Clarendon Lectures, Lecture 3
General Theory of Directed Technical Change

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Summary of Results so Far

• Importance of directed technical change.

• Relatively strong results on the equilibrium direction of technical change.

• Implications for the evolution of skill bias of technology.

• But results derived under two sets of special assumptions:
  1. Constant elasticity of substitution production functions.

• How general are the insights?
This Lecture

• Main insights will hold very generally.
• Useful to distinguish between:
  1. relative bias: about shifts of relative demand curves
  2. absolute bias: about shifts of factor demands
• Results so far about relative bias.
• Main results:
  1. Theorems on relative bias can be generalized, but only to some degree.
  2. Much more general theorems on absolute bias.
Plan

- First introduce a class of environments where we can study bias of technology.
- Then generalize results on relative bias and show their limitations.
- Most important results: weak and strong theorems on absolute bias.
- Main takeaway message: under fairly reasonable conditions, factor demand curves will be upward sloping!
Basic Environment

- Static economy consisting of a unique final good (dynamics not central to the message here).
- $N + 1$ factors of production, $Z$ and $L = (L_1, ..., L_N)$.

Inelastic supplies: $\bar{Z} \in \mathbb{R}_+$ and $\bar{L} \in \mathbb{R}^N_+$.

- Main comparative statics: changing $\bar{Z}$.

- Representative household with preferences defined over the consumption of the final good.

- A continuum of firms (final good producers) denoted by the set $\mathcal{F}$, each with an identical production function.

- Normalize the measure of $\mathcal{F}$, $|\mathcal{F}|$, to 1.

- The price of the final good is also normalized to 1.
Alternative Economies

• Consider four different environments:

  1. **Economy D**: Fully decentralized. Technologies chosen by firms themselves.

  2. **Economy C**: Centralized. Technology decided by a centralized agency (taking firms’ profit maximization is given).


  4. **Economy O**: Oligopoly. Technology decided by a set of (potentially competing) oligopolist.
**Economy D**

- For benchmark (not the most realistic economy for technology choice).
- Each firm $i \in \mathcal{F}$ has access to a production function
  \[ Y^i = G(Z^i, L^i, \theta^i), \]
- $Z^i \in \mathcal{Z} \subset \mathbb{R}_+$, $L^i \in \mathcal{L} \subset \mathbb{R}_+^N$
- $\theta^i \in \Theta \subset \mathbb{R}^K$ is the measure of technology.
- $G$: production function (throughout assumed to be twice differentiable).
- The cost of technology $\theta \in \Theta$ in terms of final goods is $C(\theta)$. 
Economy D (continued)

- Each final good producer (firm) maximizes profits:

  \[
  \max_{Z^i \in Z, L^i \in L, \theta^i \in \Theta} \pi(Z^i, L^i, \theta^i) = G(Z^i, L^i, \theta^i) - w_Z Z^i - \sum_{j=1}^{N} w_{Lj} L^i_j - C(\theta^i),
  \]

- \(w_Z\) is the price of factor \(Z\) and \(w_{Lj}\) is the price of factor \(L_j\) for \(j = 1, ..., N\).

- All factor prices taken as given by firms.

- The vector of prices for factors \(L\) denoted by \(w_L\).

- Market clearing:

  \[
  \int_{i \in F} Z^i di \leq \bar{Z} \text{ and } \int_{i \in F} L^i_j di \leq \bar{L}_j \text{ for } j = 1, ..., N.
  \]
Economy D (continued)

**Definition 1** An equilibrium in Economy D is a set of decisions \(\{Z^i, L^i, \theta^i\}_{i \in F}\) and factor prices \((w_Z, w_L)\) such that \(\{Z^i, L^i, \theta^i\}_{i \in F}\) maximize profits given prices \((w_Z, w_L)\) and market clearing conditions hold.

- Any \(\theta^i\) that is part of the set of equilibrium allocations, \(\{Z^i, L^i, \theta^i\}_{i \in F}\), is an **equilibrium technology**.

- Let us also define the **net production function**:

\[
F(Z^i, L^i, \theta^i) \equiv G(Z^i, L^i, \theta^i) - C(\theta^i).
\]
Economy D (continued)

**Assumption 1** Either \( F(Z^i, L^i, \theta^i) \) is jointly strictly concave in \((Z^i, L^i, \theta^i)\) and increasing in \((Z^i, L^i)\), and \(Z, \mathcal{L}\) and \(\Theta\) are convex; or \( F(Z^i, L^i, \theta^i) \) is increasing in \((Z^i, L^i)\) and exhibits constant returns to scale in \((Z^i, L^i, \theta^i)\), and we have \((\bar{Z}, \bar{L}) \in Z \times \mathcal{L}\).

- Main problem with Economy D: Assumption 1 overly restrictive.
- It requires concavity (strict concavity or constant returns to scale) jointly in the factors of production and technology.
Economy D (continued)

• Equilibrium characterization and welfare theorems:

**Proposition 1** Suppose Assumption 1 holds. Then any equilibrium technology $\theta$ in Economy D is a solution to

$$\max_{\theta' \in \Theta} F(\bar{Z}, \bar{L}, \theta'),$$

and any solution to this problem is an equilibrium technology.

• Equilibrium factor prices given by the marginal products of $G$ or $F$.

$$w_Z = \frac{\partial G(\bar{Z}, \bar{L}, \theta)}{\partial Z} = \frac{\partial F(\bar{Z}, \bar{L}, \theta)}{\partial Z}$$

and

$$w_{Lj} = \frac{\partial G(\bar{Z}, \bar{L}, \theta)}{\partial L_j} = \frac{\partial F(\bar{Z}, \bar{L}, \theta)}{\partial L_j}$$

for $j = 1, \ldots, N$
Economy C

• Now assume that firms maximize profits, but technologies chosen by a “welfare-maximizing” centralized research firm.

