Empirical Analysis of School Assignment Models

Nikhil Agarwal and Paulo Somaini∗

August 16, 2019

Abstract

Preferences for schools are important determinants of equity of access to high-quality education, effects of expanded choice on school improvement and the design of school choice mechanisms. Standard methods for estimating consumer preferences are not applicable in education markets where all students do not get their first choice school. This review describes recently developed methods for using rich data on the school assignment process to estimate student preferences. Our objectives are to present a unifying framework for these methods and to help applied researchers decide which techniques to use. After laying out methodological issues, we provide an overview of empirical results obtained using these models and discuss some open questions.

∗Agarwal: Department of Economics, MIT and NBER, email: agarwaln@mit.edu. Somaini: Stanford Graduate School of Business and NBER, email: soma@stanford.edu.
1 Introduction

Empirical models have been used to study a wide range of questions including socio-economic heterogeneity in preferences for schools, allocative and redistributive effects of school choice, determinants of unequal enrollment at high-quality schools and the incentives of schools to improve quality. This literature has been spurred by increasingly available administrative data on school and college applications, especially from settings that use formalized placement mechanisms. Detailed knowledge of the rules used to assign students combined with micro-data from applications to assignments provide a unique window into determinants of school demand.

But, estimating these models also requires methods that are well-tailored to the specific institutional details. Unlike traditional consumer settings, education markets typically do not use prices as the main clearing mechanisms – public schools do not levy tuition, and universities do not increase their tuition until the number of students that seek admission exactly equals planned enrollment. Empirical approaches for estimating student preferences must therefore be sensitive to the rationing system for public schools and the selection system for colleges.

This review article uses a unified framework to describe how to analyze empirical models for school preferences and provides a brief overview of empirical results based on these models.\(^1\) We hope to provide guidance to applied researchers on choosing a model and to present some open issues.

Empirical school choice models are specific about how student preferences translate to decisions or outcomes. The researcher either observes the decisions, such as a rank-order lists submitted to a choice mechanism, or outcomes, such as final enrollment. The objective is to estimate the distribution of student preferences and study how it depends on the set of student and school characteristics observed in the data. The most appropriate model for this task depends on the available data and the restrictions on behavior that best describe the empirical setting. For example, a researcher needs to decide whether or not parents and students report their ordinal preferences truthfully.

Analyzing these models require a revealed preference argument, which interpret the data on student decisions or outcomes to infer preferences. They must be tailored to the available data, the school choice mechanism used to assign students and assumptions on student behaviour. The differences in the most suitable approach emphasize the need for models that are sensitive to the institutional environment.

\(^1\)A separate literature uses quasi-experimental variation embedded in school assignment mechanisms to study the causal effects of attending particular types of schools (see Hastings et al., 2009; Deming, 2011; Abdulkadiroglu et al., 2011a, 2017a). Our review does not discuss these studies in the interest of space, although this research represents an important and innovative use of school assignment data. We believe that combining this quasi-experimental variation with empirical models of choice is a fruitful avenue for future research.
The revealed preference arguments directly suggest estimation methods. Most models provide us with a (non-parametric) likelihood function. In some cases, we obtain bounds on the likelihood. An applied researcher can use this information to estimate the parameters of a preference model. Available estimators range from classical maximum likelihood or method of moments to likelihood-based Bayesian methods or moment inequalities. But, certain estimators are better suited to specific versions of the model. We discuss the trade-offs between computational convenience and the microeconomic properties of the parametric and economic assumptions before describing some remaining challenges.

After laying out the methodological approaches, we discuss the research on empirical questions for which school choice models are important. An extensive literature that is not easily summarized studies issues about education policy, including equity in access to schools and whether school choice encourages schools to improve (see Hoxby, 2003, for a survey). Estimates of student preferences allow us to better understand whether inequality results from residential location or heterogeneity in choices conditional on location. Estimates of school demand allow us to determine the sensitivity of choices to school value-added. A finding that students would not flock to higher value-added schools under greater choice would weigh against arguments that school choice is likely to push schools to improve.

A separate theoretical literature has focused on the student assignment problem, studying trade-offs between efficiency, fairness and incentive properties (Abdulkadiroglu and Sonmez, 2003). It has found mechanisms that are on the frontier of managing these trade-offs. This literature has had a large impact in practice, resulting in several school choice reforms based on theory (see Pathak, 2017).

Empirical school choice models complement this pursuit through a quantitative counterpart by analyzing which trade-offs are most important. Such a data-driven approach is particularly attractive when theory does not yield tractable or unambiguous answers. The empirical literature has shown which trade-offs have proven to be quantitatively large, and has identified further areas of potential improvement. This progress has been made primarily based on a model in which students make optimal choices. But, a more recent push has been made towards relaxing this assumption with empirical approaches that incorporate surveys or weaker restrictions on behavior. The recent approaches hold the promise of better understanding how students interact with formal school choice mechanisms.

The review is structured as follows. Section 2 lays out the random utility model that will be used throughout the paper and describes two illustrative choice mechanisms. Section 3 uses a unified framework to present revealed preference approaches for uncovering information on preferences. The arguments depend on the data, the school choice mechanism and the desired model of behavior. Section 4 discusses identification of the model, parametric assumptions, and estimation methods. Section 5 reviews empirical findings.
2 Model

We consider a school assignment mechanism in which students indexed by \( i \in I \) are assigned to schools indexed by \( j \in J = \{1, \ldots, J\} \). Denote the outside option with school 0. Each school has \( q_j \) seats, with \( q_0 = \infty \).

2.1 Preferences

Student preferences are specified using a random utility model. Specifically, student \( i \)'s (indirect) utility for assignment to school \( j \) is given by \( v_{ij} \). The typical objective is to identify and estimate the joint distribution of the vector of random utilities \( v_i = (v_{i1}, \ldots, v_{iJ}) \) conditional on a set of observable characteristics. Let \( v_{i0} \) denote the indirect utility of the outside option or a default school.

A common approach (Abdulkadiroglu et al., 2017b, for example) is to specify a distance-metric indirect utility function

\[
v_{ij} = v(x_j, z_i, \xi_j, \gamma_i, \varepsilon_{ij}) - d_{ij}, \tag{1}\]

where \( x_j \) is a row-vector of characteristics for school \( j \) observed in the dataset, \( z_i \) is a column-vector of student \( i \)'s observed characteristics, \( \xi_j \) is a school-specific unobserved characteristic, \( \gamma_i \) is a column-vector capturing student-specific idiosyncratic tastes for program characteristics and \( \varepsilon_{ij} \) captures idiosyncratic tastes for programs.\(^2\) In the absence of tuition payments in the public school context, this specification quantifies preferences for schools in terms of “willingness to travel.” The additive separable form in distance with a normalized co-efficient embeds a scale normalization. In addition to quasi-linearity, this particular specification also assumes that, all else equal, distance to school is undesirable.

The model also needs a location normalization. If the model includes an outside option, then it is common to normalize \( v_{i0} = 0 \). Studies that omit the outside option must impose an alternative location normalization on the mean utility for a reference school.

Further, existing approaches for identification and estimation use the assumption that

\[
(\gamma_i, \varepsilon_i) \perp d_{ij}|z_i, \{x_j, \xi_j\}_{j=1}^J, \nonumber\]

where \( \varepsilon_i = (\varepsilon_{i1}, \ldots, \varepsilon_{iJ}) \). It assumes that unobserved tastes for schools are conditionally independent of distance to school given observed student and school characteristics, and school unobservables. This assumption is violated if students systematically reside near the schools for which they have idiosyncratic tastes. The approach may provide reasonable

\(^2\)Recent results in Allen and Rehbeck (2017) show how to generalize this specification to allow for a separable, but non-linear specification for distance.
approximations in settings with rich micro-data on students. Nonetheless, relaxing this assumption to incorporate residential sorting is a fruitful avenue for future research.

In addition, parametric assumptions on the functional form for \( v(\cdot) \) and the distribution of utilities are made for computational tractability and to assist estimation in finite samples. The most convenient forms depend on the mechanism analyzed, the estimation method and the size of the choice set. One tractable and flexible parametric form assumes that

\[
v_{ij} = \delta_j x_j \beta + \xi_j + x_j (\bar{\gamma}_i z_i) - d_{ij} + \varepsilon_{ij}, \tag{2}
\]

where \( \gamma_i \) and \( \varepsilon_{ij} \) have distributions known up to finite dimensional parameters and \( \bar{\gamma} \) is a matrix conformable with \( x_j \) and \( z_i \). We collect these and the other unknown parameters of the model, namely \((\beta, \bar{\gamma}, \delta_1, \ldots, \delta_J)\), in the vector \( \theta \). The revealed preference and identification results described below are for the more general specification in equation (1) and are applicable to alternative parametrizations.

There are three implicit restrictions in the model that are worth noting. First, a student’s indirect utility only depends on her own assignment, and not those of others. This rules out preferences for attending school with specific peers. It is possible to include statistics describing the students in previous years within \( x_j \) to capture preferences for the composition of the student body. However, with rare exceptions (Epple et al., 2018, for example), existing empirical approaches abstract away from equilibrium sorting based on preferences for peers. Second, welfare statements based on the willingness to travel metric makes inter-personal comparisons of utility in a non-transferable unit of measurement. This property prohibits a justification of utilitarian welfare metrics based on the Kaldor-Hicks criterion. Nonetheless, it is possible to evaluate the proportion of students that prefer various mechanisms or assignments without making interpersonal comparisons of utility. Finally, the model abstracts away from costs of acquiring information about schools and typically assumes that preferences are well-formed. An exception is Narita (2018), which considers the possibility that preferences evolve after students receive an initial assignment.

