3.3}$$

as the (signed) distance from the origin to the hyperplane tangent to $\mathcal{B}$ in the direction q. Using the arguments of (Beresteanu and Molinari (2008), Bontemps et al. (2012)), the function $\sigma(q,\mathcal{B})$ is equal to the expectation of the product of two random variables z_q and Y_q

$$\sigma(q,\mathcal{B}) = \mathbb{E}z_q Y_q, \tag{3.4}$$

where $z_q = q^\top \Sigma^{-1} V$ is a normalized projection of the covariate V onto the direction q and the variable Y_q is defined as

$$Y_q = Y_L + (Y_U - Y_L)1_{\{z_q > 0\}} := \Gamma(Y_L, Y_U - Y_L, z_q). \tag{3.5}$$

Namely, Y_q is equal to the lower bound Y_L when z_q is non-positive and equal to the upper bound Y_U otherwise. To highlight the dependence of z_q and Y_q on η, I will rewrite (3.4) as a semiparametric moment equation for $\sigma(q,\mathcal{B})$

$$\mathbb{E}[\sigma(q,\mathcal{B}) - z_q(\eta_0)Y_q(\eta_0)] = 0. \tag{3.6}$$

3.2 Orthogonality and near-orthogonality

As discussed in the introduction, the moment equation (3.6) produces a low-quality estimator of the support function. The problem arises because the moment equation (3.6) is sensitive with respect to the biased estimation of η_0. To overcome this problem, I replace (3.6) with an orthogonal moment function $g(W,p,\xi(p)) : \mathcal{W} \bigtimes \mathcal{P} \bigtimes \Xi \to \mathcal{R}$ which depends on a functional nuisance parameter $\xi = \xi(p)$, defined on a compact subset $\mathcal{P} \subset \mathcal{R}^d$ [2] . The orthogonal moment for support function takes place

$$\mathbb{E}[\sigma(q,\mathcal{B}) - g(W,p_0(q),\xi_0(p_0(q))] = 0, \tag{3.7}$$

where $p_0(q) = (\Sigma^{-1})^\top q$.

Let Ξ be a convex subset of a normed vector space that contains the functional parameter $\xi_0 = \xi_0(p)$. Define the pathwise (Gateaux) derivative map on the set $\Xi - \xi_0$ as $D_r : \Xi - \xi_0 \to \mathcal{R}$

$$D_r[\xi - \xi_0] := \partial_r \mathbb{E} g(W,p,r(\xi - \xi_0) + \xi_0), \quad \xi \in \Xi, \quad p \in \mathcal{P}, \quad r \in [0,1)$$

which I assume exists. I will also use the notation $\partial_0 \mathbb{E} g(W,p,\xi_0)[\xi - \xi_0] = D_0[\xi - \xi_0]$ for the Gateaux derivative at ξ_0. Let $\{\Xi_N, N \geq 1\}$ be a sequence of subsets of Ξ (i.e, $\Xi_N \subseteq \Xi$) and $\{\mathcal{T}_N, N \geq 1\}$ be a sequence of subsets of $\mathcal{T}$(i.e, $\mathcal{T}_N \subseteq \mathcal{T}$).

Definition 2 (Neyman-orthogonality)**.** *The moment function $g(W,p,\xi)$ obeys the orthogonality condition at ξ_0 with respect to the nuisance realization set $\Xi_N \subset \Xi$ if the following conditions hold. (1) Equation (3.7) holds. (2) The pathwide derivative map $D_r[\xi - \xi_0]$ exists for all $r \in [0,1)$ and $\xi \in \Xi_N$ and vanishes at $r = 0$ for each $p \in \mathcal{P}$*

$$\partial_\xi \mathbb{E} g(W,p,\xi_0)[\xi - \xi_0] = 0 \quad \forall p \in \mathcal{P}. \tag{3.8}$$

Definition 2 requires the expectation of the moment function $g(W,p,\xi_0)$ to have zero Gateaux derivative with respect to ξ at ξ_0 at each vector p in the projection set $\mathcal{P}$. To accommodate the moment function (3.4) that depends on η in a non-smooth way, I relax the requirement of Definition

[2]For example, the set $\mathcal{P}$ is taken to be $\mathcal{P} = \{(\Sigma^{-1})^\top q, q \in \mathcal{S}^{d-1}\}$ when Σ is known and $\mathcal{P} = \{p \in \mathcal{R}^d, \quad 0.5\lambda_{\min} \leq \|p\| \leq 2\lambda_{\max}\}$ when Σ is unknown.

2 using the notion of uniform near-orthogonality.

Definition 3 (Uniform near-orthogonality)**.** *The moment function $g(W,p,\xi)$ obeys the near-orthogonality condition at ξ_0 with respect to the nuisance realization set $\Xi_N \subset \Xi$ uniformly over $\mathcal{P}$ if the following conditions hold. (1) Equation (3.7) holds. (2) The pathwise derivative map $D_r[\xi-\xi_0]$ exists for all $r \in [0,1)$ and $\xi \in \Xi_N$. (3) There exists a sequence of positive constants $\mu_N = o(N^{-1/2})$ such that the pathwide derivative $D_r[\xi-\xi_0]$ at $r=0$ is uniformly small over the set $\mathcal{P}$*

$$\sup_{p\in\mathcal{P}} |\partial_\xi \mathbb{E} g(W,p,\xi_0)[\xi-\xi_0]| \leq \mu_N.$$

ASSUMPTION 3 (Near-orthogonality)**.** *(1) There exists a measurable moment function $g(W,p,\xi)$: $\mathcal{W}\bigtimes\mathcal{P}\bigtimes\Xi \to \mathcal{R}$ that obeys (3.7) and the near orthogonality condition uniformly over $\mathcal{P}$. (2) When Σ is unknown, there exists a moment matrix-valued function $A(W,\eta)$ that obeys (3.2) and the orthogonality condition $\partial_\eta \mathbb{E}A(W,\eta_0)[\eta-\eta_0]=0$.*

Assumption 3 is the key assumption of my paper. Assumption 3 (1) states that there exists a moment function $g(W,p,\xi)$ for the support function $\sigma(q,\mathcal{B})$ that is approximately insensitive with respect to the biased estimation of the nuisance parameter ξ at ξ_0. I show how to achieve this condition in Section 9. The second assumption states that the moment function $A(W,\eta)$ is insensitive with respect to the biased estimation of the parameter η at η_0. This assumption holds in practical applications (e.g., in Example 3). To sum up, the uniform near-orthogonal moment equation for $\sigma(q,\mathcal{B})$ is

$$\mathbb{E}\psi(W,\theta(q),\xi_0(p)) := \mathbb{E}\begin{bmatrix} \sigma(q,\mathcal{B}) - g(W,p(q),\xi_0(p(q))) \\ A(W,\eta_0)p(q) - q \end{bmatrix} = 0. \tag{3.9}$$

Definition 4 (Support Function Estimator when Σ is Known)**.** *Let $(W_i)_{i=1}^N$ be an i.i.d sample of the distribution P_W and $\xi_0(p)$ be an identified parameter of P_W. Let $\hat{\xi}(p)$ be the estimate of $\xi_0(p)$. Define an estimate of the support function $\hat{\sigma}(q,\mathcal{B})$ as follows*

$$\hat{\sigma}(q,\mathcal{B}) = \frac{1}{N}\sum_{i=1}^N g(W_i, p_0(q), \hat{\xi}_i(p)), \tag{3.10}$$

where $p_0(q) = (\Sigma^{-1})^\top q$.

Definition 5 (Support Function Estimator when Σ is Unknown)**.** *Let* $(W_i)_{i=1}^N$ *be an i.i.d. sample from a distribution* P_W*. Let* $\hat{\xi}(p), p \in \mathcal{P}$ *be the estimate of* $\xi_0(p)$*. Define an estimate of the support function* $\hat{\sigma}(q, \mathcal{B})$ *as follows*

$$\hat{\sigma}(q, \mathcal{B}) = \frac{1}{N} \sum_{i=1}^{N} g(W_i, \hat{p}(q), \hat{\xi}_i(\hat{p}(q))), \tag{3.11}$$

where $\hat{p}(q) = (\hat{\Sigma}^{-1})^\top q$ *and* $\hat{\Sigma} = \frac{1}{N} \sum_{i=1}^{N} A(W_i, \hat{\eta}_i)$.

3.3 Additional regularity conditions

Assumption 4 formalizes the speed of convergence of the estimated nuisance parameter $\hat{\xi}(p)$. It introduces the sequence of neighborhoods $\{\Xi_N, N \geq 1\}$ around the true value $\xi_0(p)$ that contain the estimate $\hat{\xi}(p)$ with probability approaching one. As the sample size N increases, the neighborhoods shrink. The following rates $r_N, r'_N, r''_N, \delta_N$ control the speed at which these neighborhoods shrink around $\xi_0(p)$.

ASSUMPTION 4 (Quality of the First-Stage Estimation and Regularity of the Moment Function)**.** *There exists a sequence* $\{\Xi_N, N \geq 1\}$ *of subsets of* Ξ *(i.e,* $\Xi_N \subseteq \Xi$*) such that the following conditions hold. (1) The true value* ξ_0 *belongs to* Ξ_N *for all* $N \geq 1$*. There exists a sequence of numbers* $\phi_N = o(1)$ *such that the first-stage estimator* $\hat{\xi}(p)$ *of* $\xi_0(p)$ *belongs to* Ξ_N *with probability at least* $1 - \phi_N$*. There exist sequences* $r_N, r'_N, r''_N, \delta_N$ *:* $\quad r''_N \log^{1/2}(1/r''_N) = o(1)$*,* $r'_N \log^{1/2}(1/r'_N) = o(1)$*,*

$r_N = o(N^{-1/2})$, *and* $\delta_N = o(N^{-1/2})$ *such that the following bounds hold*

$$\sup_{\xi\in\Xi_N}\sup_{p\in\mathcal{P}}(\mathbb{E}(g(W,p,\xi(p))-g(W,p,\xi_0(p)))^2)^{1/2}\lesssim r_N'',$$

$$\sup_{\xi\in\Xi_N}\sup_{q\in\mathcal{S}^{d-1}}\sup_{p\in\mathcal{P}:\|p-p_0(q)\|\lesssim RN^{-1/2}}(\mathbb{E}(g(W,p,\xi_0(p))-g(W,p_0,\xi_0(p_0)))^2)^{1/2}\lesssim r_N',$$

$$\sup_{r\in[0,1)}\sup_{p\in\mathcal{P}}(\partial_r^2\mathbb{E}g(W,p,r(\xi(p)-\xi_0(p))+\xi_0(p)))\le r_N,$$

$$\sup_{r\in[0,1)}\|\partial_r^2\mathbb{E}A(W,r(\eta-\eta_0)+\eta_0)\|\le r_N,$$

$$\sup_{\eta\in\mathcal{T}_N}(\mathbb{E}\|A(W,\eta)-A(W,\eta_0)\|^2)^{1/2}\lesssim\delta_N.$$

(2) The following conditions hold for the function class $\mathcal{F}_\xi=\{g(W,p,\xi(p)),p\in\mathcal{P}\}$. *There exists a measurable envelope function* $F_\xi=F_\xi(W)$ *that absolutely surely bounds all elements in the class* $\sup_{p\in\mathcal{P}}|g(W,p,\xi(p))|\le F_\xi(W)$ *a.s.. There exists* $c>2$ *such that* $\|F_\xi\|_{L_{P,c}}:=\left(\int_{w\in\mathcal{W}}(F_\xi(w))^c\right)^{1/c}<\infty$. *There exist constants* a,v *that do not depend on N such that the uniform covering entropy of the function class* $\mathcal{F}_\xi$ *is bounded* $\log\sup_Q N(\varepsilon\|F_\xi\|_{Q,2},\mathcal{F}_\xi,\|\cdot\|_{Q,2})\le v\log(a/\varepsilon),\quad$ *for all* $0<\varepsilon\le 1$.

