Player-Compatible Learning and Player-Compatible Equilibrium

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Abstract

Player-Compatible Equilibrium (PCE) imposes cross-player restrictions on the magnitudes of the players’ “trembles” onto different strategies. These restrictions capture the idea that trembles correspond to deliberate experiments by agents who are unsure of the prevailing distribution of play. PCE selects intuitive equilibria in a number of examples where trembling-hand perfect equilibrium (Selten, 1975) and proper equilibrium (Myerson, 1978) have no bite. We show that rational learning and some commonly used heuristics imply our compatibility restrictions in a steady-state setting.

Keywords: non-equilibrium learning, equilibrium refinements, trembling-hand perfect equilibrium, combinatorial bandits, Bayesian upper confidence bounds, weighted fictitious play.

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1 Introduction

Starting with Selten (1975), a number of papers have used the device of vanishingly small “trembles” to refine the set of Nash equilibria. This paper introduces player-compatible equilibrium (PCE), which extends the tremble-based approach. PCE requires that the trembles used to support an equilibrium satisfy player compatibility, which impose restrictions on how one player’s trembles compare to those of another. We say player $i$ is more player-compatible with strategy $s_i^*$ than player $j$ is with strategy $s_j^*$ if whenever $s_j^*$ is optimal for $j$ against some correlated profile $\sigma$, $s_i^*$ is optimal for $i$ against any profile $\hat{\sigma}$ matching $\sigma$ in terms of the strategies of players other than $i$ and $j$. PCE is invariant to the utility representations of players’ preferences over game outcomes, and provides a link between tremble-based refinements and learning in games. As we will explain, PCE interprets “trembles” not as errors, but as players’ deliberate experiments to learn how others play. Its cross-player tremble restrictions derive from an analysis of the relative frequencies of experiments that different players choose to undertake over time under a number of commonly used learning rules.

Section 2 defines player compatibility and PCE, studies their basic properties, and proves that PCE exist in all finite games. The compatibility relation is easiest to satisfy when $i$ and $j$ are “non-interacting,” meaning that their payoffs do not depend on each other’s play. But PCE can have bite even when all players interact with each other, provided that the interactions are not too strong. Moreover, as shown by the examples in Section 3, PCE can rule out seemingly implausible equilibria that other tremble-based refinements such as trembling-hand perfect equilibrium (Selten, 1975) and proper equilibrium (Myerson, 1978) cannot eliminate.

One of these examples is a “link-formation game,” where players are split into two sides, and each player decides whether or not to pay a cost to be Active and form links with all of the active players on the other side. Players with lower costs are more compatible with Active and so experiment with it more. In the “anti-monotonic” version of the game, players who incur a higher private cost of link formation give lower benefits to their linked partners; in the “co-monotonic” version, higher cost players give others higher benefits. In the anti-monotonic version the only PCE outcome is for all players to choose Active, because the
experimentation of the low-cost players induces all players on the other side to be Active as well. On the other hand, both “all Active” and “all Inactive” are PCE outcomes in the co-monotonic case. In contrast to PCE making different predictions in the two versions of the game, other equilibrium refinements make the same predictions whether payoffs are anti-monotonic or co-monotonic.

To provide motivation for player-compatible trembles, we study a learning framework where agents are born into different player roles of the stage game and believe that they face an unknown but time-invariant distribution of opponents’ play, as they would in a steady state of a model where a continuum of anonymous agents are randomly matched each period. We compare the experimentation behavior of agents born into different player roles who follow “index learning rules.” These rules assign an index to each strategy that depends only on data from periods when that strategy was used, and play the strategy with the highest index. We formulate an index compatibility condition for index rules, and use a coupling argument to show that any index rules for $i$ and $j$ satisfying this index-compatibility condition for strategies $s_i^*$ and $s_j^*$ will lead to $i$ experimenting relatively more with $s_i^*$ than $j$ with $s_j^*$ over their lifetimes against any distribution of opponents’ play.

Index compatibility provides a general condition for $i$ to choose $s_i^*$ more often than $j$ chooses $s_j^*$. This condition applies across a range of observation structures and (not necessarily optimal) learning rules. We link the stage-game payoff structure and player compatibility with index compatibility for some canonical learning rules. First, we consider a class of “factorable games” where choosing each strategy $s_i$ always reveals how opponents played at all the payoff-relevant information set for $s_i$, but no information about the payoffs of any other strategy of $i$. (The examples in Section 3 that we use to illustrate PCE are all factorable games for generic extensive-form payoffs, for instance.) We show that player compatibility implies index compatibility both for the rational learning rule given by the Gittins index and the algorithmically simpler Bayesian upper confidence bounds (Bayesian UCB) proposed by Kaufmann, Cappé, and Garivier (2012). In addition, we work with a broader class of “rigid” games that drops the requirement that playing one strategy of $i$ gives no information about the payoffs of another strategy. In rigid games, we consider agents who use weighted fictitious play (Cheung and Friedman, 1997), and so give more emphasis to recent
observations, and agents who use the Gittins index as a heuristic even though it need not be exactly optimal. In these games, a stronger notion of player compatibility (which is still satisfied by our examples of Section 3) leads to index compatibility. In the link-formation game, for example, our analysis of learning rules provides a foundation for the idea that low-cost agents play Active more frequently than high-cost ones.

1.1 Related Work

1.1.1 Tremble-Based Refinements

Tremble-based solution concepts date back to Selten (1975), who thanks Harsanyi for suggesting them. These solution concepts consider totally mixed strategy profiles where players do not play an exact best reply to the strategies of others, but may assign positive probability to some or all strategies that are not best replies. Different solution concepts in this class consider different kinds of “trembles,” but they all make predictions based on the limits of these non-equilibrium strategy profiles as the probability of trembling tends to zero. Since we compare PCE to these refinements below, we summarize them here for the reader’s convenience.

An $\epsilon$-perfect equilibrium is a totally mixed strategy profile where every non-best reply has weight less than $\epsilon$. A limit of $\epsilon_t$-perfect equilibria where $\epsilon_t \to 0$ is called a trembling-hand perfect equilibrium. An $\epsilon$-proper equilibrium is a totally mixed strategy profile $\sigma$ where for every player $i$ and strategies $s_i$ and $s'_i$ if $u_i(s_i, \sigma_{-i}) < u_i(s'_i, \sigma_{-i})$ then $\sigma_i(s_i) < \epsilon \cdot \sigma_i(s'_i)$. A limit of $\epsilon_t$-proper equilibria where $\epsilon_t \to 0$ is called a proper equilibrium; in this limit a more costly tremble is infinitely less likely than a less costly one, regardless of the cost difference. Approachable equilibrium (Van Damme, 1987) is also based on the idea that strategies with worse payoffs are played less often. It too is the limit of $\epsilon_t$-perfect equilibria, but where the players pay control costs to reduce their tremble probabilities. When these costs are “regular,” all of the trembles are of the same order. Because PCE does not require that the less likely trembles are infinitely less likely than more likely ones, it is closer to approachable equilibrium than to proper equilibrium. The strategic stability concept of Kohlberg and Mertens (1986) is also defined using trembles, but applies to components of Nash equilibria.
as opposed to single strategy profiles.

Unlike the central feature of PCE, proper equilibrium and approachable equilibrium do not impose cross-player restrictions on the relative probabilities of various trembles. For this reason, when each type of the sender is viewed as a different player these equilibrium concepts reduce to perfect Bayesian equilibrium in signaling games with two possible signals, such as the beer-quiche game of Cho and Kreps (1987). They do impose restrictions when applied to the ex-ante form of the game, i.e., at the stage before the sender has learned their type. However, as Cho and Kreps (1987) point out, evaluating the cost of mistakes at the ex-ante stage means that the interim losses are weighted by the prior distribution over sender types, so that less likely types are more likely to tremble. In addition, applying a different positive linear rescaling to each type’s utility function preserves every type’s preference over lotteries on outcomes, but changes the sets of proper and approachable equilibria, while such utility rescalings have no effect on the set of PCE. In light of these issues, when discussing tremble-based refinements in Bayesian games we will always apply them at the interim stage.

Like PCE, extended proper equilibrium (Milgrom and Mollner, 2017) places restrictions on the relative probabilities of tremble by different players, but it does so in a different way: An extended proper equilibrium is the limit of \((\beta, \epsilon_t)\)-proper equilibria, where \(\beta = (\beta_1, ..., \beta_I)\) is a strictly positive vector of utility re-scaling, and \(\sigma_i(s_i) < \epsilon_t \cdot \sigma_j(s_j)\) if player \(i\)’s rescaled loss from \(s_i\) (compared to the best response) is less than \(j\)’s loss from \(s_j\). In a signaling game with only two possible signals, every Nash equilibrium where each sender type strictly prefers not to deviate from their equilibrium signal is an extended proper equilibrium at the interim stage, because suitable utility rescalings for the types can lead to any ranking of their utility costs of deviating to the off-path signal. By contrast, Proposition 4 shows every PCE must satisfy the compatibility criterion of Fudenberg and He (2018), which has bite even in binary signaling games such as the beer-quiche example of Cho and Kreps (1987). So an extended proper equilibrium need not be a PCE, a fact that Examples 1 and 2 further demonstrate. Conversely, because extended proper equilibrium makes some trembles infinitely less likely than others, it can eliminate some PCE.\footnote{Example available on request.}
1.1.2 The Learning Foundations of Equilibrium

This paper builds on the work of Fudenberg and Levine (1993) and Fudenberg and Kreps (1995, 1994) on learning foundations for self-confirming and Nash equilibrium. It is also related to recent work that provides explicit learning foundations for various equilibrium concepts that reflect ambiguity aversion, misspecified priors, or model uncertainty, such as Battigalli, Cerreia-Vioglio, Maccheroni, and Marinacci (2016), Battigalli, Francetich, Lanzani, and Marinacci (2017), Esponda and Pouzo (2016), and Lehrer (2012). Unlike those papers, we focus on the very patient agents who undertake many “experiments,” and characterize the relative rates of experimentation under rational expected-utility maximization and related “near-optimal” heuristics. For this reason our analysis of learning is closer to Fudenberg and Levine (2006) and Fudenberg and He (2018).

Our investigation of learning dynamics significantly expands on that of Fudenberg and He (2018), which focused on a particular learning rule (rational Bayesians) in a restricted set of games (signaling games). In contrast, our analysis applies to a broader class of learning rules — specifically, index policies that satisfy the index compatibility condition, and to larger families of games: the factorable games defined in Section 5 and games with rigid feedback structures defined in Section 6. We develop new tools to deal with new issues that arise in these more general games. For instance, Fudenberg and He (2018) compare the Gittins indices of different sender types using the fact that any stopping time (for the auxiliary optimal-stopping problem defining the index) of the less-compatible type is also feasible for the more-compatible type. But our general setting allows player roles to interact, so it is not valid to exchange the stopping times of different players as they may condition on observed play in different parts of the game tree. We deal with this problem by considering how $i$ can nevertheless construct a feasible stopping time that mimics an unfeasible one of $j$. Moreover, when a player faces more than one opponent, their optimal experimentation policy may lead them to observe a correlated distribution of opponents’ play, even though the opponents do no actually play correlated strategies. We discuss this issue of endogenous correlation in Section 10.2; it is the reason we define PCE in terms of correlated play. Our general learning foundation in terms of abstract index policies that satisfy index compatibility
allows us to go beyond rational behavior and show that experimentation frequencies respect player compatibility (or a stronger variant thereof) for some commonly used, sub-optimal heuristics, including Bayes-UCB and weighted fictitious play.

In methodology the paper is related to other work on active learning and experimentation. In single-agent settings, these include Doval (2018), Francetich and Kreps (2018), and Fryer and Harms (2017). In multi-agent settings additional issues arise such as free-riding and encouraging others to learn, see e.g. Bolton and Harris (1999), Keller et al. (2005), Klein and Rady (2011), Heidhues, Rady, and Strack (2015), Frick and Ishii (2015), Halac, Kartik, and Liu (2016), Strulovici (2010), and the survey by Hörner and Skrzypacz (2016). Unlike most models of multi-agent bandit problems, our agents only learn from personal histories, not from the actions or histories of others. Our focus is the comparison of experimentation policies under different payoff parameters, which is central to PCE’s cross-player tremble restrictions.

2 Player-Compatible Equilibrium

In this section, we develop a concept of the relative “compatibility” between two player-strategy pairs and discuss its properties. We then introduce PCE, which embodies cross-player tremble restrictions based on this relation.

Like proper equilibrium, PCE is defined on the strategic form of a game. Of course many extensive forms can have the same strategic form, and the learning motivation for PCE and player-compatible trembles does depend on the underlying extensive form and the feedback structure, but we postpone these issues until Section 4.