### 2.2 Matchings and Assignment Mechanisms

Centralized school assignment schemes use student applications consisting of a ranking of available options, and school priorities or exam scores to match students to schools. Let \( R_i \in \mathcal{R} \) denote student \( i \)'s submitted rank-order list, where \( R_{i,k} \) denotes the school ranked in position \( k \). In most cases, the mechanism restricts the number of schools that can be ranked by a student. Denote student \( i \)’s priority or exam score for the various schools with the vector \( t_i = (t_{i,1}, \ldots, t_{i,J}) \). The priority groups may be fine so that \( t_{ij} \in \mathbb{R} \) or coarse so that \( t_{ij} \) takes on finitely many values. In the latter case, a tie-breaker is often used to order students with
the same priority score. We assume that priority-type captures all aspects that differentiates
two students from the perspective of the mechanism, except for the tie-breaker.

A mechanism $\Phi$ uses the tuple $(\mathbf{R}, \mathbf{t}) = ((R_1, \ldots, R_N), (t_1, \ldots, t_N))$ containing the rank-
order lists and priorities of all students to assign students. An assignment $\mu$ specifies the
school (if any) where each student is placed. Its $i$-$j$-component, denoted $\mu_{ij}$, is set to 1 if
student $i$ is assigned to school $j$ and 0 otherwise. This assignment is produced using the
rank order list submitted by each student together with their priorities and tie-breakers.
When the mechanism includes a tie-breaker, the assignment $\mu$ may be a (non-degenerate)
random variable conditional on $(\mathbf{R}, \mathbf{t})$. We assume that the mechanism $\Phi$ produces a feasible
matching so that no school is assigned more students than its capacity, that is, $\sum_i \mu_{ij} \leq q_j$
for all schools $j$.

We assume that the analyst knows the mechanism used and that each student’s priority type
$t_i$ is observed. The most suitable approach for estimating the model will depend on the
available data and the properties of the mechanism in place. In some cases, it is sufficient
for the analyst to observe the assignment $\mu$, and not the underlying rank order lists $\mathbf{R}$. We
discuss the various cases in greater detail in the following sections.

The formulation makes two implicit assumptions that can be directly verified based on knowl-
edge of the mechanism. All school choice mechanisms that we are aware of satisfy both these
properties. First, assignments do not depend directly on the indirect utilities $v_i$. They can
only depend on the submitted rank order lists and priorities $(R_i, t_i)$. This assumption can be
verified based on the knowledge of the mechanism used by the district to assign students. Of
course, the rank-order lists submitted by an agent may not reflect true preferences. Second,
the mechanism treats any two students of the same priority-type symmetrically. Formally,
we assume that the mechanism is semi-anonymous (see Azevedo and Budish, 2018).

**Example 1. (Student-Proposing) Deferred Acceptance Mechanism.** This popular
school choice mechanism is based on the celebrated Deferred Acceptance Algorithm of Gale
and Shapley (1962). It uses the submitted rank-order lists priorities to assign students using
the following procedure:

Step 0: If priorities are coarse so that two students may be tied at a school, then a tie-
breaker $\nu_i$ is generated. This tie-breaker can either be a single number that is
applied to all schools or a vector of school-specific tie-breakers.

Step 1: Each student $i$ applies to their highest ranked school $R_{i1}$. At each school $j$, the
applications of the students with the highest priority for that school up to the
capacity $q_j$ are tentatively held. Ties between students, if any, are resolved
according to the value of $\nu_i$ generated in step 0. The schools reject the remaining
students.
Step $k > 1$: In a general step $k$, students who's applications were rejected in step $k - 1$ apply to their highest ranked school that has not yet rejected them. At each school $j$, the previously held applications are considered along with students that apply in round $k$. The applications of the highest priority students up to capacity are tentatively held, with ties resolved according to the value of $\nu_i$ generated in step 0. The remaining applicants are rejected.

The algorithm terminates in step $k$ if no students are rejected or if all rejected students have applied to all the schools that they have ranked. Student assignments $\mu_{ij}$ are finalized at the schools where their applications are currently held if either of these terminal conditions is satisfied. Otherwise, the algorithm proceeds to step $k + 1$.

It will be useful to combine the priority of student $i$ at school $j$ into the score $e_{ij}$. In the Deferred Acceptance (DA) mechanism, the score $e_{ij}$ is determined by a function $f_j(t_i, \nu_{ij})$ that depends lexicographically on priority $t_i$ and tie-breaker $\nu_{ij}$. Azevedo and Leshno (2016) show that the DA generates an allocation that is determined by a vector of cutoff priorities $p_j$ such that each student $i$ is allocated to the highest ranked school for which $e_{ij} \geq p_j$.

The cutoff is set so that (i) no school has more students than it has capacity for, and (ii) no students who desires an assignment at school $j$ is denied one unless the school is full. It is easy to see that these two properties are satisfied if the cutoff for school $j$ is given by the lowest scoring student that is assigned to the school if the school does not have spare capacity, and the lowest possible score otherwise.

The student-proposing DA has two theoretically appealing properties. First, it is strategy-proof (Dubins and Freedman, 1981). That is, submitting a rank-order list that coincides with one’s ordinal preferences is a weakly dominant strategy. Second, the resulting assignment is stable if students submit truthful reports. In this context, stability requires that there is no student $i$ and school $j$ such that: (i) $i$ strictly prefers $j$ over the school that they is assigned to, and (ii) if school $j$ does not have spare capacity, then $i$ has a higher score than another student assigned to $j$. Azevedo and Leshno (2016) show that any stable allocation can be represented by a vector of cutoffs $p_j$ such that each student is assigned to the highest-ranked school in the set

$$S(e_i, p) = \{j : e_{ij} \geq p_j\}.$$ 

Given the cutoffs $p_j$, a student with the score $e_i$ has a score that would make them eligible for any school in this set.

**Example 2. Immediate Acceptance Mechanism**

Step 0: If student priority types are coarse so that two students may be tied at a school, then a tie-breaker $\nu_i$ is generated. This tie-breaker can either be a single number that is applied to all schools or a vector of school-specific tie-breakers.
Step 1: Each student $i$ applies to their highest ranked school $R_{i1}$. At each school $j$, the students with the highest priority for that school up to the capacity $q_j$ are assigned. The schools reject the remaining students. Ties between students, if any, are resolved according to the tie-breaker generated in step 0. The schools reject the remaining students. The remaining capacity of school with fewer applications than capacity is computed and stored in $q_{j1}$.

Step $k > 1$: In a general step $k > 1$, students who’s applications were rejected in step $k - 1$ apply to the school ranked $R_{ik}$. At each school $j$, the students with the highest priority for that school up to the capacity remaining after step $k - 1$, denoted $q_{jk-1}$, are assigned. Ties are resolved according to the tie-breaker in step 0. The remaining applicants are rejected. The remaining capacity of school with fewer applications than capacity is computed and stored in $q_{jk}$. The algorithm terminates in step $K$, denoting the maximum number of schools that can be ranked by a student.

The Immediate Acceptance (IA) mechanism also has a pivotal student that is rejected from a school. The pivotal student’s score determines the cutoff $p_j$ for that school. However, in IA, the score $e_{ij}$ is equal to $f_j(R_i, t_i, v_{ij})$ where $f_j$ is a lexicographic function of the position in which school $j$ appears in the report $R_i$, followed by the priority $t_i$, and finally the tie-breaker. Agarwal and Somaini (2018) show that this mechanism also assigns students to the highest ranked school in the set $S(e_i, p)$. Unlike DA, this set depends both on the cutoffs $p$ and directly on rankings submitted by the student through $e_i$.

An important difference between the DA and IA is that the later prioritizes students based on the submitted rank-order lists. Specifically, a student that ranks a school higher than another student effectively receives priority over them. This feature of the mechanism generates incentives to manipulate rankings. It is in the students’ interest to avoid ranking too many competitive schools. Moreover, students have an incentive to “cash” their neighborhood or sibling priority if the school is competitive by ranking it highly lest the school is oversubscribed by the time students that rank it second or third are considered. As a result, immediate acceptance is not strategy-proof and may not produce a stable assignment even if students report their preferences truthfully.

In fact, many other commonly used school choice mechanisms can be represented in this fashion. Agarwal and Somaini (2018) define a large class of mechanisms called Report-Specific Priority + Cutoff mechanisms. As the name suggests, the mechanisms use the submitted report to modify an student’s priority. In general, $e_{ij} = f_j(R_i, t_i, v_{ij})$ where the function $f_j$ modifies the priority $t_i$ depending on the report $R_i$ and $v_{ij}$ is the tie-breaker for school $j$. Each school has a cutoff priority, denoted $p_j$, and each student is placed in the highest ranked school in the set $S(e_i, p) = \{j : e_{ij} > p_j\}$. The cutoff is set so that (i) no school has more students than it has capacity for, and (ii) no student who desires an assignment at school $j$
is denied one unless the school is full. Other mechanisms that belong to this class include Serial Dictatorship; the Chinese Parallel Mechanism studied in Chen and Kesten (2013); the Pan London Admissions Scheme described in Pennell et al. (2006); First-Preferences First used in England; and the New Haven mechanism studied in Kapor et al. (2017). Each of these mechanisms differs in the use of a different function $f_j$ for the score.

