Online Appendix for “Record-Keeping and Cooperation in Large Societies”

Daniel Clark, Drew Fudenberg, and Alexander Wolitzky

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OA.1 Proof of Corollary 3

**Corollary 3.** Under any finite-partitional record system, a coordination-proof equilibrium exists if the stage game has a symmetric Nash equilibrium that is not Pareto-dominated by another (possibly asymmetric) Nash equilibrium.

Fix such a symmetric static equilibrium $\alpha^*$, and let $\sigma$ recommend $\alpha^*$ at every record pair $(r,r')$. Then $(\sigma, \mu)$ is an equilibrium for any steady state $\mu$. Moreover, note that
$$\hat{u}_{r,r'}(a,a') = (1-\gamma)u(a,a') + \gamma u(\alpha^*, \alpha^*),$$
for any $r,r', a,a'$. Thus, $(\alpha, \alpha')$ is a (possibly mixed) augmented-game Nash equilibrium if and only if it is a Nash equilibrium of the stage game. Since $(\alpha^*, \alpha^*)$ is not Pareto-dominated by another static equilibrium, there is no augmented-game Nash equilibrium $(\alpha, \alpha')$ satisfying $(u(\alpha, \alpha'), u(\alpha', \alpha)) > (u(\alpha^*, \alpha^*), u(\alpha^*, \alpha^*))$, and hence there is no augmented-game Nash equilibrium $(\alpha, \alpha')$ satisfying $(\hat{u}_{r,r'}(\alpha, \alpha'), \hat{u}_{r',r}(\alpha', \alpha)) > (\hat{u}_{r,r'}(\alpha^*, \alpha^*), \hat{u}_{r',r}(\alpha^*, \alpha^*))$ for any $r,r'$. That is, $(\sigma, \mu)$ is coordination-proof.

OA.2 Proof of Theorem 3

**Theorem 3.** Fix an action $a$. With canonical first-order records:
(i) If there exists an unprofitable punishment $b$ for $a$ and there is a strict and symmetric static equilibrium $(d,d)$, then $a$ can be limit-supported by strict equilibria.

(ii) If there exists an action $b$ such that $(b,b)$ is a strict static equilibrium and $u(a,a) > \max\{u(b,a), u(b,b)\}$, then $a$ can be limit-supported by strict equilibria.

Let $0 < \gamma < \bar{\gamma} < 1$ be such that

$$
\frac{\gamma}{1 - \gamma} > \max \left\{ \max_x \frac{u(x,a) - u(a,a)}{u(a,a) - u(c,b)}, \max_x \frac{u(x,c) - u(b,c)}{u(a,a) - u(c,b)} \right\} \quad \text{ (OA 1)}
$$

for all $\gamma \in [\gamma, \bar{\gamma}]$. Consider the strategy $\hat{\sigma}$: A player whose action has never been recorded as anything other than $a$ or $b$ is in good standing, and all other players are in bad standing. Players in good standing play $a$ against fellow good-standing players and play $b$ against bad-standing players, while bad-standing players always play $b$. described in Section 4, and let $\mu^G$ denote the share of good-standing players in a steady state. We will show that for all $\delta > 0$, there is an $\varepsilon > 0$ such that, whenever $\gamma \in [\gamma, \bar{\gamma}]$ and $\varepsilon_{x,x'} < \varepsilon$ for all $x, x' \in A$, $\hat{\sigma}$ induces strict equilibria satisfying $\mu^G > 1 - \delta$.

Thus, $\hat{\sigma}$ can be combined with threading to limit-support $a$ as $(\gamma, \varepsilon) \rightarrow (1, 0)$.

Throughout the proof, let $\bar{\varepsilon}_a = \sum_{\bar{a} \neq a,b} \bar{\varepsilon}_{a,\bar{a}}$ be the probability that a player’s action is recorded as something other than $a$ or $b$ when the player’s action action is $a$, and let $\bar{\varepsilon}_b = \sum_{\bar{a} \neq a,b} \bar{\varepsilon}_{b,\bar{a}}$ be the probability that the action is recorded as something other than $a$ or $b$ when the actual action is $b$.

