Individualistic Records Robustly Support Social Cooperation

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August 30, 2019

Abstract

Indirect reciprocity is a foundational mechanism of human cooperation.¹,²,³,⁴,⁵,⁶,⁷,⁸ Existing models of indirect reciprocity fail to robustly support social cooperation: image scoring models⁵ fail to provide robust incentives, while social standing models²,⁹,¹⁰,¹¹ are not informationally robust. Here we provide a new model of indirect reciprocity based on decentralized, individualistic records: each individual’s record depends on its own past behavior alone, and not on its partners’ or partners’ partners’ behavior. When social dilemmas exhibit a coordination motive (or strategic complementarity), tolerant trigger strategies based on individualistic records can robustly support positive social cooperation and exhibit strong stability properties. In the opposite case of strategic substitutability, positive social cooperation cannot be robustly supported. Thus, the strength of short-run coordination motives in social dilemmas determines the prospects for robust long-run cooperation.

People (and perhaps also other animals) often trust each other to cooperate even when they know they will never meet again. Such indirect reciprocity relies on individuals having some information about how their partners have behaved in the past. Existing models of indirect reciprocity fall into two paradigms. In the image scoring paradigm, each individual carries an image that improves when it helps others, and (at least some) individuals help only those with good images. In the standing paradigm, each individual carries a standing that (typically) improves when it helps others with good standing, but not when it helps those with bad standing, and individuals with good standing help only other good-standing individuals.
Neither of these paradigms provides a robust explanation for social cooperation. In image-scoring models, there is no reason for an individual to only help partners with good images: since the partner’s image does not affect one’s future payoff, helping some partners and not others is optimal only if one is completely indifferent between helping and not helping. In game-theoretic terms, individuals never have strict incentives to follow image-scoring strategies, and hence such strategies can form at best a weak equilibrium. Closely related to this point, image-scoring equilibria have been found to be unstable in several environments.\textsuperscript{9,10} Standing models do yield strict, stable equilibria, but they fail to be informationally robust: computing an individual’s standing requires knowledge of not only its own past behavior, but also its past partners’ behavior, its partners’ partners’ behaviors, and so on ad infinitum. Such infinite-order information is not likely to be widely available in many societies.\textsuperscript{6}

We develop a new theoretical paradigm for modeling indirect reciprocity that supports positive social cooperation as a strict, stable equilibrium while relying only on individualistic information: when two players meet, they observe each other’s records and nothing else, and each individual’s record depends only on its own past behavior. (Individualistic information is also called “first-order.”\textsuperscript{12,13,14,15}) We consider the prisoner’s dilemma with actions $C, D$ and the standard payoff normalization, with $g, l > 0$ and $g - l < 1$ - see the top panel in Figure 1.

![Figure 1: The prisoner's dilemma.](image)

The top panel shows how any prisoner’s dilemma can be represented by the standard normalization with $g = (T - R)/(R - P)$ and $l = (P - S)/(R - P)$. The bottom panel illustrates this for “donation games” in which choosing $G$ (Give) instead of $S$ (Shirk) incurs a cost $c$ and gives benefit $b > c$ to the opponent.
In our model, each player’s record is an integer that tracks how many times that player has defected. Newborn players have record 0. Whenever an individual plays $D$, its record increases by 1. Whenever an individual plays $C$, its record remains constant with probability $1 - \varepsilon$ and increases by 1 with probability $\varepsilon$; thus, $\varepsilon \in (0, 1)$ measures the amount of noise in the system, which can reflect either errors in recording or errors in executing the intended action. An individual’s record is considered to be “good” if the number of times the individual has been recorded as playing $D$ is less than some pre-specified threshold $K$: this individualistic scoring is similar to image-scoring models. When two individuals meet, they both play $C$ if and only if they both have good records: this conditioning on the partner’s record is similar to standing models. We call these strategies tolerant trigger strategies or Grim$K$, as they are a form of the well-known grim trigger strategies$^{16}$ with a “tolerance” of $K$ recorded plays of $D$.

We analyze the steady-state equilibria of a system where the total population size is constant, but each individual has a geometrically distributed lifetime.$^{15}$ To ensure robustness we insist that equilibrium behavior is strictly optimal at every record; in particular, this implies that the equilibrium is evolutionarily stable.

We show that Grim$K$ strategies can form a strict steady-state equilibrium if and only if the PD exhibits substantial strategy complementarity, in that the gain from playing $D$ rather than $C$ is significantly greater when the opponent plays $D$: the precise condition required in the PD payoff matrix displayed in Figure 1 is $g < l/(1 + l)$. This is likely a typical case in realistic social dilemmas: it implies that even when $D$ is selfishly optimal regardless of the partner’s action (a defining feature of the PD), the social dilemma nonetheless retains some aspect of a coordination or stag-hunt game, so that playing $C$ is substantially less costly when one’s partner also plays $C$. For example, mobbing a predator is always risky (hence costly) for each individual, but is likely much less risky when others also mob.$^{17}$

Most previous studies of indirect reciprocity restrict attention to the “donation game” instance of the PD where $g = l^8$—see the bottom panel in Figure 1. Our analysis reveals this to be a knife-edge case that obscures the distinction between
strategic complements and substitutes. We show that the threshold $K$ can be tuned so that Grim$K$ strategies robustly support positive social cooperation in the presence of sufficiently strong strategic complementarity. Note that with image-scoring strategies, an individual’s image improves when it cooperates, in contrast to our simpler strategies where cooperation only slows the deterioration of one’s image. Modifying Grim$K$ strategies by specifying that cooperation improves an individual’s image does not help support cooperation: our results for maximum cooperation under Grim$K$ strategies apply equally to this more complicated class of strategies.