3 Approaches to Revealed Preference Analysis

In this section, we discuss how to interpret data on assignments or reports depending on the mechanism and assumptions on behavior. First, we consider the implications of data on assignments under the assumption of stability. Second, we examine data on truthfully reported rank-order lists. Finally, we discuss how to interpret reports in strategic environments under various behavioral assumptions.

3.1 Using Stability

In many settings, it may be reasonable to assume that the final assignment is stable. This assumption yields useful revealed preference relations based only on assignment data and allows some flexibility on the mechanism and behavioral assumptions that generate the assignments. The arguments below assume that the researcher knows the eligibility scores $e_{ij}$ for each student and school, and that this score does not depend on reports made to the underlying mechanism. This assumption simplifies the analysis relative to approaches that must simultaneously estimate preferences for both sides of market (see Agarwal, 2015; Menzel, 2015; Diamond and Agarwal, 2017, for example).

For example, Fack et al. (2019) uses final admissions to high-schools in Paris determined by a deferred acceptance mechanism, and Akyol and Krishna (2017) use stability to study Turkish high schools that use an entrance exam. This assumption can also be used to study higher education settings that use a single centralized exam. For example, Bucarey (2018) uses stability to estimate preferences for colleges in Chile.

Assuming that assignments are stable, if student $i$ is assigned to school $j$, then we can infer that $v_{ij} > v_{ij'}$ for every other school $j'$ in $i$’s choice set $S(e_i, p) = \{ j : e_{ik} \geq p_j \}$. This follows because each student is assigned to her most preferred option for which her eligibility score $e_{ij}$ exceeds the cutoff $p_j$.

To see what can be learned from this information, consider the case with only two schools, 1 and 2, and an outside option 0. Figure 1 shows 5 different regions of indirect utilities denoted by roman numerals. Each of these regions imply different ordinal preferences, except for region V which pools the cases when $v_0 > v_1 > v_2$ and $v_0 > v_2 > v_1$. A student who is
Figure 1: Revealed Preferences – Stability – Full Choice Set

eligible in both schools will be assigned to school 1 if her indirect utilities belong to either region I or II. Therefore, the share of students that were assigned to school 1 amongst those that are eligible for both schools is an estimate of the total probability accumulated by the distribution of $v$ in regions I and II. Similarly, the share assigned to school 2 is an estimate of the total probability in regions III and IV.

A student eligible only for school 1 can either be assigned to that school or remain unassigned. In the former case, we can infer that $v_0 < v_1$ which is the darkly-shaded region in figure 2. In the later case, we infer $v_1 < v_0$ which is shaded lightly. The share of students assigned to school 1 amongst these students is an estimate of the total probability in regions I, II and III of figure 1.

These arguments are similar to those for standard discrete choice models, but differ crucially in that not all students are assigned to their first choice school. In standard discrete choice models, the fraction of consumers that are allocated good $j$ equals the fraction of consumers for which good $j$ provides the highest indirect utilities. In this context, student choice sets are constrained by their eligibility making this implication invalid. It is important to apply the revealed preference argument conditional on the set of schools that each student may achieve.

Assignments provide no information about preferences for schools that are not in a student’s choice set. Learning about the full distribution of ordinal preferences for students with a priority score of $e_i$ will require extrapolation. For example, one could use data from students
Figure 2: Revealed Preferences – Stability – Restricted Choice Set

with larger choice sets. Fack et al. (2019) do so by assuming that unobserved determinants on preferences in equation (1) are conditionally independent of the eligibility score given the observables included in the model. Formally, the assumption requires that

$$ (\gamma_i, \varepsilon_i) \perp e_i | z_i, d_i, \{x_j, \xi_j\}_{j=1}^J. $$

The assumption may be a reasonable approximation if $z_i$ contains a rich set of student characteristics. But, it can be violated, for example, if eligibility-scores are correlated with unobserved student ability.

Under the assumption above, it is possible to recover the full distribution of ordinal preferences. It is illustrative to continue the example with two schools. We obtained estimates of the probabilities of regions I, II, III and IV by adding up the share of students that were assigned to schools 1 or 2 conditional on having a full choice set. We also obtained an estimate of the probability of regions I, II and III by calculating the share of students assigned to 1 when only school 1 was in the choice set. The difference of these two probabilities is therefore an estimate of the probability of region IV. This argument can be repeated to obtain the probability mass in each of the five regions that partitions the space of indirect utilities. We are not aware of arguments that generalize this point for more than 2 schools.
3.2 Truthful Reports

A important goal in the theoretical literature on school choice is to design mechanism that are strategy-proof (Abdulkadiroglu and Sonmez, 2003). In such a mechanism, no student can benefit by submitting a list that does not rank schools in order of true preferences. A motivation for this objective is to level the playing fields between agents that are sophisticated and sincere in their behavior (see Pathak and Sonmez, 2008). This property of a school choice mechanism can also assist an empirical strategy if agents understand it and follow this recommendation.\(^3\) Specifically, if \(j = R_{ik}\) and \(j’ = R_{ik’}\) are the schools ranked in positions \(k\) and \(k’\) respectively, then we can infer that

\[ v_{ij} > v_{ij’}. \]

In contrast to schools that are ranked, it is less clear how to treat schools that are not on the list. A common approach is to assume that students rank all the schools that are acceptable, i.e. preferable to the outside option. In this case, if \(j\) is the lowest ranked school, then \(v_{ij} > v_{i0} > v_{ij’}\) if \(j’\) is not ranked. In this model, the various rank-order lists partition the space of indirect utilities as shown in figure 3 for the case when \(J = 2\). The five regions in the figure correspond the various possible ways in which two schools can be ranked when including the possibility that only one school or an empty list is submitted.

Observe that the rank-order lists provide richer information about preferences than in standard discrete choice settings in which a consumer picks only her favorite product. Specifically, if a consumer picks option 1 in a standard discrete choice setting, then we can only deduce that the consumer’s indirect utilities is in either the region labelled “Rank 1” or “Rank 1>2” in figure 3, but cannot distinguish between these two regions. The richer information in ordered lists has been shown to be important in identifying heterogeneity in preferences (Beggs et al., 1981; Berry et al., 2004). In the school choice context, students often rank many more schools, allowing for very rich specifications for the distribution of indirect utilities (see Abdulkadiroglu et al., 2017b, for example).

Assuming that agents report their true preferences can be justified for DA on theoretical grounds if there is no limit on the number of schools that a student can rank and all students have a non-zero chance of getting assigned to any of the ranked schools. Otherwise, it may be optimal for the student to omit some schools from their rank-order lists (Haeringer and Klijn, 2009). It is straightforward to modify the approach to restrict attention to the set of schools for which each student is eligible. This modification assumes that the students

\(^{3}\)Evidence from experiments and the field suggests that students are more likely to report their preferences truthfully when interacting with a strategy-proof mechanism (Chen and Sonmez, 2006; de Haan et al., 2016). Nonetheless, comprehending that a mechanism is strategyproof may be complicated (Li, 2017) and some students are liable to mistakenly submit rankings that are not truthful (Hassidim et al., 2016; Shorrer and Sovago, 2017; Rees-Jones, 2018).
understand that these are precise set of schools for which they may be eligible. But, even in this case, students may optimally avoid ranking too many competitive schools if the number of schools that can be ranked is small.

For these reasons, Fack et al. (2019) argue that relying on stability alone is more robust than assuming truthful preference reporting when the length of rank-order list is limited. They argue that stability is particularly attractive if the number of students is large relative to the number of schools. In this case, they show that the uncertainty in the cutoffs $p$ is small, allowing the students to predict the set of schools for which they will be eligible with a high degree of accuracy. If students behave optimally, only a negligible fraction of students will not be assigned to their most preferred option in the feasible set. At the same time, students will omit schools from their rank-order lists that are very unlikely to be achievable generating significant biases in approaches based on truthful reporting. Moreover, Artemov et al. (2017) show that stability is a robust prediction even when agents make mistakes because many mistakes may not affect final assignments. Indeed, many mistakes identified in the literature (Hassidim et al., 2016; Shorrer and Sovago, 2017; Rees-Jones, 2018) do not significantly affect final assignments. Of course, relying on stability alone has the drawback of providing less information on preferences and the requiring stronger assumptions for extrapolation that were discussed in the previous subsection.
3.3 Strategic Behavior

The previous subsections developed empirical approaches for the specific cases in which either the submitted rank-order lists are truthful or the final assignments are stable. However, there are many school districts that do not use strategy-proof or stable matching mechanisms. For example, the commonly used Immediate Acceptance mechanism (also known as the Boston Mechanism) prioritizes students who rank a school higher on their list and rewards students who strategically use this property of the mechanism. Evidence from the laboratory and real-world examples suggests that students do respond strategically to these incentives. This observation has been made in lab studies (Chen and Sonmez, 2006), in survey data (de Haan et al., 2016) and based on signs of strategic reporting in administrative data (Agarwal and Somaini, 2018).