2.1 Player Compatibility

Consider a game in its strategic form with a finite set of players \( i \in I, \) finite strategy sets \( |S_i| \geq 2, \) and utility functions \( u_i : S \to \mathbb{R}, \) where \( S := \times_i S_i. \) For each \( i, \) let \( \Delta(S_i) \) denote the set of mixed strategies for \( i. \) Let \( \Delta^\circ(S) \) represent the interior of \( \Delta(S), \) the set of full-support

\[ \text{If } S_i = \{s^*_i\} \text{ is a singleton, we would have } s^*_i \succeq s_j \text{ and } s_j \succeq s^*_i \text{ for any strategy } s_j \text{ of any player } j \text{ if we follow the convention that the maximum over an empty set is } -\infty. \]
correlated strategy profiles.

**Definition 1.** For player $i \neq j$ and strategies $s_i^* \in S_i$, $s_j^* \in S_j$, $i$ is more player-compatible with $s_i^*$ than $j$ is with $s_j^*$, abbreviated as $s_i^* \succeq s_j^*$, if for every totally mixed correlated strategy profiles $\sigma \in \Delta^o(S)$ with

$$\sum_{s \in S} u_j(s_j^*, s_{-j}) \cdot \sigma(s) = \max_{s_j' \in S_j} \sum_{s \in S} u_j(s_j', s_{-j}) \cdot \sigma(s),$$

we get

$$\sum_{s \in S} u_i(s_i^*, s_{-i}) \cdot \tilde{\sigma}(s) > \max_{s_i'' \in S_i \setminus \{s_i^*\}} \sum_{s \in S} u_i(s_i'', s_{-i}) \cdot \tilde{\sigma}(s)$$

for every totally mixed correlated strategy profile $\tilde{\sigma} \in \Delta^o(S)$ satisfying $\text{marg}_{-ij}(\sigma) = \text{marg}_{-ij}(\tilde{\sigma})$.

In words, if $s_j^*$ is weakly optimal for the less-compatible $j$ against $\sigma$, then $s_i^*$ is strictly optimal for the more-compatible $i$ against any $\tilde{\sigma}$ whose marginal on $-ij$’s play agrees with the marginal of $\sigma$. The compatibility condition does not depend on the particular expected utility functions used to represent the players’ preferences over probability distributions on $S$, but it does depend on the players’ preferences and is cardinal in that sense.

Since $\times_i \Delta^o(S_i) \subseteq \Delta^o(S)$, our definition of compatibility ranks fewer strategy-player pairs than an alternative definition that only considers mixed strategy profiles with independent mixing between different opponents.\footnote{Formally, this alternative definition would replace “totally mixed correlated strategy profiles” with “independently and totally mixed strategy profiles” in the definition of $s_i^* \succeq s_j^*$.} We use the more stringent definition to match the microfoundations of our compatibility-based cross-player restrictions.\footnote{Part of our microfoundation for player-compatible trembles considers rational learners who choose strategies based on their Gittins index. We use the fact that the Gittins index of a strategy is its expected payoff against a correlated strategy profile of the opponents. Even for learners who hold independent beliefs about opponents’ play at different information sets, a strategy’s Gittins index need not be its expected payoff against independent randomizations by the opponents.}

The compatibility relation is transitive, as the next proposition shows.

**Proposition 1.** Suppose $s_i^* \succeq s_j^* \succeq s_k^*$ where $s_i^*, s_j^*, s_k^*$ are strategies of $i, j, k$. Then $s_i^* \succeq s_k^*$.

The compatibility relation is also asymmetric, except in corner cases. Say that a strategy is **strictly interior dominant** if it is strictly better than any other strategy versus any strategy is strictly interior dominant if it is strictly better than any other strategy versus any other strategy. \footnote{This notation is unambiguous provided $i$ and $j$ have disjoint strategy sets. In the event that $i$ and $j$ share some strategies, we will clarify this notation by attaching player subscripts.}
totally mixed strategy profile of the opponents, and similarly say that it is strictly interior dominated\(^6\) if it is strictly dominated versus totally mixed opponent strategy profiles.

**Proposition 2.** If \(s^*_i \succeq s^*_j\), then at least one of the following is true: (i) \(s^*_j \not\succeq s^*_i\); (ii) \(s^*_i\) is strictly interior dominated for \(i\) and \(s^*_j\) is strictly interior dominated for \(j\); (iii) \(s^*_i\) is strictly interior dominant for \(i\) and \(s^*_j\) is strictly interior dominant for \(j\).

The proofs of Propositions 1 and 2 are straightforward; they can be found in the Online Appendix.

The definition of player compatibility simplifies in the following special case. A game has a multipartite structure if the set of players \(\mathcal{I}\) can be divided into \(C\) mutually exclusive classes, \(\mathcal{I} = \mathcal{I}_1 \cup \ldots \cup \mathcal{I}_C\), in such a way that whenever \(i \in \mathcal{I}_c\) and \(j \in \mathcal{I}_c\), (1) they are non-interacting, meaning neither player’s payoff depends on the other’s strategy; and (2) they have the same strategy set, \(\mathcal{S}_i = \mathcal{S}_j\). Every Bayesian game has a multipartite structure when each type is viewed as a different player. As another example, we will later use a complete-information game with a multipartite structure, the link-formation game (Example 2), to illustrate both PCE and the learning motivation for player-compatible trembles.

In a game with multipartite structure with \(i, j \in \mathcal{I}_c\), we can write \(u_i(s_c, s_{-ij})\) without ambiguity for \(s_c \in \mathcal{S}_i\), since all augmentations of the strategy profile \(s_{-ij}\) with a strategy by player \(j\) lead to the same payoff for \(i\). For \(s^*_c \in \mathcal{S}_i = \mathcal{S}_j\), the definition of \(s^*_ic \succeq s^*_jc\) reduces\(^7\) to: For every totally mixed correlated \(\sigma\) with \(\sigma_{-ij} \in \Delta^\circ(\mathcal{S}_{-ij})\),

\[
\sum_{s \in \mathcal{S}} u_j(s^*_j, s_{-ij}) \cdot \sigma(s) = \max_{s'_{ij} \in \mathcal{S}_j} \sum_{s \in \mathcal{S}} u_j(s'_j, s_{-ij}) \cdot \sigma(s)
\]

implies

\[
\sum_{s \in \mathcal{S}} u_i(s^*_i, s_{-ij}) \cdot \sigma(s) > \max_{s''_i \in \mathcal{S}_i \setminus \{s^*_i\}} \sum_{s \in \mathcal{S}} u_i(s''_i, s_{-ij}) \cdot \sigma(s).
\]

\(^6\)Recall that a strategy can be strictly dominated even though it is not strictly dominated by any pure strategy.

\(^7\)We use \(s^*_ic\) to refer to \(i\)'s copy of \(s^*_c\) and \(s^*_jc\) to refer to \(j\)'s copy.
2.2 Player-Compatible Trembles and PCE

PCE is a tremble-based solution concept. It builds on and modifies Selten (1975)'s definition of trembling-hand perfect equilibrium (in the strategic form) as the limit of equilibria of perturbed games in which agents are constrained to tremble, so we begin by defining our notation for the trembles and the associated constrained equilibria.

Definition 2. A tremble profile $\epsilon$ assigns a positive number $\epsilon(s_i \mid i) > 0$ to every player $i$ and every pure strategy $s_i \in S_i$. Given a tremble profile $\epsilon$, write $\Pi_i^{\epsilon}$ for the set of $\epsilon$-strategies of player $i$, namely:

$$
\Pi_i^{\epsilon} := \{\sigma_i \in \Delta(S_i) \text{ s.t. } \sigma_i(s_i) \geq \epsilon(s_i \mid i) \ \forall s_i \in S_i\}.
$$

We call $\sigma^0$ an $\epsilon$-equilibrium if for each $i$,

$$
\sigma_i^0 \in \arg \max_{\sigma_i \in \Pi_i^{\epsilon}} u_i(\sigma_i, \sigma_{-i}^0).
$$

Note that $\Pi_i^{\epsilon}$ is compact and convex. It is also non-empty when $\epsilon$ is close enough to 0. By standard results, whenever $\epsilon$ is small enough so that $\Pi_i^{\epsilon}$ is non-empty for each $i$, an $\epsilon$-equilibrium exists.

The key building block for PCE is $\epsilon$-PCE, which is an $\epsilon$-equilibrium where the tremble profile is “co-monotonic” with $\succsim$ in the following sense:

Definition 3. Tremble profile $\epsilon$ is player-compatible if $\epsilon(s_i^* \mid i) \geq \epsilon(s_j^* \mid j)$ for all $i, j, s_i^*, s_j^*$ such that $s_i^* \succsim s_j^*$. An $\epsilon$-equilibrium where $\epsilon$ is player-compatible is called a player-compatible $\epsilon$-equilibrium (or $\epsilon$-PCE).

The condition on $\epsilon$ says the minimum weight $i$ could assign to $s_i^*$ is no smaller than the minimum weight $j$ could assign to $s_j^*$ in the constrained game,

$$
\min_{\sigma_i \in \Pi_i^{\epsilon}} \sigma_i(s_i^*) \geq \min_{\sigma_j \in \Pi_j^{\epsilon}} \sigma_j(s_j^*).
$$

This is a “cross-player tremble restriction,” that is, a restriction on the relative probabilities of trembles by different players. Note that this restriction, like the player compatibility...
relation, depends on the players’ preferences over distributions on $S$ but not on the particular utility representation. This invariance property distinguishes player-compatible trembles from other models of stochastic behavior such as the stochastic terms in logit best responses. Our learning foundation will interpret these trembles not as mistakes, but as deliberate experiments by agents trying to learn how others play. Sections 4 presents a general sufficient condition for $i$ to experiment more with $s^*_i$ than $j$ does with $s^*_j$ over their lifetimes that is applicable across a range of feedback structures and learning policies. Sections 5 and 6 complete the story by showing that in a class of games that includes our examples, the player-compatibility condition $s^*_i \succeq s^*_j$ (or a stronger variant thereof) implies Sections 4’s sufficient condition for the rational learning policy and for a number of commonly used heuristics. For all these learning rules, we consider agents who start with the same priors over the play of their opponents. We believe we could extend this conclusion to agents with slightly different priors using a stronger notion of player compatibility, but we do not pursue this result here.\footnote{To do this, we would measure the “strength” of the compatibility ranking by saying that $i$ is $\lambda$ more player-compatible with $s^*_i$ than $j$ is with $s^*_j$ if the inequality in the definition $s^*_i \succeq s^*_j$ holds for all $\bar{\sigma} \in \Delta^c(S)$ satisfying $||\text{marg}_{-ij}(\sigma) - \text{marg}_{-ij}(\bar{\sigma})|| \leq \lambda$. We believe that our learning foundation would extend to cases where the agents’ priors are sufficiently close compared to $\lambda$.}

As is usual for tremble-based equilibrium refinements, we now define PCE as the limit of a sequence of $\epsilon$-PCE where $\epsilon \to 0$.

**Definition 4.** A strategy profile $\sigma^*$ is a *player-compatible equilibrium (PCE)* if there exists a sequence of player-compatible tremble profiles $\epsilon^{(t)} \to 0$ and an associated sequence of strategy profiles $\sigma^{(t)}$, where each $\sigma^{(t)}$ is an $\epsilon^{(t)}$-PCE, such that $\sigma^{(t)} \to \sigma^*$.

The cross-player restrictions embodied in player-compatible trembles translate into analogous restrictions on PCE, as shown in the next result.

**Proposition 3.** For any PCE $\sigma^*$, player $k$, and strategy $\bar{s}_k$ such that $\sigma^*_k(\bar{s}_k) > 0$, there exists a sequence of totally mixed strategy profiles $\sigma^{(t)}_{-k} \to \sigma^*_{-k}$ such that

(i) for every pair $i, j \neq k$ with $s^*_i \succeq s^*_j$,

$$\liminf_{t \to \infty} \frac{\sigma^{(t)}_i(s^*_i)}{\sigma^{(t)}_j(s^*_j)} \geq 1;$$
and (ii) \( \bar{s}_k \) is a best response for \( k \) against every \( \sigma^{(t)}_{-k} \).

The proof is in the Appendix, as are the proofs of subsequent results except where otherwise stated.

Treating each \( \sigma^{(t)}_{-k} \) as a totally mixed approximation to \( \sigma^*_{-k} \), in a PCE each player \( k \) essentially best responds to totally mixed opponent play that respects player compatibility.

It is easy to show that every \( \epsilon \)-PCE respects player compatibility up to the “adding up constraint” that probabilities on different strategies must sum up to 1 and \( i \) must place probability no smaller than \( \epsilon(s_i' \mid i) \) on strategies \( s_i' \neq s_i^* \). The “up to” qualification disappears in the \( \epsilon^{(t)} \to 0 \) limit because the required probabilities on \( s_i' \neq s_i^* \) tend to 0.