Claim OA.1 below shows that the steady-state share of good-standing players induced by $\hat{\sigma}$ converges to 1 uniformly over $\gamma \in [\gamma, \bar{\gamma}]$ as $\varepsilon \rightarrow 0$. For the remainder of the proof, we restrict attention to $\gamma \in [\gamma, \bar{\gamma}]$. Claim OA.2 then shows that the incentives of good-standing players are satisfied when $\varepsilon$ is sufficiently small. These two claims together complete the argument, as the incentives of bad-standing players are always satisfied since $c$ is a strict best-response to $b$ and $(d,d)$ is a strict static equilibrium.

**Claim OA.1.** For all $\delta > 0$, there is an $\varepsilon > 0$ such that, whenever $\varepsilon_{x,x'} < \varepsilon$ for all $x, x' \in A$, the steady states induced by $\hat{\sigma}$ satisfies $\mu^G > 1 - \delta$. 

Proof. Note that the inflow into good standing is $1 - \gamma$, the share of newborn players. The outflow from good standing is the sum of $(1 - \gamma)\mu^G$, the share of good-standing players who die in a given period, and $\gamma(\bar{\epsilon}_a \mu^G + \bar{\epsilon}_b(1 - \mu^G))\mu^G$, the share of good-standing players who are recorded as playing an action other than $a$ or $b$ in a given period. In a steady state, these inflows and outflows must be equal, and setting the corresponding expressions equal to each other gives

$$\mu^G = \frac{1 - \gamma}{1 - \gamma + \gamma(\bar{\epsilon}_a \mu^G + \bar{\epsilon}_b(1 - \mu^G))} \geq \frac{1 - \gamma}{1 - \gamma + \gamma \max\{\bar{\epsilon}_a, \bar{\epsilon}_b\}}.$$

The claim then follows since $\lim_{\epsilon \to 0} \inf_{\gamma \in [\gamma, \Gamma]} (1 - \gamma)/(1 - \gamma + \gamma \max\{\bar{\epsilon}_a, \bar{\epsilon}_b\}) = 1$. ■

Claim OA.2. For all $\delta > 0$, there is an $\bar{\epsilon} > 0$ such that, whenever $\epsilon_{x,x'} < \bar{\epsilon}$ for all $x, x' \in A$, the incentives of good-standing players states are satisfied.

Proof. We will use the facts that the value function of good-standing players, $V^G$, equals the average flow payoff in the population in a given period, so $\mu^G(u(a,a)) + (1 - \mu^G)u(b,c)) + (1 - \mu^G)(\mu^G(u(c,b) + (1 - \mu^G)u(d,d))$, and that the value function of bad-standing players is $V^B = \mu^G(u(c,b) + (1 - \mu^G)u(d,d)$.

When facing an opponent playing $a$, the expected payoff of a good-standing player from playing $a$ is $(1 - \gamma)u(a,a) + \gamma(1 - \bar{\epsilon}_a)V^G + \bar{\epsilon}_a V^B$ while their expected payoff from playing $b$ is $(1 - \gamma)u(b,a) + \gamma(1 - \bar{\epsilon}_b)V^G + \bar{\epsilon}_b V^B$. Thus, a good-standing player strictly prefers to play $a$ rather than $b$ precisely when

$$(1 - \gamma)(u(a,a) - u(b,b)) > \gamma(\bar{\epsilon}_a - \bar{\epsilon}_b)(V^G - V^B). \quad (OA\ 2)$$

Moreover, the expected payoff of a good-standing player from playing action $x \notin \{a, b\}$ is $(1 - \gamma)u(x,a) + \gamma(e_{x,a} + e_{x,b})V^G + \gamma(1 - e_{x,a} - e_{x,b})V^B$. Thus, a good-standing player strictly prefers to play $a$ rather than any $x \notin \{a, b\}$ precisely when

$$\frac{\gamma}{1 - \gamma} > \max_{x \notin \{a, b\}} \frac{u(x,a) - u(a,a)}{(1 - \bar{\epsilon}_a - e_{x,a} - e_{x,b})(\mu^G(u(a,a) - u(c,b)) + (1 - \mu^G)u(b,c) - u(d,d))}. \quad (OA\ 3)$$
Claim OA.1 implies that, as $\varepsilon \to 0$, the right-hand side of (OA 2) and the right-hand side of (OA 3) converge uniformly to 0 and $\max_{x \notin \{a,b\}} \left( (u(x,a) - u(a,a))/(u(a,a) - u(c,b)) \right)$, respectively. From $u(a,a) > u(b,b)$ and (OA 1), we conclude that a good-standing player strictly prefers to match $a$ with $a$ for sufficiently small noise.