This section generalizes the approaches discussed above to accommodate such assignment mechanisms. The cases of truthful reporting and stable outcomes are special cases. The approach is first developed using a model in which agents behave according to a Bayesian Nash Equilibrium. Given that this approach accords a high degree of sophistication to the agents, an active literature attempts to introduce models with behavioral biases and heterogeneous ability to game the system. We discuss these extensions after developing the baseline approach based on equilibrium play.

3.3.1 Beliefs and Behavior

In this baseline model, agents have private information about their preferences and form beliefs about their chances of getting assigned to various schools as a function of the submitted rank-order list. Agents know the mechanism being used and have a correct conjecture about the distribution of reports submitted to the mechanism by other agents. Dropping the dependence on the priority type \(t_i\) for notational simplicity, an agent with indirect utility vector \(v_i\) follows the strategy \(\sigma(v_i) \in \Delta R\). Her belief in this case are given by

\[
L_{R_i} = \mathbb{E} [\Phi (R_i, R_{-i}) | R_i, \sigma] = \int \Phi (R_i, R_{-i}) \prod_{i' \neq i} \sigma_{R_{i'}} (v_{i'}) f_V (v_{i'}) d v_{-i},
\]

where \(f_V\) is the probability density function (PDF) of the vector of indirect utilities, and \(\sigma_R (v)\) is the probability that an agent with indirect utility vector \(v\) submits rank-order list \(R\). The term \(\sigma_{R_{i'}} (v_{i'}) f_V (v_{i'})\) denotes the PDF that agent \(i'\) submits the report \(R_{i'}\). This formulation implicitly assumes that agent preference-types are independent and identically distributed from the perspective of agent \(i\).

By varying the rank-order list, an agent can implicitly choose different probabilities of assignment. Given an indirect utility vector \(v_i\), the expected utility from submitting the rank-
order-list $R_i$ is $v_i \cdot L_{R_i}$. We assume that each agent submits the report that maximizes her expected utility. That is, agent $i$ report $R_i$ only if

$$v_i \cdot L_{R_i} \geq v_i \cdot L_R$$

for all $R \in \mathcal{R}$. Below, we develop an empirical strategy based on this revealed preference inequality.

This formulation is based on assuming that agents understand the rules of the mechanism and the competitive environment that they are facing. Uncertainty in assignment arises both because of the tie-breaker, if any, and the reports of the other agents in the market place. As discussed in section 2.2, many mechanisms admit lower-dimensional statistics such as school-specific eligibility thresholds that suffice for forming expectations about the probabilities of assignment (see Azevedo and Leshno, 2016; Agarwal and Somaini, 2018).

3.3.2 A Revealed Preference Argument

An important step in the revealed preference arguments in the previous sections was to derive a set of indirect utility types that are consistent with an observed outcome. Section 3.1 was based on the assignment while section 3.2 used the reported rank-order list. Agarwal and Somaini (2018) use equation (4) to provide a similar path forward in the case of strategic behavior. We briefly explain their argument below.

Let $C_{R_i}$ be the set of indirect utilities for which the report $R_i$ maximizes expected utility, that is, $v_i \cdot (L_{R_i} - L_R) \geq 0$ for all $R \in \mathcal{R}$. Therefore, when student $i$ submits the reports $R_i$, she reveals that her indirect utility $v_i$ belongs to the set $C_{R_i}$.

Figure 4 illustrates these sets for a simplified case with two schools. In this example, $v_{R,R'}$ represents indirect utilities for which the student is indifferent between submitting $R$ and $R'$. Similarly, a student with indirect utilities given by $v_{R,R''}$ is indifferent between $R$ and $R''$. The students with indirect utility vectors in the set $C_R$ (weakly) prefer $R$ to the other reports.

Notice that each set $C_R$ in the figure is convex and defined by the indirect utility vectors for which an agent in indifferent between the report $R$ and another report $R'$. This structure is computationally useful when estimating the model because the revealed preference sets can be characterized using a small number of vectors. In fact, this structure generalizes to more than two schools because it is implied by the linearity of the expected utility formulation underlying equation (4).4

---

4The set $C_R$ is given by the convex cone $\{v_i \in \mathbb{R}^J : v_i \cdot \Delta L_{R_i} \geq 0\}$, where $\Delta L_{R_i}$ is a $J \times (|\mathcal{R}| - 1)$ dimensional matrix with columns given by $L_{R_i} - L_R$ for $R \neq R_i$. This definition can be used to show that $C_{R_i}$ is a convex cone and $C_R \cap C_{R'} \neq \emptyset$ only if $L_R = L_{R'}$. 

3.3.3 Extensions

The equilibrium approach outlined above assumes that agents have correct forecasts about the assignment probabilities in the year when they submit their applications. There are of course reasons why agents may not be able to correctly predict these probabilities. For example, school districts may publish only aggregate admissions chances that are not specific to a student’s priority-type. Moreover, The available statistics may derive from previous year’s data. Motivated by these concerns, Agarwal and Somaini (2018) estimate the model under two alternative assumptions on belief formation: coarse beliefs and adaptive expectations.

Coarse Beliefs: In a model with coarse beliefs, agents do not understand the role of priority-types in the mechanism. Formally, while subsumed in the notation, the probabilities in equation (3) depend on the priority-type \( t_i \). The coarse expectations model assumes that agent beliefs are given by

\[
L_R = \sum_t s_t L_{R,t},
\]

where \( s_t \) is the share of agents with priority type \( t \) and \( L_{R,t} \) is the vector of assignment probabilities to the various schools when an agent of priority-type \( t \) submits the report \( R \). Each of the assignment probabilities \( L_{R,t} \) are computed as in the rational expectations case.

Adaptive Expectations: In an adaptive expectations model, \( L_R \) is computed using the
distribution of reports from previous years, thereby substituting the term in equation (3) governing the distribution of reports, \( \prod_{i' \neq i} \sigma_{R_{i'}} (v_{i'}) f_V (v_{i'}) \), with the analogous previous-year quantities.

An additional concern with the baseline approach based on equilibrium behavior is that not all agents may have an equally accurate perception of the mechanism or their chances of getting admitted into the various schools. This lack of understanding leads to suboptimal behavior and violations of the revealed preference implications that we have derived.

Unfortunately, testing for optimal rational play is not straightforward. Agarwal and Somaini (2018) show that it is possible to rationalize any rank-order list submitted to a mechanism as being optimal provided that there is a non-zero probability of assignment to each of the ranked schools. As a result, any test of rationality should be based on other maintained assumptions about the distribution of preferences or on institutional knowledge that ensures that specific assignment probabilities are indeed zero.

Notwithstanding our inability to test for it, differences in the ability to game the system can also have distributional consequences (see Pathak and Sonmez, 2008). Motivated by these concerns, some papers have specified models in which agents vary in their degree of sophistication or have surveyed participants to consider models with a distribution of beliefs for the assignment probabilities \( L_{R_i} \).

**Heterogeneous Sophistication:** Specifically, motivated by the theoretical framework of Pathak and Sonmez (2008), Calsamiglia et al. (2017) and Agarwal and Somaini (2018) consider a model in which agents are either sincere or sophisticated. Sincere agents report their preferences truthfully even though it is suboptimal to do so while sophisticated agents have correct beliefs about the assignment probabilities \( L_{R_i} \). One could also consider adding more behavioral types, for example, agents whose expectations are adaptive. The assignment probabilities associated with each behavioral type can be computed following the type-appropriate procedures.

**Eliciting Beliefs over Assignment Probabilities:** One disadvantage of the aforementioned approaches is that the analyst needs to specify a belief formation model without direct empirical evidence on the most appropriate model in a given school district. Kapor et al. (2017) fill this gap by conducting extensive surveys that directly asks participants about their beliefs. Their results show that agents differ substantially in their understanding of the mechanism and the competitiveness of various schools. The paper develops an empirical approach that combines survey responses about admission chances and preference with administrative data to estimate both preferences and beliefs.
3.4 Incomplete Models of Behavior

The approaches developed above are based models in which an agent’s report or final assignment is specified as a known function of the indirect utilities. In the case of truthful reporting, the rank-order lists submitted list all schools starting from the most preferred to the least preferred. In the case of strategic behavior, the indirect utility vectors specify the optimal choice of assignment probabilities $L_R$. Finally, relying on stability is equivalent to eliciting a report consisting of the most preferred school in the student’s choice set. In all cases, we used a theoretical model to derive restrictions on preferences.

A natural question is whether much of the preference analysis is possible under weaker assumptions on behavior. He (2014) and Hwang (2016) develop approaches based on behavioral assumptions that use a subset of intuitive restrictions of optimal or truthful behavior, but are less restrictive because they do not impose all implications. This approach results in an incomplete model of behavior.

As an example, consider a district that uses the Immediate Acceptance Mechanism. Assume that students have correct beliefs about which schools are more competitive than others. In terms of the cutoffs described in section 2.2, students know which schools will have higher ex-post cutoffs than others. That is, whether or not $p_j$ will exceed $p_j'$ after the assignment has been computed. For simplicity, assume that there are no priorities and that either a single tie-breaker is used or tie-breakers for each school is drawn independently.