Note that since even individuals who always try to cooperate are sometimes recorded as playing $D$ due to noise, $K$ must be large enough that the steady-state share of the population with good records is sufficiently high: with any fixed value of $K$, a population of sufficiently long-lived players would almost all have bad records. However, $K$ cannot be too high, as otherwise an individual with a very good record (that is, a very low number of $D$’s) can safely play $D$ until its record approaches the threshold. Another constraint is that an individual with record $K - 1$ who meets a partner with a bad record must not be tempted to deviate to $C$ to preserve its own good record. These constraints lead to an upper bound on the possible share of cooperators in equilibrium. As lifetimes become long and noise becomes small, this upper bound converges to 0 whenever $g > l/(1+l)$ and to $l/(1+l)$ whenever $g < l/(1+l)$—see Table 1—and we show that this share of cooperators can in fact be attained in equilibrium in the $(\gamma, \epsilon) \to (1, 0)$ limit. Thus, greater strategic complementarity (higher $l$ and lower $g$) not only helps support some cooperation; it also increases the maximum level of cooperation in the limit, as shown in Figure 2.
Table 1: **Upper bounds on cooperation.** The entries are upper bounds on the share of cooperators possible in equilibria for various $\gamma$ and $\varepsilon$ parameters when $g = 1/2$ and $l = 2.5$, with a darker shade indicating a higher value. As we move to the bottom right, the upper bound converges to $l/(1 + l) \approx 0.7143$, which is the maximum share of cooperators sustainable in the limit, but away from the limit the upper bound can be different.

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Figure 2: **Limit performance of GrimK strategies.** (a) In the green region ($l > g/(1 - g)$), GrimK sustains a positive limit share of cooperators, which increases with $l$. In the orange region ($g < l < g/(1 - g)$), the limit share of cooperators with GrimK is 0, but other strategies may sustain positive cooperation in the limit. In the red region ($l < g$), individualistic records preclude cooperation. (b) The limit share of cooperators as a function of $l$ when $g = 1/2$. At $l = 1$, there is a discontinuity; as $l \to \infty$, the limit share of cooperators approaches 1.
GrimK strategies also satisfy desirable stability and convergence properties. These derive from a key monotonicity property of GrimK strategies: when the distribution of individual records is more favorable today, the same will be true tomorrow, because players with better records both behave more cooperatively and induce more cooperative behavior from their partners. (See Methods for a precise statement.) From this observation it can be shown that, whenever the initial distribution of records is more favorable than the best steady-state record distribution, the record distribution converges to the best steady state. Similarly, whenever the initial distribution is less favorable than the worst steady-state, convergence to the worst steady state obtains. See Figure 3.

Figure 3: Convergence of the share of cooperators. (a) depicts trajectories for the share of cooperators when $\gamma = .8$, $\varepsilon = .02$, and players use the Grim1 strategy; (b) does the same for the Grim2 strategy. In (a) all trajectories converge to the unique steady state; in (b) there are three steady states, and “high” trajectories converge to the most cooperative steady state, while “low” trajectories converge to the least cooperative steady state. See Methods for details.

These additional robustness properties are not shared by more complicated, non-monotone strategies that can sometimes support cooperation for a wider range of parameters than GrimK. For this reason, we defer consideration of such strategies to a more technical companion paper.\textsuperscript{18}

To place our work in context, note that most scoring and standing models assume
one-period memory,\textsuperscript{15} while our records can depend on arbitrarily many past actions. However, standing is computed as a recursive function of a player’s partner’s past action and standing, their partner’s action and standing, and so on; hence records, which depend only a player’s own past actions, are based on strictly less information about the outcomes of past interactions. (Scoring requires less information than records, but cannot provide robust incentives.) Further analysis of social record-keeping as a basis for robust indirect reciprocity may be directed toward specific applications such as online rating systems,\textsuperscript{19,20} credit ratings,\textsuperscript{14,21} and decentralized currencies.\textsuperscript{22,23,24}

\[\text{References}\]


Methods

Here we summarize the model and mathematical results; further details are provided in the Supplementary Information.

A Model of Cooperation with Individualistic Records

We use the steady-state equilibrium framework developed in a companion paper. Time is discrete and doubly infinite, i.e. \( t \in \{ \ldots, -2, -1, 0, 1, 2, \ldots \} \). There is a population of individuals of unit mass, with survival probability \( \gamma \in (0, 1) \); an inflow of \( 1 - \gamma \) new entrants each period keeps the total population size constant.

Each individual thus has a geometrically-distributed lifespan with mean \( 1/(1 - \gamma) \). Every period, individuals randomly match in pairs to play the PD (Figure 1). Each individuals carries a record \( k \in \mathbb{N} \). Newborns have record 0. Whenever an individual plays \( D \), its record increases by 1. Whenever an individual plays \( C \), its record remains constant with probability \( 1 - \varepsilon \) and increases by 1 with probability \( \varepsilon \); thus, \( \varepsilon \in (0, 1) \) measures the amount of noise in the system. The assumption that only plays of \( C \) are hit by noise is made only to simplify some formulas and does not affect any of our results.

When two players meet, they observe each other’s records and nothing else. A strategy is a mapping \( s : \mathbb{N} \times \mathbb{N} \to \{ C, D \} \), with the convention that the first component of the domain is a player’s own record and the second component is the current opponent’s record. We assume that all players use the same strategy, noting that this must be the case in every strict equilibrium in a symmetric, continuum-agent model like ours. (Of course, players who have different records and/or meet opponents with different records may take different actions.)