Assumption 5 is a standard requirement for a differentiable support function $\sigma(q,\mathcal{B})$ (see, e.g. Chandrasekhar et al. (2011)). The differentiability of the support function ensures that the identified set $\mathcal{B}$ is strongly convex. This property rules out the presence of the exposed faces of the identified set $\mathcal{B}$ where the bias accumulates non-trivially. When the random variable V in (2.2) is sufficiently continuous, Assumption 5 holds. When the distribution of V is discrete, adding a small amount of continuously distributed noise suffices to achieve this requirement.

ASSUMPTION 5 (Differentiable Support Function)**.** *Let* $\bar{P}\subset\mathcal{R}^d$ *be an open set that contains* $\mathcal{P}:\mathcal{P}\subset\bar{P}$. *Assume that the function* $p\to\mathbb{E}[g(W,p,\xi_0(p))],p\in\bar{P}$ *is differentiable on* $\bar{P}$. *Define its gradient*

$$G(p):=\nabla_p\mathbb{E}g(W,p,\xi_0(p)). \tag{3.12}$$

In addition, if Σ *is unknown, assume that the following bound holds uniformly on* $p_0\in\mathcal{P}$

$$\mathbb{E}[g(W,p,\xi_0(p))-g(W,p_0,\xi_0(p_0))]=G(p_0)(p-p_0)+o(\|p-p_0\|),\quad\|p-p_0\|\to 0. \tag{3.13}$$

3.4 Main Result

Theorem 1 (Limit Theory for the Support Function Estimator)**.** *Suppose Assumptions 1, 2, 3, 4, 5 hold. Let* $p_0(q) = (\Sigma^{-1})^\top q$*. Let* $\hat{\sigma}(q,\mathcal{B})$ *be the Support Function Estimator provided in Definition 4 when* Σ *is known and Definition 5 when* Σ *is unknown. Define an influence function*

$$h(W,q) := g(W, p_0(q), \xi_0(p_0(q))) - \mathbb{E}[g(W, p_0(q), \xi_0(p_0(q)))]$$

when Σ *is known and*

$$h(W,q) := g(W, p_0(q), \xi_0(p_0(q)) - \mathbb{E}[g(W, p_0(q), \xi_0(p_0(q))] - q^\top \Sigma^{-1}(A(W,\eta_0) - \Sigma)\Sigma^{-1} G(p_0(q))$$

when Σ *is unknown. Then, the Support Function Estimator* $\hat{\sigma}(q,\mathcal{B})$ *is uniformly asymptotically linear over the unit sphere* $\mathcal{S}^{d-1}$

$$\sqrt{N}(\hat{\sigma}(q,\mathcal{B}) - \sigma_0(q,\mathcal{B})) = \mathbb{G}_N[h(W_i,q)] + o_P(1). \tag{3.14}$$

Moreover, the empirical process $\mathbb{G}_N[h(W_i,q)]$ *converges to a tight Gaussian process* $\mathbb{G}[h(W_i,q)]$ *in* $L^\infty(\mathcal{S}^{d-1})$ *with the non-degenerate covariance function*

$$\Omega(q_1,q_2) = \mathbb{E}[h(W,q_1)h(W,q_2)] - \mathbb{E}[h(W,q_1)]\mathbb{E}[h(W,q_2)], \quad q_1,q_2 \in \mathcal{S}^{d-1}.$$

Theorem 1 is my first main result. It says that the Support Function Estimator is asymptotically equivalent to a Gaussian process and can be used for pointwise and uniform inference about the support function. By orthogonality, the first-stage estimation error does not contribute to the total uncertainty of the two-stage procedure. In particular, orthogonality allows me to avoid relying on any particular choice of the first-stage estimator, and, in particular, employ modern regularized techniques to estimate the first-stage. An analog of Theorem 1 was obtained in Chandrasekhar et al. (2011) in a low-dimensional setting, where the first-stage nuisance parameter, estimated by series regression, was plugged into a non-orthogonal moment equation (3.6) for the support function.

Definition 6 (Weighted Bootstrap)**.** *Let B represent a number of bootstrap repetitions. For each* $b \in \{1,2,\ldots,B\}$*, repeat*

1. *Draw N i.i.d. exponential random variables* $(e_i)_{i=1}^N : e_i \sim Exp(1)$. *Let* $\bar{e} = \mathbb{E}_N e_i$.

2. *When the matrix* Σ *is known, set* $\tilde{\Sigma} = \Sigma$. *Otherwise, estimate* $\tilde{\Sigma} = \mathbb{E}_N \frac{e_i}{\bar{e}} A(W_i, \hat{\eta}_i)$.

3. *Estimate* $\tilde{\sigma}^b(q, \mathcal{B}) = \mathbb{E}_N \frac{e_i}{\bar{e}} g(W_i, (\tilde{\Sigma}^{-1})^\top q, \hat{\xi}_i((\tilde{\Sigma}^{-1})^\top q))$.

Theorem 2 (Limit Theory for the Bootstrap Support Function Process)**.** *The bootstrap support function process* $\tilde{S}_N(q) := \sqrt{N}(\tilde{\sigma}(q, \mathcal{B}) - \hat{\sigma}(q, \mathcal{B}))$ *admits the following approximation conditional on the data* $\tilde{S}_N(q) = \mathbb{G}_N[e_i^0 h_i^0(q)] + o_{P^e}(1)$ *in* $L^\infty(\mathcal{S}^{d-1})$. *Moreover, the support function process admits an approximation conditional on the data*

$$\tilde{S}_N(q) = \tilde{\mathbb{G}}[h(q)] + o_{P^e}(1) \text{ in } L^\infty(\mathcal{S}^{d-1}),$$

where $\tilde{\mathbb{G}}[h(q)]$ *is a sequence of tight P-Brownian bridges in* $L^\infty(\mathcal{S}^{d-1})$ *with the same distributions as the processes* $\mathbb{G}_N[h(q)]$, *and independent of* $\mathbb{G}_N[h(q)]$.

Theorem 2 is my second main result. It says that weighted bootstrap of Theorem 1 can be used to approximate the support function process of Theorem 1 and conduct uniform inference about the support function. By orthogonality, I do not need to re-estimate the first-stage parameter in each bootstrap repetition. Instead, I estimate the first-stage parameter once on an auxiliary sample, and plug the estimate into the bootstrap sampling procedure. An analog of weighted bootstrap was obtained in Chandrasekhar et al. (2011), where both the first and the second stage are evaluated in each repetition.

4 Applications

In this section, I apply the asymptotic theory of Section 3 to three empirically relevant settings: Endogenous Sample Selection of Lee (2008), Average Partial Derivative of Kaido (2017), and Partially Linear Predictor. For each setting, I derive an orthogonal moment equation for the support function and provide primitive sufficient conditions for the theoretic results of Section 3 to hold.

4.1 Endogenous Sample Selection

I start this section with the original assumptions of Lee (2008), under which the bounds in (2.4) and (2.5) are derived.

ASSUMPTION 6 (Identification in Endogenous Sample Selection)**.** *The following assumptions hold. (1) The program admission D is independent of the potential employment and wage outcomes, as well as the subject covariates:* $D \perp (S_1, S_0, Y_1, Y_0, X)$. *(2) The program admission D cannot hurt selection* $S_1 \geq S_0$ *a.s.*

Orthogonal moment equation for the upper bound β_U is

$$\mathbb{E}[\beta_U - \frac{D \cdot S \cdot Y 1_{\{Y \geq Q_{\{Y|D=1,S=1,X\}}(p_0(X),X)\}} \Pr(D=0)}{\Pr(D=0,S=1)\Pr(D=1)} + \sum_{j=1}^{3} \alpha_j(W,\eta)] = 0, \tag{4.1}$$

where $W = (D, S, SY, X)$ is the vector of data and $\alpha_j(W,\eta)$, $j \in \{1,2,3\}$ is the function that corrects bias of j'th component of η. The true values of each bias correction term are

$$\begin{aligned}
\alpha_1(W,\eta) &= y_{\{1-p_0(X),X\}} \frac{\Pr(D=0)}{\Pr(D=0,S=1)} \left(\frac{(1-D)S}{\Pr(D=0)} - \eta_1(X) \right), \\
\alpha_2(W,\eta) &= y_{\{1-p_0(X),X\}} \frac{\Pr(D=0)s(0,X)}{\Pr(D=0,S=1)s(1,X)} \left(\frac{DS}{\Pr(D=1)} - \eta_2(X) \right), \\
\alpha_3(W,\eta) &= -y_{\{1-p_0(X),X\}} \frac{\Pr(D=0)s(1,X)}{\Pr(D=0,S=1)} \left(1_{\{Y \leq Q_{\{Y|D=1,S=1,X\}}(1-p_0(X),X)\}\}} - 1 + p_0(X) \right).
\end{aligned}$$

ASSUMPTION 7 (Regularity Conditions for Endogenous Sample Selection)**.** *The following assumptions hold. (1) There exist both a lower bound* $\underline{s} > 0$ *and an upper bound* $\bar{s} < \infty$ *such that the conditional employment probability* $s(d,x)$ *is bounded from above and below:* $0 < \underline{s} \leq s(D,X) \leq \bar{s} < \infty$ *a.s. (2) Let* $\mathcal{U} \subset \mathcal{R}$ *be an open set that contains the support of* $s(0,X)/s(1,X)$ *and* $1 - s(0,X)/s(1,X)$ *a.s. in X. Assume that the conditional quantile function* $u \to Q_{Y|D=1,S=1,X=x}(u,x)$ *is differentiable on* $\mathcal{U}$ *absolutely surely in X, and its derivative is bounded by some* $K_Q < \infty$ $\Pr(\sup_{u \in cl(\mathcal{U})} |\partial_u Q_{Y|D=1,S=1,X=x}(u,X)| \leq K_Q) = 1$. *(3) Assume that there exists conditional density* $\rho(y)$ *of Y given* $D = 1, S = 1, X$ *a.s. in X that is bounded and has bounded first derivative. (4) There exists a rate* $g_N = o(N^{-1/4})$ *such that the first-stage estimates* $\hat{s}(D,X)$ *and* $\hat{Q}_{Y|D=1,S=1,X}(u,x)$ *converge at* g_N *rate:* $\forall d \in \{1,0\} \quad \|\hat{s}(d,X) - s(d,X)\|_{\{P,2\}} \lesssim g_N$, $\sup_{u \in \mathcal{U}} |\hat{Q}_{Y|D=1,S=1,X}(u,x) - Q_{Y|D=1,S=1,X=x}(u,X)| \lesssim g_N$.