• Useful as an introduction to the more realistic models with monopoly and oligopoly technology suppliers.

• The research firm chooses a single technology $\theta$ and makes it available to all firms (single technology for simplicity).

• Notice that this will typically not give the social (Pareto) optimum, since employment decisions controlled by different agents.
Economy C (continued)

- The maximization problem of each final good producer is

\[
\max_{Z^i \in Z, L^i \in L} \pi(Z^i, L^i, \theta) = G(Z^i, L^i, \theta) - w_Z Z^i - \sum_{j=1}^{N} w_{Lj} L^i_j.
\]

- Notice: in contrast to Economy D, final good producers are only maximizing with respect to \((Z^i, L^i)\), not with respect to \(\theta^i\).

- The objective of the research firm is to maximize total net output:

\[
\max_{\theta \in \Theta} \Pi(\theta) = \int_{0}^{1} G(Z^i, L^i, \theta) \, di - C(\theta).
\]
Economy C (continued)

Definition 2 An equilibrium in Economy C is a set of firm decisions \( \{Z^i, L^i\}_{i \in \mathcal{F}} \), technology choice \( \theta \) and factor prices \( (w_Z, w_L) \) such that \( \{Z^i, L^i\}_{i \in \mathcal{F}} \) maximize profits given \( (w_Z, w_L) \) and \( \theta \), market clearing conditions hold, and the technology choice for the research firm, \( \theta \), maximizes its objective function.

- Major difference: we only need a weaker version of Assumption 1
- Concavity only in \( (Z, L) \):

Assumption 2 Either \( G(Z^i, L^i, \theta^i) \) is jointly strictly concave and increasing in \( (Z^i, L^i) \) and \( Z \) and \( L \) are convex; or \( G(Z^i, L^i, \theta^i) \) is increasing and exhibits constant returns to scale in \( (Z^i, L^i) \), and we have \( (\bar{Z}, \bar{L}) \in Z \times L \).
Economy C (continued)

**Proposition 2** Suppose Assumption 2 holds. Then any equilibrium technology $\theta$ in Economy C is a solution to

$$\max_{\theta' \in \Theta} F(\bar{Z}, \bar{L}, \theta') \equiv G(\bar{Z}, \bar{L}, \theta') - C(\theta')$$

and any solution to this problem is an equilibrium technology.

- Most important novel feature: while in Economy D the function $F(\bar{Z}, \bar{L}, \theta)$ is jointly concave in $(Z, \theta)$, the same is not true in Economy C.

- As in Economy D, equilibrium factor prices are given by

$$w_Z = \frac{\partial G(\bar{Z}, \bar{L}, \theta)}{\partial Z} = \frac{\partial F(\bar{Z}, \bar{L}, \theta)}{\partial Z}$$

and

$$w_{Lj} = \frac{\partial G(\bar{Z}, \bar{L}, \theta)}{\partial L_j} = \frac{\partial F(\bar{Z}, \bar{L}, \theta)}{\partial L_j}$$

for $j = 1, \ldots, N$. 
Economy M

• Now a profit-maximizing monopolist sells technologies to final good producers.

• To facilitate analysis, assume that

\[ Y^i = \alpha^{-\alpha} (1 - \alpha)^{-1} \left[ G(Z^i, L^i, \theta^i) \right]^{\alpha} q(\theta^i)^{1-\alpha}. \]

• Here \( G(Z^i, L^i, \theta^i) \) is a subcomponent of the production function.

• Productivity depends on the technology used, \( \theta^i \).

• The subcomponent \( G \) needs to be combined with an intermediate good embodying technology \( \theta^i \), denoted by \( q(\theta^i) \).

• This intermediate good will be sold by the monopolist.

• The term \( \alpha^{-\alpha} (1 - \alpha)^{-1} \) is a convenient normalization.
Economy M (continued)

• The monopolist can create technology $\theta$ at cost $C(\theta)$ from the technology menu.

• Once $\theta$ is created, the technology monopolist can produce the intermediate good embodying technology $\theta$ at constant per unit cost normalized to $1 - \alpha$ unit of the final good.

• It can then set a (linear) price per unit of the intermediate good of type $\theta$, denoted by $\chi$.

• All factor markets are again competitive, and each firm takes the available technology, $\theta$, and the price of the intermediate good embodying this technology, $\chi$, as given.
Economy M (continued)

• Final good producers’ maximization problem:

$$\max_{\substack{Z^i \in Z, L^i \in L, \\ q(\theta) \geq 0}} \alpha^{-\alpha} (1 - \alpha)^{-1} \left[ G(Z^i, L^i, \theta) \right]^\alpha q(\theta)^{1-\alpha} - w_Z Z^i - \sum_{j=1}^{N} w_L L_j - \chi q(\theta),$$

• Inverse demand for intermediates of type $\theta$ as a function of its price, $\chi$:

$$q^i(\theta, \chi, Z^i, L^i) = \alpha^{-1} G(Z^i, L^i, \theta) \chi^{-1/\alpha}.$$

• Isoelastic inverse demand.
Economy M (continued)

- The monopolist’s maximization problem:

$$\max_{\theta, \chi, [q^i(\theta, \chi, Z^i, L^i)]_{i \in \mathcal{F}}} \Pi = (\chi - (1 - \alpha)) \int_{i \in \mathcal{F}} q^i(\theta, \chi, Z^i, L^i) \, di - C(\theta)$$

subject to the inverse demand curve.