The state of the system \( \mu \in \Delta(\mathbb{N}) \) describes the share of the population with each record, where \( \mu_k \in [0, 1] \) denotes the share with record \( k \). When all players use strategy \( s \), let \( f_s : \Delta(\mathbb{N}) \to \Delta(\mathbb{N}) \) denote the resulting update map governing the evolution of the state. (The formula for \( f_s(\mu) \) may be found in the Supplementary Information.)
A *steady state* under strategy $s$ is then a state $\mu$ such that $f_{s}(\mu) = \mu$. At least one steady state exists under any strategy.\(^{25}\)

Given a strategy $s$ and state $\mu$, the expected flow payoff of a player with record $k$ is $\pi_{k}(s, \mu) = \sum_{k'} \mu_{k'} u(s(k', k), s(k', k))$, where $u$ is the PD payoff function. Denote the probability that a player with current record $k$ has record $k'$ in the future by $\phi_{k}(s, \mu)(k')$. The continuation payoff of a player with record $k$ is then $V_{k}(s, \mu) = (1 - \gamma) \sum_{t=0}^{\infty} \gamma^{t} \sum_{k'} \phi_{k}(s, \mu)(k') \pi_{k'}(s, \mu)$. Note that we have normalized continuation payoffs by $(1 - \gamma)$ to express them in per-period terms. A player’s objective to is maximize their expected lifetime payoff.

A pair $(s, \mu)$ is an *equilibrium* if $\mu$ is a steady-state under $s$ and, for each own record $k$ and opponent’s record $k'$, $s(k, k')$ maximizes $(1 - \gamma) u(a, s(k', k)) + \gamma \sum_{k''} (\rho(k', k) | k'' |) V_{k''}(s, \mu)$ over $a \in \{C, D\}$: given their record and the action their current opponent will play, a player’s action maximizes their lifetime payoff from that point on. An equilibrium is *strict* if the maximizer is unique for all pairs $(k, k')$, i.e. the optimal action is always unique. We focus on strict equilibria because they are robust, and remain equilibria under almost any “small” perturbation of the model. Note that the strategy *Always Defect*, i.e. $s(k, k') = D$ for all $(k, k')$, together with any steady state is always a strict equilibrium. Our research questions are to determine where there exist strict equilibria with positive steady-state cooperation, and how much cooperation can be supported with tolerant trigger strategies of the form *GrimK* as a function of the parameters of the PD.

**Performance of GrimK Strategies**

Under *GrimK* strategies, a matched pair of players cooperate if and only if each of their records is below a pre-specified cutoff $K$: that is, $s(k, k') = C$ if $\min\{k, k'\} < K$, and $s(k, k') = D$ if $\min\{k, k'\} \geq K$.

Call an individual a *cooperator* if its record is below $K$ and a *defector* otherwise. Note that each individual may be a cooperator for some periods of its life and a defector
for other periods, rather than being pre-programmed to cooperate or defect for its entire life.

Given an equilibrium with the tolerant trigger strategy GrimK, let $\mu^C = \sum_{k=0}^{K-1} \mu_k$ denote the corresponding steady-state share of cooperators. Note that, in a steady state with cooperator share $\mu^C$, mutual cooperation is played in share $(\mu^C)^2$ of all matches. Let $\mu^C(\gamma, \varepsilon)$ be the maximal share of cooperators in any tolerant trigger equilibrium (allowing for every possible $K$) when the survival probability is $\gamma$ and the noise level is $\varepsilon$.

Theorem 1 characterizes the performance of GrimK strategies in the double limit where the survival probability approaches 1 and the noise level approaches 0. We show that, in the double limit $(\gamma, \varepsilon) \rightarrow (1, 0)$, $\mu^C(\gamma, \varepsilon)$ converges to $l/(1 + l)$ when $g < l/(1 + l)$, and converges to 0 when $g > l/(1 + l)$. The formal statement and proof of this result are contained in the Supplementary Information.

Barring knife-edge cases, tolerant trigger strategies can thus robustly support positive cooperation in the double limit $(\gamma, \varepsilon) \rightarrow (1, 0)$ if and only if the gain from defecting against a partner who cooperates is significantly smaller than the loss from cooperating against a partner who defects: $g < l/(1 + l)$. Moreover, the maximum level of cooperation in this case is $l/(1 + l)$. Here we explain the logic of this result.

We first show that $g < \mu^C$ in any GrimK equilibrium. Newborn individuals have continuation payoff equal to the average payoff in the population, which is $(\mu^C)^2$. Thus, since a newborn player plays $C$ if and only if matched with a cooperator, $(\mu^C)^2 = (1 - \gamma)\mu^C + \gamma\mu^C V_0^C + \gamma(1 - \mu^C)V_0^D$, where $V_0^C$ and $V_0^D$ are the expected continuation payoffs of a newborn player after playing $C$ and $D$, respectively. Newborn players have the highest continuation payoff in the population, so $V_0^C < V_0 = (\mu^C)^2$. For a newborn player to prefer not to cheat a cooperative partner, it must be that $V_0^D < V_0^C - (1 - \gamma)g/\gamma$, so

$$(\mu^C)^2 < (1 - \gamma)\mu^C + \gamma(\mu^C)^2 - (1 - \gamma)(1 - \mu^C)g.$$
When \( \mu^C < 1 \) (as is necessarily the case with any noise), this inequality can hold only if \( g < \mu^C \).