Theorem 3 (Asymptotic Theory for Endogenous Sample Selection)**.** *Suppose Assumption 7 holds. Then, the estimator* $(\hat{\beta}_L, \hat{\beta}_U)$ *of Definition 5 obeys:*

$$\sqrt{N}\begin{pmatrix}\hat{\beta}_L - \beta_L \\ \hat{\beta}_U - \beta_U\end{pmatrix} \Rightarrow N(0,\Omega),$$

where Ω *is a positive-definite covariance matrix.*

Theorem 3 is my third main result. It establishes that the bounds defined by (10.12) are consistent and asymptotically normal. It extends the bounds estimator from Lee (2008), defined for a small number of discrete covariates, to the case of high-dimensional covariates that can be either discrete and continuous.

4.2 Average Partial Derivative

Consider the setup of Example 2. The constructed random variable V is equal to the derivative of the log conditional density $V = -\partial_D \log f(D|X) = \eta_0(D,X)$, the orthonormalized projection $z_q(\eta) := q^\top \eta(D,X)$ and the q-generator $Y_q = \Gamma(Y_L, Y_U - Y_L, q^\top \eta(D,X))$. The orthogonal moment equation (3.9) takes the form

$$g(W,q,\xi(q)) = z_q(\eta)Y_q(\eta) - q^\top V \gamma_q(D,X) + q^\top \partial_D \gamma_q(D,X), \tag{4.2}$$

where $z_q(\eta) = q^\top \eta(D,X)$, $Y_q(\eta) = \Gamma(Y_L, Y_U - Y_L, q^\top \eta(D,X))$, and $\gamma_q(D,X) = \gamma_L(D,X) + \gamma_{U-L}(D,X)$ $\cdot 1_{\{q^\top \partial_D \log f(D|X) > 0\}}$. The first-stage parameter of this problem is $\xi(D,X) = \{\eta(D,X), \gamma_L(D,X), \gamma_{U-L}(D,X)\}$ whose true value $\xi_0(D,X)$ is $\xi_0(D,X) = \{\partial_D \log f(D|X), \mathbb{E}[Y_L|D,X], \mathbb{E}[Y_{U-L}|D,X]\}$.

ASSUMPTION 8 (Regularity Conditions for Average Partial Derivative)**.** *The following conditions hold. (1) There exists a bound* $B_{UL} < \infty$ *such that the interval width* $Y_U - Y_L \leq B_{UL}$ *is bounded a.s. (2) The distribution of the gradient of the log-density* $\eta_0(D,X) = \partial_D \log f(D|X)$ *is sufficiently continuous. For the event* $\mathcal{E}_q := \{0 < |q^\top \partial_D \eta_0(D,X)| < |q^\top \partial_D(\eta(D,X) - \eta_0(D,X))|\}$*, the following bound holds*

$$\sup_{q \in \mathcal{S}^{d-1}} \mathbb{E}|q^\top \partial_D(\eta(D,X) - \eta_0(D,X))| 1_{\{\mathcal{E}_q\}} \leq \|\eta(D,X) - \eta_0(D,X)\|_{L_{P,2}}.$$

(3) There exist a rate $g_N = o(N^{-1/4})$ such that each component of $\hat{\xi}(D,X)$ converge at g_N rate: $\|\hat{\xi}(D,X) - \xi_0(D,X)\|_{\{P,2\}} \lesssim g_N$.

Theorem 4 (Asymptotic Theory for Average Partial Derivative with an Interval-Valued Outcome)**.** *Suppose Assumption 8 holds. Then, Theorems 1 and 2 hold for the Support Function Estimator of Definition 4 with the influence function equal to*

$$h(W,q) = g(W,q,\xi(q)) - \mathbb{E}[g(W,q,\xi(q))],$$

where $g(W,q,\xi(q))$ is given in (4.2).

Theorem 4 says that the Support Function Estimator given in (4) is uniformly consistent and asymptotically Gaussian. It extends the support function estimator of Kaido (2017), defined for a small number of covariates, to the case of high-dimensional covariates.

Remark 1. *Suppose the first-stage residual $V = D - \mathbb{E}[D|X]$ is normally distributed independently from X (i.e, $V \sim N(0,\Lambda)$). Then, Assumption 8 (2) holds. Furthermore, the estimator of the support function can be simplified as shown in the algorithm below.*

Algorithm 1 Support Function Estimator for Average Partial Derivative

Let $m_0(X) = \mathbb{E}[D|X]$. Input: a direction q on a unit sphere $\mathcal{S}^{d-1}$, an i.i.d sample $(W_i)_{i=1}^N = (D_i, X_i, Y_{L,i}, Y_{U,i})_{i=1}^N$, estimated values $(\hat{m}(X_i), \hat{\gamma}_L(D_i,X_i), \hat{\gamma}_{U-L}(D_i,X_i))_{i=1}^N$.

1: Estimate the first-stage residual for every $i \in \{1,2,\ldots,N\}$: $\hat{V}_i := D_i - \hat{m}(X_i)$.
2: Compute the sample covariance matrix of the first-stage residuals: $\hat{\Lambda} := \frac{1}{N}\sum_{i=1}^N \hat{V}_i \hat{V}_i^\top$.
3: Estimate the q-generator for every $i \in \{1,2,\ldots,N\}$

$$\hat{Y}_{q,i} := Y_{L,i} + (Y_{U,i} - Y_{L,i}) 1_{\{q^\top \hat{\Lambda}^{-1} \hat{V}_i > 0\}}.$$

4: Compute the second-stage reduced form $\hat{\gamma}_q(D_i,X_i) := \hat{\gamma}_L(D_i,X_i) + \hat{\gamma}_{U-L}(D_i,X_i) 1_{\{q^\top \hat{\Lambda}^{-1} \hat{V}_i > 0\}}$
5: Estimate $\hat{\beta}_q$ by Ordinary Least Squares with the second-stage residual of the q-generator as the dependent variable and the first-stage residual V as the regressor

$$\hat{\beta}_q = \hat{\Lambda}^{-1} \frac{1}{N} \sum_{i=1}^N \hat{V}_i [\hat{Y}_{q,i} - \hat{\gamma}_q(D_i,X_i)].$$

Return: the projection of $\hat{\beta}_q$ on the direction q: $\hat{\sigma}(q,\mathcal{B}) = q^\top \hat{\beta}_q$.

4.3 Partially Linear Predictor

Consider the setup in Example 3. The constructed variable $V_\eta = D - \eta(X)$ is equal to the first-stage residual, the orthonormalized projection $z_q(\eta)$ is equal to the inner product of this residual and the vector $p(q) = (\Sigma^{-1})^\top q$, $z_q(\eta) = q^\top \Sigma^{-1}(D - \eta(X))$, and the q-generator Y_q is equal to $Y_q(\eta) = Y_L + (Y_U - Y_L)1_{\{q^\top \Sigma^{-1}(D-\eta(X))>0\}}$. Equation (3.6) does not satisfy Assumption 3(1). The Neyman orthogonal moment equation (3.9) is

$$\psi(W, \theta(q), \xi(p(q))) = \begin{bmatrix} \sigma(q, \mathcal{B}) - p(q)^\top (D - \eta(X))(Y_q(\eta) - \mathbb{E}[Y_q(\eta)|X]) \\ (D - \eta(X))(D - \eta(X))^\top p(q) - q \end{bmatrix}, \tag{4.3}$$

where the true value of $\theta(q)$ is $\theta_0(q) = [\sigma(q, \mathcal{B}), p(q)]$ and the true value of the nuisance parameter $\xi(p) = \{\eta(X), \gamma(p, X)\}$ is

$$\xi_0(p(q)) = \{\eta_0(X), \mathbb{E}[Y_q(\eta_0)|X]\}.$$

Assumption 9 gives the regularity conditions for the Support Function Estimator.

ASSUMPTION 9 (Regularity Conditions for Partially Linear Predictor)**.** *The following regularity conditions hold. (1) There exists $\bar{D} < \infty$ such that* $\max(\|D\|, |Y_L|, |Y_U|) \leq \bar{D}$ *holds absolutely surely. (2) For all vectors $p \in \mathcal{P}$ there exists a conditional density $\rho_{\{p^\top V|X=x\}}(\cdot, x)$ absolutely surely in X. (3) A bound $K_h < \infty$ exists such that the collection of the densities in (2) $\{\rho_{\{p^\top V|X=x\}}(\cdot, x), p \in \mathcal{P}\}$ is uniformly bounded over $p \in \mathcal{P}$ a.s. in X. (4) Let $\mathcal{F}_\gamma = \{\gamma(p, x) : \mathcal{P} \bigtimes \mathcal{X} \to \mathcal{R}\}$ be a class of functions in p, x that satisfy Conditions 4(1,2). Moreover, there exists a sequence of realization sets $\mathcal{G}_N$ that are subsets of $\mathcal{F}_\gamma$ $\mathcal{G}_N \subseteq \mathcal{F}_\gamma$ such that the estimator $\hat{\gamma}(\cdot, \cdot)$ belongs to $\mathcal{G}_N$ with probability at least $1 - \phi_N$. Moreover, the nuisance realization set shrinks at a statistical rate uniformly in $p \in \mathcal{P}$* $\sup_{p \in \mathcal{P}} \sup_{\gamma(\cdot,\cdot) \in \mathcal{G}_N} \|\gamma(p, X) - \gamma_0(p, X))\|_{L_{P,2}} \leq g_N = o(N^{-1/4})$. *(5) Let $\gamma_0(p, x)$ be the conditional expectation on X of the q-generator $\gamma_0(p, x) = \mathbb{E}[Y_q|X] = \mathbb{E}[\Gamma(Y_L, Y_U - Y_L, p^\top V_{\eta_0})|X]$. Assume that there exists a sequence g'_N such that $\gamma_0(p, x)$ is continuous uniformly on $\mathcal{P}$, on average in X:* $\sup_{q \in \mathcal{S}^{d-1}} \sup_{p \in \mathcal{P}: \|p - p_0(q)\| \leq RN^{-1/2}} \|\gamma_0(p, X) - \gamma_0(p_0(q), X))\|_{L_{P,2}} \leq g'_N : g'_N \log(1/g'_N) = o(1)$. *(6) The first-stage residual $D - \eta_0(X)$ has a uniformly sufficiently smooth distribution on $\mathcal{P}$. Namely, for some m such that $0 < m \leq 1$, the following bound holds:* $\sup_{p \in \mathcal{P}} \Pr(0 < \frac{p^\top (D - \eta_0(X))}{\|D - \eta_0(X)\|} < \delta) = O(\delta^m), \delta \to 0$.