**Definition 3** An equilibrium in Economy M is a set of firm decisions $\{Z^i, L^i, q^i(\theta, \chi, Z^i, L^i)\}_{i \in \mathcal{F}}$, technology choice $\theta$, and factor prices $(w_Z, w_L)$ such that $\{Z^i, L^i, q^i(\theta, \chi, Z^i, L^i)\}_{i \in \mathcal{F}}$ maximizes profits given $(w_Z, w_L)$ and technology $\theta$, market clearing conditions hold, and the technology choice and pricing decision of the monopolist, $(\theta, \chi)$, maximize monopoly profits subject to the inverse demand curve.

- As in Economy C, factor demands and technology are decided by different agents; the former by the final good producers, the latter by the technology monopolist.
Economy M (continued)

• Note that the inverse demand function has constant elasticity.
• Profit-maximizing price will be a constant markup over marginal cost
  \[ \chi = 1 \]
• Consequently, \( q^i (\theta) = q^i (\theta, \chi = 1, \bar{Z}, \bar{L}) = \alpha^{-1} G(\bar{Z}, \bar{L}, \theta) \) for all \( i \in \mathcal{F} \).
• Therefore, the monopolist’s problem becomes
  \[
  \max_{\theta \in \Theta} \Pi (\theta) = G(\bar{Z}, \bar{L}, \theta) - C (\theta) .
  \]

Proposition 3 Suppose Assumption 2 holds. Then any equilibrium technology \( \theta \) in Economy M is a solution to
\[
\max_{\theta' \in \Theta} F(\bar{Z}, \bar{L}, \theta') \equiv G(\bar{Z}, \bar{L}, \theta') - C (\theta')
\]
and any solution to this problem is an equilibrium technology.
• Relative to Economies D and C, the presence of the monopoly markup implies greater distortions in this economy.
Economy M (continued)

- However, qualitatively equilibrium similar to that in Economy C.
- It is given by the maximization of

\[ F(\bar{Z}, \bar{L}, \theta) \equiv G(\bar{Z}, \bar{L}, \theta) - C(\theta) \]

- Most important: as in Economy C, \( F(\bar{Z}, \bar{L}, \theta) \) need not be concave in \((Z, \theta)\), even in the neighborhood of the equilibrium.

- Factor prices again given by:

\[ w_Z = \frac{\partial G(\bar{Z}, \bar{L}, \theta)}{\partial Z} = \frac{\partial F(\bar{Z}, \bar{L}, \theta)}{\partial Z} \]

and

\[ w_{Lj} = \frac{\partial G(\bar{Z}, \bar{L}, \theta)}{\partial L_j} = \frac{\partial F(\bar{Z}, \bar{L}, \theta)}{\partial L_j} \]

for \( j = 1, \ldots, N \).
Economy O

- Same as Economy M, except that multiple technologies supplied by competing oligopolists.
- Let $\theta^i$ be the vector $\theta^i \equiv (\theta^i_1, \ldots, \theta^i_S)$.
- Suppose that output is now given by

$$Y^i = \alpha^{-\alpha} (1 - \alpha)^{-1} \left[ G(Z^i, L^i, \theta^i) \right]^\alpha \sum_{s=1}^{S} q_s \left( \theta^i_s \right)^{1-\alpha},$$

- $\theta^i_s \in \Theta_s \subset \mathbb{R}^{K_s}$: technology supplied by technology producer $s = 1, \ldots, S$;
- $q_s \left( \theta^i_s \right)$: intermediate good produced and sold by technology producer $s$, embodying technology $\theta^i_s$. 
Economy O (continued)

• Essentially the same result as in Economy M.

Proposition 4 Suppose Assumption 2 holds. Then any equilibrium technology in Economy O is a vector \((\theta_1^*, ..., \theta_S^*)\) such that \(\theta_s^*\) is solution to

\[
\max_{\theta_s \in \Theta_s} G(\bar{Z}, \bar{L}, \theta_1^*, ..., \theta_s^*, ..., \theta_S^*) - C_s(\theta_s)
\]

for each \(s = 1, ..., S\), and any such vector gives an equilibrium technology.

• Main difference: equilibrium technology no longer given by maximization, but by a fixed point problem.

• Nevertheless, general insights continue to apply.
Relative Bias

- Let us first study relative bias.
- Two factors $Z$ and $L$.
- Defined factor prices as:

$$w_Z(Z, L, \theta) = \frac{\partial G(Z, L, \theta)}{\partial Z} \quad \text{and} \quad w_L(Z, L, \theta) = \frac{\partial G(Z, L, \theta)}{\partial L},$$
Definitions

**Definition 4** An increase in technology $\theta_j$ for $j = 1, \ldots, K$ is relatively biased towards factor $Z$ at $(\bar{Z}, \bar{L}, \theta) \in Z \times L \times \Theta$ if $\partial (w_Z/w_L)/\partial \theta_j \geq 0$.

**Definition 5** Denote the equilibrium technology at factor supplies $(\bar{Z}, \bar{L}) \in Z \times L$ by $\theta^* (\bar{Z}, \bar{L})$, and assume that $\partial \theta^*_j/\partial Z$ exists at $(\bar{Z}, \bar{L})$ for all $j = 1, \ldots, K$. Then there is weak relative equilibrium bias at $(\bar{Z}, \bar{L}, \theta^* (\bar{Z}, \bar{L}))$ if

$$\sum_{j=1}^{K} \frac{\partial (w_Z/w_L)}{\partial \theta_j} \frac{\partial \theta^*_j}{\partial Z} \geq 0.$$ 