We next show that \( \gamma(1-\varepsilon)\mu^C < l/(1+l) \) in any \( \text{GrimK} \) equilibrium. The continuation payoff \( V_{K-1} \) of an individual with record \( K-1 \) satisfies \( V_{K-1} = (1-\gamma)\mu^C + \gamma(1-\varepsilon)\mu^C V_{K-1} \), or \( V_{K-1} = (1-\gamma)\mu^C/(1-\gamma(1-\varepsilon)\mu^C) \). A necessary condition for an individual with record \( K-1 \) to prefer to play \( D \) against a defector partner is \( (1-\gamma)(-l) + \gamma(1-\varepsilon)V_{K-1} < 0 \), or \( l > \gamma(1-\varepsilon)V_{K-1}/(1-\gamma) \). Combining this inequality with the expression for \( V_{K-1} \) just derived yields \( \gamma(1-\varepsilon)\mu^C < l/(1+l) \), which in the \((\gamma,\varepsilon) \to (1,0)\) limit gives \( \mu^C \leq l/(1+l) \).

We have established that tolerant trigger strategies can support positive cooperation in the \((\gamma,\varepsilon) \to (1,0)\) limit only if \( g \leq l/(1+l) \), and that the maximum cooperation share cannot exceed \( l/(1+l) \). The proof of Theorem 1 is completed by showing that, by carefully choosing the tolerance level \( K \), cooperation shares arbitrarily close to \( l/(1+l) \) can be supported in equilibrium whenever \( g < l/(1+l) \), the survival probability is close to 1, and the noise level is close to 0.

**Convergence of \( \text{GrimK} \) Strategies**

Fix an arbitrary initial record distribution \( \mu^0 \in \Delta(\mathbb{N}) \). When all individuals use \( \text{GrimK} \) strategies, the population share with record \( k \) at time \( t \), \( \mu^t_k \), evolves according to

\[
\begin{align*}
\mu^{t+1}_0 &= 1 - \gamma + \gamma(1-\varepsilon)\mu^{C,t}\mu^t_0, \\
\mu^{t+1}_k &= \gamma(1-(1-\varepsilon)\mu^{C,t})\mu^{t-1}_k + \gamma(1-\varepsilon)\mu^{C,t}\mu^t_k \quad \text{for } 0 < k < K, \\
\end{align*}
\]

where \( \mu^{C,t} = \sum_{k=0}^{K-1} \mu^t_k \).

Fixing \( K \), we say that distribution \( \mu \) *dominates* distribution \( \tilde{\mu} \) if, for every \( k < K \), \( \sum_{k=0}^{k} \mu^t_k \geq \sum_{k=0}^{k} \tilde{\mu}^t_k \); that is, if for every \( k \) the share of the population with record no worse than \( k \) is greater under distribution \( \mu \) than under distribution \( \tilde{\mu} \). Informally we also say that \( \mu \) is more favorable than \( \tilde{\mu} \) when this is the case. Under the \( \text{GrimK} \)
strategy, let $\bar{\mu}$ denote the steady state with the largest share of cooperators, and let $\underline{\mu}$ denote the steady state with the smallest share of cooperators.

Theorem 2 in the Supplementary Information shows that, if the initial record distribution is more favorable than $\bar{\mu}$, then the record distribution converges to $\bar{\mu}$; similarly, if the initial record distribution is less favorable than $\underline{\mu}$, then the record distribution converges to $\underline{\mu}$. Formally, if $\mu^0$ dominates $\bar{\mu}$, then $\lim_{t \to \infty} \mu^t = \bar{\mu}$; similarly, if $\mu^0$ is dominated by $\underline{\mu}$, then $\lim_{t \to \infty} \mu^t = \underline{\mu}$.

In Figure 3(a) the blue trajectory corresponds to the initial distribution where all players have record 0, the yellow trajectory is constant at the unique steady-state value $\mu_C \approx 0.2484$, and the red trajectory corresponds to the initial distribution where all players have defector records. Here all the trajectories converge to the unique steady state. In Figure 3(b), the red trajectory is constant at the largest steady-state value $\mu_C \approx 0.9855$, the yellow trajectory is constant at the intermediate steady-state value $\mu_C \approx 0.9184$, and the green trajectory is constant at the smallest steady-state value $\mu_C \approx 0.6471$. The blue trajectory corresponds to the initial distribution where all players have record 0 and converges to the largest steady-state share of cooperators. The purple trajectory corresponds to the initial distribution where all players have defector records and converges to the smallest steady-state share of cooperators.
Supplementary Information

Here we provide the formula for the update map and the mathematical proofs of the results.

Update Map and Steady States

When all players use strategy \( s \), the distribution over next-period records of a player with record \( k \) who meets a player with record \( k' \) is given by

\[
\phi_{k,k'}(s) = \begin{cases} 
  k \text{ w/ prob. } 1 - \varepsilon, & k + 1 \text{ w/ prob. } \varepsilon \\
  k + 1 & \text{if } s(k,k') = D
\end{cases}.
\]

The evolution of the state over time under strategy \( s \) is described by the update map \( f_s : \Delta(\mathbb{N}) \to \Delta(\mathbb{N}) \), given by

\[
f_s(\mu)[0] := 1 - \gamma + \gamma \sum_{k'} \sum_{k''} \mu_{k'} \mu_{k''} \phi_{k',k''}(s)[0],
\]

\[
f_s(\mu)[k] := \gamma \sum_{k'} \sum_{k''} \mu_{k'} \mu_{k''} \phi_{k',k''}(s)[k] \text{ for } k \neq 0.
\]

A steady state under strategy \( s \) is a state \( \mu \) such that \( f_s(\mu) = \mu \).