Theorem 5 (Asymptotic Theory for Partially Linear Predictor with an Interval-Valued Outcome)**.** *Suppose Assumption 2 and 9 holds. Then, Theorem 1 and 2 hold for the Support Function Estimator with the influence function $h(W,q)$ equal to*

$$\begin{aligned} h(W,q) = {} & g(W,p_0(q),\xi_0(p_0(q),X)) - \mathbb{E}[g(W,p_0(q),\xi_0(p_0(q),X))] \\ & - q^\top \Sigma^{-1} \left((D-\eta_0(X))(D-\eta_0(X))' - \Sigma\right) \Sigma^{-1} \mathbb{E}(D-\eta_0(X)) Y_q, \end{aligned}$$

where $p_0(q) = (\Sigma^{-1})^\top q$ and $Y_q = Y_L + (Y_U - Y_L) 1_{q^\top \Sigma^{-1}(D-\eta_0(X))>0}$.

Theorem 5 is my fifth main result. It establishes that the Support Function Estimator given in Algorithm 2 is uniformly consistent and asymptotically normal.

I describe the computation steps of the Support Function Estimator $\hat{\sigma}(q,\mathcal{B})$ in the following algorithm.

Algorithm 2 Support Function Estimator for Partially Linear Predictor

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Input: a direction q on a unit sphere $\mathcal{S}^{d-1}$, an i.i.d sample $(W_i)_{i=1}^N = (D_i, X_i, Y_{L,i}, Y_{U,i})_{i=1}^N$, estimated values $(\hat{\eta}(X_i), \hat{\gamma}(p,X_i))_{i=1}^N, p \in \mathcal{P}$.

1: Estimate the first-stage residual for every $i \in \{1,2,\ldots,N\}$: $\hat{V}_i := D_i - \hat{\eta}(X_i)$.

2: Compute the sample covariance matrix of the first-stage residuals: $\hat{\Sigma} := \frac{1}{N}\sum_{i=1}^N \hat{V}_i \hat{V}_i^\top$.

3: Estimate the q-generator for every $i \in \{1,2,\ldots,N\}$

$$\hat{Y}_{q,i} := Y_{L,i} + (Y_{U,i} - Y_{L,i}) 1_{\{q^\top \hat{\Sigma}^{-1} \hat{V}_i > 0\}}.$$

4: Estimate $\hat{\beta}_q$ by Ordinary Least Squares with the second-stage residual of the q-generator as the dependent variable and the first-stage residual V as the regressor

$$\hat{\beta}_q = \hat{\Sigma}^{-1} \frac{1}{N} \sum_{i=1}^N \hat{V}_i [\hat{Y}_{q,i} - \hat{\gamma}(\hat{\Sigma}^{-1} q^\top, X_i)].$$

Return: the projection of $\hat{\beta}_q$ on the direction q: $\hat{\sigma}(q,\mathcal{B}) = q^\top \hat{\beta}_q$.

ASSUMPTION 10 (Simple Sufficient Conditions for Partially Linear Predictor)**.** *The following conditions hold. (1) The first-stage residual is independent from the covariates X and has a symmetric continuous distribution around zero (i.e., $\Pr(p^\top V > 0) = \frac{1}{2} \quad \forall p \in \mathcal{R}^d$). (2) Conditional on X, the interval width $Y_U - Y_L$ and the covariate D are mean independent*

$$\mathbb{E}[(Y_U - Y_L) 1_{\{p'V>0\}} | X] = \mathbb{E}[Y_U - Y_L | X] \Pr(p'V > 0 | X) = \frac{1}{2} \mathbb{E}[Y_U - Y_L | X].$$

(3) There exists a sequence of realization sets $\{\mathrm{Z}_{L,N}, N \geq 1\}$ *and* $\mathrm{Z}_{U-L,N}, N \geq 1\}$ *that are shrinking neighborhoods of* $\gamma_{L,0}(X) := \mathbb{E}[Y_L|X]$ *and* $\gamma_{U-L,0}(X) := \mathbb{E}[Y_U - Y_L|X]$ *obeying the following conditions. For some sequence* $\phi_N = o(1)$ *the estimate* $\hat{\gamma}_L$ *of* $\gamma_{L,0}$ *and* $\hat{\gamma}_{U-L}$ *of* $\gamma_{U-L,0}$ *belong to respective sets* $\mathrm{Z}_{L,N}$ *and* $\mathrm{Z}_{U-L,N}$ *with probability at least* $1-\phi_N$*. There exist rates* $\zeta_{L,N} = o(N^{-1/4})$ *and* $\zeta_{U-L,N} = o(N^{-1/4})$ *such that the following bounds hold*

$$\sup_{\gamma_L \in \mathrm{Z}_{L,N}} (\mathbb{E}(\gamma_L(X) - \gamma_{L,0}(X))^2)^{1/2} \lesssim \zeta_{L,N}, \qquad \sup_{\gamma_{U-L} \in \mathrm{Z}_{U-L,N}} (\mathbb{E}(\gamma_{U-L}(X) - \gamma_{U-L,0}(X))^2)^{1/2} \lesssim \zeta_{U-L,N}.$$

Remark 2. *Simple Sufficient Conditions for Partially Linear Predictor Suppose Assumption 10 holds. Then, the conditional expectation function* $\gamma_0(q,X)$ *no longer depends on* q*:*

$$\gamma_0(p,X) := \gamma_0(X) = \mathbb{E}[Y_L|X] + \frac{1}{2}\mathbb{E}[Y_U - Y_L|X] = \gamma_{L,0}(X) + \frac{1}{2}\gamma_{U-L,0}(X).$$

Therefore, Assumptions 9 (4)-(6) are satisfied as long as the functions $\gamma_{L,0}(X)$ *and* $\gamma_{U-L,0}(X)$ *are estimated at the* $o(N^{-1/4})$ *rate.*

5 Empirical Application

In this section, I re-examine the effectiveness of Colombia PACES program, a voucher initiative established in 1991 to subsidize private school education in low-income population, studied in Angrist et al. (2002) and in Angrist et al. (2006). After being admitted to a private school, a student participates in a lottery to win a voucher that partially covers his tuition fee. Each year, a student can renew an existing voucher if he passes to the next grade. After high school graduation, some students take a centralized test to enter a college. Following Angrist et al. (2006), I am interested in the average effect of winning the private school voucher today on the college admission test scores several years later.

I use the notation of Example 1 to define the voucher's effect. The variable $D = 1$ is a dummy for whether a student has won a voucher, $S_0 = 1$ is a dummy for whether a student would have participated in a test after losing the voucher, $S_1 = 1$ is a dummy for whether a student would have participated in a test after winning the voucher. Similarly, the potential test scores Y_0 and Y_1 are the scores a student would have had after losing and winning the lottery, respectively. I am interested

in the average voucher's effect on the group of students who would have taken the test regardless of receiving the voucher

$$\mathbb{E}[Y_1 - Y_0 | S_1 = 1, S_0 = 1],$$

or, briefly, the always-takers. The data contain the voucher status D, observed test participation[3] S, test score $S \cdot Y$ observed only if a student takes a test, and the covariates. The covariates X include age, phone access, gender, and four indicators of having an invalid or inaccurately recorded ID constructed by Angrist et al. (2006) by matching PACES records to administrative data.

Because test participation may be endogenous, the average voucher effect is not point-identified. To bound the effect, Angrist et al. (2006) make two assumptions: receiving a voucher can neither deter the test participation $S_1 \geq S_0$ a.s. , nor hurt the test scores $Y_1 \geq Y_0$ a.s. . Angrist et al. (2006) state that the monotonicity assumption about the test scores may not hold if private school applicants anticipated educational gains that did not materialize. To relax this assumption, I use Lee (2008)'s bounds that are based only on the first assumption. I describe the construction of Lee (2008)' bounds in Example 1.

I estimate Lee (2008)'s bounds with all covariates in two stages. In the first stage, I estimate the probability of receiving the voucher given covariates X (i.e, $\Pr(D = 1)$), the probability of test participation given the voucher status D and covariates X (i.e, $s(D,X) = \mathbb{E}[S = 1|D,X]$), and the quantile function of the winners' test scores given the covariates. I estimate the first two functions using logistic lasso algorithm of Belloni et al. (2016) with the penalty choice described in Chernozhukov et al. (2018) package. Assuming that winners' test scores are determined by age and gender only, I estimate the quantile function by taking an empirical quantile[4] in the relevant group. In the second stage, I plug the estimates into Neyman-orthogonal moment equations for the bounds given in (10.12).

My empirical findings are as follows. Lee (2008)'s bounds based on the choice of covariates, made by Angrist et al. (2006), have opposite signs and cannot determine the direction of the effect

[3]The test participation S is not explicitly recorded in the data. I conclude that a student comes to a test if and only if his test score is positive $S = 1_{\{Y>0\}}$. My conclusion is based on two facts. For a given subject, Angrist et al. (2006) interprets the subset of voucher losers with positive test scores as the always-takers (page 14). To arrive at this interpretation, one needs to assume that $S = 1_{\{Y>0\}}$. Second, the test scores have a 66% point mass at zero value for both subjects.

[4]Because the test scores' distribution had multiple point masses, I added a small amount of $N(0, 0.01)$ distributed noise in order to compute the exact quantiles.

(Table 1, Column 2). An attempt to include all 7 covariates into Lee (2008)'s bounds runs into perfect multicollinearity problem. In particular, certain cells determined by the partition of the samples using a subset of binary covariates, have no variation in treatment assignment. Because I apply logistic lasso to estimate selection equation, some of the redundant covariates are dropped, and variation in treatment assignment is restored. The bounds produced by my method (Table 1, Column 3) are positive, four times narrower than the bounds in Column 2, and significant in case of Language.

