First price auctions
with general information structures:
Implications for bidding and revenue

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Premises

1. Classical auction theory makes stylized assumptions about information.
2. Assumptions about information are hard to test.
3. Equilibrium behavior can depend a lot on how we specify information.
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- Other results on max revenue, min bidder surplus, min efficiency
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- Two bidders
- Pure common value $v \sim U[0, 1]$
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- Allocation of good is always efficient, total surplus $1/2$
- Seller’s expected revenue is $R = \mathbb{E}\left[ \max\{b_1, b_2\} \right]$
- Bidder surplus $U = 1/2 - R$
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- Seller’s expected revenue is $R = \mathbb{E}[\max\{b_1, b_2\}]$
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- What predictions can we make about $U$ and $R$ in equilibrium?
Filling in beliefs

- What do bidders know about the value?
- What do they know about what others know?
Filling in beliefs

- What do bidders know about the value?
- What do they know about what others know?
- Assume beliefs are consistent with a common prior

\[
\begin{align*}
\text{Unique equilibrium:} & \quad b_1 = b_2 = R = \frac{1}{2} \\
\text{Bidders observe everything:} & \quad b_1 = b_2 = v, R = \frac{1}{2} \\
\text{True information structure is likely somewhere in between:} & \\
\text{Bidders have some information about } v, \text{ but not perfect} \\
\text{But exactly how much information do they have?}
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\]
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  - But exactly how much information do they have?
Lower revenue?

- Bidder 1 observes $\nu$, bidder 2 observes nothing
  - $b_1 = \nu/2$, $b_2 \sim U[0, 1/2]$ and independent of $\nu$
Lower revenue?

- Bidder 1 observes \( v \), bidder 2 observes nothing
  - \( b_1 = \frac{v}{2}, \; b_2 \sim U[0, 1/2] \) and independent of \( v \)
- Bidder 2 is indifferent:
  With a bid of \( b_2 \in [0, 1/2] \), will win whenever \( v \leq 2b_2 \)
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  - Bidder 1 wins with a bid of $b_1$ with probability $2b_1$
    Surplus is $(\nu - b_1)2b_1$
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  $\implies$ optimal to bid $b_1 = v/2$!
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    Surplus is \( (v - b_1)2b_1 \)
    \( \implies \) optimal to bid \( b_1 = v/2! \)
  - \( U_1 = \int_{v=0}^{1} v(v - v/2)dv = 1/6, \ U_2 = 0, \ R = 1/3 \)
How we model beliefs matters

- Welfare outcomes are sensitive to modelling of information
- Why? Optimal bid depends on distribution of others’ bids, and on correlation between others’ bids and values
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How we model beliefs matters

- Welfare outcomes are sensitive to modelling of information
- Why? Optimal bid depends on distribution of others’ bids, and on correlation between others’ bids and values
- Problem: hard to say which specification is “correct”
- What welfare predictions do not depend on how we model information?
Uniform example continued

- Can we characterize minimum revenue?
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- Must be greater than zero!
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- At min $R$, winning bids have been pushed down “as far as they can go”
- Force pushing back must be incentive to deviate to higher bids
Uniform example continued

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- At min $R$, winning bids have been pushed down “as far as they can go”
- Force pushing back must be incentive to deviate to higher bids
- In EMW, informed bidder strictly prefers equilibrium bid
Towards a bound

- Consider symmetric equilibria in which winning bid is an increasing function $\beta(v)$ of $v$.
- Which $\beta$ could be incentive compatible in equilibrium?
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- Consider the following *uniform upward deviation*: Whenever equilibrium bid is $x < b$, bid $b$ instead
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- Uniform deviation up to $b = \beta(v)$ is not attractive if

$$\frac{1}{2} \int_{x=0}^{v} (\beta(v) - \beta(x)) \, dx$$

*loss when would have won*
Towards a bound

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- Uniform deviation up to $b = \beta(v)$ is not attractive if

$$\frac{1}{2} \int_{x=0}^{\beta(v)} (\beta(v) - \beta(x))dx \geq \frac{1}{2} \int_{x=0}^{\beta(v)} (x - \beta(v))dx$$

\[\text{loss when would have won} \quad \text{gain when would have lost}\]
Restrictions on $\beta$

- Rearranges to

$$\beta(v) \geq \frac{1}{2v} \int_{x=0}^{v} (x + \beta(x)) dx \quad (IC)$$
Restrictions on $\beta$