We now handle the incentives of a good-standing player to play $b$ against an opponent who plays $c$. When facing an opponent playing $c$, the expected payoff of a good-standing player from playing $a$ is $(1 - \gamma)u(a,c) + \gamma(1 - \tilde{\varepsilon}_a)V^G + \tilde{\varepsilon}_a V^B$ while their expected payoff from playing $b$ is $(1 - \gamma)u(b,c) + \gamma(1 - \tilde{\varepsilon}_b)V^G + \tilde{\varepsilon}_b V^B$. Thus, a good-standing player strictly prefers to play $b$ rather than $a$ precisely when

$$(1 - \gamma)(u(b,c) - u(a,c)) > \gamma(\tilde{\varepsilon}_b - \tilde{\varepsilon}_a)(V^G - V^B). \quad \text{(OA 4)}$$

Moreover, the expected payoff of a good-standing player from playing action $x \notin \{a,b\}$ is $(1 - \gamma)u(x,c) + \gamma(\varepsilon_{x,a} + \varepsilon_{x,b})V^G + \gamma(1 - \varepsilon_{x,a} - \varepsilon_{x,b})V^B$. Thus, a good-standing player strictly prefers to play $b$ rather than any $x \notin \{a,b\}$ precisely when

$$\frac{\gamma}{1 - \gamma} > \max_{x \notin \{a,b\}} \frac{u(x,c) - u(b,c)}{u(x,c) - u(c,b)(\mu^G(u(a,a) - u(c,b)) + (1 - \mu^G)(u(b,c) - u(d,d)))}. \quad \text{(OA 5)}$$

Claim OA.1 implies that as $\varepsilon \to 0$, the right-hand side of (OA 4) and the right-hand side of (OA 5) converge uniformly to 0 and $\max_{x \notin \{a,b\}} \left( (u(x,c) - u(b,c))/(u(a,a) - u(c,b)) \right)$, respectively. From $u(b,c) > u(a,c)$ and (OA 1), we conclude that a good-standing player strictly prefers to play $b$ rather than any other action against an opponent playing $c$ for sufficiently small noise.

\section*{OA.3 Proofs of Lemmas for Theorem 5(ii)}

\subsection*{OA.3.1 Proof of Lemma 12}

\textbf{Lemma 12.} There is a $D_1P_KS_1D_\infty$ equilibrium with shares $\mu^{D_1}$, $\mu^P$, $\mu^S$, and $\mu^{D_2}$ if and only if the following conditions hold:
1. Feasibility: 
\[\mu^{D_1} = 1 - \alpha(\gamma, 1 - \varepsilon_D)^J,\]
\[\mu^P = \alpha(\gamma, 1 - \varepsilon_D)^J(1 - \beta(\gamma, \varepsilon, \mu^D)^K),\]
\[\mu^S = \alpha(\gamma, 1 - \varepsilon_D)^J\beta(\gamma, \varepsilon, \mu^D)^K(1 - \alpha(\gamma, \varepsilon_C)),\]
\[\mu^{D_2} = \alpha(\gamma, 1 - \varepsilon_D)^J\beta(\gamma, \varepsilon, \mu^D)^K\alpha(\gamma, \varepsilon_C).\]

2. Incentives:
\[\begin{align*}
(C|C)_J & : \frac{(1 - \varepsilon_C - \varepsilon_D)\mu^D}{\varepsilon_C + (1 - \varepsilon_C - \varepsilon_D)\mu^D} \left( \frac{\mu^S}{1 - \mu^{D_1}} l + \frac{\mu^{D_2}}{1 - \mu^{D_1}} (\mu^P - \mu^S g) \right) > g, \\
(D|D)_{J+K-1} & : \frac{\gamma(1 - \varepsilon_C - \varepsilon_D)(1 - \alpha(\gamma, \varepsilon_C))\mu^P l + \alpha(\gamma, \varepsilon_C)(\mu^P - \mu^S g)}{1 - \gamma + \gamma(\varepsilon_C + (1 - \varepsilon_C - \varepsilon_D)\mu^D)} < l, \\
(C|D)_{J+K} & : \left( \frac{\mu^P - \mu^S g - \mu^P l}{1 - \gamma + \gamma\varepsilon_C} \right) > 1.
\end{align*}\]

We will derive the feasibility conditions and then derive the incentive conditions. The feasibility conditions of Lemma 12 are a consequence of the following lemma.