Bounds become narrow because Angrist et al. (2006) data set contains a very informative covariate whose power to predict test-taking decision have been previously overlooked. This covariate, referred to as ID validity, is a binary indicator for whether ID number recorded at the moment of lottery registration, can be matched with an administrative database (i.e., corresponds to a valid ID), and, therefore, is pre-determined. This covariate explains 96% of the total variance in test participation, while age and gender, that have been previously used, explain only 35%. Once ID validity is taken into account, voucher has little effect on the test-taking decision, resulting in tighter bounds.

6 Conclusion

In this paper, I incorporate machine learning tools into set-identification and harness their predictive power to tighten an identified set. I focus on the set-identified models with high-dimensional covariates and provide two-stage estimation and inference methods for an identified set. In the first stage, I select covariates (or estimate a nonparametric function of them) using machine learning tools. In the second stage, I plug the estimates into the moment equation for the identified set's boundary that is insensitive, or, formally, Neyman-orthogonal, to the bias in the first-stage estimates. I establish the uniform limit theory for the proposed estimator and the weighted bootstrap procedure and provide a general recipe to construct a Neyman-orthogonal moment function starting from a non-orthogonal one.

My method's main application is to estimate Lee (2008) nonparametric bounds on the average treatment effect in the presence of endogenous selection. I derive a Neyman-orthogonal moment equation for Lee (2008)'s bounds and provide primitive sufficient conditions for their validity.

Moreover, I substantially tighten Lee (2008)'s bounds in the data from Angrist et al. (2006). In addition, I also provide the low-level sufficient conditions to estimate sharp identified sets for two other parameters - the causal parameter in the partially linear model and the average partial derivative when the outcome variable is interval-censored.

7 Appendix

Notation. We use the standard notation for vector and matrix norms. For a vector $v \in \mathcal{R}^d$, denote the ℓ_2 norm of a as $\|v\|_2 := \sqrt{\sum_{j=1}^d v_j^2}$. Denote the ℓ_1 norm of v as $\|v\|_1 := \sum_{j=1}^d |v_j|$, the ℓ_∞ norm of v as $\|v\|_\infty := \max_{1\le j\le d} |v|_j$, and ℓ_0 norm of v as $\|v\|_0 := \sum_{j=1}^d 1_{\{a_j \neq 0\}}$. Denote a unit sphere as $\mathcal{S}^{d-1} = \{\alpha \in \mathcal{R}^d : \|\alpha\| = 1\}$. For a matrix M, denote its operator norm by $\|M\|_2 = \sup_{\alpha \in \mathcal{S}^{d-1}} \|M\alpha\|$. We use standard notation for numeric and stochastic dominance. For two numeric sequences $\{a_n, b_n\}, n \ge 1$ $a_n \lesssim b_n$ stands for $a_n = O(b_n)$. For two sequences of random variables $\{a_n, b_n, n \ge 1\}$: $a_n \lesssim_P b_n$ stands for $a_n = O_P(b_n)$. Finally, let $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. For a random variable ξ, $\quad (\xi)^0 := \xi - \mathbb{E}[\xi]$.

Fix a partition k in a set of partitions $[K] = \{1, 2, \ldots, K\}$. Define the sample average of a function $f(\cdot)$ within this partition as: $\mathbb{E}_{n,k}[f] = \frac{1}{n}\sum_{i \in J_k} f(x_i)$ and the scaled normalized sample average as:

$$\mathbb{G}_{n,k}[f] = \frac{\sqrt{n}}{n} \sum_{i \in J_k} [f(x_i) - \mathbb{E}[f(x_i)|J_k^c]],$$

where $[\cdot|J_k^c] := [\cdot|(W_i, i \in J_k^c)]$. For each partition index $k \in [K]$ define an event $\mathcal{E}_{n,k} := \{\hat{\xi}_k \in \Xi_n\}$ as the nuisance estimate $\hat{\xi}_k$ belonging to the nuisance realization set $\mathcal{G}_N$. Define $\mathcal{E}_N = \cap_{k=1}^K \mathcal{E}_{n,k}$ as the intersection of such events.

7.1 Proofs

This section contains the proofs of main results of this paper. Section 7.1.1 contains the proofs of Theorem 1 and Theorem 2 from Section 3. Section 7.1.2 contains the proofs of Theorem 5. Other Lemmas can be found in Supplementary Appendix.

7.1.1 Proofs of Section 3

Proof of Theorem 1, Σ is known. To simplify notation, we assume $\Sigma = I_d$. The proof holds for any invertible matrix Σ. Let us focus on the partition $k \in [K]$.

$$\begin{aligned}\sqrt{n}|\mathbb{E}_{n,k}[g(W_i,q,\hat{\xi}(q)) - g(W_i,q,\xi_0(q))]| &\leq \sqrt{n}|\mathbb{E}\big[g(W_i,q,\hat{\xi}(q)) - g(W_i,q,\xi_0(q))\big]| \\ &\quad + |\mathbb{G}_{n,k}[g(W_i,q,\hat{\xi}(q)) - g(W_i,q,\xi_0(q))]| \\ &=: |i(q)| + |ii(q)|.\end{aligned}$$

Step 1. Recognize that $|i(q)|$ converges to zero conditionally on the partition complement J_k^c and the event $\mathcal{E}_N$:

$$\begin{aligned}|i(q)| &:= \sup_{q\in\mathcal{S}^{d-1}} \sqrt{n}|\mathbb{E}[g(W_i,q,\hat{\xi}(q)) - g(W_i,q,\xi_0(q))|\mathcal{E}_N \cup J_k^c]| \\ &\leq \sup_{q\in\mathcal{S}^{d-1}} \sup_{\xi\in\Xi_n} \sqrt{n}|\mathbb{E}[g(W_i,q,\xi(q)) - g(W_i,q,\xi_0(q))|\mathcal{E}_N \cup J_k^c]| \\ &\leq \sup_{q\in\mathcal{S}^{d-1}} \sup_{\xi\in\Xi_n} \sqrt{n}|\mathbb{E}[g(W_i,q,\xi(q)) - g(W_i,q,\xi_0(q))]| \\ &\leq \sqrt{n}\mu_n = o(1).\end{aligned}$$

By Lemma 6.1 of Chernozhukov et al. (2017a), the term $i(q) = O(\mu_n) = o(1)$ unconditionally.

Step 2. To bound the second quantity, consider the function class

$$\mathcal{F}_{\hat{\xi}\xi_0} = \{g(W_i,q,\hat{\xi}(q)) - g(W_i,q,\xi_0(q)), \quad q \in \mathcal{S}^{d-1}\}.$$

for some fixed $\hat{\xi}$. By definition of the class,

$$\mathbb{E}\sup_{q\in\mathcal{S}^{d-1}} |ii(q)| := \mathbb{E}\sup_{f\in\mathcal{F}} |\mathbb{G}_{n,k}[f]|.$$

We apply Lemma 6.2 of Chernozhukov et al. (2017a) conditionally on the hold-out sample J_k^c and the event $\mathcal{E}_N$ so that $\hat{\xi}(q) = \hat{\xi}_k$ can be treated as a fixed member of Ξ_n. The function class $\mathcal{F}_{\hat{\xi}\xi_0}$ is obtained as the difference of two function classes: $\mathcal{F}_{\hat{\xi}\xi_0} := \mathcal{F}_{\hat{\xi}} - \mathcal{F}_{\xi_0}$, each of which has an

integrable envelope and bounded logarithm of covering numbers by Assumption 4. In particular, one can choose an integrable envelope as $F_{\hat{\xi}\xi_0} := F_{\hat{\xi}} + F_{\xi_0}$ and bound the covering numbers as:

$$\log \sup_Q N(\varepsilon \|F_{\hat{\xi}\xi_0}\|_{Q,2}, \mathcal{F}_{\hat{\xi}\xi_0}, \|\cdot\|) \leq \log \sup_Q N(\varepsilon \|F_{\hat{\xi}}\|_{Q,2}, \mathcal{F}_{\hat{\xi}}, \|\cdot\|) + \log \sup_Q N(\varepsilon \|F_{\xi_0}\|_{Q,2}, \mathcal{F}_{\xi_0}, \|\cdot\|)$$
$$\leq 2v \log(a/\varepsilon), \quad \text{for all } 0 < \varepsilon \leq 1.$$

Finally, we can choose the speed of shrinkage $(r')_n^2$ such that

$$\sup_{q \in \mathcal{S}^{d-1}} \sup_{\xi \in \Xi_n} \left(\mathbb{E}[g(W_i, q, \xi(q)) - g(W_i, q, \xi_0(q))]^2 \right)^{1/2} \leq r'_n,$$

the application of Lemma 6.2 of Chernozhukov et al. (2017a) gives with $M := \max_{i \in I_k^c} F_{\hat{\xi}\xi_0}(W_i)$

$$\begin{aligned}
\sup_{q \in \mathcal{S}^{d-1}} |ii(q)| &\leq \sup_{q \in \mathcal{S}^{d-1}} |\mathbb{G}_{n,k}[g(W_i, q, \hat{\xi}(q)) - g(W_i, q, \xi_0(q))]| \\
&\leq \sqrt{v(r')_n^2 \log(a\|F_{\hat{\xi}\xi_0}\|_{P,2}/r'_n)} + v\|M\|_{P,c'}/\sqrt{n} \log(a\|F_{\hat{\xi}\xi_0}\|_{P,2}/r'_n) \\
&\lesssim_P r'_n \log^{1/2}(1/r'_n) + n^{-1/2+1/c'} \log^{1/2}(1/r'_n)
\end{aligned}$$

where a constant $\|M\|_{P,c'} \leq n^{1/c'}\|F\|_{P,c'}$ for the constant $c' \geq 2$ in Assumption 4.