Performance of GrimK Strategies

Theorem 1.

\[
\lim_{(\gamma,\varepsilon) \to (1,0)} \mu^C(\gamma, \varepsilon) = \begin{cases} 
  \frac{t}{1+t} & \text{if } g < \frac{t}{1+t} \\
  0 & \text{if } g > \frac{t}{1+t}
\end{cases}.
\]

Let \( \beta : (0,1) \times (0,1) \times (0,1) \to (0,1) \) be the function given by

\[
\beta(\gamma, \varepsilon, \mu^C) = \frac{\gamma(1 - (1 - \varepsilon)\mu^C)}{1 - \gamma(1 - \varepsilon)\mu^C}.
\]
When players use GrimK strategies and the share of cooperators is \( \mu^C \), \( \beta(\gamma, \varepsilon, \mu^C) \) is the probability that a player with cooperator record \( k \) survives to reach record \( k + 1 \). (This probability is the same for all \( k < K \).)

**Lemma 1.** There is a trigger strategy equilibrium with tolerance \( K \) and cooperator share \( \mu^C \) if and only if the following conditions hold:

1. **Feasibility:**
   \[
   \mu^C = 1 - \beta(\gamma, \varepsilon, \mu^C)^K. \tag{3}
   \]

2. **Incentives:**
   \[
   \frac{(1 - \varepsilon)(1 - \mu^C)}{1 - (1 - \varepsilon)\mu^C - \mu^C} > g, \tag{4}
   \]
   \[
   \mu^C < \frac{1}{\gamma(1 - \varepsilon)} \frac{l}{1 + l}. \tag{5}
   \]

The upper bounds on the equilibrium share of cooperators in Table 1 are the suprema of the \( \mu^C \in (0, 1) \) that satisfy (4) and (5) for the corresponding \( (\gamma, \varepsilon) \) parameters. When no \( \mu^C \in (0, 1) \) satisfy (4) and (5), the upper bound is 0, since Grim0 (where everyone plays D) is always a strict equilibrium.

To see how Part 2 of Theorem 1 comes from Proposition 1, note that

\[
\frac{(1 - \varepsilon)(1 - \mu^C)}{1 - (1 - \varepsilon)\mu^C} \leq 1.
\]

Thus, (4) requires \( \mu^C > g \). Moreover, combining \( \mu^C > g \) with (5) gives \( \gamma(1 - \varepsilon)g < l/(1 + l) \). Taking the \( (\gamma, \varepsilon) \to (1, 0) \) limit of this inequality gives \( g \leq l/(1 + l) \). Thus, when \( g > l/(1 + l) \), it follows that \( \lim_{(\gamma, \varepsilon) \to (1, 0)} \mu^C(\gamma, \varepsilon) = 0 \).

All that remains is to show that \( \lim_{(\gamma, \varepsilon) \to (1, 0)} \mu^C(\gamma, \varepsilon) = l/(1 + l) \) when \( g > l/(1 + l) \).

Since \( \lim_{\varepsilon \to 0} (1 - \varepsilon)(1 - \mu^C)/(1 - (1 - \varepsilon)\mu^C) = 1 \) for any fixed \( \mu^C \) and \( \lim_{(\gamma, \varepsilon) \to (1, 0)} 1/(\gamma(1 - \varepsilon)) = 1 \), it follows that values of \( \mu^C \) smaller than, but arbitrarily close to, \( l/(1 + l) \) satisfy (4) and (5) in the double limit. Thus, the only difficulty is the feasibility of \( \mu^C \).
as a steady-state level of cooperation: because $K$ must be an integer, some values of $\mu^C$ cannot be generated by any $K$, for given values of $\gamma$ and $\varepsilon$. The following result shows that this “integer problem” becomes irrelevant in the limit. That is, any value of $\mu^C \in (0, 1)$ can be approximated arbitrarily closely by a feasible steady-state share of cooperators for trigger strategies with some tolerance $K$ as $(\gamma, \varepsilon) \to (1, 0)$.

**Lemma 2.** Fix any $\mu^C \in (0, 1)$. For all $\Delta > 0$, there exist $\overline{\gamma} < 1$ and $\overline{\varepsilon} > 0$ such that, for all $\gamma > \overline{\gamma}$ and $\varepsilon < \overline{\varepsilon}$, there exists $\hat{\mu}^C$ that satisfies (3) for some $K$ such that $|\hat{\mu}^C - \mu^C| < \Delta$.

**Proof of Lemma 1**

The feasibility condition of Lemma 1 comes from the following lemma.

**Lemma 3.** In a trigger strategy equilibrium with tolerance $K$ and cooperator share $\mu^C$, $\mu_k = \beta(\gamma, \varepsilon, \mu^C)^k(1 - \beta(\gamma, \varepsilon, \mu^C))$ for all $k < K$.