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(\text{IC})
$$

- What is the smallest $\beta$ subject to (IC) and $\beta \geq 0$?
- Must solve (IC) with equality for all $v$
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- What is the smallest $\beta$ subject to (IC) and $\beta \geq 0$?
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- Solution is

$$\beta(v) = \frac{1}{\sqrt{v}} \int_{x=0}^{v} x \frac{1}{2\sqrt{x}} \, dx$$
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$$\beta(v) = \frac{1}{\sqrt{v}} \int_{x=0}^{v} x \frac{1}{2\sqrt{x}} dx$$

$$= \frac{v}{3}$$
A lower bound on revenue

- Induced distribution of winning bids is $U[0, 1/3]$
- Revenue is $1/6$
A lower bound on revenue

- Induced distribution of winning bids is $U[0, 1/3]$
- Revenue is $1/6$
- In fact, symmetry/deterministic winning bid are not needed
- Distribution of winning bid has to FOSD $U[0, 1/3]$ in all equilibria under any information
- $1/6$ is a global lower bound on equilibrium revenue
Bound is tight

- Can construct information/equilibrium that hits bound
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- Bidders get i.i.d. signals $s_i \sim F(x) = \sqrt{x}$ on $[0, 1]$
- Value is highest signal
- Distribution of highest signal is $U[0, 1]$
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- Equilibrium bid: $\sigma_i(s_i) = s_i/3$
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- Defer proof until general results
Beyond the example

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  - In general model, only depends on a one-dimensional statistic of the value profile
  - Bound is characterized by binding uniform upward incentive constraints
The plan

- Detailed exposition of minimum bidding
- Maximum revenue/minimum bidder surplus
- Restrictions on information
- Other directions in welfare space (e.g., efficiency)
General model

- $N$ bidders
- Distribution of values: $P(dv_1, \ldots, dv_N)$
- Support of marginals $V = [\underline{v}, \overline{v}] \subseteq \mathbb{R}_+$
General model

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- Distribution of values: $P(dv_1, \ldots, dv_N)$
- Support of marginals $V = [\underline{v}, \overline{v}] \subseteq \mathbb{R}_+$
- An information structure $S$ consists of
  - A measurable space $S_i$ of signals for each player $i$, $S = \times_{i=1}^N S_i$
  - A conditional probability measure

$$\pi : V^N \to \Delta(S)$$
Equilibrium

- Bidders’ strategies map signals to distributions over bids in $[0, v]$

$$\sigma_i : S_i \rightarrow \Delta(B)$$

- Assume “weakly undominated strategies”: bidder $i$ never bids strictly above the support of first-order beliefs about $v_i$
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- Bidder $i$’s payoff given strategy profile $\sigma = (\sigma_1, \ldots, \sigma_N)$:

$$U_i(\sigma, S) = \int_{v \in V} \int_{s \in S} \int_{b \in B^N} (v_i - b_i) \frac{\mathbb{I}b_i \geq b_j \ \forall j}{|\arg \max_j b_j|} \sigma(db|s)\pi(ds|v)P(dv)$$
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- $\sigma$ is a Bayes Nash equilibrium if

$$U_i(\sigma, S) \geq U_i(\sigma'_i, \sigma_{-i}, S) \ \forall i, \sigma'_i$$
Other welfare outcomes

Bidder surplus: $U(\sigma, S) = \sum_{i=1}^{N} U_i(\sigma, S)$

Revenue: $R(\sigma, S) = \int_{v \in V^N} \int_{s \in S} \int_{b \in B^N} \max_i b_i \sigma(b|s) \pi(ds|v) P(dv)$

Total surplus: $T(\sigma, S) = R(\sigma, S) + U(\sigma, S)$

Efficient surplus: $\overline{T} = \int_{v \in V} \max_i v_i P(dv)$
As we generalize, minimum bidding continues to be characterized by a *deterministic winning bid* given values: 
\[ \beta(v_1, \ldots, v_N) \]

\[ \beta \] has an explicit formula.
General common values

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- $\beta$ has an explicit formula
- Consider pure common values with $v \sim P \in \Delta([\underline{v}, \overline{v}])$
As we generalize, minimum bidding continues to be characterized by a deterministic winning bid given values: \[ \beta(v_1, \ldots, v_N) \] 

\( \beta \) has an explicit formula.

Consider pure common values with \( v \sim P \in \Delta([v, \bar{v}]) \).