**Definition 6** Denote the equilibrium technology at factor supplies $(\bar{Z}, \bar{L}) \in Z \times L$ by $\theta^* (\bar{Z}, \bar{L})$, and assume that $\partial \theta^*_j/\partial Z$ exists at $(\bar{Z}, \bar{L})$ for all $j = 1, \ldots, K$. Then there is strong relative equilibrium bias at $(\bar{Z}, \bar{L}, \theta^* (\bar{Z}, \bar{L}))$ if

$$\frac{d (w_Z/w_L)}{dZ} = \frac{\partial (w_Z/w_L)}{\partial Z} + \sum_{j=1}^{K} \frac{\partial (w_Z/w_L)}{\partial \theta_j} \frac{\partial \theta^*_j}{\partial Z} > 0.$$
Generalized Relative Bias Theorem

**Theorem 1**  Consider Economy C, M or O with two factors, $Z, L$, and two factor-augmenting technologies, $A_Z, A_L$. Assume that $G(A_Z Z, A_L L)$ is twice differentiable, concave and homothetic, and the cost of producing technologies $C(A_Z, A_L)$, is twice differentiable, strictly convex and homothetic. Let

$$\sigma = -\frac{\partial \ln(Z/L)}{\partial \ln(w_Z/w_L)} \bigg|_{A_Z A_L}$$

be the (local) elasticity of substitution between $Z$ and $L$, and

$$\delta = \frac{\partial \ln(C_Z/C_L)}{\partial \ln(A_Z/A_L)}.$$  Then:

$$\frac{\partial \ln (A_Z/A_L)}{\partial \ln (Z/L)} = \frac{\sigma - 1}{1 + \sigma \delta},$$

and

$$\frac{\partial \ln (w_Z/w_L)}{\partial \ln (A_Z/A_L)} \frac{\partial \ln (A_Z/A_L)}{\partial \ln (Z/L)} \geq 0,$$

so that there is always **weak relative equilibrium bias**. Moreover,

$$\frac{d \ln (w_Z/w_L)}{d \ln (Z/L)} = \frac{\sigma - 2 - \delta}{1 + \sigma \delta},$$

so that there is **strong relative equilibrium bias** if and only if $\sigma - 2 - \delta > 0$. 
Idea of the Proof

• Essentially the same as the simple example in Lecture 1.

• Locally, the economy behaves as if the elasticity of substitution is constant.

• Important that the result is for Economy, C, M or O, since the maximization problem choosing all of $Z, L, A_Z$ and $A_L$ is not concave.

• In fact, this non-concavity is essential for strong bias as we will see shortly.
Can This Result Be Generalized Further?

- **None** of the assumptions of Theorem 1 can be relaxed (for sufficiency).
- In particular, with non-factor augmenting technologies, increase in relative supply of $Z$ can induced technological changes biased against $Z$.
- This does **not** mean that this “contrarian” result will apply in general.
- But it does mean that we cannot guarantee induced biased to go in the “right direction”.
Counterexample 1

• Suppose

\[ G(Z, L, \theta) = \left[ Z^\theta + L^\theta \right]^{1/\theta} \]

and \( C(\theta) \) convex and differentiable.

• The choice of \( \theta \) again maximizes \( F(Z, L, \theta) \equiv G(Z, L, \theta) - C(\theta) \):

\[ \frac{\partial G(\bar{Z}, \bar{L}, \theta^*)}{\partial \theta} - \frac{\partial C(\theta^*)}{\partial \theta} = 0 \]

and

\[ \frac{\partial^2 G(\bar{Z}, \bar{L}, \theta^*)}{\partial \theta^2} - \frac{\partial^2 C(\theta^*)}{\partial \theta^2} < 0 \]

• A counterexample would correspond to a situation where

\[ \Delta(w_Z/w_L) \equiv \frac{\partial (w_Z/w_L)}{\partial \theta} \frac{\partial \theta^*}{\partial Z} = -\frac{\partial (w_Z/w_L)}{\partial \theta} \frac{\partial^2 F/\partial \theta \partial Z}{\partial^2 F/\partial \theta^2} < 0. \]
Counterexample 1 (continued)

• Here:

\[ w_Z/w_L = (Z/L)^{\theta - 1} \]

increasing in \( \theta \) as long as \( Z > L \), so that higher \( \theta \) is relatively biased towards \( Z \).

• Now choose \( C(\cdot) \) such that \( \theta^* \) is sufficiently small, e.g., \( \bar{L} = 1, \bar{Z} = 2 \), and \( \theta^* = 0.1 \).

• In this case, it can be verified that \( \partial^2 F(\bar{Z}, \bar{L}, \theta^*) / \partial \theta \partial Z < 0 \).

• From the second-order conditions \( \partial^2 F / \partial \theta^2 < 0 \).

• Therefore \((\partial^2 F / \partial \theta \partial Z) \times (\partial^2 F / \partial \theta^2) > 0\).

• Conclusion: an increase in \( Z/L \) reduces \( \theta^* \) and induces technological change technology relatively biased against \( Z \).
Counterexample 2

• Suppose

\[ G(Z, L, \theta) = Z\theta + L\theta^2, \]

and

\[ C(\theta) = C_0\theta^2/2 \]

for all \( \theta \in \Theta = \mathbb{R} \) and \( L \in \mathcal{L} \subset (0, C_0/2) \).

• The equilibrium technology \( \theta^* \) is given by

\[ \theta^* (\bar{Z}, \bar{L}) = \frac{\bar{Z}}{C_0 - 2\bar{L}}, \]

• This is increasing in \( \bar{Z} \) for any \( \bar{L} \in \mathcal{L} \).

• The relative price of factor \( Z \) is decreasing in \( \theta \):

\[ w_Z (\theta) / w_L (\theta) = \theta^{-1} \]

• \( \bar{Z} \uparrow \Rightarrow \) technological change relatively biased against \( Z \).
Why the Counterexamples?