Step 3. Asymptotic Normality. By Theorem 19.14 from van der Vaart (1998), Assumption 4 implies that the function class $\mathcal{F}_{\xi_0} = \{g(W, q, \xi_0(q)), \quad q \in \mathcal{S}^{d-1}\}$ is P-Donsker. Therefore, the asymptotic representation follows from the Skorohod-Dudley-Whichura representation, assuming the space $L^\infty(\mathcal{S}^{d-1})$ is rich enough to support this representation. □

Proof of Theorem 1, Σ is unknown. Step 1.$\sqrt{n}$-Convergence of Matrix Estimator. Let us show that there exists $\phi_N = o(1)$ and a constant R such that with probability at least $1 - \phi_N$,

$$\|\hat{\Sigma} - \Sigma\| \leq RN^{-1/2},$$

where

$$\hat{\Sigma} := \mathbb{E}_N A(W_i, \hat{\eta}_i) = \frac{1}{K}\sum_{k=1}^{K} \underbrace{\mathbb{E}_{n,k} A(W_i, \hat{\eta}_i) - \mathbb{E}_{n,k} A(W_i, \eta_0)}_{I_{1,k}} + \mathbb{E}_N A(W_i, \eta_0) - \mathbb{E} A(W_i, \eta_0).$$

Recognize that the first and the second moments of $\sqrt{N} I_{1,k}$ converge to zero conditionally on the partition complement J_k^c and the event $\mathcal{E}_N$. The first moment is bounded as:

$$\begin{aligned}
\sqrt{n}\|\mathbb{E} I_{1,k} | \mathcal{E}_N \cup J_k^c]\| &:= \sqrt{n}\|\mathbb{E}[A(W_i, \hat{\eta}) - A(W_i, \eta_0)] | \mathcal{E}_N \cup J_k^c]\| \\
&\leq \sup_{\eta \in \mathcal{T}_N} \sqrt{n}\|\mathbb{E}[A(W_i, \eta) - A(W_i, \eta_0)] | \mathcal{E}_N \cup J_k^c]\| \\
&\leq \sqrt{n}\mu_n = o(1).
\end{aligned}$$

By Assumption 4, the bound on the second moment $\|I_{1,k}\|^2$ applies:

$$n\mathbb{E}[\|I_{1,k}\|^2 | \mathcal{E} \cup J_k^c] \leq \sup_{\eta \in \mathcal{T}_n} \mathbb{E}\|A(W_i, \eta) - A(W_i, \eta_0)\|^2 \leq \delta_n = o(1).$$

Applying the Markov inequality conditionally on $J_k^c, \mathcal{E}_N$ yields: $ii = o_P(\delta_n)$. By Lemma 6.1 of Chernozhukov et al. (2017a), conditional convergence to zero implies unconditional convergence. Therefore, for each $k \in [K]$, $I_{1,k} = o_P(1)$. Since the number of partitions K is finite, $\frac{1}{K}\sum_{k=1}^{K} I_{1,k} = o_P(1)$. The application of the Law of Large Numbers for Matrices to the term $\mathbb{E}_N[A(W_i, \eta_0) - \Sigma]$ yields: $\|\mathbb{E}_N A(W_i, \eta_0) - \Sigma\| = O_P(N^{-1/2})$.

Step 2. Decomposition of the error. Fix a generic element of this set $p \in \mathcal{P}$ and $\xi \in \Xi_N$.

$$\begin{aligned}
\mathbb{E}_N g(W_i, p, \xi(p)) &= \mathbb{E}_N g(W_i, p_0(q), \xi_0(p_0(q))) \\
&+ \underbrace{\mathbb{E}_N[g(W_i, p, \xi(p)) - g(W_i, p_0(q), \xi_0(p_0(q)))]^0}_{I_3} \\
&+ \underbrace{\mathbb{E}[g(W_i, p, \xi(p)) - g(W_i, p_0(q), \xi_0(p_0(q)))]}_{I_4}.
\end{aligned} \tag{7.1}$$

Consider the following expansion:

$$
\begin{aligned}
I_4 &= \mathbb{E}[g(W_i,p,\xi(p)) - g(W_i,p,\xi_0(p))] + \mathbb{E}[g(W_i,p,\xi_0(p)) - \mathbb{E}g(W_i,p_0(q),\xi_0(p_0(q)))] \\
&= \underbrace{\mathbb{E}[g(W_i,p,\xi(p)) - \mathbb{E}g(W_i,p,\xi_0(p))]}_{I_{4,1}} + J_0(p_0(q))[p - p_0(q)] + R(p,p_0(q)),
\end{aligned}
$$

where the gradient $J_0(p_0(q)) = \nabla_{p_0(q)}\mathbb{E}g(W_i,p_0(q),\xi_0(p))$. By Assumption 5, the remainder term $R(p,p_0) = o(\|p - p_0(q)\|)$ uniformly over $q \in \mathcal{S}^{d-1}$. By Assumption 4 applied on $\mathcal{P}$, the term $I_{4,1}$ in (7.1) is bounded as:

$$
\begin{aligned}
\sqrt{n}I_{4,1} &= \sqrt{n}|\mathbb{E}[g(W_i,p,\xi(p)) - g(W_i,p,\xi_0(p))]| \\
&\leq \sqrt{n}\sup_{p\in\mathcal{P}}\sup_{\xi\in\Xi_n}|\mathbb{E}[g(W_i,p,\xi(p)) - g(W_i,p,\xi_0(p))]| \\
&\leq \sqrt{n}r_n'' = o(1).
\end{aligned}
$$

Step 4. The bound on I_3. We have

$$
\mathcal{F}_{\xi\xi_0} := \{g(\cdot,p,\xi(p)) - g(\cdot,p_0(q),\xi(p_0(q))), p \in \mathcal{P}, q \in \mathcal{S}^{d-1}, \|p - p_0(q)\| \leq RN^{-1/2}\}.
$$

We apply Lemma 6.2 of Chernozhukov et al. (2017a) conditionally on J_k^c and $\mathcal{E}_N$, so that $\hat{\xi}$ can be treated as fixed. By Assumption 4, the class $\mathcal{F}_{\hat{\xi}\xi_0}$ has a measurable envelope $F_{\hat{\xi}\xi_0} := F_{\hat{\xi}} + F_{\xi_0}$:

$$
\begin{aligned}
&\sup_{q\in\mathcal{S}^{d-1}}\sup_{p\in\mathcal{P},\|p-p_0(q)\|\leq RN^{-1/2}}|g(W_i,p,\xi(p)) - g(W_i,p_0(q),\xi(p_0(q)))| \\
&\leq \sup_{q\in\mathcal{S}^{d-1}}\sup_{p\in\mathcal{P},\|p-p_0(q)\|\leq RN^{-1/2}}|g(W_i,p,\xi(p))| + \sup_{q\in\mathcal{S}^{d-1}}\sup_{p\in\mathcal{P},\|p-p_0(q)\|\leq RN^{-1/2}}|g(W_i,p,\xi(p))| \\
&\leq F_\xi + F_{\xi_0}.
\end{aligned}
$$

Moreover, the uniform covering entropy of the function class $F_{\xi\xi_0} := F_\xi + F_{\xi_0}$ is bounded by the

sum of the entropies of F_ξ and F_{ξ_0}. Finally,

$$\sup_{q\in\mathcal{S}^{d-1}}\sup_{p\in\mathcal{P},\|p-p_0(q)\|\le RN^{-1/2}}\sup_{\xi\in\Xi_n}\left(\mathbb{E}(g(W_i,p,\xi(p))-g(W_i,p_0(q),\xi_0(p_0(q))))^2\right)^{1/2}$$
$$\lesssim r_n'\log^{1/2}(1/r_n')+n^{-1/2+1/c}\log^{1/2}n.$$

□

Lemma 6. *Let $\{D_i(p), p\in\mathcal{P}\}$ be a function class with a bounded uniform covering entropy and $\mathbb{E}D_i(p)=0 \quad \forall p\in\mathcal{P}$. Let $(D_i(p))_{i=1}^N$ be an i.i.d sequence of random functions. Let $(e_i)_{i=1}^N$ be an i.i.d sequence of $Exp(1)$ random variables independent from $(D_i(p))_{i=1}^N$. Then uniformly in $\mathcal{P}$*

$$\sqrt{N}\mathbb{E}_N\frac{e_i}{\bar{e}}D_i(p)=\sqrt{N}\mathbb{E}_N e_iD_i(p)+\sqrt{N}\mathbb{E}_N e_iD_i(p)\underbrace{\frac{1-\bar{e}}{\bar{e}}}_{o_P(1)}=\sqrt{N}\mathbb{E}_N e_iD_i(p)(1+o_P(1)).$$

Lemma 7. *Let Assumptions 3 - 4 hold. Then*

$$\sqrt{N}\mathbb{E}_N\big(g(W_i,(\tilde{\Sigma}^{-1})^\top q,\hat{\xi}((\tilde{\Sigma}^{-1})^\top q))-g(W_i,(\tilde{\Sigma}^{-1})^\top q,\xi_0((\tilde{\Sigma}^{-1})^\top q))\big)\frac{e_i}{\bar{e}}=o_P(1).$$

Proof. Step 1. Decompose the sample average into the sample averages within each partition:

$$\mathbb{E}_N\big(g(W_i,(\tilde{\Sigma}^{-1})^\top q,\hat{\xi}(p))-g(W_i,(\tilde{\Sigma}^{-1})^\top q,\xi_0((\tilde{\Sigma}^{-1})^\top q))\big)\frac{e_i}{\bar{e}}$$
$$=\frac{1}{K}\sum_{k=1}^K\mathbb{E}_{n,k}\big(g(W_i,(\tilde{\Sigma}^{-1})^\top q,\hat{\xi}(p))-g(W_i,(\tilde{\Sigma}^{-1})^\top q,\xi_0((\tilde{\Sigma}^{-1})^\top q))\big)\frac{e_i}{\bar{e}}.$$

Since the number of partitions K is finite, it suffices to show that the bound holds on every partition:

$$\mathbb{E}_{n,k}\big(g(W_i,(\tilde{\Sigma}^{-1})^\top q,\hat{\xi}(p))-g(W_i,(\tilde{\Sigma}^{-1})^\top q,\xi_0((\tilde{\Sigma}^{-1})^\top q))\big)\frac{e_i}{\bar{e}}=o_P(1).$$

Let $\mathcal{E}_N:=\cap_{k=1}^K\{\hat{\xi}_{I_k^c}\in\Xi_n\}$. By Assumption 4 $\Pr(\mathcal{E}_N)\ge 1-K\phi_N=1-o(1)$. The analysis below is conditionally on $\mathcal{E}_N$ for some fixed element $\hat{\xi}_k\in\Xi_n$. Since the probability of $\mathcal{E}_N$ approaches one, the statements continue to hold unconditionally, which follows from the Lemma 6.1 of Chernozhukov et al. (2017a).