Minimum winning bid generalizes to

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\beta(v) = \frac{1}{\sqrt{P(v)}} \int_{x=v}^{x} \frac{P(dx)}{2\sqrt{P(x)}}
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General common values

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- \( \underline{\beta} \) has an explicit formula
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\underline{\beta}(v) = \frac{1}{\sqrt{P(v)}} \int_{x=v}^{\bar{v}} P(dx) \times \frac{P(dx)}{2\sqrt{P(x)}}
\]

- Minimum revenue:

\[
R = \int_{v=v}^{\bar{v}} \underline{\beta}(v)P(dv)
\]
$N = 2$ and general value distributions

- Write $P(dv_1, dv_2)$ for value distribution
\( N = 2 \) and general value distributions

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- Write $P(dv_1, dv_2)$ for value distribution
- Similarly, lots of binding uniform upward IC
- Incentive to deviate up depends on value when you lose
- On the whole, efficient allocation reduces gains from deviating up
- Suggests minimizing equilibrium is efficient, winning bid is constrained by loser’s (i.e., lowest) value
General bounds for $N = 2$

- Similar $\beta$, but now depends on *lowest* value
- $Q(dm)$ is distribution of $m = \min\{v_1, v_2\}$ (assume non-atomic)
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- Minimum revenue:

$$
\overline{R} = \int_{m=v}^{\bar{v}} \beta(m) Q(dm)
$$
Losing values when $N > 2$

- With $N > 2$, bid minimizing equilibrium should still be efficient
- Intuition: coarse information about losers’ values lowers revenue
Losing values when \( N > 2 \)

- With \( N > 2 \), bid minimizing equilibrium should still be efficient.
- Intuition: coarse information about losers’ values lowers revenue.
- Consider complete information, all values are common knowledge.
- High value bidder wins and pays second highest value.
Average losing values I

- Simple variation: Bidders only observe
  (i) High value bidder’s identity
  (ii) *Distribution* of values

\[
\mu(v_1, \ldots, v_N) = \frac{1}{N-1} \left( \sum_{i=1}^{N} v_i - \max_i v_i \right)
\]
Average losing values 1

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- Winner is still high value bidder, but losing bidders don’t know who has which value

- If prior is symmetric, believe they are equally likely to be at any point in the distribution except the highest

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- Winner is still high value bidder, but losing bidders don’t know who has which value
- If prior is symmetric, believe they are equally likely to be at any point in the distribution except the highest
- In equilibrium, winner pays average of $N - 1$ lowest values:

$$
\mu(v_1, \ldots, v_N) = \frac{1}{N - 1} \left( \sum_{i=1}^{N} v_i - \max_i v_i \right)
$$
General bounds

- $Q(dm)$ is distribution of $m = \mu(\nu)$ (assume non-atomic)
General bounds

- $Q(dm)$ is distribution of $m = \mu(v)$ (assume non-atomic)
- Minimum winning bid and revenue:

$$
\beta(m) = \frac{1}{Q \frac{N-1}{N}(v)} \int_{x=v}^{v} \frac{N - 1}{N} \frac{Q(dx)}{Q \frac{1}{N}(x)}
= \frac{1}{Q \frac{N-1}{N}(v)} \int_{x=v}^{v} \frac{Q^{\frac{N-1}{N}}(dx)}{Q^{\frac{1}{N}}(x)}
$$
General bounds

- $Q(dm)$ is distribution of $m = \mu(v)$ (assume non-atomic)
- Minimum winning bid and revenue:

$$
\beta(m) = \frac{1}{Q^{N-1}(v)} \int_{x=v}^{v} x \frac{N - 1}{N} \frac{Q(dx)}{Q^{1/N}(x)}
= \frac{1}{Q^{N-1}(v)} \int_{x=v}^{v} x \frac{N - 1}{N} (dx)
$$

- Minimum revenue:

$$
R = \int_{m=v}^{v} \beta(m) Q(dm)
$$
General bounds

- \( Q(dm) \) is distribution of \( m = \mu(v) \) (assume non-atomic)
- Minimum winning bid and revenue:
  
  \[
  \beta(m) = \frac{1}{Q \frac{N-1}{N} (v)} \int_{x=v}^{\bar{v}} \frac{N-1}{N} \frac{Q(dx)}{Q \frac{1}{N} (x)} \\
  = \frac{1}{Q \frac{N-1}{N} (v)} \int_{x=v}^{\bar{v}} Q \frac{N-1}{N} (dx)
  \]

- Minimum revenue:
  
  \[
  R = \int_{m=v}^{\bar{v}} \beta(m) \cdot Q(dm)
  \]

- Let \( \widetilde{H}(b) = Q(\beta^{-1}(b)) \)
Main result

Theorem (Minimum winning bids)

1. *In any equilibrium under any information structure in which the marginal distribution of values is* $P$, *the distribution of winning bids must first-order stochastically dominate* $H$. 
Main result

**Theorem (Minimum winning bids)**

1. *In any equilibrium under any information structure in which the marginal distribution of values is $P$, the distribution of winning bids must first-order stochastically dominate $H$."

2. *Moreover, there exists an information structure and an efficient equilibrium in which the distribution of winning bids is exactly $H$."