- In both cases, the increase in $\bar{Z}$ increases $w_Z$ (at given factor proportions).
- But it increases $w_L$ even more so that $w_Z/w_L$ declines at given factor proportions.
- Perhaps looking at absolute bias more natural.
Absolute Bias: Definitions

• Straightforward definitions of absolute bias (in light of the definitions for relative bias above).

**Definition 7** An increase in technology \( \theta_j \) for \( j = 1, \ldots, K \) is **absolutely biased** towards factor \( Z \) at \((\bar{Z}, \bar{L}) \in \mathcal{Z} \times \mathcal{L} \) if \( \frac{\partial w_Z}{\partial \theta_j} \geq 0 \).

**Definition 8** Denote the equilibrium technology at factor supplies \((\bar{Z}, \bar{L}) \in \mathcal{Z} \times \mathcal{L} \) by \( \theta^* (\bar{Z}, \bar{L}) \) and assume that \( \frac{\partial \theta^*_j \theta}{\partial \bar{Z}} \) exists at \((\bar{Z}, \bar{L}) \) for all \( j = 1, \ldots, K \). Then there is **weak absolute equilibrium bias** at \((\bar{Z}, \bar{L}, \theta^* (\bar{Z}, \bar{L}) \) if

\[
\sum_{j=1}^{K} \frac{\partial w_Z}{\partial \theta_j} \frac{\partial \theta^*_j}{\partial \bar{Z}} \geq 0.
\]
**Absolute Bias: Local Theorem**

**Theorem 2**  Consider Economy D, C or M. Suppose that $\Theta$ is a convex subset of $\mathbb{R}^K$ and $F(Z, L, \theta)$ is twice continuously differentiable in $(Z, \theta)$. Let the equilibrium technology at factor supplies $(\bar{Z}, \bar{L})$ be $\theta^* (\bar{Z}, \bar{L})$ and assume that $\theta^* (\bar{Z}, \bar{L})$ is in the interior of $\Theta$ and that $\partial \theta^*_j / \partial Z$ exists at $(\bar{Z}, \bar{L})$ for all $j = 1, \ldots, K$. Then, there is **weak absolute equilibrium bias** at all $(\bar{Z}, \bar{L}) \in Z \times L$, i.e.,

$$\sum_{j=1}^{K} \frac{\partial w_Z}{\partial \theta_j} \frac{\partial \theta^*_j}{\partial Z} \geq 0 \text{ for all } (\bar{Z}, \bar{L}) \in Z \times L,$$

with strict inequality if $\partial \theta^*_j / \partial Z \neq 0$ for some $j = 1, \ldots, K$. 

Sketch of the Proof

• The result follows from the Implicit Function Theorem.

• Consider the special case where $\theta \in \Theta \subset \mathbb{R}$.

• Since $\theta^*$ is in the interior of $\Theta$, we have $\partial F/\partial \theta = 0$ and $\partial^2 F/\partial \theta^2 \leq 0$.

• The Implicit Function Theorem then implies:

$$\frac{\partial \theta^*}{\partial Z} = -\frac{\partial^2 F / \partial \theta \partial Z}{\partial^2 F / \partial \theta^2} = -\frac{\partial w_Z / \partial \theta}{\partial^2 F / \partial \theta^2}, \quad (2)$$

• Therefore:

$$\frac{\partial w_Z}{\partial \theta} \frac{\partial \theta^*}{\partial Z} = -\left(\frac{\partial w_Z / \partial \theta}{\partial^2 F / \partial \theta^2}\right)^2 \geq 0, \quad (3)$$

establishing the weak inequality.

• Moreover, if $\partial \theta^*/\partial Z \neq 0$, then $\partial w_Z / \partial \theta \neq 0$, so the strict inequality applies.

• The general result somewhat more involved, but a similar intuition.
Intuition

• Again the **market size effect**.

• Locally, an increase in \( Z \) makes technologies that the value of marginal product of \( Z \) more profitable.

• The result applies in all four economies.

• Once again, similarity to LeChatelier Principle.

• Major differences to come soon.
Local Bias Does Not Imply Global Bias

• Theorem 2 is for small changes.

• A natural question is whether it also holds for “large” (non-infinitesimal) changes.

• Interestingly, the answer is No.

• The reason is intuitive: technological change biased towards an particular factor at some factor proportion may be biased against that factor at some other (not too far) factor proportion.

• The next example illustrates this.
No Global Bias without Further Assumptions

• Suppose that $F(Z, \theta) = Z + \left(Z - Z^2/8\right) \theta - C(\theta)$ and $Z \in \mathcal{Z} = [0, 6]$ and $\Theta = [0, 2]$ so that $F$ is everywhere increasing in $Z$.

• Suppose also that $C(\theta)$ is a strictly convex and differentiable function with $C'(0) = 0$ and $C'(2) = \infty$.

• Note that $F(Z, \theta)$ satisfies all the conditions of Theorem 2 at all points $Z \in \mathcal{Z} = [0, 6]$ (since $F$ is strictly concave in $\theta$ everywhere on $\mathcal{Z} \times \Theta = [0, 6] \times [0, 2]$).
No Global Bias without Further Assumptions (continued)

• Now consider $\bar{Z} = 1$ and $\bar{Z}' = 5$ as two potential supply levels of factor $Z$.

• It can be easily verified that $\theta^*(1)$ satisfies $C'(\theta^*(1)) = 7/8$ while $\theta^*(5)$ is given by $C'(\theta^*(5)) = 15/8$

• The strict convexity of $C(\theta)$ implies that $\theta^*(5) > \theta^*(1)$.

• Moreover, $w_Z(Z, \theta) = 1 + (1 - Z/4) \theta$, therefore $w_Z(5, \theta^*(5)) = 1 - \theta^*(5)/4 < 1 - \theta^*(1)/4 = w_Z(5, \theta^*(1))$.