Lemma OA.1. In a \(D_PK_S\) steady state with total share of defectors \(\mu^D\),
\[\mu_k = \begin{cases} 
\alpha(\gamma, 1 - \varepsilon_D)^k(1 - \alpha(\gamma, 1 - \varepsilon_D)) & \text{if } 0 \leq k \leq J - 1, \\
\alpha(\gamma, 1 - \varepsilon_D)^J\beta(\gamma, \varepsilon, \mu^D)^k(1 - \beta(\gamma, \varepsilon, \mu^D)) & \text{if } J \leq k \leq J + K - 1, \\
\alpha(\gamma, 1 - \varepsilon_D)^J\beta(\gamma, \varepsilon, \mu^D)^K(1 - \alpha(\gamma, \varepsilon_C)) & \text{if } k = J + K.
\end{cases}\]

To see why Lemma OA.1 implies the feasibility conditions of Lemma 12, note that
\[\mu^{D_1} = \sum_{k=0}^{J-1} \alpha(\gamma, 1 - \varepsilon_D)^k(1 - \alpha(\gamma, 1 - \varepsilon_D)) = 1 - \alpha(\gamma, 1 - \varepsilon_D)^J,\]
\[\mu^P = \sum_{k=J}^{J+K-1} \alpha(\gamma, 1 - \varepsilon_D)^J\beta(\gamma, \varepsilon, \mu^D)^k(1 - \beta(\gamma, \varepsilon, \mu^D)) = \alpha(\gamma, 1 - \varepsilon_D)^J(1 - \beta(\gamma, \varepsilon, \mu^D)^K),\]
\[\mu^S = \mu_{J+K} = \alpha(\gamma, 1 - \varepsilon_D)^J\beta(\gamma, \varepsilon, \mu^D)^K(1 - \alpha(\gamma, \varepsilon_C)),\]

which also gives \(\mu^{D_2} = 1 - \mu^{D_1} - \mu^P - \mu^S = \alpha(\gamma, 1 - \varepsilon_D)^J\beta(\gamma, \varepsilon, \mu^D)^K\alpha(\gamma, \varepsilon_C).\)

Proof of Lemma OA.1. The inflow into score 0 is \(1 - \gamma\), while the outflow from score...
0 is \((1 - \gamma + \gamma(1 - \varepsilon_D))\mu_0\). Setting these equal gives

\[
\mu_0 = \frac{1 - \gamma}{1 - \gamma + \gamma(1 - \varepsilon_D)} = 1 - \alpha(\gamma, 1 - \varepsilon_D).
\]

Additionally, for every \(0 < k < J\), both score \(k\) and score \(k - 1\) are defectors. Thus, the inflow into score \(k\) is \(\gamma(1 - \varepsilon_D)\mu_{k-1}\), while the outflow from score \(k\) is \((1 - \gamma + \gamma(1 - \varepsilon_D))\mu_k\). Setting these equal gives

\[
\mu_k = \frac{\gamma(1 - \varepsilon_D)}{1 - \gamma + \gamma(1 - \varepsilon_D)}\mu_{k-1} = \alpha(\gamma, 1 - \varepsilon_D)\mu_{k-1}.
\]

Combining these facts gives \(\mu_k = \alpha(\gamma, 1 - \varepsilon_D)^k(1 - \alpha(\gamma, 1 - \varepsilon_D))\) for \(0 \leq k \leq J - 1\).

Since record \(J - 1\) is a defector and record \(J\) is a preciprocator, the inflow into record \(J\) is \(\gamma(1 - \varepsilon_D)\mu_{J-1}\), while the outflow from record \(J\) is \((1 - \gamma + \gamma(\varepsilon_C + (1 - \varepsilon_C - \varepsilon_D)\mu^D))\mu_J\). Setting these equal and using the fact that \(\mu_{J-1} = \alpha(\gamma, 1 - \varepsilon_D)^{J-1}(1 - \alpha(\gamma, 1 - \varepsilon_D))\) gives

\[
\mu_J = \alpha(\gamma, 1 - \varepsilon_D)^J(1 - \alpha(\gamma, 1 - \varepsilon_D))\frac{\gamma(1 - \varepsilon_D)}{1 - \gamma + \gamma(\varepsilon_C + (1 - \varepsilon_C - \varepsilon_D)\mu^D)}
= \alpha(\gamma, 1 - \varepsilon_D)^J\frac{1 - \gamma}{1 - \gamma + \gamma(\varepsilon_C + (1 - \varepsilon_C - \varepsilon_D)\mu^D)}
= \alpha(\gamma, 1 - \varepsilon_D)^J(1 - \beta(\gamma, \varepsilon, \mu^D)).
\]