In this setting, it is not optimal for student to rank a competitive school $j$ above a less competitive school $j'$ unless the student prefers school $j$ to school $j'$. The reason is that if a student prefers $j'$ to $j$, then replacing $j'$ with $j$ increases their chances of assignment at a more preferred school. Therefore, if we observe student $i$ rank school $j$ in position $k$, then we can infer that $v_{ij}$ exceeds $v_{ij'}$ for all schools $j'$ that are less competitive than $j$ and are either ranked below $k$ or unranked. But, notice that the model only allows for partial comparisons. Specifically, if a competitive school $j$ is not ranked, we cannot directly infer whether or not $v_{ij}$ exceeds $v_{ij'}$ if $j'$ is less competitive than $j$. Doing so requires a model similar to the one developed in section 3.3 that is based on quantitative comparisons between the competitiveness of schools $j$ and $j'$.

Figure 5 illustrates an example with two schools in which a student can rank only one of the schools. If they do not rank a school, then they are assigned to a default school other than these two. In this example school 2 is more competitive than school 1. A student who prefers only one of these schools to the default option should only rank that school. This observation determines the behavior for students with indirect utility vectors in the lower-right or upper-left quadrants. The model also has unambiguous revealed preference implications for agents who submit the null rank: their vector of indirect utility is in the lower-left orthant. Now, let us turn our attention to the upper-right quadrant. If school 1 is preferred to school 2, then the student should rank school 1 because it is less competitive and preferable. However,
if school 2 is preferred to school 1, then the model is silent about which school should be ranked. This fact makes the model incomplete in the sense defined in Tamer (2003) and Manski (1988).

We now derive a set of inequalities implied by revealed preference relations. This approach adapts arguments in Ciliberto and Tamer (2009) and Pakes (2010). Specifically, consider a model that defines a set of reports that are permissible for an agent to submit given her indirect utility $v_i$. We focus on models that do not place any other restrictions on (the frequency with) which reports are chosen from within this set.

We say that report $R$ is **rationalizable** for $v_i$ under the model if $R$ belongs the set of permissible reports. Let $C_R$ be the set of indirect utilities $v_i$ such that $R$ is rationalizable for $v_i$ and let $C^*_R$ be the set of indirect utilities such that $R$ is the *only* rationalizable report. By definition, $C^*_R$ is a subset of $C_R$.

In the example of figure 5, $R_1$ or “rank 1” is rationalizable for all the solid shaded area, while $R_2$ or “rank 2” is rationalizable by for all the hashed region. Both reports are permissible for indirect utilities in the intersection of the two regions. That implies that $C^*_{R_2}$ is the upper-left quadrant and $C_{R_2}$ is the entire hashed region.
4 Identification and Estimation

We now show how to use the arguments in section 3 to identify and estimate the random utility model presented in section 2. Each approach for interpreting school choice data arrived at restrictions on the vector of indirect utilities implied by an agent’s choice or assignment. Except for the incomplete model of behavior, (almost) every indirect utility vector corresponds to a unique report or assignment. Specifically, we derived a partition of the set of indirect utilities into regions or sets \( \{C_1, \ldots, C_K\} \). Each \( C_k \) is a convex set defined by a collection of linear inequalities on \( v_i \) so that \( C_k \) contains all vectors of indirect utilities that rationalize the report \( R_k \) (or assignment to school \( k \)). This allows us to determine the probability of the observed outcome or report given agent \( i \)'s observable characteristics and the parameters of the model:

\[
P(v_i \in C_k | x_j, z_i, \xi_j, d_i; F_V),
\]

where \( F_V \) governs the distribution of indirect utilities conditional on the covariates.

The incomplete model of behavior will also yield similar restrictions, but equality between the probability of observing report \( k \) and the expression above does not hold. This is the case because the model admits more than one possible choice for many vectors of \( v_i \). Nonetheless, our discussion obtained the sets \( C_k \) and \( C_k^* \). For some of these sets, the model yields unambiguous revealed preferences implications. That is, report \( k \) that is rationalized if and only if the vector of indirect utilities belong to the region \( C_k \). In other cases, observing \( k \) only implies that \( v_i \in C_k \), but does not exclude the possibility that \( v_i \in C_{k'} \) for some \( k' \).

In all cases, the sets \( C_k \) (and \( C_k^* \)) show what can be learned about indirect utilities from reports or outcomes. Under stability, truthful reporting, or the incomplete model discussed above, the sets associated with each allocation or report can be derived directly from the revealed preference arguments. If reports are strategic, as in the model developed in section 3.3, the sets \( C_k \) are constructed from assignment probabilities that need to be estimated.

The rest of this section proceeds as follows. We start by discussing how these probabilities can be estimated in a first-step. Then, we discuss how to estimate the preference parameters for the complete models in a second step. Finally, we consider incomplete models of behavior.

4.1 Estimating Assignment Probabilities

The first objective is to estimate the assignment probabilities \( L_R \) for each report \( R \). Because these probabilities only depend on reports and the mechanism, these estimates can be obtained separately from the preference parameters in most of the models discussed earlier. We start by discussing the case in which agents have correct beliefs about the assignment
probabilities. That is, we would like to estimate the assignment probabilities from various reports in a Bayesian Nash Equilibrium.

At first blush, this object is extremely high dimensional because $L_R$ is a $J$-dimensional vector for each $R$ and the number of possible reports is given by the number of ways $J$ schools can be ranked in $K$ positions, where $K$ is the maximum length of the rank-order lists. However, the problem is simplified for the class of Report-Specific Priority + Cutoff mechanisms described in section 2. In these models, the probability that $i$ is assigned to $j$ if they report $R_i$ is given by

$$L_{R,t_i,j} = \mathbb{P}(e_{ij} \geq p_j \text{ and } e_{ij'} < p_{j'} \text{ if } j' \text{ is ranked above } j),$$

where $e_{ij} = f_j(R_i, t_i, \nu_{ij})$ combines the report, student priority and the tie-breaker to construct the eligibility score. The dependence of $L_R$ on the priority-type $t_i$ has been reintroduced for clarity.

The key unknown in the expression above are the cutoffs $p_j$ because distribution of the tie-breaker and the function $f$ are known to the researcher. Agarwal and Somaini (2018) develop a resampling estimator motivated by the auctions literature (Guerre et al., 2000; Hortacsu and McAdams, 2010; Cassola et al., 2013) for this task. This estimator constructs bootstrapped samples of students with their reports and priority draws. It computes an assignment and the corresponding cutoffs. This bootstrap distribution of cutoffs is used to estimate $L_R$. In a market with many students relative to the number of schools, the cutoffs $p_j$ converge to a limit because aggregate uncertainty in the reports submitted by the agents vanish and any given student has a negligible effect on the cutoffs. In any finite sample, each student faces uncertainty because of the draw of their tie-breaker and because of the specific reports submitted by the other agents.\(^5\)

Assignment probabilities for the cases of coarse expectations, adaptive expectations and heterogeneous sophistication introduced in section 3.3 can be computed analogously. Unfortunately, assignment probabilities cannot be separately estimated when the belief model allows for heterogeneity and is estimated using survey data.

### 4.2 Identification for Complete Models using variation in Distance

This subsection describes how variation in $d$ within a market, fixing $z_i$ and $(x_j, \xi_j)$ for every school $j$, can be used to learn about the distribution of indirect utilities. Quasilinearity of distance in equation (1), implies that $v_i = u_i - d_i$ where $u_i = (u_{i1}, \ldots, u_{iJ})$ and $u_{ij} = v(x_j, z_i, \xi_j, \gamma_i, \epsilon_{ij})$. The distribution of $u_i$ conditional on a given value of $z_i$ and $(x_j, \xi_j)$

---

\(^5\)In some cases, researchers assume away finite sample uncertainty due to the reports submitted by other agents (Calsamiglia et al., 2017; Fack et al., 2019). This approximation has only a minor effect on the assignment probabilities when the number of students is large relative to the number of schools.
for every school in the market describes the distribution of indirect utilities conditional on observables. *Agarwal and Somaini (2018)* derive conditions on the sets $C_k$ that guarantee the non-parametric identification of the distribution of $u_i$ assuming that

$$\langle \gamma_i, \varepsilon_i \rangle \perp d_i | z_i, \{x_j, \xi_j\}_{j=1}^J.$$ 

The identification argument can be illustrated using figure 6. Consider a report or allocation $k$ and the set $C_k$ associated with it. By the revealed preference implications of the model, the probability that a student reports or is assigned to $k$ is equal to the probability that $u_i$ belongs to the set $C_k + d_i$, denoted $\mathbb{P}(u_i - d_i \in C_k)$. This set is represented in the figure by the lightly shaded set with $d$ as a vertex. Similarly, by focusing on the set of students with $d_i \in \{d_i', d_i'', d_i'''\}$, we can determine the probability that a student has indirect utilities in the corresponding regions (see figure 6). By appropriately adding and subtracting these probabilities, we can learn the proportion of students with utilities in the parallelogram defined by $(d, d', d'', d''')$. This allows us to learn the total weight placed by the distribution of $u_i$ on such parallelograms of arbitrarily small size. It turns out that we can learn the density this distribution around any point focusing on variation in the neighborhood around it.

While this argument is non-parametric, it becomes necessary in empirical applications with finite datasets to parametrize the distribution of preferences due to the high-dimensional nature of the non-parametric problem. Below, we describe some of the most commonly used methods and discuss the types of models for which they are applicable or particularly
tractable.