To see why the feasibility condition of Lemma 1 comes from Lemma 3, note that

$$\mu^C = \sum_{k=0}^{K-1} \beta(\gamma, \varepsilon, \mu^C)^k(1 - \beta(\gamma, \varepsilon, \mu^C)) = 1 - \beta(\gamma, \varepsilon, \mu^C)^K.$$

**Proof of Lemma 3.** The inflow into record 0 is $1 - \gamma$, while the outflow from record 0 is $(1 - \gamma(1 - \varepsilon)\mu^C)\mu_0$. Setting these equal gives

$$\mu_0 = \frac{1 - \gamma}{1 - \gamma(1 - \varepsilon)\mu^C} = 1 - \beta(\gamma, \varepsilon, \mu^C).$$

Additionally, for every $0 < k < K$, the inflow into record $k$ is $\gamma(1 - (1 - \varepsilon)\mu^C)\mu_{k-1}$, while the outflow from record $k$ is $(1 - \gamma(1 - \varepsilon)\mu^C)\mu_k$. Setting these equal gives

$$\mu_k = \frac{\gamma(1 - (1 - \varepsilon)\mu^C)}{1 - \gamma(1 - \varepsilon)\mu^C} \mu_{k-1} = \beta(\gamma, \varepsilon, \mu^C)\mu_{k-1}.$$
The incentive constraints of Lemma 1 come from the following lemma.

**Lemma 4.** In a trigger strategy equilibrium with tolerance $K$ and cooperator share $\mu^C$,

$$V_k = \begin{cases} 
(1 - \beta(\gamma, \varepsilon, \mu^C)^{K-k})\mu^C & \text{if } k < K \\
0 & \text{if } k \geq K.
\end{cases}$$

To see why the incentive constraints of Lemma 1 come from Lemma 4, note that the expected continuation payoff of a record-0 player from playing $C$ is $(1 - \varepsilon)V_0 + \varepsilon V_1$, while the expected continuation payoff from playing $D$ is $V_1$. Thus, a record 0 player strictly prefers to play $C$ against an opponent playing $C$ iff $(1 - \varepsilon)\gamma(V_0 - V_1)/(1 - \gamma) > g$. Combining Lemmas 3 and 4 gives

$$\frac{(1 - \varepsilon)}{1 - \gamma}(V_0 - V_1) = \frac{1 - \varepsilon}{1 - (1 - \varepsilon)\mu^C} \beta(\gamma, \varepsilon, \mu^C)^K \mu^C = \frac{(1 - \varepsilon)(1 - \mu^C)}{1 - (1 - \varepsilon)\mu^C} \mu^C,$$

so (4) follows. Moreover, the expected continuation payoff of a record $K - 1$ player from playing $C$ is $(1 - \varepsilon)V_{K-1} + \varepsilon V_K$, while the expected continuation payoff from playing $D$ is $V_K$. Thus, a record $K - 1$ player strictly prefers to play $D$ against an opponent playing $D$ iff $(1 - \varepsilon)\gamma(V_{K-1} - V_K)/(1 - \gamma) < l$. Lemma 4 gives

$$\frac{(1 - \varepsilon)}{1 - \gamma}(V_{K-1} - V_K) = \frac{\gamma(1 - \varepsilon)\mu^C}{1 - \gamma(1 - \varepsilon)\mu^C},$$

and setting this to be less than $l$ gives (5).

**Proof of Lemma 4.** The flow payoff for any record $k \geq K$ is 0, so $V_k = 0$ for $k \geq K$. For $k < K$, $V_k = (1 - \gamma)\mu^C + \gamma(1 - \varepsilon)\mu^C V_k + \gamma(1 - (1 - \varepsilon)\mu^C) V_{k+1}$, which gives $V_k = (1 - \beta(\gamma, \varepsilon, \mu^C))\mu^C + \beta(\gamma, \varepsilon, \mu^C) V_{k+1}$. Combining this with $V_K = 0$ gives $V_k = (1 - \beta(\gamma, \varepsilon, \mu^C)^{K-k})\mu^C$ for $k < K$. □
Proof of Lemma 2

Let \( \tilde{K} : (0, 1) \times (0, 1) \times (0, 1) \to \mathbb{R}_+ \) be the function given by

\[
\tilde{K}(\gamma, \varepsilon, \mu^C) = \frac{\ln(1 - \mu^C)}{\ln(\beta(\gamma, \varepsilon, \mu^C))}.
\]

(6)

By construction, \( \tilde{K}(\gamma, \varepsilon, \mu^C) \) is the unique \( K \in \mathbb{R}_+ \) such that \( \mu^C = 1 - \beta(\gamma, \varepsilon, \mu^C)^K \).

Let \( d : (0, 1] \times [0, 1) \times (0, 1) \to \mathbb{R} \) be the function given by

\[
d(\gamma, \varepsilon, \mu^C) = 
\begin{cases} 
1 + \ln(1 - \mu^C)(1 - \mu^C) & \text{if } \gamma < 1 \\
1 + \frac{(1 - \varepsilon) \ln(1 - \mu^C)}{1 - (1 - \varepsilon) \mu^C} \frac{\partial \beta}{\partial \mu^C}(\gamma, \varepsilon, \mu^C) \ln(\beta(\gamma, \varepsilon, \mu^C)) & \text{if } \gamma = 1 
\end{cases}
\]

The \( \mu^C \) derivative of \( \tilde{K}(\gamma, \varepsilon, \mu^C) \) is related to \( d(\gamma, \varepsilon, \mu^C) \) by the following lemma.