Additionally, for every \(J < k < J + K\), both record \(k\) and record \(k - 1\) are preciprcators. Thus, the inflow into record \(k\) is \(\gamma(\varepsilon_C + (1 - \varepsilon_C - \varepsilon_D)\mu^D)\mu_{k-1}\), while the outflow from record \(k\) is \((1 - \gamma + \gamma(\varepsilon_C + (1 - \varepsilon_C - \varepsilon_D)\mu^D))\mu_k\). Setting these equal gives

\[
\mu_k = \frac{\gamma(\varepsilon_C + (1 - \varepsilon_C - \varepsilon_D)\mu^D)}{1 - \gamma + \gamma(\varepsilon_C + (1 - \varepsilon_C - \varepsilon_D)\mu^D)}\mu_{k-1} = \beta(\gamma, \varepsilon, \mu^D)\mu_{k-1}.
\]

Combining these facts gives \(\mu_k = \alpha(\gamma, 1 - \varepsilon_D)^J\beta(\gamma, \varepsilon, \mu^D)^k(1 - \beta(\gamma, \varepsilon, \mu^D))\) for \(J \leq k \leq J + K - 1\).

Since record \(J + K - 1\) is a preciprocator and record \(J + K\) is a supercooperator,
the inflow into record $J + K$ is $\gamma(\varepsilon_C + (1 - \varepsilon_C - \varepsilon_D)\mu^D)\mu_{J+K-1}$, while the outflow is $(1 - \gamma + \gamma\varepsilon_C))\mu_K$. Setting these equal and using the fact that $\mu_{J+K-1} = \alpha(\gamma, 1 - \varepsilon_D)^J \beta(\gamma, \varepsilon, \mu^D)^{K-1}(1 - \beta(\gamma, \varepsilon, \mu^D))$, we have

$$
\mu_{J+K} = \alpha(\gamma, 1 - \varepsilon_D)^J \frac{\gamma(\varepsilon_C + (1 - \varepsilon_C - \varepsilon_D)\mu^D)}{1 - \gamma + \gamma\varepsilon_C} \beta(\gamma, \varepsilon, \mu^D)^{K-1}(1 - \beta(\gamma, \varepsilon, \mu^D))
$$

$$
= \alpha(\gamma, 1 - \varepsilon_D)^J \beta(\gamma, \varepsilon, \mu^D)^K \frac{1 - \gamma}{1 - \gamma + \gamma\varepsilon_C}
$$

$$
= \alpha(\gamma, 1 - \varepsilon_D)^J \beta(\gamma, \varepsilon, \mu^D)^K(1 - \alpha(\gamma, \varepsilon, \mu^D)).
$$

\[\blacksquare\]

Now we establish the incentive conditions in Lemma 12. We first handle the incentives of the score $J$ preciprocator to play $C$ against an opponent playing $C$. (When this incentive condition is satisfied, all other preciprocators play $C$ against an opponent playing $C$.) Since $V_J$ equals the average payoff in the population of players with score greater than $J$, we have

$$
V_J = \frac{\mu^P}{1 - \mu^D_1} \mu^C + \frac{\mu^S}{1 - \mu^D_1} (\mu^C - \mu^D l) + \frac{\mu^D_2}{1 - \mu^D_1} \mu^S(1 + g).
$$

Since the flow payoff to a preciprocator is $\mu^C$, Lemma 7 along with the fact that $\mu^D_k = \varepsilon_C + (1 - \varepsilon_C - \varepsilon_D)\mu^D$ for any preciprocator implies that a score $J$ preciprocator plays $C$ against $C$ iff

$$
\frac{1 - \varepsilon_C - \varepsilon_D}{\varepsilon_C + (1 - \varepsilon_C - \varepsilon_D)} \left( \mu^C - \frac{\mu^P}{1 - \mu^D_1} \mu^C - \frac{\mu^S}{1 - \mu^D_1} (\mu^C - \mu^D l) - \frac{\mu^D_2}{1 - \mu^D_1} \mu^S(1 + g) \right) > g.
$$