4.3 Estimating Preference Parameters

We now describe some of the most commonly used parametrization and estimation methods. Logit models are particularly convenient in cases where the allocation is stable or reports are truthful. In these models, choice probabilities have a tractable closed form. In the case of strategic reports, the sets that rationalize different choices have less structure, and the logit models do not necessarily yield closed-form expressions. Another disadvantage of the logit model is that it is harder to relax the independence of irrelevant alternatives property. Both these issues can be solved using probit models. They can be used in any of the complete models and allow for random co-efficients relatively easily. Their convenience arises from the possibility of employing Bayesian estimation techniques that do not even require the computation of choice probabilities.

4.3.1 Logit Models

Section 3 shows that both stability and truthful reporting imply pairwise inequalities between components of the indirect utility vectors. Specifically, in the case of stability, the indirect utility of the school that student $i$ is assigned to is higher than the indirect utility of all other schools for which they is eligible. Similarly, in the case of truthful reporting, the indirect utility of the highest ranked school exceeds that of the second-highest, which in turn exceeds that of the third-highest. Moreover, the indirect utilities of the unranked schools are lower than those of the lowest ranked school. Perhaps the most tractable parametrization of indirect utilities for these two models results from the logit random utility model. This model uses the special case of equation (1) where

$$v_{ij} = \delta_j + x_j \gamma z_i - d_{ij} + \epsilon_{ij}$$

and $v_{i0} = \epsilon_{i0}$, where $\epsilon_{ij}$ follows an extreme-value type I distribution with location parameter 0 and scale parameter $\sigma$. In addition to the distributional assumption on $\epsilon_{ij}$, this specification excludes the terms $\gamma_i$.

When final assignments are stable, the probability that student $i$ is assigned to school $j$ conditional on the observable characteristics and the parameters of the model is given by

$$\Pr(\text{i is assigned to } j | x_j, z_i; \theta) = \frac{\exp \left( \frac{1}{\sigma} (\delta_j + x_j \gamma - d_{ij}) \right)}{1 + \sum 1 \{k \in S(e_i, p)\} \exp \left( \frac{1}{\sigma} (\delta_k + x_k \gamma - d_{ik}) \right)}, \quad (7)$$
where \( e_i \) is the observed eligibility score vector for student \( i \) and \( p \) is the eligibility cutoff vector.

This formula is similar to the logit choice probabilities from the standard discrete choice model (McFadden, 1973; Train, 2009). The only difference is that the summation in the denominator only includes terms for schools that are achievable by the student.

The functional form described above also turns out to be convenient for the case in which rank-order lists are assumed to be truthful. Under the parameterization described in equation (6), the probability that student \( i \) submits the rank-order list \( R_i = (R_{i1}, \ldots, R_{iK_i}) \), where \( R_{ik} \) is the school ranked \( k \) and \( K_i \) is the number of schools ranked, is given by

\[
P(i \text{ submits } R_i | x_j, z_i; \theta) = \prod_{k=1}^{K_i} \frac{\exp \left( \frac{1}{\sigma} (\delta_{R_{ik}} + x_{R_{ik}} z_i \gamma - d_{R_{ik}}) \right)}{1 + \sum_{j \neq R_{ik}'} \exp \left( \frac{1}{\sigma} (\delta_j + x_j z_i \gamma - d_{ij}) \right)} \cdot \exp \left( \frac{1}{\sigma} (\delta_{R_{ik}} + x_{R_{ik}} z_i \gamma - d_{R_{ik}}) \right).
\] (8)

This probability is therefore the product of choice probabilities identical to those derived from the logit discrete choice model. The term corresponding \( k = 1 \) is the probability that the school ranked first, \( R_{i1} \), has the highest indirect utility. Similarly, the term corresponding to \( k \) is the probability that the school ranked in position \( k \) has the highest indirect utility amongst the schools not ranked any higher. The product structure of the closed form solution depends on the parametric assumption of the model because of the independence of irrelevant alternatives (IIA) property in the logit model. Fack et al. (2019) used this approach in the school choice literature to estimate preferences for high-schools in Paris.

The IIA property of the logit model described above is an important limitation as it does not allow students to have unobserved tastes for school characteristics. In many contexts, it is natural to expect that a student that ranks a school with, say, good math outcomes at the highest position will also rank other schools with good math outcomes near the top of their list. Such patterns motivate the introduction of the random co-efficients \( \gamma_i \) in equation (1). In these models, students with a high co-efficient on a particular school characteristic will tend to rank many schools with high values of that characteristic.

A challenge with specifications that include random co-efficients is that closed-form expressions for the probability of submitting a particular rank are not typically available. Estimation techniques for these models typically require simulation, even in the simpler discrete choice context. For example, one approach is to estimate the mixed logit model using simulated maximum likelihood. The model assumes a distribution for \( \gamma_i \) and maintains the extreme-value type I assumption on \( \varepsilon_{ij} \). Conditional on a realization of \( \gamma_i \), the probability of choice \( k \) is given by \( P(v_i \in C_k | x_j, z_i, \gamma_i; \theta) \) which follows the functional form of equation (7) and (8) for the stable and truthful case. If \( \gamma_i \) is assumed to be distributed \( \gamma_i \sim \mathcal{N}(0, \Sigma_\gamma) \) with density \( \phi(\cdot; \Sigma_\gamma) \) as is common practice (Berry et al., 1995),

\[
P(v_i \in C_k | x_j, z_i; \theta, \Sigma_\gamma) = \int P(v_i \in C_k | x_j, z_i, \gamma; \theta) \phi(\gamma; \Sigma_\gamma) d\gamma.
\] (9)
Provided that the number of random co-efficients is small, this expression can be computed by numerical integration or Monte Carlo simulation. The parameters in \( \theta \) and \( \Sigma_{\gamma} \) can be estimated by simulated maximum likelihood of simulated method of moments, but a large number of draws is recommended in the former case.

Logit models are not particularly useful in the case of strategic reports. As we saw above, the set of utilities that rationalize the different reports do not correspond to the sets describing ordinal preferences. As a result, an analytic closed form, which is the main advantage of the logit model, is not available.

### 4.3.2 Probit Models

In this section we argue that Probit models are convenient both for cases where the allocation is stable, or when we have a complete model of behavior, irrespective of whether reports are truthful or strategic. Probit models do not provide a close form solution to the probability of a particular report or allocation. Instead, their convenience arises from using Bayesian estimation techniques that do not require computing these probabilities.

Probit models assume following distributional assumptions on \( \gamma_i \) and \( \varepsilon_{ij} \):

\[
\gamma_i \sim \mathcal{N}(0, \Sigma_{\gamma}) \quad \text{and} \quad \varepsilon_{ij} \sim \mathcal{N}\left(0, \sigma^2_{\varepsilon}\right).
\]

This model can be estimated via Gibbs’ sampling with appropriate conditional distributions. The Gibbs’ sampler iterates between drawing the parameters of the model (including the random co-efficients) conditional on simulated indirect utilities \( v_{ij} \), and drawing the indirect utilities \( v_{ij} \) conditional on the parameters.\(^6\) A key step in Gibbs’ sampling procedures requires a tractable solution for the distribution of the indirect utility of each school \( v_{ij} \) conditional on the parameters of the model, the random co-efficients \( \gamma_i \), and the indirect utility of the remaining schools \( v_{ij'} \) for \( j' \neq j \). Because indirect utilities are jointly normal in probit models, \( v_{ij} | v_{ij'}, \gamma, \theta \) is normally distributed, and \( v_{ij} | v_{ij'}, \gamma_i, \theta \) conditional on a particular choice has a truncated normal distribution.

This Bayesian technique yields estimates that are asymptotically equivalent to the maximum likelihood estimator (van der Vaart, 2000, Theorem 10.1 (Bernstein-von Mises)). Moreover, the sampling method yields both the point estimates and confidence sets simultaneously. These features of the model have made it popular method for discrete choice models in the marketing literature (McCulloch and Rossi, 1994). Recall that estimation under stability is just a particular case of a discrete choice model with individual specific choice sets. (Abdulkadiroglu et al., 2017b) discuss how to implement the Gibbs sampler in the case of truthful reporting. They employ it to estimate preferences for high schools in New York City where students can list up to twelve out of several hundred schools. The size of the district results

\(^6\)The draw of the indirect utilities in this second step is known as data-augmentation.
in an extremely large number of possible reports. The application stresses one attractive feature of the Gibbs sampler in the case of truthful reports: it is computationally tractable even with a large number of potential choices.

Agarwal and Somaini (2018) extended the use of the Gibbs’ Sampler for the strategic report case. The key insight is that the step of drawing $v_{ij} | v_{ij}', \gamma_i, \theta$ conditional on a particular choice reduces to drawing from a truncated normal with truncation points determined by differences in probabilities of assignment between the chosen report and every alternative. That is, the truncation points depend on $L_{R_i} - L_R$ for all $R$ other than the report $R_i$, which was chosen by $i$. This feature of the conditional distributions enables estimation using a Gibbs’ sampler.

In the case of strategic reports, the Gibbs sampling procedure can be computationally burdensome when there are many possible reports that can be submitted. For example, if a student can rank $K$ out of $J$ schools, then the possible rank order lists exceed the number of $K$–permutations of $J$. In some cases, a number of constraints are redundant and can be eliminated. One way to eliminate redundant constraints is by using a linear programming solver. However, this approach still requires computing a large number of differences $L_{R_i} - L_R$.