Implications

Corollary (Minimum revenue)

Minimum revenue over all information structures and equilibria is $R$. 
Corollary (Minimum revenue)

*Minimum revenue over all information structures and equilibria is \( R \).*

Corollary (Maximum bidder surplus)

*Maximum total bidder surplus over all information structures and equilibria is \( \bar{T} - R \).*
Proof methodology

1. Obtain a bound via relaxed program
Proof methodology

1. Obtain a bound via relaxed program
2. Construct information and equilibrium that attain the bounds
Proof methodology

1. Obtain a bound via relaxed program
2. Construct information and equilibrium that attain the bounds
   (start with #2)
Minimizing equilibrium and information

- Bidders receive independent signals \( s_i \sim Q^{1/N}(s_i) \)
  \[ \Rightarrow \] distribution of highest signal is \( Q(s) \)
Minimizing equilibrium and information

- Bidders receive independent signals $s_i \sim Q^{1/N}(s_i)$
  \[\implies\] distribution of highest signal is $Q(s)$
- Signals are correlated with values s.t.
  - Highest signal is true average lowest value, i.e.,
    \[\mu(v_1, \ldots, v_n) = \max\{s_1, \ldots, s_n\}\]
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  - Bidder with highest signal is also bidder with highest value, i.e.,
    \[ \arg \max_i s_i \subseteq \arg \max_i v_i \]
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    \mu(v_1, \ldots, v_n) = \max\{s_1, \ldots, s_n\}
    \]
  - Bidder with highest signal is also bidder with highest value, i.e.,
    \[
    \arg\max_i s_i \subseteq \arg\max_i v_i
    \]
- All bidders use the monotonic pure-strategy $\beta(s_i)$
Proof of equilibrium

- \( \beta \) is the equilibrium strategy for an “as-if” IPV model, in which \( v_i = s_i \)
Proof of equilibrium

- $\beta$ is the equilibrium strategy for an “as-if” IPV model, in which $v_i = s_i$

- IC for IPV model with independent draws from $Q^{1/N}$:

  $$(s_i - \sigma(s_i))Q^{\frac{N-1}{N}}(s_i)$$
Proof of equilibrium

- $\beta$ is the equilibrium strategy for an “as-if” IPV model, in which $v_i = s_i$
- IC for IPV model with independent draws from $Q^{1/N}$:
  \[
  (s_i - \sigma(s_i))Q^{\frac{N-1}{N}}(s_i) \geq (s_i - \sigma(m))Q^{\frac{N-1}{N}}(m)
  \]
Proof of equilibrium

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- Local IC:

$$\left(s_i - \sigma(s_i)\right)Q^{\frac{N-1}{N}}(ds_i) - \sigma'(s_i)Q^{\frac{N-1}{N}}(s_i) = 0$$
Proof of equilibrium

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  (s_i - \sigma(s_i)) Q^{\frac{N-1}{N}} (ds_i) - \sigma'(s_i) Q^{\frac{N-1}{N}} (s_i) = 0
  \]
- Solution is precisely
  \[
  \sigma(s_i) = \frac{1}{Q^{\frac{N-1}{N}} (s_i)} \int_{x=v}^{s_i} x Q^{\frac{N-1}{N}} (dx) = \beta(s_i)
  \]
Downward deviations

- Expectation of the bidder with the highest signal is $\tilde{v}(s_i) \geq s_i$
- Downward deviator obtains surplus

\[ (\tilde{v}(s_i) - \beta(m)) Q^{\frac{N-1}{N}}(m) \]

and

\[ (\tilde{v}(s_i) - \beta(m)) Q^{\frac{N-1}{N}}(dm) - \beta'(m) Q^{\frac{N-1}{N}}(m) \]

\[ \geq (s_i - \beta(m)) Q^{\frac{N-1}{N}}(dm) - \beta'(m) Q^{\frac{N-1}{N}}(m) \]
Downward deviations

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\[
(\tilde{v}(s_i) - \beta(m)) \frac{N-1}{N} (m)
\]

and

\[
(\tilde{v}(s_i) - \beta(m)) \frac{N-1}{N} (dm) - \beta'(m) \frac{N-1}{N} (m)
\geq (s_i - \beta(m)) \frac{N-1}{N} (dm) - \beta'(m) \frac{N-1}{N} (m)
\]

- Well-known that IPV surplus is single peaked: if $m < s_i$,

\[
\implies (s_i - \beta(m)) \frac{N-1}{N} (dm) - \beta'(m) \frac{N-1}{N} (dm) \geq 0
\]
Average losing values II

- Winning bids depend on avg of lowest values
  = average of losing bids (since equilibrium is efficient)
Average losing values II

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  \(=\) average of losing bids (since equilibrium is efficient)

- Suppose winning bid in equilibrium is \(\beta(m) > \beta(s_i)\)
  \(\implies \mu(v) = m\) for true values \(v\)