Since

$$
\mu^C - \frac{\mu^P}{1 - \mu^D_1} \mu^C - \frac{\mu^S}{1 - \mu^D_1} (\mu^C - \mu^D l) - \frac{\mu^D_2}{1 - \mu^D_1} \mu^S(1 + g)
$$

$$
= \mu^D \left( \frac{\mu^S}{1 - \mu^D_1} l + \frac{\mu^D_2}{\mu^D_1 (1 - \mu^D_1)} (\mu^P - \mu^S g) \right),
$$
it follows that the \((C|C)_J\) constraint is equivalent to
\[
\frac{(1 - \varepsilon_C - \varepsilon_D) \mu^D}{\varepsilon_C + (1 - \varepsilon_C - \varepsilon_D) \mu^D} \left( \frac{\mu^S}{1 - \mu^D l} + \frac{\mu^D_2}{\mu^D_1 (1 - \mu^D_1)} (\mu^P - \mu^S g) \right) > g.
\]

To handle the incentives of a score \(J + K\) supercooperator, note that
\[
V_{J+K} = (1 - \gamma) (\mu^C - \mu^D l) + \gamma (1 - \varepsilon_C) V_K + \gamma \varepsilon_C V_{J+K+1}.
\]

Combining this with the fact that \(V_k = \mu^S (1 + g)\) for all \(k > K + J\) gives
\[
V_{J+K} = (1 - \alpha(\gamma, \varepsilon_C))(\mu^C - \mu^D l) + \alpha(\gamma, \varepsilon_C) \mu^S (1 + g).
\]

Thus, we have
\[
\gamma \frac{(1 - \varepsilon_C - \varepsilon_D)}{1 - \gamma} (V_{J+K} - V_{J+K+1}) = \frac{\gamma (1 - \varepsilon_C - \varepsilon_D)}{1 - \gamma + \gamma \varepsilon_C} (\mu^P - \mu^S g - \mu^D l),
\]
from which the \((C|D)_{J+K}\) constraint in Lemma 12 immediately follows.

Finally, we show that a record \(J + K - 1\) preciprocator prefers to play \(D\) against an opponent playing \(D\). (This implies that all other preciprocators play \(D\) against an opponent playing \(D\).) Note that
\[
V_{J+K-1} = (1 - \gamma) \mu^C + \gamma (1 - \varepsilon_C - (1 - \varepsilon_C - \varepsilon_D) \mu^D) V_{K-1} + \gamma (\varepsilon_C + (1 - \varepsilon_C - \varepsilon_D) \mu^D) V_{J+K},
\]
so
\[
\gamma \frac{(1 - \varepsilon_C - \varepsilon_D)}{1 - \gamma} (V_{J+K-1} - V_{J+K}) = \frac{\gamma (1 - \varepsilon_C - \varepsilon_D)}{1 - \gamma + \gamma (\varepsilon_C + (1 - \varepsilon_C - \varepsilon_D) \mu^D)} (\mu^C - V_{J+K}).
\]

Combining this with the expression for \(V_{J+K}\) in Equation OA 6 gives
\[
\gamma \frac{(1 - \varepsilon_C - \varepsilon_D)}{1 - \gamma} (V_{J+K-1} - V_{J+K}) = \frac{\gamma (1 - \varepsilon_C - \varepsilon_D)((1 - \alpha(\gamma, \varepsilon_C)) \mu^D l + \alpha(\gamma, \varepsilon_C) (\mu^P - \mu^S g))}{1 - \gamma + \gamma (\varepsilon_C + (1 - \varepsilon_C - \varepsilon_D) \mu^D)}.
\]
which implies the form of the \((D|D)_{J+K-1}\) constraint in Lemma 12.