• Intuition: reversal in the meaning of bias.
A Global Theorem

- For a global result, we need to rule out “reversals in the meaning of bias”
- Somewhat stronger assumptions are necessary.
- Fortunately, reasonable assumptions suffice for this purpose.
- What we need to ensure is that “complements” do not become “substitutes”.
- Natural assumption: supermodularity.
Globality

**Definition 9** Let $\theta^*$ be the equilibrium technology choice in an economy with factor supplies $(\bar{Z}, \bar{L})$. Then there is **global absolute equilibrium bias** if for any $\bar{Z}', \bar{Z} \in \mathcal{Z}$, $\bar{Z}' \geq \bar{Z}$ implies that

$$w_Z \left( \tilde{Z}, \bar{L}, \theta^*(\bar{Z}', \bar{L}) \right) \geq w_Z \left( \tilde{Z}, \bar{L}, \theta^*(\bar{Z}, \bar{L}) \right)$$

for all $\tilde{Z} \in \mathcal{Z}$ and $\bar{L} \in \mathcal{L}$.

- Two notions of globality.
  1. the increase from $\bar{Z}$ to $\bar{Z}'$ is not limited to small changes;
  2. the change in technology induced by this increase is required to raise the price of factor $Z$ for all $\bar{Z} \in \mathcal{Z}$.

- The same economic forces will take care of both types of globality.
Supermodularity and Increasing Differences

**Definition 10** Let \( x = (x_1, \ldots, x_n) \) be a vector in \( X \subset \mathbb{R}^n \), and suppose that the real-valued function \( f(x) \) is twice continuously differentiable in \( x \). Then \( f(x) \) is **supermodular** on \( X \) if and only if \( \frac{\partial^2 f(x)}{\partial x_i \partial x_i'} \geq 0 \) for all \( x \in X \) and for all \( i \neq i' \).

**Definition 11** Let \( X \) and \( T \) be partially ordered sets. Then a function \( f(x, t) \) defined on a subset \( S \) of \( X \times T \) has **increasing differences** (strict increasing differences) in \((x, t)\), if for all \( t'' > t \), \( f(x, t'') - f(x, t) \) is nondecreasing (increasing) in \( x \).
Absolute Bias: The Global Theorem

Theorem 3  Suppose that $\Theta$ is a lattice, let $\bar{Z}$ be the convex hull of $Z$, let $\theta^* (\bar{Z}, \bar{L})$ be the equilibrium technology at factor proportions $(\bar{Z}, \bar{L})$, and suppose that $F (Z, L, \theta)$ is continuously differentiable in $Z$, supermodular in $\theta$ on $\Theta$ for all $Z \in \bar{Z}$ and $L \in \mathcal{L}$, and exhibits strictly increasing differences in $(Z, \theta)$ on $\bar{Z} \times \Theta$ for all $L \in \mathcal{L}$, then there is global absolute equilibrium bias, i.e., for any $\bar{Z}', \bar{Z} \in Z$, $\bar{Z}' \geq \bar{Z}$ implies

$$\theta^* (\bar{Z}', \bar{L}) \geq \theta^* (\bar{Z}, \bar{L})$$

for all $\bar{L} \in \mathcal{L}$,

and

$$w_Z \left( \tilde{Z}, \bar{L}, \theta^* (\bar{Z}', \bar{L}) \right) \geq w_Z \left( \tilde{Z}, \bar{L}, \theta^* (\bar{Z}, \bar{L}) \right)$$

for all $\tilde{Z} \in Z$ and $\bar{L} \in \mathcal{L}$, with strict inequality if $\theta^* (\bar{Z}', \bar{L}) \neq \theta^* (\bar{Z}, \bar{L})$.
Proof Idea

• The proof basically follows from Topkis’s Monotone Comparative Statics Theorem.

• An increase in $Z$ is complementary to technologies that are biased towards $Z$.

• Therefore, the increase in $Z$ will cause globally (weak) absolute bias.
Global Absolute Bias with Multiple Factors

- The same result generalizes to the case where the supply of a subset of complementary factors increases.
- In this case, technology becomes biased towards all of these factors.
- Let now $Z$ denote a vector of inputs.

**Theorem 4** Consider Economy D, C or M. Suppose that $Z$ and $\Theta$ are lattices, let $\bar{Z}$ be the convex hull of $Z$, let $\theta(\bar{Z}, \bar{L})$ be the equilibrium technology at factor proportions $(\bar{Z}, \bar{L})$, and suppose that $F(Z, L, \theta)$ is continuously differentiable in $Z$, supermodular in $\theta$ on $\Theta$ for all $Z \in \bar{Z}$ and $L \in \mathcal{L}$, and exhibits strictly increasing differences in $(Z, \theta)$ on $\bar{Z} \times \Theta$ for all $L \in \mathcal{L}$, then there is global absolute equilibrium bias, i.e., for any $\bar{Z}'$, $\bar{Z} \in Z$, $\bar{Z}' \geq \bar{Z}$ implies

$$\theta(\bar{Z}', \bar{L}) \geq \theta(\bar{Z}, \bar{L}) \quad \text{for all } \bar{L} \in \mathcal{L}$$

and

$$w_{Z,j}\left(\tilde{Z}, \bar{L}, \theta(\bar{Z}', \bar{L})\right) \geq w_{Z,j}\left(\tilde{Z}, \bar{L}, \theta(\bar{Z}, \bar{L})\right) \quad \text{for all } (\tilde{Z}, \bar{L}) \in Z \times \mathcal{L} \text{ and for all } j.$$
Strong Bias

- Much more interesting and surprising are the results on strong bias.
- The main result will show that strong bias is quite ubiquitous.
Definition of Strong Bias