- By symmetry, all permutations of \(v\) are in \(\mu^{-1}(m)\) and equally likely
Winning bids depend on avg of lowest values
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If you only know that

(i) you lose in equilibrium and
(ii) \( v \in \mu^{-1}(m), \)

you expect your value to be \( m! \)
Average losing values II

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- Suppose winning bid in equilibrium is $\beta(m) > \beta(s_i)$
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- By symmetry, all permutations of $v$ are in $\mu^{-1}(m)$ and equally likely
- If you only know that
  (i) you lose in equilibrium and
  (ii) $v \in \mu^{-1}(m)$,
  you expect your value to be $m$!
- By deviating up to win on this event, gain $m$ in surplus
Upward deviations

- Upward deviator’s surplus

\[
(\tilde{v}(s_i) - \beta(m))Q^{\frac{N-1}{N}}(s_i) + \int_{x=s_i}^{m} (x - \underline{\beta}(m))Q^{\frac{N-1}{N}}(dx)
\]
Upward deviations

- Upward deviator’s surplus

\[
(\tilde{v}(s_i) - \beta(m))Q^{\frac{N-1}{N}}(s_i) + \int_{x=s_i}^{m} (x - \beta(m))Q^{\frac{N-1}{N}}(dx)
\]

- Derivative w.r.t. \( m \):

\[
(m - \beta(m))Q^{\frac{N-1}{N}}(dm) - \beta(m)'Q^{\frac{N-1}{N}}(m) = 0!
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Upward deviations

- Upward deviator’s surplus

\[ (\tilde{v}(s_i) - \beta(m))Q^{\frac{N-1}{N}}(s_i) + \int_{x=s_i}^{m} (x - \beta(m))Q^{\frac{N-1}{N}}(dx) \]

- Derivative w.r.t. \( m \):

\[ (m - \beta(m))Q^{\frac{N-1}{N}}(dm) - \beta(m)'Q^{\frac{N-1}{N}}(m) = 0! \]

- In effect, correlation between others bids’ and losing values induces adverse selection s.t. losing bidders are indifferent to deviating up
Towards a general bound

- Claim is that construction attains a lower bound
- Show this via relaxed program
- Minimum CDF of winning bids subject to uniform upward IC
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- Key WLOG properties of solution (and minimizing equilibrium):
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  2. Winning bid depends on average losing value
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  2. Winning bid depends on average losing value
  3. Efficiency
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- Minimum CDF of winning bids subject to uniform upward IC
- Key WLOG properties of solution (and minimizing equilibrium):
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  2. Winning bid depends on average losing value
  3. Efficiency
  4. Monotonicity of winning bids in losing values
Towards a general bound

- Claim is that construction attains a lower bound
- Show this via relaxed program
- Minimum CDF of winning bids subject to uniform upward IC
- Key WLOG properties of solution (and minimizing equilibrium):
  1. Symmetry
  2. Winning bid depends on average losing value
  3. Efficiency
  4. Monotonicity of winning bids in losing values
  5. All uniform upward IC bind
Winning bid distributions

- Choice variables: Measure over \( i \)'s winning bids given values:

\[
H_i(db|v_1, ..., v_n)
\]
Winning bid distributions

- Choice variables: Measure over $i$’s winning bids given values:

$$H_i(db|v_1, ..., v_n)$$

- Feasibility:

$$H_i(b|v) \geq 0, \quad \sum_i H_i(b|v) \leq 1 \quad \text{(Feas)}$$
Winning bid distributions

- Choice variables: Measure over i’s winning bids given values:

\[ H_i(db|v_1, \ldots, v_n) \]

- Feasibility:

\[ H_i(b|v) \geq 0, \quad \sum_i H_i(b|v) \leq 1 \quad \text{(Feas)} \]

- Note

\[
H(b) = \int_{v \in V^N} \sum_{i=1}^{N} H_i(b|v) P(dv)
\]
Relaxed program

- Also impose *uniform upward incentive constraints* (IC):

\[
\int_{v \in V^N} \int_{x=v}^b (b - x) H_i(dx|v) P(dv) \\
\geq \int_{v \in V^N} \int_{x=v}^b (v_i - b) \sum_{j \neq i} H_j(dx|v) P(dv)
\]

loss when would have won

gain when would have lost

Relaxed program: for fixed \( f(b) \) that is weakly increasing,

\[
\min_{v \in V^N} \sum_{i=1}^{N} \int_{v \in V^N} \int_{x=v}^b H_i(dx|v) P(dv)
\]

over \( \{H_i(b|v)\} \) subject to (Feas) and (IC)

Note: Objective and constraints are linear in \( H_i \)
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- Also impose *uniform upward incentive constraints* (IC):