Step 2. Consider the function class $\mathcal{F}_{\xi\xi_0}:=\{(g(W_i,p,\xi(p))-g(W_i,p,\xi_0(p))),\quad p\in\mathcal{P}\}$. Con-

sider the function class $\mathcal{F}^e_{\xi\xi_0} := \{(g(W_i,p,\xi(p)) - g(W_i,p,\xi_0(p)))e_i, \quad p \in \mathcal{P}\}$. The function class is obtained by the multiplication of a random element of class $\mathcal{F}_{\xi\xi_0}$ by an integrable random variable e_i. Therefore, $\mathcal{F}^e_{\xi\xi_0}$ is also P-Donsker and has bounded uniform covering entropy. The expectation of the random element of the class $\mathcal{F}^e_{\xi\xi_0}$ is bounded as:

$$\begin{aligned}\sqrt{n}\sup_{p\in\mathcal{P}}|\mathbb{E}[g(W_i,p,\hat{\xi}(p)) - g(W_i,p,\xi_0(p))|\mathcal{E}_N]| &\lesssim \sup_{\xi\in\Xi_n}\mathbb{E}[g(W_i,p,\xi(p)) - g(W_i,p,\xi_0(p))] \\ &\lesssim \sqrt{n}\mu_n = o(1).\end{aligned}$$

The variance of each element of the class $\mathcal{F}^e_{\xi\xi_0}$ is bounded as:

$$\begin{aligned}\sup_{p\in\mathcal{P}}\sup_{\xi\in\Xi_n}\mathbb{E}((g(W_i,p,\xi(p)) - g(W_i,p,\xi_0(p)))^0 e_i)^2 &= \sup_{p\in\mathcal{P}}\sup_{\xi\in\Xi_n}\mathbb{E}\big((g(W_i,p,\xi(p)) - g(W_i,p,\xi_0(p)))^0\big)^2\mathbb{E}e_i^2 \\ &\le 2\sup_{p\in\mathcal{P}}\sup_{\xi\in\Xi_n}\mathbb{E}((g(W_i,p,\xi(p)) - g(W_i,p,\xi_0(p))))^2 \\ &\lesssim r_n'',\end{aligned}$$

where the bound follows from the conditional independence of e_i from W_i, $\mathbb{E}e_i^2 = 2$ for $e_i \sim \text{Exp}(1)$, and Assumption 4. By Lemma 6

$$\mathbb{E}[\sup_{p\in\mathcal{P}}\mathbb{G}_{n,k}[(g(W_i,p,\xi(p)) - g(W_i,p,\xi_0(p)))^0]\frac{e_i(1-\bar{e})}{\bar{e}}|\mathcal{E}_N] = o_P(1).Q.E.D.$$

□

Lemma 8. *Let Assumptions 3 - 4 hold. Consider the function class* $\mathcal{F}_{pp_0} = \{g(W_i,p,\xi_0(p)) - g(W_i,p_0,\xi_0(p_0)), \quad \|p - p_0(q)\| \le N^{-1/2}, \quad q \in \mathcal{S}^{d-1}, p \in \mathcal{P}\}$. *Then uniformly on* $\mathcal{F}_{pp_0}$,

$$\begin{aligned}&\sqrt{N}\mathbb{E}_N\big(g(W_i,(\tilde{\Sigma}^{-1})^\top q,\xi_0((\tilde{\Sigma}^{-1})^\top q) - g(W_i,(\Sigma^{-1})^\top q,\xi_0((\Sigma^{-1})^\top q))\big)\frac{e_i}{\bar{e}} \\ &= \sqrt{N}q'\Sigma^{-1}(\tilde{\Sigma}-\Sigma)\Sigma^{-1}G(\Sigma^{-1}q) + o_P(1),\end{aligned}$$

where the gradient $G(p)$ *is defined in Assumption 5.*

Proof. Consider the function class $\mathcal{F}^e_{pp_0} = \{e_i(g(W_i,p,\xi_0(p)) - g(W_i,p_0,\xi_0(p_0))), \quad \|p - p_0(q)\| \le N^{-1/2}, \quad q \in \mathcal{S}^{d-1}, p \in \mathcal{P}\}$. The function class is obtained by the multiplication of a random ele-

ment of class $\mathcal{F}_{pp_0}$ by an integrable random variable e_i. Therefore, $\mathcal{F}^e_{pp_0}$ is also P-Donsker and has bounded uniform covering entropy. The expectation of the random element is as decomposed as:

$$\mathbb{E}[(g(W_i,p,\xi_0(p))-g(W_i,p_0,\xi_0(p_0)))] = (p-p_0)\cdot G(p_0)+R(p,p_0).$$

By Assumption 5, the remainder term $R(p,p_0)=o(\|p-p_0(q)\|)$ uniformly over $q\in\mathcal{S}^{d-1}$. The variance of each element of the class $\mathcal{F}^e_{pp_0}$ is bounded as:

$$\mathbb{E}[(g(W_i,p,\xi_0(p))-g(W_i,p_0,\xi_0(p_0)))^0 e_i^2]^2 \leq 2\mathbb{E}[g(W_i,p,\xi_0(p))-g(W_i,p_0,\xi_0(p_0))]^2 \lesssim r_n',$$

where the bound follows from the conditional independence of e_i from W_i, $\mathbb{E}(e_i)^2=2$ for $e_i\sim$ Exp(1), and Assumption 4. By Lemma 6

$$\mathbb{E}[\sup_{p\in\mathcal{P}}\mathbb{G}_{n,k}[(g(W_i,p,\xi_0(p))-g(W_i,p_0,\xi_0(p_0)))^0]\frac{e_i(1-\bar{e})}{\bar{e}}|\mathcal{E}_N] = o_P(1).Q.E.D.$$

□

Proof of Theorem 2. The difference between the bootstrap and the true support function as follows:

$$\begin{aligned}\sqrt{N}(\tilde{\sigma}(q,\mathcal{B})-\sigma(q,\mathcal{B})) = &\underbrace{\frac{1}{\sqrt{N}}\sum_{i=1}^N \frac{e_i}{\bar{e}}\big(g(W_i,(\tilde{\Sigma}^{-1})^\top q,\hat{\xi}((\tilde{\Sigma}^{-1})^\top q))-g(W_i,(\tilde{\Sigma}^{-1})^\top q,\xi_0((\tilde{\Sigma}^{-1})^\top q))\big)}_{K_{\xi\xi_0}(q)}\\ &+\underbrace{\frac{1}{\sqrt{N}}\sum_{i=1}^N \frac{e_i}{\bar{e}}(g(W_i,(\tilde{\Sigma}^{-1})^\top q,\xi_0((\tilde{\Sigma}^{-1})^\top q))-g(W_i,(\Sigma^{-1})^\top q,\xi_0((\Sigma^{-1})^\top q)))}_{K_{pp_0}(q)}\\ &+\underbrace{\frac{1}{\sqrt{N}}\sum_{i=1}^N \frac{e_i}{\bar{e}}(g(W_i,(\Sigma^{-1})^\top q,\xi_0((\Sigma^{-1})^\top q))-\sigma(q,\mathcal{B}))}_{K_e}\end{aligned}$$

By Lemma 7 $\sup_{q\in\mathcal{S}^{d-1}}|K_{\xi\xi_0}|=o_P(1)$. By Lemma 8 $K_{pp_0}(q)=-q'\Sigma^{-1}(\mathbb{E}_N e_i A(W_i,\eta_0)-$

$\Sigma)\Sigma^{-1}G((\Sigma^{-1})^\top q)+o_P(1)$ uniformly in q. By Lemma 6,

$$\begin{aligned}
K_e &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{e_i}{\bar{e}}(g(W_i,(\Sigma^{-1})^\top q,\xi_0((\Sigma^{-1})^\top q))-\sigma(q,\mathcal{B})) \\
&=^{i} \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{e_i}{\bar{e}}(g(W_i,(\Sigma^{-1})^\top q,\xi_0((\Sigma^{-1})^\top q)))^0 \\
&=^{ii} \frac{1}{\sqrt{N}}\sum_{i=1}^{N} e_i(g(W_i,(\Sigma^{-1})^\top q,\xi_0((\Sigma^{-1})^\top q)))^0+o_P(1),
\end{aligned}$$

where i follows from (3.7) and ii from Lemma 6. A similar argument applies to the leading term of K_{pp_0}

$$q'\Sigma^{-1}(\mathbb{E}_N\frac{e_i}{\bar{e}}A(W_i,\eta_0)-\Sigma)\Sigma^{-1}G((\Sigma^{-1})^\top q)=q'\Sigma^{-1}(\mathbb{E}_N e_i A(W_i,\eta_0)-\Sigma)\Sigma^{-1}G((\Sigma^{-1})^\top q)+o_P(1).$$

The first statement of Theorem 2 follows from:

$$\begin{aligned}
\tilde{S}_N(q) &=^{i} \sqrt{N}(\tilde{\sigma}(q,\mathcal{B})-\hat{\sigma}(q,\mathcal{B})) \\
&= \sqrt{N}(\tilde{\sigma}(q,\mathcal{B})-\sigma(q,\mathcal{B})-(\hat{\sigma}(q,\mathcal{B})-\sigma(q,\mathcal{B}))) \\
&=^{ii} \frac{1}{\sqrt{N}}\sum_{i=1}^{N} e_i h(W_i,q)+o_P(1) \\
&= \frac{1}{\sqrt{N}}\sum_{i=1}^{N} e_i^0(h(W_i,q))^0,
\end{aligned}$$

where i is by definition of the bootstrap support function process and ii is by definition of $h(W,q)$ in Theorem 1. Once the asymptotic approximation conditional on the data has been established all further statements follows from Steps 2 and 3 of the Proof of Theorem 3 Chandrasekhar et al. (2011).

□

7.1.2 Proof of Theorem 5

The following Lemma is useful for proving Theorem 5.

Lemma 9 (Indicator Function)**.** *Suppose the following statements hold. (1) There exists a bound $B_{UL}<\infty$ such that the interval width is absolutely surely bounded for all nuisance parameter*

values $\eta \in \mathcal{T}_N$ $\sup_{\eta\in\mathcal{T}_N} Y_{U,\eta} - Y_{L,\eta} \leq B_{UL} < \infty$. *(2) A collection of distributions of* $\{p^\top V_{\eta_0}, p \in \mathcal{P}\}$ *is uniformly continuous on* $\mathcal{P}$

$$\sup_{p\in\mathcal{P}} \sup_{\eta\in\mathcal{T}_N} \mathbb{E}|p^\top V_\eta - p^\top V_{\eta_0}| 1_{\{0<|p^\top V_{\eta_0}|\leq|p^\top V_\eta - p^\top V_{\eta_0}|\}} \lesssim \mathbb{E}\|V_\eta - V_{\eta_0}\|^2.$$

(3) The following convergence bound applies

$$\sup_{\eta\in\mathcal{T}_N} \left(\mathbb{E}\|V_\eta - V_{\eta_0}\|^2\right)^{1/2} \lesssim \sup_{\eta\in\mathcal{T}_N} \left(\mathbb{E}\|\eta - \eta_0\|^2\right)^{1/2}.$$

(4) The nuisance parameter η *is estimated at least at* $o(N^{-1/4})$ *rate*

$$\sup_{\eta\in\mathcal{T}_N} \left(\mathbb{E}\|\eta - \eta_0\|^2\right)^{1/2} = o(N^{-1/4}).$$

Then replacing (3.6) by its smooth analog produces a bias of order $o(N^{-1/2})$

$$\sup_{p\in\mathcal{P}} \sup_{\eta\in\mathcal{T}_N} |\mathbb{E}[m(W,p,\eta) - m_0(W,p,\eta)]| = o(N^{-1/2}).$$

Proof is left to Supplementary Appendix.