Definition 12 Denote the equilibrium technology at factor supplies 
$(\bar{Z}, \bar{L}) \in \mathcal{Z} \times \mathcal{L}$ by $\theta^*(\bar{Z}, \bar{L})$ and suppose that $\partial\theta^*_j/\partial Z$ exists at $(\bar{Z}, \bar{L})$ for all $j = 1, \ldots, K$. Then there is strong absolute equilibrium bias at 
$(\bar{Z}, \bar{L}) \in \mathcal{Z} \times \mathcal{L}$ if

$$\frac{dw_Z}{dZ} = \frac{\partial w_Z}{\partial Z} + \sum_{j=1}^{K} \frac{\partial w_Z}{\partial \theta_j^*} \frac{\partial \theta_j^*}{\partial Z} > 0.$$
Main Theorem

**Theorem 5** Consider Economy D, C or M. Suppose that $\Theta$ is a convex subset of $\mathbb{R}^K$, $F$ is twice continuously differentiable in $(Z, \theta)$, let $\theta^* (\bar{Z}, \bar{L})$ be the equilibrium technology at factor supplies $(\bar{Z}, \bar{L})$ and assume that $\theta^*$ is in the interior of $\Theta$ and that $\partial \theta^*_j (\bar{Z}, \bar{L}) / \partial Z$ exists at $(\bar{Z}, \bar{L})$ for all $j = 1, ..., K$. Then there is strong absolute equilibrium bias at $(\bar{Z}, \bar{L})$ if and only if $F(Z, L, \theta)$'s Hessian in $(Z, \theta)$, $\nabla^2 F_{(Z, \theta)}(Z, \theta)$, is not negative semi-definite at $(\bar{Z}, \bar{L}, \theta^* (\bar{Z}, \bar{L}))$. 
Sketch of the Proof

• Let us again focus on the case where $\Theta \subset \mathbb{R}$.

• By hypothesis, $\partial F/\partial \theta = 0$, $\partial^2 F/\partial \theta^2 \leq 0$.

• Then the condition for strong absolute equilibrium bias can be written as:

$$\frac{dw_Z}{dZ} = \frac{\partial w_Z}{\partial Z} + \frac{\partial w_Z}{\partial \theta} \frac{\partial \theta^*}{\partial Z},$$

$$= \frac{\partial^2 F}{\partial Z^2} - \left( \frac{\partial^2 F/\partial \theta \partial Z}{\partial^2 F/\partial \theta^2} \right)^2 > 0.$$  

• From Assumption 1 or 2, $F$ is concave in $Z$, so $\partial^2 F/\partial Z^2 \leq 0$, and from the fact that $\theta^*$ is a solution to the equilibrium maximization problem

$$\partial^2 F/\partial \theta^2 < 0.$$
Sketch of the Proof (continued)

• Then the fact that $F$’s Hessian, $\nabla^2 F(z, \theta)(z, \theta)$, is not negative semi-definite at $(\bar{Z}, \bar{L}, \theta^* (\bar{Z}, \bar{L}))$ implies that

$$\frac{\partial^2 F}{\partial Z^2} \times \frac{\partial^2 F}{\partial \theta^2} < \left( \frac{\partial^2 F}{\partial Z \partial Z \theta} \right)^2,$$

• Since at the optimal technology choice $\frac{\partial^2 F}{\partial \theta^2} < 0$, this immediately yields $dw_Z/dZ > 0$, establishing strong absolute bias at $(\bar{Z}, \bar{L}, \theta (\bar{Z}, \bar{L}))$.

• Conversely, if $\nabla^2 F(z, \theta)(z, \theta)$ is negative semi-definite at $(\bar{Z}, \bar{L}, \theta^* (\bar{Z}, \bar{L}))$, then the previous relationship does not hold and this together with $\frac{\partial^2 F}{\partial \theta^2} < 0$ implies that $dw_Z/dZ \leq 0$. 
Intuition

• When $F(Z, L, \theta)$ is not jointly concave in $Z$ and $\theta$, the equilibrium corresponds to a saddle point of $F$ in the $Z, \theta$ space.

• This implies that there exists direction in which output and hence monopoly profits for technology suppliers can be increased.

• Nevertheless, the saddle point is an equilibrium, since $Z$ and $\theta$ are chosen by different agents.

• When $Z$ changes by a small amount, then $\theta$ can be changed in the direction of ascent.

• This not only increases output but also the marginal product of factor $Z$ that has become more abundant.

• The result is an upward-sloping demand curve for $Z$. 
Simple Example

• Let us suppose $\Theta = \mathbb{R}$ and $F(Z, L, \theta) = 4Z^{1/2} + Z\theta - \theta^2/2 + B(L)$ with the cost of creating new technologies incorporated into this function.

• Clearly $F$ is not jointly concave in $Z$ and $\theta$ (for $Z > 1$) but is strictly concave in $Z$ and $\theta$ individually.

• Consider a change from $\bar{Z} = 1$ to $\bar{Z} = 4$.

• The first-order necessary and sufficient condition for technology choice gives $\theta(\bar{Z}, \bar{L}) = \theta(\bar{Z}) = \bar{Z}$.

• Therefore, $\theta(\bar{Z} = 1) = 1$ while $\theta(\bar{Z} = 4) = 4$.

• Moreover, for any $\bar{L} \in \mathcal{L}$, $w_Z(\bar{Z}, \bar{L}, \theta) = 2Z^{-1/2} + \theta$

• Therefore, $w_Z(\bar{Z} = 1, \bar{L}, \theta(1)) = 3 < w_Z(\bar{Z} = 4, \bar{L}, \theta(4)) = 5$, establishing strong (absolute) equilibrium bias between $\bar{Z} = 1$ to $\bar{Z} = 4$. 
How Likely Is This?