**Lemma 5.** \( \tilde{K} : (0, 1) \times (0, 1) \times (0, 1) \to \mathbb{R}_+ \) is differentiable in \( \mu^C \) with derivative given by

\[
\frac{\partial \tilde{K}}{\partial \mu^C}(\gamma, \varepsilon, \mu^C) = -\frac{d(\gamma, \varepsilon, \mu^C)}{(1 - \mu^C) \ln(\beta(\gamma, \varepsilon, \mu^C))}.
\]

**Proof of Lemma 5.** From (6), it follows that \( \tilde{K}(\gamma, \varepsilon, \mu^C) \) is differentiable in \( \mu^C \) with derivative given by

\[
\frac{\partial \tilde{K}}{\partial \mu^C}(\gamma, \varepsilon, \mu^C) = -\frac{\ln(\beta(\gamma, \varepsilon, \mu^C)) + \frac{\ln(1 - \mu^C) \frac{\partial \beta}{\partial \mu^C}(\gamma, \varepsilon, \mu^C)}{\beta(\gamma, \varepsilon, \mu^C)^2}}{(1 - \mu^C) \ln(\beta(\gamma, \varepsilon, \mu^C))}
\]

\[
= -\frac{1 + \ln(1 - \mu^C)(1 - \mu^C)}{(1 - \mu^C) \ln(\beta(\gamma, \varepsilon, \mu^C))} \frac{\partial \beta}{\partial \mu^C}(\gamma, \varepsilon, \mu^C)
\]

\[
= -\frac{d(\gamma, \varepsilon, \mu^C)}{(1 - \mu^C) \ln(\beta(\gamma, \varepsilon, \mu^C))}. 
\]

■

The following two lemmas concern properties of \( d(\gamma, \varepsilon, \mu^C) \) that will be useful for the proof of Lemma 2.
Lemma 6. $d : (0, 1] \times [0, 1) \times (0, 1) \rightarrow \mathbb{R}$ is well-defined and continuous.

Proof of Lemma 6. Since $\beta(\gamma, \varepsilon, \mu^C)$ is differentiable and takes only values in $(0, 1)$, it follows that $d(\gamma, \varepsilon, \mu^C)$ is well-defined. Moreover, since $\beta(\gamma, \varepsilon, \mu^C)$ is continuously differentiable for all $\mu^C \in (0, 1)$, $d(\gamma, \varepsilon, \mu^C)$ is continuous for $\gamma < 1$. All that remains is to check that $d(\gamma, \varepsilon, \mu^C)$ is continuous for $\gamma = 1$.

First, note that $d(1, \varepsilon, \mu^C)$ is continuous in $(\varepsilon, \mu^C)$. Thus, we need only check the limit in which $\gamma$ approaches 1, but never equals 1. Note that

$$\frac{\partial \beta}{\partial \mu^C}(\gamma, \varepsilon, \mu^C) = -\frac{\gamma(1-\varepsilon)(1-\gamma)}{(1-\gamma(1-\varepsilon)\mu^C)^2} \frac{1}{\beta(\gamma, \varepsilon, \mu^C) \ln(\beta(\gamma, \varepsilon, \mu^C))}$$

(7)

It is clear that

$$\lim_{\gamma \rightarrow 1, \gamma \neq 1} \frac{\gamma(1-\varepsilon)}{\beta(\gamma, \varepsilon, \mu^C)(1-\gamma(1-\varepsilon)\mu^C)} = \frac{1-\varepsilon}{(1-(1-\varepsilon)\mu^C)}$$

(8)

for all $(\varepsilon, \mu^C) \in [0, 1) \times (0, 1)$. For $\gamma$ close to 1,

$$\ln(\beta(\gamma, \varepsilon, \mu^C)) = \beta(\gamma, \varepsilon, \mu^C) - 1 + O((\beta(\gamma, \varepsilon, \mu^C) - 1)^2).$$

Thus,

$$\lim_{\gamma \rightarrow 1, \gamma \neq 1} \frac{1-\beta(\gamma, \varepsilon, \mu^C)}{\ln(\beta(\gamma, \varepsilon, \mu^C))} = -1$$

(9)

for all $(\varepsilon, \mu^C) \in [0, 1) \times (0, 1)$. Equations 7, 8, and 9 together imply that $d(\gamma, \varepsilon, \mu^C)$ is continuous for $\gamma = 1$.

Lemma 7. $d(1, 0, \mu^C)$ has precisely one zero in $\mu^C \in (0, 1)$, and the zero is located at $\mu^C = 1 - 1/e$.

Proof of Lemma 7. This follows from the fact that $d(1, 0, \varepsilon) = 1 + \ln(1 - \mu^C)$.  

With these preliminaries established, we now present the proof of Lemma 2.
Proof of Lemma 2. Fix some $\tilde{\mu}^C \in (0, 1)$ such that $\tilde{\mu}^C \neq 1 - 1/e$. Lemma 7 says $d(1, 0, \tilde{\mu}^C) \neq 0$. Because of this and the continuity of $d$, there exists some $\lambda > 0$ and some $\delta > 0$, $\overline{\gamma}' < 1$, and $\overline{\varepsilon} > 0$ such that $|d(\gamma, \varepsilon, \mu^C)| > \lambda$ for all $\gamma > \overline{\gamma}'$, $\varepsilon < \overline{\varepsilon}$, and $|\mu^C - \tilde{\mu}^C| < \delta$.