Since PCE is defined as the limit of \( \epsilon \)-equilibria for a restricted class of trembles, the set of PCE form a subset of trembling-hand perfect equilibria; the next result shows this subset is not empty. It uses the fact that the tremble profiles with the same lower bound on the probability of each action satisfy the compatibility condition in any game.

**Theorem 1.** A PCE exists in every finite strategic-form game.

### 2.3 Some Remarks about PCE

A tremble profile \( \epsilon \) is *uniform* if for all \( i \) and \( s_i \in S_i \), we have \( \epsilon(s_i \mid i) = \epsilon \) for the same \( \epsilon > 0 \). A trembling-hand perfect equilibrium is a *uniform THPE* if it is the limit of \( \epsilon \)-equilibria where \( \epsilon \to 0 \) and each tremble profile \( \epsilon \) is uniform. The proof of Theorem 1 in fact establishes the existence of uniform THPE, which form a subset of PCE since uniform trembles are always player-compatible regardless of the stage game.

One drawback of uniform THPE is that there is no clear microfoundation for uniform trembles. In addition to the cross-player restrictions of the compatibility condition, these uniform trembles impose the same lower bound on the tremble probabilities for all strategies of each player. PCE and the learning foundation we develop for player-compatible trembles allow for more complicated patterns of experimentation that respect the compatibility structure. We study a more permissive refinement than uniform THPE where we can offer a learning story for the tremble restrictions. PCE is a fairly weak solution concept that nevertheless has bite in some cases of interest, as we will discuss in Section 3.
We think of PCE as primarily a solution concept for games with three or more players, where the relative tremble probabilities of $i \neq j$ affect some third party’s best response. In two-player games, $s_i^* \succeq s_j^*$ only when $s_j^*$ is strictly dominated or $s_i^*$ is strictly dominant (in each case, we test dominance against strictly mixed opponent strategies).

In Section 7, we expand the game to include duplicate copies of some of the original strategies, where two strategies are duplicates if they provide exactly the same payoff and exactly the same information.\(^9\) If $s_i^* \succeq s_j^*$ in the original game, then in the expanded game we impose the cross-player tremble restriction that the probability of $i$ trembling onto the set of copies of $s_i^*$ is larger than the probability of $j$ trembling onto the set of copies of $s_j^*$. (In the learning model, the sum of a rational $i$’s lifetime experimentation frequencies with all duplicates of $s_i^*$ should be independent of the number of duplicates, so the foundation for player-compatible trembles in games without duplicates applies to these sums of trembles.)

We show that the set of PCE in the expanded game with these new tremble restrictions is the same as the set of PCE in the original game.

3 Examples of PCE

In this section, we study examples of games where PCE rules out unintuitive Nash equilibria. We will also use these examples to distinguish PCE from existing refinements.

3.1 The Restaurant Game

We start with a complete-information game where PCE differs from other solution concepts.

Example 1. There are three players in the game: a food critic, a regular diner, and a restaurant. Simultaneously, the restaurant decides between ordering high-quality (H) or low-quality (L) ingredients, while the critic and the diner decide whether to go eat at the restaurant (R) or order pizza (Z) and eat at home. The utility from Z is normalized to 0. If both customers choose Z, the restaurant also gets 0 payoff. Otherwise, the restaurant’s payoff depends on the ingredient quality and clientele. Choosing L yields a profit of +2 per

\(^9\)Two strategies with the same payoffs that give different information about opponents’ play are not equivalent in our learning model.
customer while choosing $H$ yields a profit of +1 per customer. In addition, if the food critic is present she will write a review based on ingredient quality, which affects the restaurant’s payoff by ±2.5. Each customer gets a payoff of $x < -1$ from consuming food made with low-quality ingredients and a payoff of $y > 0.5$ from consuming food made with high-quality ingredients, while the critic gets an additional +1 payoff from going to the restaurant and writing a review (regardless of food quality). Customers each incur a 0.5 congestion cost if they both go to the restaurant. We depict this situation in the game tree below, with $c$ and $d$ subscripts denoting strategies of the critic and the diner.

The strategies of the two customers affect each other’s payoffs, so the critic and the diner are not non-interacting players. In particular, they cannot be mapped into two types of the same agent in a Bayesian game.

The strategy profile $(Z_c, Z_d, L)$ is a proper equilibrium, sustained by the restaurant’s belief that when at least one customer plays $R$, it is far more likely that the diner deviated to patronizing the restaurant than the critic, even though the critic has a greater incentive to go to the restaurant since she gets paid for writing reviews. It is also an extended proper equilibrium.\(^\text{10}\)

We claim that $R_c \succ R_d$. Note that for any profile $\sigma$ of totally mixed, correlated play that makes the diner indifferent between $Z_d$ and $R_d$, we must have $u_1(R_c, \tilde{\sigma}; c) > 0.5$ for any profile $\tilde{\sigma}$ that agrees with $\sigma$ in terms of the restaurant’s play. The critic’s utility from $R_c$ is minimized when the diner chooses $R_d$ with probability 1, but even then the critic gets

\(^{10}(Z_c, Z_d, L)$ is an extended proper equilibrium, because scaling the critic’s payoff by a large positive constant makes it more costly for the critic to deviate to $R1$ than for the diner to deviate to $R2$.\)
0.5 higher utility from going to a crowded restaurant than the diner gets from going to an empty restaurant, holding fixed food quality at the restaurant. This shows $R_c \succeq R_d$.

Whenever $\sigma_c^{(t)}(R_c)/\sigma_d^{(t)}(R_d) > \frac{1}{4}$, the restaurant strictly prefers H over L. Thus by Proposition 3, there is no PCE where the restaurant plays L with positive probability.

### 3.2 The Link-Formation Game

In the next example, PCE makes different predictions in two versions of a game with different payoff parameters, while all other solution concepts we know of make the same predictions in both versions.

**Example 2.** There are 4 players in the game, split into two sides: North and South. The players are named North-1, North-2, South-1, and South-2, abbreviated as N1, N2, S1, and S2.

These players engage in a strategic link-formation game. Each player simultaneously takes an action: either Inactive or Active. An Inactive player forms no links. An Active player forms a link with every Active player on the opposite side. (Two players on the same side cannot form links.) For example, suppose N1 plays Active, N2 plays Active, S1 plays Inactive, and S2 plays Active. Then N1 creates a link to S2, N2 creates a link to S2, S1 creates no links, and S2 creates links to both N1 and N2.

![Diagram of link-formation game](image)

Each player $i$ is characterized by two parameters: cost ($c_i$) and quality ($q_i$). Cost refers to the private cost that a player pays for each link they create. Quality refers to the benefit that a player provides to others when they link to her. A player who forms no links gets
a payoff of 0. In the above example, the payoff to North-1 is \( q_{S2} - c_{N1} \) and the payoff to South-2 is \( (q_{N1} - c_{S2}) + (q_{N2} - c_{S2}) \).

We consider two specifications of the payoff functions. In the anti-monotonic version on the left, players with a higher cost have a lower quality. In the co-monotonic version on the right, players with a higher cost have a higher quality. There are two pure-strategy Nash outcomes for each version: all links form or no links form. “All links form” is the unique PCE outcome in the anti-monotonic case, while both “all links” and “no links” are PCE outcomes under co-monotonicity.

<table>
<thead>
<tr>
<th></th>
<th>Anti-Monotonic</th>
<th>Co-Monotonic</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Player</strong></td>
<td><strong>Cost</strong></td>
<td><strong>Quality</strong></td>
</tr>
<tr>
<td>North-1</td>
<td>14</td>
<td>30</td>
</tr>
<tr>
<td>North-2</td>
<td>19</td>
<td>10</td>
</tr>
<tr>
<td>South-1</td>
<td>14</td>
<td>30</td>
</tr>
<tr>
<td>South-2</td>
<td>19</td>
<td>10</td>
</tr>
<tr>
<td>North-1</td>
<td>14</td>
<td>10</td>
</tr>
<tr>
<td>North-2</td>
<td>19</td>
<td>30</td>
</tr>
<tr>
<td>South-1</td>
<td>14</td>
<td>10</td>
</tr>
<tr>
<td>South-2</td>
<td>19</td>
<td>30</td>
</tr>
</tbody>
</table>

PCE makes different predictions in these two versions of the game because the compatibility structure with respect to own quality is reversed. In both versions, Active_{N1} \succ Active_{N2}, but N1 has high quality in the anti-monotonic version, and low quality in the co-monotonic version. Thus, in the anti-monotonic version but not in the co-monotonic version, player-compatible trembles lead to the high-quality counterparty choosing Active at least as often as the low-quality counterparty, which means Active has a positive expected payoff even when one’s own cost is high.

In contrast, the set of equilibria that satisfy extended proper equilibrium, proper equilibrium, trembling-hand perfect equilibrium, \( p \)-dominance, Pareto efficiency, and strategic stability do not depend on whether payoffs are anti-monotonic or co-monotonic, as shown in the Online Appendix.

3.3 Signaling Games

Recall that a signaling game is a two-player Bayesian game, where P1 is a sender who knows their own type \( \theta \), and P2 only knows that P1’s type is drawn according to the distribution
\( \lambda \in \Delta(\Theta) \) on a finite type space \( \Theta \). After learning their type, the sender sends a signal \( s \in S \) to the receiver. Then, the receiver responds with an action \( a \in A \). Utilities depend on the sender’s type \( \theta \), the signal \( s \), and the action \( a \).

Fudenberg and He (2018)’s compatibility criterion is defined only for signaling games. It does not use limits of games with trembles, but instead restricts the beliefs that the receiver can have about the sender’s type. That sort of restriction does not seem easy to generalize beyond games with observed actions, while using trembles allows us to define PCE for general strategic-form games. As we will see, the more general PCE definition implies the compatibility criterion in signaling games.

With each sender type viewed as a different player, this game has \( |\Theta| + 1 \) players, \( I = \Theta \cup \{2\} \), where the strategy set of each sender type \( \theta \) is \( S_\theta = S \) while the strategy set of the receiver is \( S_2 = A^S \), the set of signal-contingent plans. So a mixed strategy of \( \theta \) is a possibly mixed signal choice \( \sigma_1(\cdot \mid \theta) \in \Delta(S) \), while a mixed strategy \( \sigma_2 \in \Delta(A^S) \) of the receiver is a mixed plan about how to respond to each signal.

Fudenberg and He (2018) define type compatibility for signaling games. A signal \( s^* \) is more type-compatible with \( \theta \) than with \( \theta' \) if for every behavioral strategy \( \sigma_2 \),

\[
\max_{s', s \neq s^*} u_1(s', s^*; \sigma_2 \cdot \theta') \geq u_1(s^*, s^*; \sigma_2 \cdot \theta')
\]

implies

\[
u_1(s^*, \sigma_2; \theta) > \max_{s', s \neq s^*} u_1(s', s^*; \sigma_2; \theta).
\]

They also define the compatibility criterion, which imposes restrictions on off-path beliefs in signaling games. Consider a Nash equilibrium \( \sigma_1^*, \sigma_2^* \). For any signal \( s^* \) and receiver action \( a \) with \( \sigma_2^*(a \mid s^*) > 0 \), the compatibility criterion requires that \( a \) best responds to some belief \( p \in \Delta(\Theta) \) about the sender’s type such that, whenever \( s^* \) is more type-compatible with \( \theta \) than with \( \theta' \) and \( s^* \) is not equilibrium dominated\(^\text{11}\) for \( \theta \), \( p \) satisfies \( \frac{p(\theta')}{p(\theta)} \leq \frac{\lambda(\theta')}{\lambda(\theta)} \).

Since every totally mixed strategy of the receiver is payoff-equivalent to a behavioral strategy, it is easy to see that type compatibility implies \( s^* \preceq s_{\theta'}^* \).\(^\text{12}\) The next result shows

\(^{11}\) Signal \( s^* \) is not equilibrium dominated for \( \theta \) if \( \max_{a \in A} u_1(s^*, a; \theta) > u_1(s_1, \sigma_2^*; \theta) \) for every \( s_1 \) with \( \sigma_1^*(s_1 \mid \theta) > 0 \).

\(^{12}\) The converse does not hold. We defined type compatibility to require testing against all receiver strategies.
that when specialized to signaling games, all PCE pass the compatibility criterion.

**Proposition 4.** In a signaling game, every PCE \( \sigma^* \) is a Nash equilibrium satisfying the compatibility criterion of Fudenberg and He (2018).

This proposition in particular implies that in the beer-quiche game of Cho and Kreps (1987), the quiche-pooling equilibrium is not a PCE, as it does not satisfy the compatibility criterion.