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\]

loss when would have won

\[
\geq \int_{v \in V^N} \int_{x=v}^{b} (v_i - b) \sum_{j \neq i} H_j(dx|v) P(dv)
\]
\[
\sum_{i=1}^{N} \int_{b=v_i}^{V} f(b) H_i(db|v) P(dv)
\]

over \( \{H_i(b|v)\} \) subject to (Feas) and (IC)
Relaxed program

- Also impose *uniform upward incentive constraints* (IC):

\[
\int_{v \in V^N} \int_{x=v}^b (b - x) H_i(dx|v)P(dv) \geq \\
\int_{v \in V^N} \int_{x=v}^b (v_i - b) \sum_{j \neq i} H_j(dx|v)P(dv)
\]

- Relaxed program: for fixed \( f(b) \) that is weakly increasing,

\[
\min \int_{v \in V^N} \sum_{i=1}^N \int_{b=v}^\bar{v} f(b) H_i(db|v)P(dv)
\]

over \( \{H_i(b|v)\} \) subject to (Feas) and (IC)

- Note: Objective and constraints are *linear* in \( H_i \)
Symmetry

- WLOG to consider *symmetric* solutions in which

\[ H_i(\cdot|\nu) = H_{\xi(i)}(\cdot|\xi(\nu)) \]

for all permutations \( \xi \)
WLOG to consider symmetric solutions in which

\[ H_i(\cdot|\nu) = H_{\xi(i)}(\cdot|\xi(\nu)) \]

for all permutations \( \xi \)

For example, with \( N = 2 \), can create symmetric solution:

\[
\tilde{H}_1(b|v_1, v_2) = \frac{1}{2} (H_1(b|v_1, v_2) + H_2(b|v_2, v_1))
\]

\[
\tilde{H}_2(b|v_1, v_2) = \frac{1}{2} (H_2(b|v_1, v_2) + H_1(b|v_2, v_1))
\]
Average losing values III

- Consider a bidder who uniformly deviates up, so they *always* win when the equilibrium winning bid is \( b \).
- Say there is a value profile \( v \) at which \( b \) is sometimes the winning bid.
- Symmetry \( \implies b \) is equally likely to be the winning bid when values are permutations of \( v \), \( \xi(v) \).
Consider a bidder who uniformly deviates up, so they always win when the equilibrium winning bid is $b$.

Say there is a value profile $v$ at which $b$ is sometimes the winning bid.

Symmetry $\implies b$ is equally likely to be the winning bid when values are permutations of $v$, $\xi(v)$.

Upward deviator can only control equivalence classes $[v] = \{\xi(v)\}$ on which they win, and expected value on $[v]$ is average value.

But someone has to win in equilibrium...
Average losing values III

- Consider a bidder who uniformly deviates up, so they *always* win when the equilibrium winning bid is $b$
- Say there is a value profile $v$ at which $b$ is sometimes the winning bid
- Symmetry $\implies b$ is equally likely to be the winning bid when values are permutations of $v$, $\xi(v)$
- Upward deviator can only control *equivalence classes* $[v] = \{\xi(v)\}$ on which they win, and expected value on $[v]$ is *average* value
- But someone has to win in equilibrium...
- Incremental gain from winning when you would lose in equilibrium is the *average losing value* given $[v]$:

$$\mu(v) = \frac{1}{N - 1} \left( \sum_{i=1}^{N} v_i - \text{expected winner’s value} \right)$$
Efficiency

- Can rewrite gain from upward deviating as

\[
\int_{v \in V^N} \int_{x = v} (\mu(v) - b) \frac{N - 1}{N} \sum_i H_i(dx|v) P(dv)
\]

- Incentive to deviate is weaker if \(\mu(v)\) is smaller
Efficiency

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  \[
  \int_{v \in V^N} \int_{x = v} (\mu(v) - b) \frac{N - 1}{N} \sum_i H_i(dx|v)P(dv)
  \]

- Incentive to deviate is weaker if \( \mu(v) \) is smaller

- \( \mu(v) \) is minimized by *efficient allocation*

\[
\mu(v) = \frac{1}{N - 1} \left( \sum_{i=1}^{N} v_i - \max_i v_i \right)
\]
Efficiency

- Can rewrite gain from upward deviating as

\[
\int_{v \in V^N} \int_{x = v} (\mu(v) - b) \frac{N - 1}{N} \sum_i H_i(dx|v)P(dv)
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- \(\mu(v)\) is minimized by efficient allocation

\[
\mu(v) = \frac{1}{N - 1} \left( \sum_{i=1}^{N} v_i - \max_i v_i \right)
\]

- Can always induce efficient allocation without changing \(H(b)\):
If \(v_i = \max v\), set

\[
\tilde{H}_i(b|v) = \frac{1}{|\arg \max v|} \sum_{j=1}^{N} H_j(b|v)
\]
Relaxed program II