One solution to this problem is to theoretically derive the set of deviations from the report $R_i$ that are necessary and sufficient for optimality. Larroucau and Rios (2019) develop this approach for settings when admission chances across schools are independent. In this case, the set of alternative reports that need to be considered are characterized by one-shot swaps. This observation vastly simplifies computation as the number of alternatives that need to be considered are now of the order $K \times J$, which is much smaller than the total number of possible reports. A more general solution to this problem is unknown.

4.3.3 Other Parametrizations and Simulated Maximum Likelihood

A general alternative when the simplifications above are not available is to use a simulation estimator. For example, some researchers have used simulated maximum likelihood methods. This approach simulates the right-hand side of equation (5) by drawing indirect utilities many times given the parameter vector $\theta$ and computing the optimal reports for each draw to compute the probability that any given $R_i$ is optimal. This procedure has two problems when $J$ is large. The first is the well-known issue that simulated maximum likelihood is biased unless the number of simulations is much larger than the number of choices (Train, 2009, Chapter 10). This problem creates a computational burden in settings with many choices – or in the school choice case, many reports. The second problem is that computing the optimal report, or even verifying that a report is optimal, can be a hard problem if there are many schools. This is because one must, in principle, check many potential deviations.
A solution to the second problem with simulation methods requires a tractable algorithm for computing optimal reports in environments with many choices. There are two existing approaches to this problem. The first, due to Calsamiglia et al. (2017), is applicable for a broad class of mechanisms when there is limited or no uncertainty about the cutoff at various schools. It uses a backwards induction method to check whether a particular report is optimal for a given indirect utility vector by starting from the lowest ranked school and working its way up to the highest ranked school. It checks whether the school ranked in each position is the optimal one to rank given that the student was rejected from all schools ranked at a higher position.

The second approach, due to Ajayi and Sidibe (2017), accommodates uncertainty in the cutoffs for a given school and formulates the student’s problem as a portfolio choice problem. The paper proposes an approximation to the portfolio choice problem based on the marginal improvement algorithm of Chade and Smith (2006). While the solution is not guaranteed to be optimal, it produces rank-order lists with very similar schools listed. The approximate solutions are used in a simulated method of moments approach to estimate the preference parameters.

### 4.4 Incomplete Models

An alternative to the likelihood-based techniques discussed above is to develop approaches based on the method of moments. The technique is particularly useful for incomplete models of behavior.

We begin by describing the moment equality methods for the simpler models. Specifically, the approaches for stable assignments, truthful reporting and strategic reporting yield the following moment equality restrictions:

\[
\mathbb{E} \left[ 1 \{ R_i = R \} \right] - \mathbb{P} \left( v_i \in C_R | x_j, z_i, \xi_j; \theta \right) = 0,
\]

where \( 1 \{ R_i = R \} \) denotes the event that student \( i \) submits rank-order list \( R_i \); \( \theta \) parametrizes the distribution of preferences; and expectations are taken over individuals. Because the expectation of the indicator is equal to the second term when \( \theta \) is equal to the true value, this moment equality holds at \( \theta_0 \) for each value of \( R \) and each value of the exogeneous variables \( (x_j, z_i, \xi_j) \). The identification condition requires that there are no other value of \( \theta \) under which this equality is satisfied. The moment condition presented here is analogous to the one for the standard discrete choice models.

During estimation, the typical practice is to convert the conditional moment equality to unconditional versions by interacting the moment function with the observed characteristics. In principle, the parameters can be estimated by simulating the probability \( \mathbb{P} \left( v_i \in C_R | x_j, z_i, \xi_j; \theta \right) \) if it is hard to compute in closed form (McFadden, 1989; Pakes and Pollard, 1989). The
method of simulated moments is consistent with a small number of simulated draws even though simulated maximum likelihood is not (see Train, 2009, Chapter 10). Therefore, this particular formulation of the problem solves the first problem with simulated maximum likelihood mentioned earlier. For this reason, this approach can be more tractable than simulated maximum likelihood for the case of strategic reporting and a large number of choices.\footnote{A challenge in implementing the simulated method of moments in this formulation is that the resulting objective function may not be smooth, creating potential problems with standard derivative based optimizers. This issue is important for the simulated maximum likelihood procedure as well.}

Instead of delivering moment equalities, the incomplete model of ranking behavior presented in section 3.4 delivers moment inequalities. This approach is used in both Fack et al. (2019) and Hwang (2016). Specifically, the revealed preference arguments imply the following inequalities on the probability of reports:

$$P \left( v_i \in C^*_{R_i} | x_j, z_i, \xi_j; F_V \right) \leq P \left( R_i | x_j, z_i, \xi_j \right) \leq P \left( v_i \in C_{R_i} | x_j, z_i, \xi_j; F_V \right).$$

Both these inequalities follow from set-inclusion. The first inequality follows from the fact that if $v_i \in C^*_{R_i}$, then the agent must submit report $R_i$ as it is the only rationalizable report. The second inequality follows from the fact that if agent $i$ reports $R_i$, then $v_i$ must belong to $C_{R_i}$ because $R_i$ must be rationalizable for $v_i$.\footnote{It is possible that one of these inequalities is trivial, for example, if $C^*_{R_i}$ is the empty set.}

In the example illustrated in figure 5, the shaded region below the 45-degree line belongs to $C^*_{R_1}$; the hashed upper-right quadrant belongs to $C^*_{R_2}$; and the region that is both shaded and hashed belongs to both $C_{R_1}$ and $C_{R_2}$. The inequalities derived for the two-school example are a special case of those discussed above.

The analogous moment restrictions are:

$$E \left[ 1 \{ R_i = R \} - P \left( v_i \in C^*_{R_i} | x_j, z_i, \xi_j; \theta \right) \right] x_j, z_i, \xi_j \geq 0$$

$$E \left[ P \left( v_i \in C_{R_i} | x_j, z_i, \xi_j; \theta \right) - 1 \{ R_i = R \} \right] x_j, z_i, \xi_j \geq 0.$$
5 Review of Empirical Findings

5.1 Education Policy

School demand has important implications on equality of access and the efficacy of school choice. The literature studying these issues is vast (see Hoxby, 2003) and we do not attempt to summarize it. Instead, we discuss a few papers that shed light on the implications of school choice on access to high-quality education using estimated models of student preferences.

Preferences for schools influence access to and enrollment at high quality schools. A common finding in the literature is that student preferences are correlated with both proximity to school and measures of school performance. The preferences for proximity can limit access to high quality schools due to residential location (Burgess et al., 2015). This literature also finds significant heterogeneity in preferences across socio-economic groups and across baseline academic achievement. Students from higher income families are typically willing to travel further to attend schools with better outcomes (Hastings et al., 2009; Burgess et al., 2015).

The demand for schools can also affect the pressure for school improvement (Hoxby, 2003). For example, Hastings et al. (2009) find that demand for schools with good educational outcomes is more elastic than demand for schools with lower outcomes. This finding suggests that disparate pressures to improve schools following greater choice may lead to further stratification of school quality.

Motivated by these findings, Neilson (2013) augments the demand framework with a supply-side model in which schools choose quality in order to maximize profits. The model is used to study the effects of a voucher reform on access to quality schools. In a similar vein, Dinerstein and Smith (2019) use a choice model study the responses of private schools to reduced demand following increased funding and improvement in the public school sector. A constraint faced in these two studies is that micro-data on applications is not available, limiting the ability to investigate rationing that occurs if there is excess demand in certain schools.

Choices and the school choice mechanism can also influence educational mobility. Ayaji (2017) finds that students from low-performing elementary schools are less likely to apply to selective secondary schools, partly because of stronger preferences for proximity priority. However, the school choice environment is also a contributor as students from lower-performing schools do not navigate the mechanisms as effectively, reducing their chances of getting admission into selective schools.

Instead of value-added, the student preferences described above are typically based on peer-quality of school outcomes. Therefore, it is unclear whether demand-side pressures are aligned with increasing quality as measured by value-added. Abdulkadiroglu et al. (2017c) show that while mean preferences for schools are correlated with both high-achieving peers and value-
added, they are not significantly correlated with value-added once controls for the quality of the peers have been included.

A related question is the extent to which choices induces selection into schools based on academic ability. Strong selection through choices can affect value-added estimates from models that only control for measured student ability. Akyol and Krishna (2017) estimate a value-added model for higher-education that corrects for selection induced by application decisions and the university entrance exam scores, assuming that the latter are a noisy measure of true-ability. They find that selectivity and value-added are only loosely correlated as student preferences are influences by a number of other factors.

The emphasis placed on proximity in preferences suggests that school districts should pay attention to the geographical distribution of high quality schools. The decisions on where to invest in new schools or which schools to close should therefore consider the resulting assignment of students. Epple et al. (2018) model the problem of a school district superintendent who has to decide which schools to close when confronted with a contracting student population. An important component of the model is a demand-side in which students weigh proximity, school quality and peer-characteristics when choosing a school.

5.2 Mechanism Design

A separate, now well-developed literature has investigated the trade-offs between various school choice systems. This literature has provided a quantitative analog to the vast theoretical literature (Abdulkadiroglu and Sonmez, 2003; Pathak, 2017) studying the school assignment problem from a mechanism design perspective.