### 4 Index Learning Rules and Index Compatibility

This section characterizes a general class of “index learning rules” that lead \( i \) to experiment more with \( s_i^* \) than \( j \) does with \( s_j^* \). The next two sections show that many natural learning rules belong to this class when \( s_i^* \succeq s_j^* \) (or a stronger related condition) holds, including optimal learning behavior in certain settings, some recently developed heuristics for dealing with computationally complex exploration-and-exploitation problems, and weighted fictitious play. Together, these sections link the player-compatibility relation with agents’ learning behavior for various learning rules, providing a learning foundation for the tremble restrictions central to PCE.

The learning problem the players face depends on what they observe about the play of others, which in turn depends on the extensive form of the game, denoted by \( \Gamma \). This game has a set of players \( i \in I \) and also a player 0 that we will use to model Nature’s moves. The collection of information sets of player \( i \in I \) is written as \( H_i \). At each \( h \in H_i \), player \( i \) chooses an action \( a_h \) from the finite set of possible actions \( A_h \). An extensive-form pure strategy of \( i \) specifies an action at each information set \( h \in H_i \). We denote by \( S_i \) the set of all such strategies. Let \( Z \) be the set of terminal vertices of \( \Gamma \). Also, let \( Z(s) \) denote the terminal vertex reached under the pure strategy profile \( s \in \times_i S_i \).

We consider an agent born into player role \( i \) who maintains this role throughout their life. They have a geometrically distributed lifetime with \( 0 \leq \gamma < 1 \) probability of survival and not just the totally mixed ones, so it is possible that \( s_\theta^* \succeq s_{\theta'}^* \) but \( s^* \) is not more type-compatible with \( \theta \) than with \( \theta' \), so type-compatibility is harder to satisfy than player compatibility. We now realize that we could have restricted type compatibility to only consider totally mixed strategies, and all of the results of Fudenberg and He (2018) would still hold.
between periods. Each period, the agent plays the stage game $\Gamma$, choosing a strategy $s_i \in S_i$. Then, with probability $\gamma$, they continue into the next period and play the stage game again. With complementary probability they exit the system.

Each player is equipped with a finite set of observations $O_i$ and a feedback function $o_i : Z \rightarrow O_i$ that maps the terminal node reached to an observation. We assume each player has perfect recall and remembers their chosen strategy.

**Definition 5.** The set of all finite histories of all lengths for $i$ is $Y_i := \bigcup_{t \geq 0} (S_i \times O_i)^t$. For a history $y_i \in Y_i$ and $s_i \in S_i$, the subhistory $y_{i, s_i}$ is the (possibly empty) subsequence of $y_i$ where the agent played $s_i$.

To compare players $i$ and $j$’s relative experimentation probabilities, we need the agents in these two player roles to face “equivalent” learning environments. The next definition formalizes what this means; it is a joint restriction on the game tree $\Gamma$ and the feedback structures $(O_i, o_i), (O_j, o_j)$ of the two player roles.

**Definition 6.** Agents $i$ and $j$ face isomorphic learning problems if:

- There exists an isomorphism $\varphi : S_i \rightarrow S_j$

- For each $s_i \in S_i$, the union of histories of $i$ after $s_i$ and histories of $j$ after $s_j = \varphi(s_i)$ can be partitioned into a set of equivalence classes $E_{s_i}$, where elements $E \in E_{s_i}$ are disjoint subset of $\left(\{s_i\} \times O_i\right) \cup \left(\{s_j\} \times O_j\right)$.

- For each pure strategy profile $\tilde{s}$ and $s_i \in S_i$ with $s_j = \varphi(s_i)$, $(s_i, o_i(Z(s_i, \tilde{s}_{-i})))$ and $(s_j, o_j(Z(s_j, \tilde{s}_{-j})))$ belong to the same equivalence class in $E_{s_i}$.

If $i$ and $j$ face isomorphic learning problems, there is a bijection $\varphi$ between the strategies of the two agents and a notion of equivalence between the histories they might observe after choosing $s_i$ and $s_j$, with $s_j = \varphi(s_i)$. This equivalence puts $i$’s observation after $(s_i, \tilde{s}_{-i})$ in the same equivalence class as $j$’s observation after $(s_j, \tilde{s}_{-j})$, for any profile $\tilde{s}$.

Consider Example 1 when the Critic and the Diner observe all other players’ actions if they choose $R$, but observe nothing if they choose $Z$. That is,

$$O_C = O_D = \{(L, R), (L, Z), (H, R), (H, Z), \varnothing\}.$$
Consider the natural isomorphism \( \varphi(R_c) = R_d \) and \( \varphi(Z_c) = Z_d \), and partition histories after \( R_c \) and \( R_d \) as

\[
\mathbb{E}_{R_c} = \{(R_c, (L, R)), (R_d, (L, R)), (R_c, (L, Z)), (R_d, (L, Z))\},
\]

\[
\{(R_c, (H, R)), (R_d, (H, R)), (R_c, (H, Z)), (R_d, (H, Z))\}\}.
\]

The two equivalence classes in \( \mathbb{E}_{R_c} \) represent whether the restaurant is observed to play \( L \) or \( H \). Also let \( \mathbb{E}_{Z_c} = \{(Z_c, \emptyset), (Z_d, \emptyset)\} \) contain just one equivalence class, which corresponds to no observations. It is clear that given any pure strategy profile \( s, (R_c, s_{-c}) \) and \( (R_d, s_{-d}) \) lead to the same histories, up to equivalence classes (i.e., the same observation of the restaurant’s play.)

The agent decides which strategy to use in each period based on their history so far. We assume that this learning rule is a deterministic map (which is w.l.o.g. for expected-utility maximizers), and denote it \( r_i : Y_i \rightarrow S_i \).

**Definition 7.** A learning rule is an index policy if there exist index functions \( \iota_{s_i} \) with each \( \iota_{s_i} \) mapping subhistories of \( s_i \) to real numbers, such that \( r_i(y_i) \in \arg \max_{s_i \in S_i} \{\iota_{s_i}(y_i, s_i)\} \).

If an agent uses an index policy, we can think of their behavior in the following way. At each history, they compute an index for each strategy \( s_i \in S_i \) based on the subhistory of those periods where they chose \( s_i \), and play a strategy with the highest index.\(^\text{13}\) The best-known example of an index policy is the Gittins index (Gittins, 1979). Some heuristics for learning problems, such as UCB (Lai, 1987; Katehakis and Robbins, 1995) and Bayes-UCB (Kaufmann, Cappé, and Garivier, 2012) are also index policies. The key restriction in an index policy is that each strategy’s index depends only on the observations when that strategy was played. Note that index policies are deterministic, unlike some heuristics such as Thompson sampling (Thompson, 1933).

We extend the notion of equivalence to histories with more than one period in the natural way.

**Definition 8.** Suppose \( i \) and \( j \) face isomorphic learning problems with \( \varphi(s_i) = s_j \). Say

\(^{13}\)To handle possible ties, we can introduce a strict order over each agent’s strategy set, and specify that if two strategies have the same index the agent plays the one that is higher ranked.
′s subhistory $y_{i,s_i}$ is equivalent to $j$’s subhistory $y_{j,s_j}$, written as $y_{i,s_i} \sim y_{j,s_j}$, if they are equivalent period by period according to the equivalence classes $\mathbb{E}_{s_i}$.

Equivalence of $y_{i,s_i}$ and $y_{j,s_j}$ says $i$ has played $s_i$ as many times as $j$ has played $s_j$, and that the sequence of observations that $i$ encountered from experimenting with $s_i$ are the “same” as those that $j$ encountered from experimenting with $s_j$.

The following histories for the critic and the diner of the restaurant game are equivalent for the strategy $R$ (under the strategy isomorphism and observation equivalence classes previously given). This equivalence arises because the subhistories $y_{c,R}$ and $y_{d,R}$ contain the same sequences of the restaurant’s play (even though the two agents have different observations in terms of how often the other patron goes to the restaurant).

<table>
<thead>
<tr>
<th>period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_c$:</td>
<td>own strategy</td>
<td>$R$</td>
<td>$Z$</td>
<td>$Z$</td>
<td>$Z$</td>
</tr>
<tr>
<td></td>
<td>others’ play</td>
<td>$(L, Z)$</td>
<td>$\varnothing$</td>
<td>$\varnothing$</td>
<td>$\varnothing$</td>
</tr>
<tr>
<td>$y_d$:</td>
<td>own strategy</td>
<td>$Z$</td>
<td>$R$</td>
<td>$Z$</td>
<td>$R$</td>
</tr>
<tr>
<td></td>
<td>others’ play</td>
<td>$\varnothing$</td>
<td>$(L, R)$</td>
<td>$\varnothing$</td>
<td>$(H, Z)$</td>
</tr>
</tbody>
</table>

Table 1: The two histories $y_c$ (for the critic, with length 5) and $y_d$ (for the diner, with length 4) have equivalent subhistories for $R$.

Finally, we define a notion of the relative compatibility of index policies $r_i$ and $r_j$ with various strategies. Section 5 links this new notion of index-compatibility with the player-compatibility defined earlier. We show that, in factorable games, some natural learning procedures satisfy the index-compatibility condition for $s^*_i$ and $s^*_j$ exactly when $s^*_i \succ s^*_j$.

Section 6 shows that in a broader class of games with rigid feedback structures, a stronger notion of player compatibility implies index compatibility.

**Definition 9.** Suppose $i$ and $j$ face isomorphic learning problems with $\varphi(s^*_i) = s^*_j$. For two index policies $r_i$ and $r_j$, $r_i$ is more index-compatible with $s^*_i$ than $r_j$ is with $s^*_j$ if for any histories $y_i, y_j$ and any strategy $s'_i \in \mathbb{S}_i$, $s'_i \neq s^*_i$ satisfying

- $y_{i,s'_i} \sim y_{j,s^*_j}$ and $y_{i,s'_i} \sim y_{j,\varphi(s'_i)}$
- $s^*_j$ has weakly the highest index for $j$,

then $s'_i$ does not have the weakly highest index for $i$. 

20
Suppose the agent is randomly matched with agents in other player roles, and suppose that the play of these other agents corresponds to a fixed mixed strategy profile \( \sigma \), as it would in a steady state. We call \( \sigma \) the \textit{social distribution}. It, together with the agent’s learning rule, generates a stochastic process \( X^t_i \) describing \( i \)’s strategy in period \( t \) with distribution \( \mathbb{P}_{r_i,\sigma} \). Our analysis only depends on \(-i\)’s play being drawn i.i.d. across periods from \( \sigma_{-i} \).

**Definition 10.** Let \( X^t_i \) be the \( S_i \)-valued random variable representing \( i \)’s play in period \( t \). Player \( i \)’s \textit{induced response} to \( \sigma \) under learning rule \( r_i \) is \( \phi_i(\cdot; r_i, \sigma) : S_i \to [0, 1] \), where for each \( s_i \in S_i \)

\[
\phi_i(s_i; r_i, \sigma) := (1 - \gamma) \sum_{t=1}^{\infty} \gamma^{t-1} \cdot \mathbb{P}_{r_i,\sigma}\{X^t_i = s_i\}.
\]

Thus \( \phi_i(\cdot; r_i, \sigma) \) describes \( i \)’s weighted lifetime average play, where the weight on \( X^t_i \), the strategy they use in period \( t \) of their life, is proportional to the probability \( \gamma^{t-1} \) of surviving into that period.

The next result provides a general learning foundation for player-compatible trembles when learners use index rules: it restricts the relative experimentation frequencies generated by a pair of index-compatible policies. This result is not obvious because the index-compatibility relation only applies when two agents have equivalent histories, which typically does not hold during the dynamic process of experimentation.

**Proposition 5.** Suppose \( i \) and \( j \) face isomorphic learning problems with \( \varphi(s^*_i) = s^*_j \), and \( r_i, r_j \) are index rules, with \( r_i \) more index-compatible with \( s^*_i \) than \( r_j \) is with \( s^*_j \). Then \( \phi_i(s^*_i; r_i, \sigma_{-i}) \geq \phi_j(s^*_j; r_j, \sigma_{-j}) \) for any \( 0 \leq \gamma < 1 \) and \( \sigma \in \times_k \Delta(S_k) \).