- Can write $H(b|m)$ for CDF of winning bid given $\mu(v) = m$
- Recall $Q(dm)$ is distribution of $m$
Relaxed program II

- Can write $H(b|m)$ for CDF of winning bid given $\mu(v) = m$
- Recall $Q(dm)$ is distribution of $m$
- Relaxed program:

$$\min \int_{m=v}^{\bar{v}} \int_{b=v}^{\bar{v}} f(b)H(db|m)Q(dm)$$

subject to

$$0 \leq H(b|m) \leq 1 \quad \text{(Feas)}$$

and

$$\frac{1}{N} \int_{m=v}^{\bar{v}} \int_{x=v}^{b} (b - x)H(dx|m)Q(dm) \geq \frac{N - 1}{N} \int_{m=v}^{\bar{v}} (m - b)H(b|m)Q(dm) \quad \text{(IC)}$$
Monotonicity

- Only part of (IC) that depends on correlation between $b$ and $m$ is

\[
\hat{m}(b) = \int_{m=\bar{v}}^{\bar{Y}} m \, H(b|m) \, Q(dm),
\]

i.e., average losing value when winning bid is less than $b$

- Incentive to deviate up is weaker if $\hat{m}(b)$ is lower
Monotonicity

- Only part of (IC) that depends on correlation between $b$ and $m$ is

$$\hat{m}(b) = \int_{m=\hat{m}}^{\bar{V}} m \, H(b|m) \, Q(dm),$$

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- Incentive to deviate up is weaker if $\hat{m}(b)$ is lower

- Which correlation structure minimizes $\hat{m}(b)$?
Monotonicity

- Only part of (IC) that depends on correlation between $b$ and $m$ is

$$\hat{m}(b) = \int_{m=\nu}^{\nu} m H(b|m)Q(dm),$$

i.e., average losing value when winning bid is less than $b$

- Incentive to deviate up is weaker if $\hat{m}(b)$ is lower

- Which correlation structure minimizes $\hat{m}(b)$?

- Can minimize $\hat{m}(b)$ **pointwise** by making $b$ and $m$ comonotonic,

  i.e., the $\alpha$ lowest $m$ are associated with the $\alpha$ lowest $b$
Monotonicity

- Only part of (IC) that depends on correlation between \( b \) and \( m \) is

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\]

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- Incentive to deviate up is weaker if \( \hat{m}(b) \) is lower

- Which correlation structure minimizes \( \hat{m}(b) \)?

- Can minimize \( \hat{m}(b) \) pointwise by making \( b \) and \( m \) comonotonic,
  i.e., the \( \alpha \) lowest \( m \) are associated with the \( \alpha \) lowest \( b \)

- Implies a deterministic winning bid \( \beta(m) \) s.t. for all \( m \),

\[
H(\beta(m)) = Q(m)
\]
Relaxed program III

- Relaxed program is reduced to what we assumed in example:

\[
\min \int_{m=v}^{\bar{V}} f(\beta(m)) Q(dm)
\]

subject to \( \beta(m) \geq v \) and

\[
\frac{1}{N} \int_{x=v}^{m} (\beta(m) - \beta(x)) Q(dx)
\geq \frac{N - 1}{N} \int_{x=v}^{\bar{V}} (x - \beta(m)) Q(dx)
\]  

(\text{IC})
Relaxed program III

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\]

\[
\geq \frac{N - 1}{N} \int_{x=v}^{\bar{v}} (x - \beta(m)) Q(dx)
\]

- Minimize \(\beta(m)\) pointwise by \(\beta(\bar{v}) = \bar{v}\) and (IC) binding everywhere
Relaxed program III

- Relaxed program is reduced to what we assumed in example:

\[
\min \int_{m=\nu}^{\overline{V}} f(\beta(m))Q(dm)
\]

subject to \(\beta(m) \geq \nu\) and

\[
\frac{1}{N} \int_{x=\nu}^{m} (\beta(m) - \beta(x))Q(dx) \geq \frac{N - 1}{N} \int_{x=\nu}^{\overline{V}} (x - \beta(m))Q(dx)
\]

(\text{IC})

- Minimize \(\beta(m)\) pointwise by \(\beta(\nu) = \nu\) and (\text{IC}) binding everywhere

- Solution is precisely \(\beta\)!
Wrapping up

- $H$ solves the relaxed program for an arbitrary $f(\max b)$
- Must therefore be FOSD by any equilibrium $H(b)$
- Construction attains $H$, so proof of theorem is complete
Maximum revenue

- With pure common value, no-information and complete information induce full surplus extraction
- Not true with idiosyncratic values:
  - No-information induces inefficiency
  - Complete information gives rents to bidders
- Nonetheless...
Maximum revenue

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- Not true with idiosyncratic values:
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  - Complete information gives rents to bidders
- Nonetheless...