**OA.3.2 Proof of Lemma 13**

**Lemma 13.** There are \(0 < \underline{\gamma} < \overline{\gamma} < 1\) and \(\overline{\varepsilon} > 0\) such that, for all \(\gamma \in [\underline{\gamma}, \overline{\gamma}]\) and \(\varepsilon_C, \varepsilon_D < \overline{\varepsilon}\), there is a \(D_JP_KS_1D_\infty\) strategy with a steady state whose shares satisfy \(|\mu^P - \delta|, |\mu^P - \overline{\mu^P}|, |\mu^S - \overline{\mu^S}| \leq \eta\), and are such that the \((C|D)_{J+K}\) constraint in Lemma 12 is satisfied.

Let \(J(\gamma, \delta) = \lceil \ln(1 - \delta)/\ln(\gamma) \rceil\) be the smallest integer greater than \(\ln(1 - \delta)/\ln(\gamma)\). Let \(K(\gamma, \delta) = \lceil (\ln(\gamma^{J(\gamma, \delta)} - \overline{\mu^P}) - \ln(\gamma^{J(\gamma, \delta)}) / \ln(\beta(\gamma, 0, \delta)) \rceil\). Let \(\overline{\varepsilon} \in ((1 + \delta)/2, 1)\) be such that

\[
\begin{align*}
|\overline{\varepsilon}^{J(\gamma, \delta)} - (1 - \delta)| & \leq \frac{\eta}{6}, \\
|\overline{\varepsilon}^{J(\gamma, \delta)}(1 - \beta(\overline{\varepsilon}, 0, \delta)^{K(\gamma, \delta)}) - \overline{\mu^P}| & \leq \frac{\eta}{6}, \\
|\overline{\varepsilon}^{J(\gamma, \delta)}(1 - \beta(\overline{\varepsilon}, 0, \delta + 2(1 - \gamma)^{K(\gamma, \delta)}) - \overline{\mu^P}| & \leq \frac{\eta}{6}, \\
\frac{\overline{\varepsilon}}{1 - \overline{\varepsilon}}(\overline{\mu^P} - \eta - (\overline{\mu^S} + \eta)g - (\delta + 2\eta)l) & > l.
\end{align*}
\]  

(\text{OA 7})

To see that such a \(\overline{\varepsilon}\) exists, note that \(\lim_{\gamma \to 1} \gamma^{J(\gamma, \delta)} = 1 - \delta\) and \(\lim_{\gamma \to 1} \beta(\gamma, 0, \delta)^{K(\gamma, \delta)} = 1 - \overline{\mu^P}/(1 - \delta)\), so \(\lim_{\gamma \to 1} \gamma^{J(\gamma, \delta)}(1 - \beta(\gamma, 0, \delta)^{K(\gamma, \delta)}) = \overline{\mu^P}\). Additionally, since \(\overline{\mu^P} - \eta - (\overline{\mu^S} + \eta)g - (\delta + 2\eta)l > 0\), the left-hand side of the fourth inequality approaches infinity as \(\gamma \to 1\). The argument for the third inequality is a little more involved. Let \(K'(\gamma, \delta) = \lceil (\ln(\gamma^{J(\gamma, \delta)} - \overline{\mu^P}) - \ln(\gamma^{J(\gamma, \delta)}) / \ln(\beta(\gamma, 0, \delta + 2(1 - \gamma))) \rceil\). It can be shown that \(\lim_{\gamma \to 1} K'(\gamma, \delta)/K'(\gamma, \delta) = 1\). Moreover, \(\lim_{\gamma \to 1} \beta(\gamma, 0, \delta + 2(1 - \gamma))^{K'(\gamma, \delta)} = 1 - \overline{\mu^P}/(1 - \delta)\), so it follows that \(\lim_{\gamma \to 1} \beta(\gamma, 0, \delta + 2(1 - \gamma))^{K'(\gamma, \delta)} = \lim_{\gamma \to 1} (\beta(\gamma, 0, \delta + 2(1 - \gamma))^{K'(\gamma, \delta)})^{K(\gamma, \delta)/K'(\gamma, \delta)} = 1 - \overline{\mu^P}/(1 - \delta)\). Combining this with \(\lim_{\gamma \to 1} \gamma^{J(\gamma, \delta)} = 1 - \delta\) gives \(\lim_{\gamma \to 1} \gamma^{J(\gamma, \delta)}(1 - \beta(\gamma, 0, \delta + 2(1 - \gamma)))^{K'(\gamma, \delta)} = \overline{\mu^P}\).