The proof extends the coupling argument in the proof of Fudenberg and He (2018)’s Lemma 2, which only applies to the Gittins index in signaling games, and also fills in a missing step (Lemma 5) that the earlier proof implicitly assumed. To deal with the issue that \( i \) and \( j \) learn from endogenous data that diverge as they undertake different experiments, we couple the learning problems of \( i \) and \( j \) using what we call \textit{response paths} \( \mathcal{G} \in ((S^N)^{\infty} \) where \( N = \max_i |S_i| \). We can think of \( \mathcal{G} \) as a two-dimensional array of strategy profiles, \( \mathcal{G} = ((S_{1,1}, S_{1,2}, \ldots, S_{1,N}), (S_{2,1}, S_{2,2}, \ldots, S_{2,N}), \ldots) \), where \( S_{t,n_i} \in S \) for every \( t \geq 1, 1 \leq n_i \leq N \). We may enumerate each player’s strategy set \( S_i \) and interchangeably refer to each
strategy \( s_i \in S_i \) with its assigned number \( n_{s_i} \in \{1, \ldots, N\} \). For a given path and learning rule \( r_i \) for player \( i \), imagine running the rule against the data-generating process where the \( t \)-th time \( i \) plays the \( n_i \)-th strategy in \( S_i \), \( i \) is matched up with opponents who play the strategies \( S_{t,n_i} \). Given a learning rule \( r_i \), each \( \mathcal{S} \) induces a deterministic infinite history of \( i \)'s strategies \( y_i(\mathcal{S}, r_i) \in (S_i)^\infty \). (For \( n_i > |S_i| \), the values of \( (S_{t,n_i})_{t \geq 1} \) do not matter for the induced history.) We show that under the hypothesis that \( r_i \) is more index-compatible with \( s_i^* \) than \( r_j \) is with \( s_j^* \), the weighted lifetime frequency of \( s_i^* \) in \( y_i(\mathcal{S}, r_i) \) is larger than the frequency of \( s_j^* \) in \( y_j(\mathcal{S}, r_j) \) for every \( \mathcal{S} \), where play in different periods of the infinite histories \( y_i(\mathcal{S}, r_i), y_j(\mathcal{S}, r_j) \) are weighted by the probabilities of surviving into these periods, just as in the definition of induced responses.

Lemma 5 in the Appendix shows that when \( i \) and \( j \) face i.i.d. draws of opponents’ play from a fixed learning environment \( \sigma \), the induced responses are the same as if they each faced a random response path \( \mathcal{S} \) drawn at birth according to the (infinite) product measure over \(( (S)^N)^\infty \) whose marginals correspond to \( \sigma^{|S_i|} \).

## 5 Index Compatibility and Player Compatibility in Factorable Games

Section 4 proves that whenever index-strategy pairs \((r_i, s_i^*)\) and \((r_j, s_j^*)\) satisfy index compatibility (for \( i, j \) facing isomorphic learning problems), index rule \( r_i \) uses \( s_i^* \) more often than \( r_j \) uses \( s_j^* \) against any social distribution \( \sigma \). Index compatibility is a joint restriction on the agents’ learning policy and the game’s feedback structure (which gives the domain that the policies are defined on). This section shows that player compatibility implies index compatibility for rational behavior and Bayes-UCB in a class of factorable games. The following section shows that a stronger version of player compatibility implies index compatibility for the Gittins index and for weighted fictitious play in a broader class of games with rigid feedback structures. All of these conditions apply to the examples discussed in Section 3 for players ranked by compatibility.
5.1 Factorability and Isomorphic Factoring

Let $O_i \subseteq \mathbb{R}$ be the set of payoffs $i$ could receive in the stage game, and let $a_i(s_i, s_{-i}) = u_i(s_i, s_{-i})$. So each player only observes their own payoffs. In general, these payoffs need not perfectly reveal other players’ actions at all information sets. Factorability roughly says that playing strategy $s_i$ against any strategy profile of $-i$ identifies all of the opponents’ actions that can be payoff-relevant for $s_i$, but does not reveal any information about the payoff consequences of any other strategy $s'_i \neq s_i$.

For an information set $h$ of $j$ with $j \neq i$, write $P_h$ for the partition on $S_{-i}$ where two strategies $s_{-i}, s'_{-i}$ are in the same element of the partition if they prescribe the same play on $h$. That is, the partition elements in $P_h$ are $\{s_{-i} \in S_{-i} : s_{-i}(h) = a_h\}$ for $a_h \in A_h$. Thus partition $P_h$ contains perfect information about play on $h$, but no other information.

**Definition 11.** For each player $i$ and strategy $s_i \in S_i$, let $\Pi_i[s_i]$ be the coarsest partition of $S_{-i}$ that makes $s_{-i} \mapsto u_i(s_i, s_{-i})$ measurable. The game $\Gamma$ is factorable for $i$ if:

1. For each $s_i \in S_i$ there exists a (possibly empty) collection of $-i$’s information sets $F_i[s_i] \subseteq \mathcal{H}_{-i}$ so that $\Pi_i[s_i] = \bigwedge_{h \in F_i[s_i]} P_h$. (The meet across an empty collection is the coarsest possible partition on $S_{-i}$, i.e. no information).

2. For two strategies $s_i \neq s'_i$, $F_i[s_i] \cap F_i[s'_i] = \emptyset$.

When $\Gamma$ is factorable for $i$, we refer to $F_i[s_i]$ as the $s_i$-relevant information sets, a terminology we now justify. In general, $i$’s payoff from playing $s_i$ can depend on the profile of $-i$’s actions at all opponent information sets. Condition (1) implies that only opponents’ actions on $F_i[s_i]$ matter for $i$’s payoff after choosing $s_i$, and furthermore this dependence is one-to-one. That is,

$$u_i(s_i, s_{-i}) = u_i(s_i, s'_{-i}) \iff (\forall h \in F_i[s_i] \quad s_{-i}(h) = s'_{-i}(h)).$$

The one-to-one mapping from $s_{-i}$ to $i$’s payoff implies that $i$’s learning cannot be blocked by another player: By choosing $s_i$, $i$ can always use their own payoffs to identify actions on $F_i[s_i]$.
regardless of what happens elsewhere in the game tree.\footnote{\textit{It is easy but expositionally costly to extend this to the case where several actions on }A_h\textit{ lead to the same payoff for }i.} It also shows that if $\Gamma$ is factorable for $i$, then $F_i[s_i]$ are uniquely defined for all $s_i$. Suppose there were two collections $(F_i[s_i])_{s_i \in S_i}$ and $(\tilde{F}_i[s_i])_{s_i \in S_i}$ with $F_i[s_i] \setminus \tilde{F}_i[s_i] \neq \emptyset$ for some $s_i \in S_i$ that both satisfy Condition (1) of Definition 11. Then there are two $-i$ profiles $s_{-i}, s'_{-i}$ that match on $\tilde{F}_i[s_i]$ but not on $F_i[s_i]$. But then we get both $u_i(s_i, s_{-i}) = u_i(s_i, s'_{-i})$ and $u_i(s_i, s_{-i}) \neq u_i(s_i, s'_{-i})$, a contradiction.

Condition (2) implies that $i$ does not learn about the payoff consequence of a different strategy $s'_i \neq s_i$ through playing $s_i$ (provided $i$’s prior is independent about opponents’ play on different information sets). This is because there is no intersection between the $s_i$-relevant information sets and the $s'_i$-relevant ones. In particular this means that player $i$ cannot “free ride” on others’ experiments and learn about the consequences of various risky strategies while playing a safe one that is myopically optimal.

If $F_i[s_i]$ is empty, then $s_i$ is a kind of “opt out” action for $i$. After choosing $s_i$, $i$ receives the same utility from every reachable terminal node and gets no information about the payoff consequences of any of their other strategies.

We now illustrate factorability using the examples from Section 3 and some other general classes of games.

5.1.1 The Restaurant Game

Consider the restaurant game from Example 1. Since $x < -1$ and $y > 0.5$, we have $x \neq y$ and $x \neq y + 0.5$. By choosing $R$, the customer’s payoff perfectly reveals others’ play. By choosing $Z$, the customer always gets 0 payoff (these nodes are colored in the diagram below) and so cannot infer anyone else’s play.
The restaurant game is factorable for the Critic and the Diner. Let $F_i[R_i]$ consist of the two information sets of $-i$ and let $F_i[Z_i]$ be the empty set for each $i \in \{1, 2\}$. It is easy to verify that the two conditions of factorability are satisfied.

It is important for factorability that a customer who takes the “outside option” of ordering pizza gets the same payoff regardless of the restaurant’s play, and does not observe the restaurant’s quality choice even if the other customer patronizes the restaurant. Factorability rules out this sort of “free information,” so that when we analyze the non-equilibrium learning problem we know that each agent can only learn a strategy’s payoff consequences by playing it herself.

5.1.2 The Link-Formation Game

Consider the link-formation game from Example 2. The payoff for a player choosing Inactive is always 0, whereas the payoff for a player choosing Active exactly identifies the play of the two players on the opposite side. We can let $F_i[Active_i]$ consist of the information sets of the other two agents on the other side of $i$ and let $F_i[Inactive_i]$ be empty. This specification of the $s_i$-relevant information sets shows the stage game is factorable for every player.

5.1.3 Binary Participation Games

More generally, $\Gamma$ is factorable for $i$ whenever it is a binary participation game for $i$.

Definition 12. $\Gamma$ is a binary participation game for $i$ if the following are satisfied.

1. $i$ has a unique information set with two actions, without loss labeled In and Out.
2. All paths of play in $\Gamma$ pass through $i$’s information set.

3. All paths of play where $i$ plays $\text{In}$ pass through the same information sets.

4. Terminal vertices associated with $i$ playing $\text{Out}$ all have the same payoff for $i$.

5. Terminal vertices associated with $i$ playing $\text{In}$ all have different payoffs for $i$.

Action $\text{Out}$ is an outside option for $i$ that leads to a constant payoff regardless of others’ play. We are implicitly assuming in part (5) of the definition that the game has generic payoffs for $i$ after choosing $\text{In}$, in the sense that changing the action at any one information set on the path of play will change $i$’s payoff.

If $\Gamma$ is a binary participation game for $i$, let $F_i[\text{In}]$ be the collection of $-i$ information sets encountered in paths of play where $i$ chooses $\text{In}$. Let $F_i[\text{Out}]$ be the empty set. We see that $\Gamma$ is factorable for $i$. Clearly $F_i[\text{In}] \cap F_i[\text{Out}] = \emptyset$, so Condition (2) of factorability is satisfied. When $i$ chooses the strategy $\text{In}$, the tree structure of $\Gamma$ implies different profiles of play on $F_i[\text{In}]$ must lead to different terminal nodes, and the generic payoff condition means Condition (1) of factorability is satisfied for strategy $\text{In}$. When $i$ plays $\text{Out}$, $i$ gets the same payoff regardless of the others’ play, so Condition (1) of factorability is satisfied for strategy $\text{Out}$.

The restaurant game is a binary participation game for the critic and the diner, where ordering pizza is the outside option. The link-formation game is a binary participation game for every player, where $\text{Inactive}$ is the outside option.

5.1.4 Signaling to Multiple Audiences

To give a different class of examples of factorable games, consider a game of signaling to one or more audiences. To be precise, Nature moves first and chooses a type for the sender, drawn according to some known distribution over a finite set of types, $\Theta$. The sender then chooses a signal $s \in S$, observed by all receivers $r_1, \ldots, r_n$. Each receiver then simultaneously chooses an action. The profile of receiver actions, together with the sender’s type and signal, determine payoffs for all players. Viewing different types of senders as different players, this game is factorable for all sender types, provided payoffs are generic. This factorability arises
because for each type $i$, $F_i[s]$ is the set of $n_r$ information sets for the receivers after seeing signal $s$.

5.1.5 Examples of Non-Factorable Games

The next result gives a necessary condition for factorability, which we then use to provide examples of non-factorable games. Suppose $H$ is an information set of player $j \neq i$. Player $i$’s payoff is independent of $h$ if $u_i(a_h, a_{-h}) = u_i(a_h', a_{-h})$ for all $a_h, a_h', a_{-h}$, where $a_h, a_h'$ are actions on information set $h$, and $a_{-h}$ is a profile of actions on all other information sets in the game tree. In games where each player moves at most once along any path of play, if $i$’s payoff is not independent of the action taken at some information set $h$, then $i$ can always put $h$ onto the path of play via a unilateral deviation at one of their information sets.

Proposition 6. Suppose each player moves at most once along any path of play in $\Gamma$, the game is factorable for $i$, and $i$’s payoff is not independent of $h^*$. For any strategy profile, either $h^*$ is on the path of play, or $i$ has a deviation at one of their information sets that puts $h^*$ onto the path of play.

This result follows from two lemmas.

Lemma 1. For any game that is factorable for $i$ and any information set $h^*$ for player $j \neq i$ where $j$ has at least two different actions, if $h^* \in F_i[s_i]$ for some extensive-form strategy $s_i \in S_i$, then $h^*$ is always on the path of play when $i$ chooses $s_i$.

Lemma 2. For any game that is factorable for $i$ and any information set $h^*$ of player $j \neq i$, suppose $i$’s payoff is not independent of $h^*$. Then 1) $j$ has at least two different actions on $h^*$; and (2) there exists some extensive-form strategy $s_i \in S_i$ so that $h^* \in F_i[s_i]$.