Additionally, note that $\lim_{\gamma \to 1} \inf_{(\varepsilon, \mu^C) \in (0, \overline{\varepsilon}) \times (\mu^C - \delta, \mu^C + \delta)} \beta(\gamma, \varepsilon, \mu^C) = 1$. Together these facts imply that there exists some $\gamma < 1$ such that

$$\left| \frac{\partial \tilde{K}}{\partial \mu^C} (\gamma, \varepsilon, \mu^C) \right| = \left| \frac{d(\gamma, \varepsilon, \mu^C)}{(1 - \mu^C) \ln(\beta(\gamma, \varepsilon, \mu^C))} \right| > \frac{2}{\min\{\delta, \Delta\}}$$

and $\tilde{K}(\gamma, \varepsilon, \mu^C) \geq 1$ for all $\gamma > \overline{\gamma}$, $\varepsilon < \overline{\varepsilon}$, and $|\mu^C - \tilde{\mu}^C| < \delta$. It thus follows that

$$\sup_{|\mu^C - \tilde{\mu}^C| \leq \min\{\delta, \Delta\}} |\tilde{K}(\gamma, \varepsilon, \mu^C) - \tilde{K}(\gamma, \varepsilon, \mu^C)| > 1$$

for all $\gamma > \overline{\gamma}$, $\varepsilon < \overline{\varepsilon}$. Hence, there exists some $\hat{\mu}^C$ within $\Delta$ of $\tilde{\mu}^C$ and some non-negative integer $\hat{K}$ such that $\tilde{K}(\gamma, \varepsilon, \hat{\mu}^C) = \hat{K}$, which implies that $\hat{\mu}^C$ is feasible since $\hat{\mu}^C = 1 - \beta(\gamma, \varepsilon, \hat{\mu}^C)$. □

Convergence of GrimK Strategies

Theorem 2.

1. If $\mu^0$ dominates $\tilde{\mu}$, then $\lim_{t \to \infty} \mu^t = \tilde{\mu}$.

2. If $\mu^0$ is dominated by $\tilde{\mu}$, then $\lim_{t \to \infty} \mu^t = \mu$.

Let $x_k = \sum_{k=0}^k \mu_k$ denote the share of the population with record no worse than $k$. From Equation 1, it follows that

$$x_0^{t+1} = 1 - \gamma + \gamma(1 - \varepsilon)x_{K-1}^t x_0^t,$$

$$x_k^{t+1} = 1 - \gamma + \gamma x_{k-1}^t + \gamma(1 - \varepsilon)x_{K-1}^t (x_k^t - x_{k-1}^t) \text{ for } 0 < k < K.$$

(10)
To see this, note that \( x_0 = \mu_0 \) and \( x_{K-1} = \mu^C \), so rewriting the first line in Equation 1 gives the first line in Equation 10. Additionally, for \( 0 < k < K \), Equation 1 gives

\[
x_{k+1}^t = \sum_{k \leq k} \mu_k^{t+1} = 1 - \gamma + \gamma \sum_{k \leq k-1} \mu_{k-1}^t + \gamma (1 - \varepsilon) \mu_0^C \mu_k^t,
\]

\[
= 1 - \gamma + \gamma x_{k-1}^t + \gamma (1 - \varepsilon) x_{K-1}^t (x_k^t - x_{k-1}^t).
\]

**Lemma 8.** The update map in Equation 10 is weakly increasing in that if \((x_0, ..., x_{K-1}) \geq (\tilde{x}_0, ..., \tilde{x}_{K-1})\), then \((x_0^{t+1}, ..., x_{K-1}^{t+1}) \geq (\tilde{x}_0^{t+1}, ..., \tilde{x}_{K-1}^{t+1})\).

**Proof.** The right-hand side of the first line in Equation 10 depends only on the product of \( x_0^t \) and \( x_{K-1}^t \), and it is strictly increasing in this product. The right-hand side of the second line in Equation 10 depends only on \( x_{k-1}^t, x_k^t, \) and \( x_{K-1}^t \), and, holding fixed any two of these variables, it is weakly increasing in the third variable.

**Proof of Theorem 2.** We prove the first statement of Theorem 2. A similar argument handles the second statement. Let \((\tilde{x}_0^t, ..., \tilde{x}_{K-1}^t)\) denote the time-path corresponding to the highest possible initial conditions, i.e. \((\tilde{x}_0^0, ..., \tilde{x}_{K-1}^0) = (1, ..., 1)\). By Lemma 8, \((\tilde{x}_0^t, ..., \tilde{x}_{K-1}^t) \leq (\tilde{x}_0^t, ..., \tilde{x}_{K-1}^t)\) for all \( t \). Thus, it follows that \( \lim_{t \to \infty} (\tilde{x}_0^t, ..., \tilde{x}_{K-1}^t) = \inf_t (\tilde{x}_0^t, ..., \tilde{x}_{K-1}^t) \), so in particular \( \lim_{t \to \infty} (\tilde{x}_0^t, ..., \tilde{x}_{K-1}^t) \) exists. Since the update rules in Equation 10 are continuous, it follows that \( \lim_{t \to \infty} (\tilde{x}_0^t, ..., \tilde{x}_{K-1}^t) \) must be a steady state of the system. By Lemma 8 and the fact that \((\tilde{x}_0^0, ..., \tilde{x}_{K-1}^0)\) is the steady state with the highest share of cooperators, it follows that \( \lim_{t \to \infty} (\tilde{x}_0^t, ..., \tilde{x}_{K-1}^t) = (\tilde{x}_0, ..., \tilde{x}_{K-1}) \).

Now, fix some \((x_0^0, ..., x_{K-1}^0) \geq (\tilde{x}_0, ..., \tilde{x}_{K-1})\). By Lemma 8,

\[
(x_0^0, ..., x_{K-1}^0) \leq (x_0^0, ..., x_{K-1}^0) \leq (\tilde{x}_0^t, ..., \tilde{x}_{K-1}^t)
\]

for all \( t \), so it follows that \( \lim_{t \to \infty} (x_0^t, ..., x_{K-1}^t) = (\tilde{x}_0, ..., \tilde{x}_{K-1}) \).

\[\blacksquare\]