Theorem (Maximum revenue and minimum bidder surplus)

For every $\epsilon > 0$, there exists an information structure and equilibrium such that revenue is at least $\overline{T} - \epsilon$ and bidder surplus is at most $\epsilon$. 
Additional restrictions on information

- We refer to above model as *unknown values*: bidder need not know anything about value
- Sensible starting point in common value models
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- Sensible starting point in common value models
- Often, want to model values with an idiosyncratic component
- Reasonable to suppose that bidders are more informed about own value than others’ values
- The *known values* model: own value is known exactly
Additional restrictions on information

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- Sensible starting point in common value models
- Often, want to model values with an idiosyncratic component
- Reasonable to suppose that bidders are more informed about own value than others’ values
- The *known values* model: own value is known exactly
- Weak dominance: players do not bid more than own value
A lower bound bidder surplus

- If bid $b$, always win when others’ values are less than $b$
- Lower bound on bidder surplus $U_i(v_i)$ from best responding to “worst case” in which others bid their values:

$$U_i(v_i) = \max_b \left\{ (v_i - b) \int_{\{v_{-i} \mid \max_{j \neq i} v_j \leq b\}} P(dv_{-i} \mid v_i) \right\}$$

- Integrate over values to obtain an ex-ante bound $U_i$
Maximum revenue/minimum bidder surplus

Theorem (Known values)

1. There exists an equilibrium in which every bidder receives surplus \( U_i \), thus attaining minimum bidder surplus with known values.
Maximum revenue/minimum bidder surplus

Theorem (Known values)

1. There exists an equilibrium in which every bidder receives surplus $U_i$, thus attaining minimum bidder surplus with known values.

2. Moreover, this equilibrium is efficient, thus attaining maximum revenue with known values.
Proof sketch

- Bidders with $v_i < \max v$ see entire profile $v$
- Known they will lose to some $b_j \geq v_i$
- $\implies$ losers bid $b_i = v_i$
Proof sketch

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- High valuation bidder learns he has the high value
- Receives partial information about losers’ values such that
  
  (i) He outbids the others with probability 1
  (ii) Indifferent between equilibrium bid and the bid that generates $U_i$
Proof sketch

- Bidders with \( v_i < \max \nu \) see entire profile \( \nu \)
- Known they will lose to some \( b_j \geq v_i \)
- \( \implies \) losers bid \( b_i = v_i \)
- High valuation bidder learns he has the high value
- Receives partial information about losers’ values such that
  (i) He outbids the others with probability 1
  (ii) Indifferent between equilibrium bid and the bid that generates \( U_i \)
- Uses ideas from “The Limits of Price Discrimination”, BBM 2015
Known values: Minimum revenue

- Learning own value from bid is no longer an issue
- Instead, bid is informative about others’ values
Known values: Minimum revenue

- Learning own value from bid is no longer an issue
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- Also, with unknown values, likelihood of you winning in equilibrium at a winning bid $b$ is always $1/N$
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Known values: Minimum revenue

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- Instead, bid is informative about others’ values
- Also, with unknown values, likelihood of you winning in equilibrium at a winning bid $b$ is always $1/N$
- With unknown values, likelihood may depend on $b$ and distribution of others’ values
- Example: higher winning bids occur when values are higher on average
- If equilibrium is efficient and $v_i$ is low, I am unlikely to win in equilibrium at high bids
- Increase in probability of winning from upward deviation varies with $v_i$
Binary known values

- Case we can solve completely: $v_i \in \{v_L, v_H\}$
- Setting first considered by Maskin and Riley (1985)
- $v_L$ types are in Bertrand competition
  $\Rightarrow$ essentially always bid $v_L$, lose to $v_H$
Binary known values

- Case we can solve completely: $v_i \in \{v_L, v_H\}$
- Setting first considered by Maskin and Riley (1985)
- $v_L$ types are in Bertrand competition
  $\implies$ essentially always bid $v_L$, lose to $v_H$
- All uniform upward constraints bind
- Winning bids are higher when average value is higher
Binary known values

- Case we can solve completely: $v_i \in \{v_L, v_H\}$
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- General known values minimum revenue is an open question
Other directions

- We talked about max/min revenue, max/min bidder surplus
- What about weighted sums? Minimum efficiency?
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- More broadly, what is the *whole set* of possible \((U, R)\) pairs?
- Solved numerically for two bidder i.i.d. \(U[0, 1]\) model
Welfare set

Note: Lower bound on efficiency
What can we do with this?

- Applications/extensions:
  - Many bidder limit
  - Impact of reserve prices/entry fees
  - Identification
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- Context:
  - Part of a larger agenda on robust predictions and information design
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Thank you!