We can combine these two lemmas to prove the proposition. Consider the centipede game for three players below.
Each player moves at most once on each path, and 1 and 2’s payoffs are not independent of the (unique) information set of player 3. But, if both 1 and 2 choose “drop”, then no one step deviation by either 1 or 2 can put the information set of 3 onto the path of play. Proposition 6 thus implies the centipede game is not factorable for either 1 or 2. Moreover, Fudenberg and Levine (2006) showed that in this game even very patient player 2’s may not learn to play a best response to player 3, so that the strategy profile (drop, drop, pass) can persist even though it is not trembling-hand perfect. Intuitively, if the player 1’s only play pass as experiments, then when the fraction of new players is very small, the player 2’s may not get to play often enough to make experimentation with pass worthwhile.

As another example, the Selten’s horse game displayed above is not factorable for 1 or 2 if the payoffs are generic, even though the conclusion of Proposition 6 is satisfied. The information set of 3 must belong to both \( F_1[\text{Down}] \) and \( F_1[\text{Across}] \) because 3’s play can affect 1’s payoff even if 1 chooses Across, since 2 could choose Down. This violates the factorability requirement that \( F_1[\text{Down}] \cap F_1[\text{Across}] = \emptyset \). The same argument shows the information set of 3 must belong to both \( F_2[\text{Down}] \) and \( F_2[\text{Across}] \), since when 1 chooses Down the play of 3 affects 2’s payoff regardless of 2’s play. So, again, \( F_2[\text{Down}] \cap F_2[\text{Across}] = \emptyset \) is violated.

Condition (2) of factorability also rules out games where \( i \) has two strategies that give the same information, but one strategy always has a worse payoff under all profiles of opponents’
play. In this case, we can think of the worse strategy as an informationally equivalent but more costly experiment than the better strategy. Reasonable learning rules (including rational learning) will not use such strategies, but we do not capture this feature in the general definition of PCE because our setup there only consider abstract strategy spaces $S_i$ and not an extensive-form game tree.\footnote{It would be interesting to try to refine the definition of PCE to capture this, perhaps using the “signal function” approach of Battigalli and Guaitoli (1997) and Rubinstein and Wolinsky (1994).}

### 5.1.6 Isomorphic Factoring

In order to compare the learning behavior of agents $i$ and $j$, we must ensure that they face isomorphic learning problems. To do this we use the notion of isomorphic factoring.

**Definition 13.** When $\Gamma$ is factorable for both $i$ and $j$, the factoring is isomorphic for $i$ and $j$ if there exists a bijection $\varphi : S_i \to S_j$ such that $F_i[s_i] \cap \mathcal{H}_{-ij} = F_j[\varphi(s_i)] \cap \mathcal{H}_{-ij}$ for every $s_i \in S_i$.

This says the $s_i$-relevant information sets (for $i$) are the same as the $\varphi(s_i)$-relevant information sets (for $j$), insofar as the actions of $-ij$ are concerned. For example, the restaurant game is isomorphically factorable for the critic and the diner (under the isomorphism $\varphi(R1)=R2, \varphi(Z1)=Z2$) because $F_1[\text{In}1] \cap \mathcal{H}_3 = F_2[\text{In}2] \cap \mathcal{H}_3 = \{\text{the singleton set containing the unique information set of the restaurant}\}$. As another example, all signaling games (with possibly many receivers as in Section 5.1.4) are isomorphically factorable for the different types of the sender. Similarly, the link-formation game is isomorphically factorable for pairs $(N1, N2)$, and $(S1, S2)$, but note that it is not isomorphically factorable for $(N1, S1)$.

### 5.2 Rational Learning and Bayes-UCB under Isomorphic Factoring

We study two index learning rules in factorable games: maximizing expected discounted utility and the Bayes upper confidence bound heuristic. With both rules, agents form a Bayesian belief over opponents’ play, independent at different information sets. More precisely, we assume that each agent $i$ starts with a regular independent prior:
Definition 14. Agent $i$ has a regular independent prior if their beliefs $g_i$ on $\times_{h \in \mathcal{H}_{-i}} \Delta(A_h)$ can be written as the product of full-support marginal densities on $\Delta(A_h)$ across different $h \in \mathcal{H}_{-i}$, so that $g_i((\alpha_h)_{h \in \mathcal{H}_{-i}}) = \prod_{h \in \mathcal{H}_{-i}} g^h_i(\alpha_h)$ with $g^h_i(\alpha_h) > 0$ for all $\alpha_h \in \Delta^o(A_h)$.

Thus, the agent holds a belief about the distribution of actions at each $-i$ information set $h$, and thinks actions at different information sets are generated independently, whether the information sets belong to the same player or to different players. Furthermore, the agent holds independent beliefs about the randomizing probabilities at different information sets.\footnote{We assume that agents do not know Nature’s mixed actions, which must be learned just as the play of other players. If agents know Nature’s move, then a regular independent prior would be a density $g_i$ on $\times_{h \in \mathcal{H}_{\{i\}}} \Delta(A_h)$, so that $g_i((\alpha_h)_{h \in \mathcal{H}_{\{i\}}}) = \prod_{h \in \mathcal{H}_{\{i\}}} g^h_i(\alpha_h)$ with $g^h_i(\alpha_h) > 0$ for all $\alpha_h \in \Delta^o(A_h)$. As Fudenberg and Kreps (1993) point out, an agent who believes two opponents are randomizing independently may nevertheless have subjective correlation in their uncertainty about the randomizing probabilities of these opponents. Here we study the natural special case where the agents’ prior beliefs about the opponents are independent, i.e., a product measure.}

The agent updates $g_i$ by applying Bayes rule to their history $y_i$. If the stage game is a signaling game, for example, this independence assumption means that the senders only update their beliefs about the receiver’s response to a given signal $s$ based on the responses received to that signal, and that the senders’ beliefs about this response do not depend on the responses they have observed to other signals $s' \neq s$.

If $i$ starts with independent prior beliefs in a stage game factorable for $i$, their learning problem is a combinatorial bandit problem. A combinatorial bandit consist of a set of basic arms, each with an unknown distribution of outcomes, together with a collection of subsets of basic arms called super arms. Each period, the agent must choose a super arm, which results in pulling all of the basic arms in that subset and obtaining a utility based on the outcomes of these pulls. To translate into our language, each basic arm corresponds to an $-i$ information set $h$ and the super arms are identified with strategies $s_i \in S_i$. The subset of basic arms in $s_i$ are the $s_i$-relevant information sets, $F_i[s_i]$. The collection of outcomes from these basic arms, i.e. the action profile $(a_h)_{h \in F_i[s_i]}$, determines $i$’s payoff, $u_i(s_i; (a_h)_{h \in F_i[s_i]})$.

A special case of combinatorial bandits is where the outcome from pulling each basic arm is simply a $\mathbb{R}$-valued reward, and the payoff from choosing a super arm is the sum of these rewards. This corresponds to the stage game being additively separable for $i$.

Definition 15. A factorable game $\Gamma$ is additively separable for $i$ if there is a collection of
auxiliary payoff functions \( v_{i,h} : A_h \rightarrow \mathbb{R} \) such that \( u_i(s_i, (a_h)_{h \in F_i[s_i]}) = \sum_{h \in F_i[s_i]} v_{i,h}(a_h) \).

The term \( v_{i,h}(a_h) \) is the “reward” of action \( a_h \) towards \( i \)'s payoff. The total payoff from \( s_i \) is the sum of such rewards over all \( s_i \)-relevant information sets. A factorable game is not additively separable for \( i \) when the opponents’ actions on \( F_i[s_i] \) interact in some way to determine \( i \)'s payoff following \( s_i \). All the examples discussed in Section 3 are additively separable for the players ranked by compatibility. While we provide our learning foundation for rational agents in any factorable game, our analysis of the Bayes upper confidence bound algorithm is restricted to additively separable games.

### 5.2.1 Expected Discounted Utility and the Gittins Index

Consider a rational agent who maximizes discounted expected utility. In addition to the survival chance \( 0 \leq \gamma < 1 \) between periods, the agent further discounts future payoffs according to their patience \( 0 \leq \delta < 1 \), so their overall effective discount factor is \( 0 \leq \delta \gamma < 1 \).

Given a belief about the distribution of play at each opponent information set, we can calculate the Gittins index of each strategy \( s_i \in S_i \), corresponding to a super arm in the combinatorial bandit problem.

Let \( \nu_{s_i} \times_{h \in F_i[s_i]} \Delta(\Delta(A_h)) \) be a belief over opponents’ mixed actions at the information sets in \( F_i[s_i] \). The Gittins index of \( s_i \) under belief \( \nu_{s_i} \) is given by the maximum value of the following auxiliary optimization problem:

\[
\sup_{\tau \geq 1} \mathbb{E}_{\nu_{s_i}} \left\{ \frac{\sum_{t=1}^\tau (\delta \gamma)^{t-1} \cdot u_i(s_i, (a_h(t))_{h \in F_i[s_i]})}{\sum_{t=1}^\tau (\delta \gamma)^{t-1}} \right\},
\]

where the supremum is taken over all positive-valued stopping times \( \tau \geq 1 \). Here \((a_h(t))_{h \in F_i[s_i]}\) means the profile of actions that \(-i\) plays on \( F_i[s_i] \) the \( t \)-th time that \( i \) uses \( s_i \) — by assumption about factorable games, only these actions and not actions elsewhere in the game.

---

\( ^{17} \) Additive separability is trivially satisfied whenever \( |F_i[s_i]| \leq 1 \) for each \( s_i \), so that there is at most one \( s_i \)-relevant information set for each strategy \( s_i \) of \( i \). So, every signaling game is additively separable for every sender type. It is also satisfied in the link-formation game in Section 5.1.2 even though here \( |F_i[\text{Active } i]| = 2 \), as each agent computes their payoff by summing their linking costs/benefits with respect to each potential counterparty. Additive separability is also satisfied in the restaurant game in Section 5.1.1 for each customer \( i \). \( F_i[R_i] \) contains two information sets, corresponding to the play of the Restaurant and the other customer. The play of the other customer additively contributes either 0 or -0.5 to \( i \)'s payoff, depending on whether they choose R or not.
tree determine $i$’s payoff from playing $s_i$, and $i$ can always infer these actions from their own payoffs. The distribution over the infinite sequence of profiles $(a_h(t))_{t=1}^{\infty}$ is given by $i$’s belief $\nu_{s_i}$, that is, there is some fixed mixed action in $\times_{h \in F_i[s_i]} \Delta(A_h)$ that generates profiles $(a_h(t))$ i.i.d. across periods $t$. The event $\{\tau = T\}$ for $T \geq 1$ corresponds to using $s_i$ for $T$ periods, observing the first $T$ elements $(a_h(t))_{t=1}^{T}$, then stopping.

We write the solution to the rational agent’s dynamic optimization problem as $\text{OPT}_i$, which involves playing the strategy $s_i$ with the highest Gittins index after each history $y_i$.

The drawback of this learning rule is that the Gittins index is computationally intractable even in relatively simple bandit problems. The combinatorial structure of our bandit problem makes computing the index even more complex.

### 5.2.2 Bayesian Upper-Confidence Bound

The Bayesian upper confidence bound (Bayes-UCB) procedure was first proposed by Kaufmann, Cappé, and Garivier (2012) as a computationally tractable algorithm for dealing with the exploration-exploitation trade-off in bandit problems.

We restrict attention to games additively separable for $i$ and adopt a variant of Bayes-UCB. Every $y_{i,h}$ subhistory of play on $h \in F_i[s_i]$ induces a posterior belief $g_i(\cdot \mid y_{i,h})$ over play on $h$, so $g_i(\cdot \mid y_{i,h})$ is an element of $\Delta(\Delta(A_h))$. By an abuse of notation, we use $v_{i,h}(g_i(\cdot \mid y_{i,h})) \in \Delta(\mathbb{R})$ to mean the distribution over contributions for play distributed according to $g_i(\cdot \mid y_{i,h})$. As a final bit of notation, when $F$ is a distribution on $\mathbb{R}$, $Q(F; q)$ is the $q$-quantile of $F$.

**Definition 16.** Let prior $g_i$ and quantile-choice function $q : \mathbb{N} \rightarrow [0, 1]$ be given for $i$. The *Bayes-UCB index for $s_i$ after history $y_i$ (relative to $g_i$ and $q$)* is

$$\sum_{h \in F_i[s_i]} Q(v_{i,h}(g_i(\cdot \mid y_{i,h}) \mid q(#(s_i \mid y_i)))$$

where $#(s_i \mid y_i)$ is the number of times $s_i$ has been used in history $y_i$.