Proof of Theorem 5. Step 1. Verification of Assumption 3(1). Consider the original moment function:

$$m(W,p,\eta) = p^\top(D - \eta(X))\Gamma(Y_L, Y_U - Y_L, p^\top(D - \eta(X))).$$

First, we apply Lemma 9 (verified in Step 2) to shift to a smoothened moment function

$$m_0(W,p,\eta) = p^\top(D - \eta(X))\Gamma(Y_L, Y_U - Y_L, p^\top(D - \eta_0(X))).$$

Second, we apply Lemma 11 to the moment function $m_0(W,p,\eta)$ and derive the bias correction term

$$\alpha_0(W,p,\eta) = -\gamma(p,X)[D - \eta(X)],$$

where for each $p \in \mathcal{P}$ the function $\gamma(p,\cdot)$ is a P-square integrable function.The true value of the

nuisance parameter $\xi(p,X) = \{\eta(X), \gamma(p,X)\}$ is:

$$\xi_0(p,X) = \{\mathbb{E}[D|X], \mathbb{E}[(Y_U - Y_L)1_{\{p^\top(D-\eta_0(X))>0\}}|X]\}.$$

Therefore,

$$g(W,p,\xi(p)) = p^\top(D-\eta(X))[\Gamma(Y_L, Y_U - Y_L, p^\top(D-\eta(X))) - \gamma(p,X)]$$

has zero Gateaux derivative with respect to $\xi(p)$. Since the function $m_0(W,p,\eta)$ is linear in η, its second derivative w.r.t η is equal to zero. Therefore, Assumption 4 holds. Assumption 17 implies that Assumption 3 is satisfied. Step 2. Verification of Lemma 9. Assumption 1 of Lemma 9 follows from Assumption 7 (3). Assumptions 3 and 4 follows from the Assumption 7 (6). Assumption 2 is verified below:

$$\begin{aligned}
&\sup_{p\in\mathcal{P}}\sup_{\eta\in\mathcal{T}_N} \mathbb{E}|p^\top V_\eta - p^\top V_{\eta_0}| 1_{0<|p^\top V_{\eta_0}|<|p^\top V_\eta - p^\top V_{\eta_0}|} \\
&= \sup_{p\in\mathcal{P}}\sup_{\eta\in\mathcal{T}_N} \mathbb{E}|p^\top(\eta(X)-\eta_0(X))| 1_{0<|p^\top(D-\eta_0(X))|<|p^\top(\eta(X)-\eta_0(X))|} \\
&\leq \mathbb{E}_X|p^\top(\eta(X)-\eta_0(X))| \int_0^{|p^\top(\eta(X)-\eta_0(X))|} \rho_{p^\top(D-\eta_0(X))|X}(t,X)dt \qquad (\mathbb{E}[\cdot] = \mathbb{E}_X[\cdot]\mathbb{E}_X[\cdot|X]) \\
&\leq K_h \mathbb{E}_X(p^\top(\eta(X)-\eta_0(X)))^2 \qquad \text{(Assumption 7 (4))} \\
&\leq K_h\|p\|^2 \mathbb{E}_X\|\eta(X)-\eta_0(X)\|^2 \qquad \text{(Cauchy Schwartz)} \\
&\lesssim o(N^{-1/2}). \qquad \text{(Assumption 7 (6))}
\end{aligned}$$

Step 3. Verification of Assumption 3(2). The moment condition for Σ is insensitive to the biased estimation of η:

$$\partial_\eta \mathbb{E}A(W,\eta_0) = 2\partial_\eta \mathbb{E}(D-\eta_0(X))(\eta(X)-\eta_0(X))^\top = 0.$$

Step 4. Verification of Assumption 5. The moment function $g(W,p,\xi(p))$ takes the form:

$$g(W,p,\xi(p)) = p^\top(D-\eta(X))(\Gamma(Y_L, Y_U - Y_L, p^\top(D-\eta(X))) - \gamma(p,X)).$$

The expectation of $g(W,p,\xi(p))$ evaluated at $\xi_0(p) = \{\eta_0(X), \gamma_0(p,X)\}$ is equal to:

$$L(p) := \mathbb{E}g(W,p,\xi_0(p)) = p^\top \mathbb{E}(D-\eta_0(X))(\Gamma(Y_L, Y_U - Y_L, p^\top(D-\eta_0(X))),$$

since $\mathbb{E}\gamma_0(p,X)(D-\eta_0(X)) = 0$. According to Lemma 3 of Chandrasekhar et al. (2011), the function $L(p)$ is differentiable on $\mathcal{P}$ with a uniformly continuous derivative:

$$\nabla L(p) := \mathbb{E}(D-\eta_0(X))(\Gamma(Y_L, Y_U - Y_L, p^\top(D-\eta_0(X))) =: G(p).$$

Moreover, the gradient $G(p)$ is uniformly continuous on $\mathcal{P}$. Verification of (3.13). The intermediate Value Theorem implies:

$$L(p) - L(p_0) = \nabla L(p_0')(p-p_0) = G(p_0')(p-p_0),$$

where p_0' is a point on the interval $[p_0, p]$. Therefore,

$$\begin{aligned} G(p_0')(p-p_0) &= G(p_0)(p-p_0) + (G(p_0') - G(p_0))(p-p_0) \\ &= G(p_0)(p-p_0) + o(\|p-p_0\|), \end{aligned}$$

where the last equality follows from the uniform continuity of $G(p)$ that is established in Lemma 3 of Chandrasekhar et al. (2011).

Step 5. Verification of Assumption 4. Consider the setting of Lemma 19. Consider the function class $\mathcal{A}_\xi = \{p^\top(D-\eta(X))\gamma(p,X), p \in \mathcal{P}\}$. Consider an envelope function $F_g(X) = \|p\|\|D - \eta(X)\|B_{UL}$. By Assumption 9, this function is integrable. Since the class is obtained by multiplying a linear function $p \to p^\top(D-\eta(X))$ by a Lipshitz function $\gamma(p,X)$, its uniform covering entropy is bounded. The application of Lemma 19 with the function class $\mathcal{A}_\xi$ implies that Assumptions 4(4) hold.

Step 6. Verification of Assumption 4. We use the following notation: $Y_{p,\eta} := \Gamma(Y_L, Y_U -$

$Y_L, p^\top(D-\eta(X))), \quad p \in \mathcal{P}, \eta \in \mathcal{T}_N$. Let us decompose the difference of $g(W,p,\xi(p))$ and $g(W,p,\xi_0(p))$

$$\begin{aligned} g(W,p,\xi(p)) - g(W,p,\xi_0(p)) &:= p^\top(D-\eta(X))(Y_{p,\eta} - \gamma(p,X)) - p^\top(D-\eta_0(X))(Y_{p,\eta_0} - \gamma_0(p,X)) \\ &= \underbrace{p^\top(\eta(X)-\eta_0(X))\gamma(p,X)}_{I_1} + \underbrace{p^\top(D-\eta_0(X))(\gamma_0(p,X)-\gamma(p,X))}_{I_2} \\ &+ \underbrace{p^\top(D-\eta_0(X))(Y_{p,\eta} - Y_{p,\eta_0})}_{I_3} + \underbrace{p^\top(\eta_0(X)-\eta(X))(Y_{p,\eta} - Y_{p,\eta_0})}_{I_4}. \end{aligned}$$

Under Assumptions 7, the terms $I_k, k \in \{1,2,3,4\}$ exhibit mean square convergence with the rate $o(N^{-1/4})$:

$$\begin{aligned} (\mathbb{E}I_1^2)^{1/2} &\lesssim \lambda_{\min}^{-1} Y_{UL} \|\eta - \eta_0\|_{L_{P,2}} = o(N^{-1/4}), \\ (\mathbb{E}I_2^2)^{1/2} &\lesssim \lambda_{\min}^{-1} D \|\xi(p) - \xi_0(p)\|_{L_{P,2}} = o(N^{-1/4}), \\ (\mathbb{E}I_3^2)^{1/2} &\lesssim \mathbb{E}(p^\top(D-\eta_0(X)))^2 1_{0<|p^\top(D-\eta_0(X)))|<|p^\top(\eta(X)-\eta_0(X))|} B_{UL} \\ &\leq (\mathbb{E}|p^\top(\eta(X)-\eta_0(X))|^2)^{1/2} = o(N^{-1/4}), \\ (\mathbb{E}I_4^2)^{1/2} &\lesssim \lambda_{\min}^{-1} 2 Y_{UL} \|\eta - \eta_0\|_{L_{P,2}}. \end{aligned}$$

Under Assumption 7, the terms $S_k, k \in \{1,2,3,4\}$ exhibit mean square convergence:

$$\begin{aligned} g(W,p,\xi_0(p)) - g(W,p_0,\xi_0(p_0)) &:= p^\top(D-\eta_0(X))(Y_{p,\eta_0} - \gamma_0(p,X)) - p_0'(D-\eta_0(X))(Y_{p_0,\eta_0} - \gamma_0(p_0,X)) \\ &= -\underbrace{p_0^\top(D-\eta_0(X))(\gamma_0(p,X)-\gamma_0(p_0,X))}_{S_1} - \underbrace{(p-p_0)^\top(D-\eta_0(X))\gamma_0(p,X)}_{S_2} \\ &+ \underbrace{p_0^\top(D-\eta_0(X))(Y_{p,\eta_0} - Y_{p_0,\eta_0})}_{S_3} + \underbrace{(p-p_0)^\top(D-\eta_0(X))Y_{p,\eta_0}}_{S_4}. \end{aligned}$$

By Assumption 7(9),

$$(\mathbb{E}S_1^2)^{1/2} \lesssim g_N' : \quad g_N' \log(1/g_N') = o(1)$$

holds. The bound on the mean square convergence of the other terms is as follows:

$$
\begin{aligned}
(\mathbb{E}S_2^2)^{1/2} &\lesssim \|p-p_0\| DY_{UL} = O(N^{-1/2}),\\
(\mathbb{E}S_3^2)^{1/2} &\leq (\mathbb{E}[p_0'(D-\eta_0(X))]^2 1_{0<|p^\top(D-\eta_0(X))|<|(p-p_0)^\top(D-\eta_0(X)|})^{1/2}\\
&\leq \|p-p_0\| D = O(N^{-1/2}),\\
(\mathbb{E}S_4^2)^{1/2} &\lesssim \|p-p_0\| DY_{UL} = O(N^{-1/2}).
\end{aligned}
$$

□

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