• The key requirement is that technologies and factor demands are not decided by the same agent.

• Once we are in such an equilibrium situation, there is no guarantee that the equilibrium point corresponds to a global maximum.

• Thus the requirements are not very restrictive.

• However, naturally, $F$ cannot be globally concave in all of its arguments.

• Thus some degree of increasing returns is necessary.
How Likely Is This? (continued)

- Therefore an immediate corollary:

**Corollary 1** Suppose that $\Theta$ is a convex subset of $\mathbb{R}^K$, $F$ is twice continuously differentiable in $(Z, \theta)$, let the equilibrium technology at factor supplies $(\bar{Z}, \bar{L})$ be $\theta^* (\bar{Z}, \bar{L})$, and assume that $\partial \theta_j^* / \partial Z$ exists at $(\bar{Z}, \bar{L})$ for all $j = 1, \ldots, K$. Then there cannot be strong absolute equilibrium bias in Economy D.

- Intuitively, in Economy D, $F$ must be negative semi-definite in $Z$ and $\theta$, since the same firms choose both $Z$ and $\theta$.

- However, interestingly, one can construct examples where there is strong bias in Economy D if $\Theta$ is a finite set.
How Likely Is This? (continued)

- However, outside of Economy D, strong equilibrium bias easily possible.
- Let $\mathcal{C}^2 [B]$ denote the set of twice continuously differentiable functions over $B$.
- Let $\mathcal{C}^2_+ [B] \subset \mathcal{C}^2 [B]$ be the set of such functions that are strictly convex.
- Let $\mathcal{C}^2_- [B] \subset \mathcal{C}^2 [B]$ be the set of such functions that are strictly concave in each of their arguments (though not necessarily jointly so).

**Theorem 6** Suppose that $\Theta \subset \mathbb{R}$ and $\mathcal{Z} \subset \mathbb{R}_+$ are compact, and denote the equilibrium technology by $\theta^*$, and for fixed $\bar{L} \in \mathcal{L}$, let $G (\bar{Z}, \bar{L}, \theta) \in \mathcal{C}^2_- [\mathcal{Z} \times \Theta]$. For each $C (\cdot) \in \mathcal{C}^2_+ [\Theta]$, let $\mathcal{D}_C \subset \mathcal{C}^2_- [\Theta]$ be such that for all $G (\bar{Z}, \bar{L}, \theta) \in \mathcal{D}_C$ there is strong absolute equilibrium bias. Then we have:

1. For each $C (\cdot) \in \mathcal{C}^2_+ [\Theta]$, $\mathcal{D}_C$ is a nonempty open subset of $\mathcal{C}^2_- [\Theta]$.

2. Suppose that $\theta^*$ is an equilibrium technology for both $C_1 (\cdot), C_2 (\cdot) \in \mathcal{C}^2_+ [\Theta]$ and that $\partial^2 C_1 (\theta^*) / \partial \theta^2 < \partial^2 C_2 (\theta^*) / \partial \theta^2$, then $\mathcal{D}_{C_2} \subset \mathcal{D}_{C_1}$ (and $\mathcal{D}_{C_2} \neq \mathcal{D}_{C_1}$).
Global Strong Bias

- In contrast to the weak bias absolute theorem, not much more is necessary for a global version of the strong absolute bias theorem.

- Technical intuition: Fundamental Theorem of Calculus.
Global Strong Bias Theorem

**Theorem 7** Suppose that $\Theta$ is a convex subset of $\mathbb{R}^K$ and that $F$ is twice continuously differentiable in $(Z, \theta)$. Let $\bar{Z}, \tilde{Z}' \in \mathcal{Z}$, with $\tilde{Z}' > \bar{Z}$, $\bar{L} \in \mathcal{L}$, and let $\theta^* \left( \tilde{Z}, \bar{L} \right)$ be the equilibrium technology at factor supplies $\left( \tilde{Z}, \bar{L} \right)$ and assume that $\theta^* \left( \tilde{Z}, \bar{L} \right)$ is in the interior of $\Theta$ and that $\partial \theta_j^* / \partial Z$ exists at $(\tilde{Z}, \bar{L})$ for all $j = 1, ..., K$ and all $\tilde{Z} \in [\bar{Z}, \tilde{Z}']$. Then there is **strong absolute equilibrium bias** at $\left( \{\bar{Z}, \tilde{Z}'\}, \bar{L} \right)$ if $F(\tilde{Z}, \bar{L}, \theta)$'s Hessian, $\nabla^2 F(\tilde{Z}, \bar{L}, \theta)$, fails to be negative semi-definite at $(\tilde{Z}, \bar{L}, \theta^* \left( \tilde{Z}, \bar{L} \right))$ for all $\tilde{Z} \in [\bar{Z}, \tilde{Z}']$. 
Conclusions

• Study of direction and bias of technology important both for practical and theoretical reasons.

• Surprisingly tractable framework and many strong results are possible.

• Most interestingly:
  1. In contrast to previous non-micro-founded models, a strong force towards induced bias in favor of factors becoming more abundant (weak bias theorems).
  2. Under fairly reasonable conditions, demand curves can slope upward (strong bias theorems).
Conclusions (continued)

- Many applications of endogenous bias:
  1. Endogenous skill bias (both recently and industry).
  2. Why is long-run technological change labor augmenting?
  3. Technological sources of unemployment persistence in Europe.
  4. Demographics and evolution on innovations in the pharmaceutical industry.
  5. A theory of cross-country income differences.
  6. Possible perspectives on “lost decades”.
  7. The effect of international trade on the nature of innovation and on cross-country income differences.