In words, our Bayes-UCB index computes the $q$-th quantile of $v_{i,h}(a_h)$ under $i$’s belief about $-i$’s play on $h$, then sums these quantiles to return an index of the strategy $s_i$. The
Bayes-UCB policy UCBi prescribes choosing the strategy with the highest Bayes-UCB index after every history.

This procedure embodies a kind of wishful thinking for \( q \geq 0.5 \). The agent optimistically evaluates the payoff consequence of each \( s_i \) under the assessment that opponents will play a favorable response to \( s_i \) at each of the \( s_i \)-relevant information sets, where greater \( q \) corresponds to greater optimism in this evaluation procedure. Indeed, if \( q \) approaches 1 for every \( s_i \), the Bayes-UCB procedure approaches picking the strategy with the highest potential payoff.

If \( F_i[s_i] \) consists of only a single information set for every \( s_i \), then the procedure we define is the standard Bayes-UCB policy. In general, our procedure differs from the usual Bayes-UCB procedure, which would instead compute

\[
Q \left( \sum_{h \in F_i[s_i]} v_{i,h}(g_i(\cdot \mid y_{i,h})); q(\#(s_i \mid y_i)) \right).
\]

Instead, our procedure computes the sum of the quantiles, which is easier than computing the quantile of the sum, a calculation that requires taking the convolution of the associated distributions.

This variant of the Bayesian UCB is analogous to variants of the non-Bayesian UCB algorithm\(^{18}\) that separately computes an index for each basic arm and chooses the super arm maximizing sum of the basic arm indices – see e.g., Gai, Krishnamachari, and Jain (2012) and Chen, Wang, and Yuan (2013).\(^{19}\)

### 5.2.3 Index-Compatibility of Rational Learning and Bayes-UCB

**Theorem 2.** Suppose \( s_i^* \succeq s_j^* \) and the game is isomorphically factorable for \( i \) and \( j \) with \( \varphi(s_i^*) = s_j^* \). Consider two learning agents in the roles of \( i \) and \( j \) with equivalent independent regular priors.\(^{20}\) For any common survival chance \( 0 \leq \gamma < 1 \) and any social distribution \( \sigma \),

\(^{18}\)The non-Bayesian UCB index of a basic arm is an “optimistic” estimate of its mean reward that combines its empirical mean in the past with a term inversely proportional to the number of times the basic arm has been pulled.

\(^{19}\)Kveton, Wen, Ashkan, and Szepesvari (2015) have established tight \( O(\sqrt{n \log n}) \) regret bounds for this kind of algorithm across \( n \) periods.

\(^{20}\)The theorem easily generalizes to the case where \( i \) starts with one of \( L \geq 2 \) possible priors \( g_i^{(1)}, ..., g_i^{(L)} \) with probabilities \( p_1, ..., p_L \) and \( j \) starts with priors \( g_j^{(1)}, ..., g_j^{(L)} \) with the same probabilities, and each \( g_i^{(l)}, g_j^{(l)} \)
we have $\phi_i(s^*_i; r_i, \sigma_{-i}) \geq \phi_j(s^*_j; r_j, \sigma_{-j})$ under either of the following conditions:

- $r_i = \text{OPT}_i$, $r_j = \text{OPT}_j$, and $i$ and $j$ have the same patience $0 \leq \delta < 1$.
- The stage game is additively separable for $i$ and $j$; at every $h \in \mathcal{H}_{-ij}$ the auxiliary payoff functions $v_{i,h}, v_{j,h}$ rank $\alpha \in \Delta(A_h)$ in the same way; $r_i = \text{UCB}_i$, $r_j = \text{UCB}_j$; $i$ and $j$ have the same quantile-choice function $q_i = q_j$.

This result provides learning foundations for player-compatible trembles in a number of games, including the restaurant game from Section 5.1.1 and the link-formation game from Section 5.1.2, where the additive separability and same-ranking assumptions are satisfied for players ranked by compatibility.

## 6 Index Compatibility and Strong Player Compatibility with Rigid Feedback

### 6.1 Rigid Feedback Structures and Strong Player Compatibility

We will show that with rigid feedback structures, a stronger notion of player compatibility implies index compatibility for learners who use the following index rules: (1) the Gittins index as a heuristic, even in settings where it isn’t fully optimal, and (2) weighted fictitious play, where the agent expects opponents to play according to the weighted empirical frequency of past actions, with more weight on recent observations.

**Definition 17.** $(\mathcal{O}_i, o_i)$ is rigid if there exist $F_i[s_i] \subseteq \mathcal{H}_{-i}$ for each $i, s_i \in S_i$, such that $\mathcal{O}_i = \bigcup_{s_i \in S_i} \times_{h \in F_i[s_i]} A_h$, $o_i(s_i, s_{-i}) = (s_{-i}(h))_{h \in F_i[s_i]}$, and $u_i(s_i, s'_i) = u_i(s_i, s''_i)$ whenever $o_i(s_i, s'_i) = o_i(s_i, s''_i)$.

Like factorable games, rigid feedback structures consistently provide $i$ the same kind of information from a given strategy $s_i$ independent of others’ play. But these structures do not require that different strategies provide information about play on disjoint opponent information sets ($F_i[s_i] \cap F_i[s'_i] = \emptyset$) or that different $-i$ action profiles on $F_i[s_i]$ lead to

is a pair of equivalent regular priors for $1 \leq l \leq L$. 

different payoffs for $i$. Starting with a factorable game with $s_i$-relevant information sets $(F_i[s_i])_{s_i \in S_i}$, any collection $(\tilde{F}_i[s_i])_{s_i \in S_i}$ with $\tilde{F}_i[s_i] \supseteq F_i[s_i]$ for each $s_i$ leads to a rigid feedback structure with $\mathcal{O}_i = \bigcup_{s_i \in S_i} \times_{h \in F_i[s_i]} A_h$. In general there can be multiple rigid feedback structures for the same extensive-form game, as we show in Example 3. In particular, letting $F_i[s_i] = \mathcal{H}_{-i}$ and having $i$ perfectly observe others’ extensive-form strategies always leads to a rigid feedback structure.

If $i$ and $j$ have rigid feedback structures, then given any isomorphism $\varphi : S_i \rightarrow S_j$, $i$ and $j$ face isomorphic learning problems under the following notion of equivalence: for $s_j = \varphi(s_i)$, the union of their histories $\big((s_i) \times \mathcal{O}_i\big) \cup \big((s_j) \times \mathcal{O}_j\big)$ is partitioned with $(s_i, \times_{h \in F_{-i}[s_i]} a_h)$ and $(s_j, \times_{h \in F_{-j}[s_j]} h')$ equivalent if and only if $a_h = a'_h$ for all $h \in F_i[s_i] \cap F_j[s_j]$. So observations of $i$ and $j$ are equivalent if and only if they correspond to the same play by $-ij$ on the common information sets.

**Definition 18.** Given an isomorphism $\varphi : S_i \rightarrow S_j$ and collections of $-i, -j$ information sets $F_i = (F_i[s_i])_{s_i \in S_i}$, $F_j = (F_j[s_j])_{s_j \in S_j}$, $i$ is strongly more player-compatible with $s_i^* \in S_i$ than $j$ is with $s_j^* \in S_j$ (with respect to $\varphi, F_i, F_j$) if for any pair of lists of strategy profiles $(\tilde{\sigma}[s_i])_{s_i \in S_i}$, $(\sigma[s_j])_{s_j \in S_j}$ with $\tilde{\sigma}[s_i], \sigma[s_j] \in \Delta^o(S)$ such that:

- $\tilde{\sigma}[s_i]_{F_i[s_i] \cap F_j[\varphi(s_i)]} = \sigma[\varphi(s_i)]_{F_i[s_i] \cap F_j[\varphi(s_i)]}$ for all $s_i \in S_i$, and

$$\sum_{s_j \in S_j} u_j(s_j^*, s_{-j}) \cdot [\tilde{\sigma}[s_j]](s) = \max_{s_j' \in S_j} \sum_{s_j' \in S_j} u_j(s_{j}', s_{-j}) \cdot [\sigma[s_j']](s),$$

We have $\sum_{s_i \in S_i} u_i(s_i^*, s_{-i}) \cdot \tilde{\sigma}[s_i^*](s) > \max_{s_i'' \in S_i \setminus \{s_i^*\}} \sum_{s_i'' \in S_i} u_i(s_{i''}, s_{-i}) \cdot \tilde{\sigma}[s_i^*](s)$.

Player compatibility evaluates strategies based on pairs of strategy profiles of $-i$ and $-j$ that match on their $-ij$ marginals. Strong player compatibility supposes that $i$ first chooses a strategy, then their opponents pick a strategy profile conditional on $i$’s choice, and similarly for $j$. It then requires that whenever $j$ finds it weakly optimal to play $s_j^*$, $i$ finds it strictly optimal to play $s_i^*$, provided the way that $i$’s opponents play conditional on $s_i$ matches the way that $j$’s opponents play conditional on the isomorphic strategy $\varphi(s_i)$ on the $-ij$ information sets $F_i[s_i] \cap F_j[\varphi(s_i)]$.

To illustrate the difference between strong player compatibility and player compatibility, consider a game where players 1, 2, and 3 simultaneously choose $L$ or $R$. Player 3 always
gets 0. Player 1 and 2’s payoffs only depend on their own action and player 3’s action, as show in the table below.

<table>
<thead>
<tr>
<th>P3’s action</th>
<th>P1’s action</th>
<th>P2’s action</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>R</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
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<table>
<thead>
<tr>
<th>P3’s action</th>
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<tbody>
<tr>
<td>L</td>
</tr>
<tr>
<td>R</td>
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</tbody>
</table>

P1 is more player-compatible with L than P2 is with L. If P2 weakly prefers L, it means \( \sigma_3(L) \geq 0.5 \), in which case P1 must get strictly more from L than from R. But now consider the isomorphism mapping \( L, R \) of players 1 and 2 to each other, and the two strategies of P3 where \( \sigma_3^{[L]}(L) = 1/2 \), \( \sigma_3^{[R]}(L) = 0 \). This makes P2 weakly prefer L, but P1 strictly prefers R. So P1 is not strongly more player-compatible with L than P2 is with L.

**Example 3.** Consider the restaurant game, which is factorable as seen in Section 5.1.1. The same information sets \( F_c[^c] \), \( F_d[^d] \) also give rigid feedback structures for the critic and the diner. The critic is strongly more player compatible with \( R_c \) than the diner is with \( R_d \). Consider two lists of correlated profiles, \((\tilde{\sigma}^{[R_c]}, \tilde{\sigma}^{[Z_c]}) \) and \((\sigma^{[R_d]}, \sigma^{[Z_d]}) \), so that the diner prefers playing \( R_d \) against \( \sigma^{[R_d]} \) over \( Z_d \) against \( \sigma^{[Z_d]} \). Since the payoff to \( Z_d \) is always 0, the diner gets utility at least 0 from \( (R_d, \sigma^{[Z_d]}) \). But this means whenever \( \tilde{\sigma}^{[R_c]} \) matches \( \sigma^{[R_d]} \) on \( F_c[R_c] \cap F_d[R_d] \), i.e. the restaurant’s information set, the critic gets utility at least 0.5 from \( (R_c, \tilde{\sigma}^{[Z_c]}) \), so must strictly prefer this over playing \( Z_c \) against \( \tilde{\sigma}^{[Z_c]} \).

Also, note that if \( F_c[Z_c] \) and \( F_d[Z_d] \) were expanded to include all opponent information sets, then the feedback structure would remain rigid and the strong player compatibility relation continues to hold.

### 6.2 The Heuristic Gittins Index

The Gittins index is the optimal policy in multi-armed bandit problems with independent arms. We consider agents who apply this index as a heuristic in the steady-state learning setting of Section 4 in a stage game with rigid feedback, treating different extensive-form strategies as different bandit arms. This index is optimal when the game is factorable for
i, but may not be optimal in all rigid feedback structures, since different strategies of the agent may give correlated information about the social distribution of play.

The agents’ learning rules depend on their perception of the learning environment. To model this, we assume that agent $i$ believes that a fixed mixed action is played at each opponent information set $h \in F_i[s_i]$ after each $s_i$. Agents are uncertain about what these actions are, and have prior beliefs about them that are independent over $h$ and over $s_i$. Thus even if two strategies of $i$ lead to the same information set of an opponent, $i$ believes that observing play on this information set after choosing one strategy is uninformative about how the same opponent would have played if $i$ had chosen a different strategy.