One role of empirical work is to weigh-in when theory does not provide unambiguous answers. For example, consider the choice between the IA and DA. While DA is celebrated for its incentives properties, Abdulkadiroglu et al. (2011b) show that the mechanism may not efficiently sort students if they differ in the intensity of their preferences. Strategic incentives in an Immediate Acceptance mechanism encourage students to only apply to a competitive school if they have strong preferences for it. However, this potential advantage of IA requires that students understand and respond optimally to these incentives. Moreover, IA could also disadvantage students that are not sophisticated in their preference reporting (Pathak and Sonmez, 2008).

The empirical literature has shed light on the welfare effects of the two mechanisms and the extent to which students understand the mechanisms. This trade-off has been estimated using models in which behavior is described by equilibrium play (Agarwal and Somaini, 2018); a mix of sincere and sophisticated players (Calsamiglia et al., 2017; Agarwal and Somaini, 2018); weak restrictions on beliefs (He, 2014; Hwang, 2016); and heterogeneous beliefs estimated using survey data (Kapor et al., 2017). The papers largely find that the
average student welfare is higher under IA if students behaviour is described by equilibrium play, but the difference is small at best in economic terms (Kapor et al., 2017; Agarwal and Somaini, 2018). Survey evidence suggests that many students are mistaken about their admission chances or the mechanism used (He, 2014; Kapor et al., 2017). These biases can result in an overstatement of potential, although already small, advantages of IA (Kapor et al., 2017; Agarwal and Somaini, 2018).

Empirical approaches have also been used to investigate variants of the DA mechanism motivated by theory. For example, de Haan et al. (2016) compared a DA mechanism with a single tie-breaker to one in which a student gets independent tie-breakers for each school. This comparison has been a longstanding open theoretical question (Pathak and Sethuraman, 2011; Ashlagi and Nikzad, 2017). Another example is a variant of DA devised in Che and Tercieux (2015) to obtain assignments that are both approximately efficient and stable in large markets.

The experience with implementing school choice mechanisms in practice have also motivated new issues that can be addressed using empirical work. An example is the problem of a school district which has to manage transportation costs, and therefore offers students a menu of school choice options as opposed to unrestricted choice. Estimates from the models described above have been used to predict and guide the assignments that result from various menu designs (Shi, 2015, 2019). An open question in this realm is the extent to which choice models accurately predict the assignments following such reforms (Pathak and Shi, 2015).

Other papers have investigated how to manage the choices a student can submit. In principle, the desirable incentive properties of the Deferred Acceptance mechanism rely on placing no restrictions on the number of schools that can be ranked (Haeringer and Klijn, 2009). However, with rare exceptions, choice plans limit the number of schools that can be ranked. Ajayi and Sidibe (2017) argue that these limitations can constrain welfare substantially and result in redistribution.

Part of the concern with limiting the length of rank order lists is that uncertainty about admissions chances makes students liable to rank too many competitive schools, resulting in some being left out. Luflade (2017) studies a sequential implementation that is possible when students are ordered according to a single exam score. In this system, students with high scores are approached first and asked to submit preferences with a limit on the number of options that can be ranked. Then, lower priority students are approached and are given information about the remaining seats. This implementation reduces uncertainty and improves welfare even with a small number of rounds.

Perhaps the primary goal of a school choice system coordinated via a mechanism is to increase allocative efficiency. But, comparing them with decentralized systems requires data on a system without a formal school choice system. Such data are hard to come by. Abdulkadiroglu et al. (2017b) make use of the implementation of the New York City High School
assignment system to quantify the welfare effects of centralized school assignment. They find that, following the reform that centralized the assignment process, students were placed at more desirable schools, were more likely to enroll in their assigned school and exit from the public school system fell. Moreover, they compared the new system based on DA to the old system and alternatives motivated by matching theory. The new system realized 80% of the way from a no-choice neighborhood assignment to the utilitarian optimal, whereas the old system was at most one-third of the way on this scale. Other ordinal mechanisms studied in the theoretical literature were within a few percentage points, suggesting that the primary gains arise from coordination of assignments.

5.3 Information, Beliefs and Behavior

Many of the issues discussed above depend on the information available to students (or parents acting on their behalf), how they process the information to form beliefs and how those beliefs ultimately translate to behavior. For example, preferences may not welfare relevant if students are not familiar with the schools when deciding where to apply. Similarly, school choice is ineffective at improving schools if students do not know which schools are good.

There are several studies that show that students are not perfectly informed about school quality. For example, Hastings and Weinstein (2008) use a field experiment to show that providing families with information about school performance changes choice behavior. More students apply to higher-performing schools. The finding emphasizes the need for greater information provision if school choice is to pressure lower-performing schools to raise their standards. This reasoning also raises questions about the type of information that should be provided if performance does not perfectly correlate with test-score gains or value-added.

In an innovative and ambitious paper, Allende et al. (2019) propose a framework for extrapolating from the estimated effects of information provision experiments on assignments to large-scale information interventions. There are two important issues to consider. First, these interventions steer students to specific schools, creating or increasing excess demand. This brings the method for rationing slots at the schools into focus. Second, schools may endogenously respond by changing their investment in quality and adjusting school capacity. Allende et al. (2019) consider these two issues in the Chilean context using a model in which schools adjust quality and tuition to maximize profits while ensuring that demand exactly matches the number of seats available prior to the intervention. Therefore, school characteristics adjust so that all students are ultimately placed in their first-choice school.

An important area of research is to better understand the objectives of the school and to use detailed application data to better capture the rationing of school seats.

Information about the school system also has an important consequences for how to design the school choice system. Abdulkadiroglu et al. (2017b) estimate a significant gap between a
Pareto optimal assignment produced using the strategyproof top-trading cycles mechanism and the utilitarian optimal assignment. Bridging this gap may require devising mechanisms that directly utilize cardinal information. If strategyproof solutions are not available, then one must consider the information environment and whether students understand how the mechanism works. In particular, there may also be redistributive consequences if some students are unable to effectively navigate the system (Pathak and Sonmez, 2008).

Navigating a large school district with many choices can be daunting, opening the possibility of formal choice mechanisms to result in ex-post mismatch. For example, students and parents may invest more time investigating a school after they have been initially assigned to it. Indeed, Narita (2018) documents significant cases of preference reversals in which students in New York appeal the assignment obtained through the main mechanism. Another cause for concern is that one out of every ten students unexpectedly remained unassigned after the main rounds of the New York City school choice mechanism (Abdulkadiroglu et al., 2017b). These students were placed by an administrator, usually to the closest school with available space.

These issues highlight the importance of organizing aftermarkets that enable students to trade if they learn that they are mismatched. An investigation into these questions requires a deeper understanding of the process through which students acquire information about schools. It also highlights the need for better information about the school assignment process as an important complement to reforming choice systems.

6 Conclusion

Estimating models of school demand differs from methods for standard consumer choice because education markets typically involve rationing. Not all students get admitted to their top choice school. Capacity constraints, not demand, often determines total enrollment.

We describe recently developed methods for using rich data derived from formal student assignment systems to estimate models of student preferences. The choice of technique depends on the properties of the school choice mechanism in place, the available data and the assumptions a researcher is willing to make on student behavior.

The newly developed methods yielded a wealth of insights into educational markets. One consistent finding is that student preferences are correlated with both proximity to school and measures of school performance. This suggests that proximity is an important barrier to access. Moreover, students from disadvantaged backgrounds are less likely to apply to selective secondary schools, partly because of stronger preferences for proximity.

These results suggest that different pressures to improve schools after students are granted greater choice may lead to further stratification of school quality. Importantly, school districts should pay attention to the geographical distribution of high-quality schools.
The models have also been used to investigate the trade-offs between various school choice systems. An important result is that the benefits from centralizing allocations dwarf the differences across well-coordinated mechanisms. Within coordinated systems, a common conclusion is that average student welfare is higher under the Immediate Acceptance Mechanism if students’ behavior is described by equilibrium play. This difference is small at best in economic terms. These small gain shrink or disappear altogether if agents fail to behave optimally or have biased beliefs about their assignment probabilities.

There are several fruitful areas for improving the approaches described above. First, existing models hold residential choice fixed. Evidence suggests that school district boundaries are an important determinant of residential decisions (Black, 1999; Bayer et al., 2007). It would be valuable to extend the framework presented in this paper to jointly consider within district residential choice and school choice. Second, the methods abstract away from peer effects. While summary measures of peer quality can be included as school characteristics, these cannot be interpreted as equilibrium quantities.

Perhaps an important open question is the extent to which preferable assignments also result in preferable outcomes. While some studies mentioned in our review have measured the correlations between choices and school value-added, it is unknown whether improvements in school quality cause increases in demand. Further, while there is some evidence of positive selection into schools based on heterogeneous and idiosyncratic effects on test-score gains (Walters, 2013), the extent to which school choice enables better sorting on unobserved test-score gains is unknown. These questions deserve further research.

There are many open avenues for using student preference models to guide education policy. These include guiding investment in school infrastructure and quality; managing transportation investments to expand access to high-quality schools; developing models of how school demand influences school investment decisions in the presence of rationing; and understanding how the choice environment can be improved to provide more equitable access.

References


He, Yinghua, “Gaming the Boston School Choice Mechanism in Beijing,” Toulouse School of Economics, mimeo, 2014.


Neilson, Christopher, “Targeted Vouchers, Competition Among Schools, and the Academic Achievement of Poor Students,” 2013.