Let \(\overline{J} = J(\overline{\varepsilon}, \delta)\) and \(\overline{K} = K(\overline{\varepsilon}, \delta)\). There exists some \(\underline{\gamma} \in ((1 + \delta)/2, \overline{\varepsilon})\) such that \(\overline{J} - 1 \leq \ln(1 - \delta)/\ln(\gamma) \leq \overline{J}\) for all \(\gamma \in [\underline{\gamma}, \overline{\varepsilon}]\). Moreover, continuity, combined with the inequalities in (OA 7), implies that this \(\underline{\gamma}\) can be chosen along with some \(\overline{\varepsilon} > 0\) such
that

\[
|\alpha(\gamma, 1 - \varepsilon_D)\overline{J} - (1 - \delta)| \leq \frac{\eta}{3},
\]

\[
|\alpha(\gamma, 1 - \varepsilon_D)\overline{J}(1 - \beta(\gamma, \varepsilon, \delta)\overline{K}) - \overline{P}^C| \leq \frac{\eta}{3},
\]

\[
|\alpha(\gamma, 1 - \varepsilon_D)\overline{J}\left(1 - \beta(\gamma, \varepsilon, \delta + 2(1 - \gamma))\overline{K}\right) - \overline{P}^C| \leq \frac{\eta}{3},
\]

\[
\frac{\gamma(1 - \varepsilon_C - \varepsilon_D)}{1 - \gamma + \gamma \varepsilon_C}(\overline{P} - \eta - (\overline{P}^S + \eta)g - (\delta + 2\eta)l) > l,
\]

for all \(\gamma \in [\gamma, \overline{\gamma}]\) and \(\varepsilon_C, \varepsilon_D < \varepsilon\).

Since \(\mu^{D_2} \leq \alpha(\gamma, \varepsilon_C)\) and \(\alpha(\gamma, \varepsilon_C) \to 0\) as \(\varepsilon_C \to 0\) uniformly over \(\gamma \in [\gamma, \overline{\gamma}]\), we can take \(\varepsilon\) to be such that \(\mu^{D_2} \leq \min\{\eta/3, (1 - \gamma)/2\}\) for all \(\gamma \in [\gamma, \overline{\gamma}]\) and \(\varepsilon_C, \varepsilon_D < \varepsilon\). Moreover, as \(\overline{J} - 1 \leq \ln(1 - \delta)/\ln(\gamma) \leq \overline{J}\), it follows that \(\gamma \overline{J} \in [\gamma(1 - \delta), 1 - \delta]\) for all \(\gamma \in [\gamma, \overline{\gamma}]\). Because \(\alpha(\gamma, 1 - \varepsilon_D) \leq \gamma\) and \(\alpha(\gamma, 1 - \varepsilon_D) \to \gamma\) as \(\varepsilon_D \to 0\) uniformly over \(\gamma \in [\gamma, \overline{\gamma}]\), we can take \(\varepsilon\) to be such that \(\mu^{D_1} = 1 - \alpha(\gamma, 1 - \varepsilon_D)\overline{J} \in [\delta, \delta + 3(1 - \gamma)/2]\) for all \(\gamma \in [\gamma, \overline{\gamma}]\) and \(\varepsilon_C, \varepsilon_D < \varepsilon\). Thus, \(\mu^D \in [\delta, \delta + 2(1 - \gamma)]\) for all \(\gamma \in [\gamma, \overline{\gamma}]\) and \(\varepsilon_C, \varepsilon_D < \varepsilon\). As \(\beta(\gamma, \varepsilon, \mu^D)\) is increasing in \(\mu^D\), the first three inequalities in (OA 8) imply that, for all \(\gamma \in [\gamma, \overline{\gamma}]\) and \(\varepsilon_C, \varepsilon_D < \varepsilon\), there are feasible steady states with \(|\mu^{D_1} - \delta|, |\mu^P - \overline{P}^C|, \mu^{D_2} \leq \eta/3\). Additionally, since \(\overline{P}^S = 1 - \delta - \overline{P}^P\) and \(\mu^S = 1 - \mu^{D_1} - \mu^P - \mu^{D_2}\), it follows that all such steady states must have \(|\mu^S - \overline{P}^S| \leq \eta\).

Finally, note that these facts, along with the fourth inequality in (OA 8), imply that the \((C|D)_{J+K}\) constraint in Lemma 12 is satisfied.