All newcomer agents draw their prior beliefs from a common collection $(g^h)_{h \in H}$, with each $g^h : \Delta(A_h) \rightarrow \mathbb{R}_+$ a density function. This means that if $i$ and $j$’s strategies lead to the same information set of $k \neq i, j$, then $i$ and $j$ share the same prior belief about $k$’s play there. Agents maximize the expected sum of payoffs, discounting future payoffs by $\delta \gamma$, where $0 \leq \delta < 1$ is the common patience parameter for all agents.

This perceived environment leads each agent to act as if they are facing a multi-armed bandit problem, where the reward after each “arm” $s_i$ is generated by a fixed mixed play $\alpha_{-i} \in \times_{h \in F_i[s_i]} \Delta(A_h)$, with a prior distribution $\times_{h \in F_i[s_i]} g^h$. We denote the Gittins index rules in this bandit problem (with respect to the priors defined by $(g^h)$ and parameters $0 \leq \delta, \gamma < 1$) as $HG_i$ for “Heuristic Gittins,” since it is not in general the optimal policy for $i$ in the actual learning setting of Section 4.

The next result says the index-strategy pairs $(HG_i, s^*_i)$, $(HG_j, s^*_j)$ satisfy index-compatibility provided $i, j$ satisfy strong player compatibility with respect to $s^*_i, s^*_j$.

**Proposition 7.** Suppose the game has rigid feedback for both $i$ and $j$. If $i$ is strongly more player-compatible with $s^*_i \in S_i$ than $j$ is with $s^*_j \in S_j$ (with respect to $\varphi, F_i, F_j$), then $HG_i$ is more index-compatible with $s^*_i$ than $HG_j$ is with $s^*_j$ under the observation and feedback structures constructed from $F_i, F_j$.

By combining Propositions 5 and 7, we see that strong player compatibility implies $\phi_i(s^*_i; HG_i, \sigma_{-i}) \geq \phi_j(s^*_j; HG_j, \sigma_{-j})$ for any social distribution $\sigma$. 

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6.3 Weighted Fictitious Play

Next we consider the weighted fictitious play heuristic, a generalization of Brown (1951)’s fictitious play.\footnote{This heuristic was first estimated on lab data by Cheung and Friedman (1997). It was generalized by Camerer and Ho (1999) and later analyzed by Benaim, Hofbauer, and Hopkins (2009).}

We consider a variant of the weighted fictitious play heuristic where $i$ separately keeps track of opponents’ play at different information sets $F_i[s_i]$ and after different strategies $s_i$. There is a collection of \textit{initial counts},

$$\{N^a_h(0) \in \mathbb{R}_{++} : h \in \mathcal{H}, a_h \in A_h\}.$$ 

After history $y_i$ of $i$ where $s_i$ has been used $T \geq 0$ times, $i$’s subhistory for $s_i$ is $y_{i,s_i} = (s_i, s_{-i}^{(t)}(h)_{h \in F_{-i}[s_i]})_{t=1}^T$ where $s_{-i}^{(t)}(h)_{h \in F_{-i}[s_i]}$ is the observed $-i$’s play on $F_i[s_i]$ the $t$-th time that $s_i$ was used. The updated count on $(h, a_h)$ for $h \in F_i[s_i]$ and $a_h \in A_h$ is

$$N^a_h(y_i) = \sum_{t=1}^T 1(s_{-i}^{(t)}(h) = a_h) \cdot \rho^{T-t} + \rho^T N^a_h(0)$$

for some $\rho \in [0, 1]$. That is, $i$ discounts past observations on $F_i[s_i]$ at a rate $\rho$ between periods of playing $s_i$. Following history $y_i$, $i$ assigns an index to $s_i$ equal to its expected payoff when opponents play the mixed action $\alpha_h(a_h; y_i) = \frac{N^a_h(y_i)}{\sum_{a_h' \in A_h} N^{a_h'}_h(y_i)}$ on information sets $h \in F_i[s_i]$.\footnote{Cooper and Kagel (2008) estimated a similar model of weighted fictitious play in a signaling-game experiment.}

This formulation differs from weighted fictitious play in normal-form games in that we suppose $i$ separately keeps track of counts on information sets after different $s_i$, even if two $s_i$ lead to observing play on the same $-i$ information set. Also, note that when $\rho = 1$, weights are updated according to the usual fictitious play, and $\rho = 0$ corresponds to myopically best replying to the observed play when each strategy was most recently used. The special case of the heuristic Gittins index where each $g^h$ is a Dirichlet prior and $\delta = 0$ is equivalent to the special case of unweighted fictitious play (i.e. $\rho = 1$). But in general $HG_i$ differs from $WFP_i$ except in these corner cases.

We denote the policies of weighted fictitious play learners by $WFP_i$. The next result
says the index-strategy pairs \((WFP_i, s_i^*)\) and \((WFP_j, s_j^*)\) satisfy index-compatibility when \(i\) and \(j\) satisfy strong player compatibility with respect to \(s_i^*, s_j^*\).

**Proposition 8.** Suppose the game has rigid feedback for both \(i\) and \(j\). If \(i\) is strongly more player-compatible with \(s_i^* \in S_i\) than \(j\) is with \(s_j^* \in S_j\) (with respect to \(\varphi, F_i, F_j\)), then \(WFP_i\) is more index-compatible with \(s_i^*\) than \(WFP_j\) is with \(s_j^*\) under the observation and feedback structures constructed from \(F_i, F_j\).

If \(i\) and \(j\) have equivalent subhistories after \(s_i\) and \(\varphi(s_i)\), then they have observed the same sequence of \(-ij\)'s play on the information sets \(F_i[s_i] \cap F_j[\varphi(s_i)]\), and thus have the same time-weighted counts about opponents’ play on these information sets. The definition of strong player compatibility allows us to compare \(i\) and \(j\)'s preferences over strategies given two lists of opponent strategy profiles that agree on \(F_i[s_i] \cap F_j[\varphi(s_i)]\), and hence establish index-compatibility.

By combining Propositions 5 and 8, we see that strong player compatibility implies \(\phi_i(s_i^*; WFP_i, \sigma_{-i}) \geq \phi_j(s_j^*; WFP_j, \sigma_{-j})\) for any social distribution \(\sigma\).

### 7 Replication Invariance of PCE

Fix a base game with game tree \(\Gamma\), observations \((O_i)_{i \in I}\), and feedback functions \((o_i)_{i \in I}\). The finite set of strategies of \(i\) is denoted \(S_i\) and \(i\)'s utility function is \(u_i : S \rightarrow \mathbb{R}\). An extended game with duplicates is a game tree \(\tilde{\Gamma}\) with the same players and same observations \((O_i)_{i \in I}\) as the base game, where the set of extensive-form strategies of \(i\), \(\tilde{S}_i\), can be viewed as a finite subset of \(S_i \times \mathbb{N}\), such that for all \((s_i, n_i) \in \tilde{S}_i\) and \((s_j, n_j)_{j \neq i} \in \tilde{S}_{-i}\),

- the payoff in the new game is \(\tilde{u}_i((s_i, n_i), (s_j, n_j)_{j \neq i}) = u_i(s_i, s_j)\);
- the feedback in the new game is \(\tilde{o}_i((s_i, n_i), (s_j, n_j)_{j \neq i}) = o_i(s_i, s_j)\).

We require that for every \(s_i \in S_i\), there exists some \(n_i \in \mathbb{N}\) so that \((s_i, n_i) \in \tilde{S}_i\). The interpretation is that \(i\) has a non-zero but finite number of copies of every strategy they had in the base game, and could have different numbers of copies of different strategies. These duplicate strategies lead to the same payoffs and give the same information about others’
play. As an example, suppose that in the Restaurant Game the diner can arrive at the restaurant by taking the red bus or the blue bus, and the color of the bus is not observed by other players, does not change anyone’s payoffs, and does not change what the diner observes. We can then replace $R^d$ with two actions $R^d_{\text{red}}, R^d_{\text{blue}}$ at every node in the diner’s information set, letting $R^d_{\text{red}}$ and $R^d_{\text{blue}}$ both have the same payoff consequences as $R^d$ in the original game. When agents learn from their own payoffs (i.e., $\mathcal{O}_i \subseteq \mathbb{R}$ is the set of possible payoffs of $i$), this modified game is an extended game with duplicates for the original game.

A tremble profile of the extended game $\bar{\epsilon}$ assigns a positive number $\bar{\epsilon}((s_i, n_i) | i) > 0$ to every player $i$ and every pure strategy $(s_i, n_i) \in \bar{S}_i$. We define $\bar{\epsilon}$-strategies of $i$ and $\bar{\epsilon}$-equilibrium of the extended game in the usual way, relative to the strategy sets $\bar{S}_i$.

**Definition 19.** Tremble profile $\bar{\epsilon}$ is player-compatible in the extended game if
\[
\sum_{n_i} \bar{\epsilon}((s_i^*, n_i) | i) \geq \sum_{n_j} \bar{\epsilon}((s_j^*, n_j) | j)
\]
for all $i, j, s_i^*, s_j^*$ such that $s_i^* \succeq s_j^*$. An $\bar{\epsilon}$-equilibrium where $\bar{\epsilon}$ is player-compatible is called a player-compatible $\bar{\epsilon}$-equilibrium (or $\bar{\epsilon}$-PCE).

While we do not formally pursue the microfoundation of player-compatible trembles in the extended game, note that a rational $i$ facing a stationary distribution of opponents’ play is exactly indifferent between all duplicate copies of the same base-game strategy $s_i$ after all histories. Holding fixed their initial belief and the distribution of opponents’ play, $i$’s weighted lifetime average play of $s_i$ in the base game equals the sum of their weighted lifetime average plays of all duplicate copies of $s_i$ in the extended game. In the example above, the learning foundation that says the critic experiments more with the restaurant than the diner in the base game also implies that the critic plays $R^c$ more often than the sum of frequencies of the diner playing $R^d_{\text{red}}$ and $R^d_{\text{blue}}$ in the extended game.

We now relate $\bar{\epsilon}$-equilibria in the extended game to $\epsilon$-equilibria in the base game. Recall the following constrained optimality condition that applies to both the extended game and the base game:

**Fact 1.** A feasible mixed strategy of $i$ is **not** a constrained best response to a $-i$ profile if and only if it assigns more than the required weight to a non-optimal response.

We associate with a strategy profile $\bar{\sigma} \in \times_{i \in I} \Delta(\bar{S}_i)$ in the extended game a consolidated strategy profile $\mathcal{C}(\bar{\sigma}) \in \times_{i \in I} \Delta(S_i)$ in the base game, given by adding up the probabilities
assigned to all copies of each base-game strategy. More precisely, \( C(\sigma)_i(s_i) := \sum_{n_i} \sigma_i(s_i, n_i) \). Similarly, \( C(\bar{\epsilon}) \) is the consolidated tremble profile, given by \( C(\bar{\epsilon})(s_i | i) := \sum_{n_i} \bar{\epsilon}(s_i, n_i | i) \).

Conversely, given a strategy profile \( \sigma \in \times_{i \in \mathbb{I}} \Delta(S_i) \) in the base game, the extended strategy profile \( \varepsilon(\sigma) \in \times_{i \in \mathbb{I}} \Delta(\bar{S}_i) \) is defined by \( \varepsilon(\sigma)_i(s_i, n_i) := \sigma_i(s_i) / N(s_i) \) for each \( (s_i, n_i) \in \bar{S}_i \), where \( N(s_i) \) is the number of copies of \( s_i \) that \( \bar{S}_i \) contains. Similarly, \( \varepsilon(\epsilon) \) is the extended tremble profile, given by \( \varepsilon(\epsilon)((s_i, n_i) | i) := \epsilon(s_i | i) / N(s_i) \).

**Lemma 3.** If \( \bar{\sigma} \) is an \( \bar{\epsilon} \)-equilibrium in the extended game, then \( C(\bar{\sigma}) \) is an \( C(\bar{\epsilon}) \)-equilibrium in the base game. If \( \sigma \) is an \( \epsilon \)-equilibrium in the base game, then \( \varepsilon(\sigma) \) is an \( \varepsilon(\epsilon) \)-equilibrium in the extended game.

The proof of results in this section can be found in the Online Appendix.

PCE is defined as usual in the extended game.

**Definition 20.** A strategy profile \( \bar{\sigma}^* \) is a player-compatible equilibrium (PCE) in the extended game if there exists a sequence of player-compatible tremble profiles \( \bar{\epsilon}^{(t)} \to 0 \) and an associated sequence of strategy profiles \( \bar{\sigma}^{(t)} \), where each \( \bar{\sigma}^{(t)} \) is an \( \bar{\epsilon}^{(t)} \)-PCE, such that \( \bar{\sigma}^{(t)} \to \bar{\sigma}^* \).

These PCE correspond exactly to PCE of the base game.

**Proposition 9.** If \( \bar{\sigma}^* \) is a PCE in the extended game, then \( C(\bar{\sigma}^*) \) is a PCE in the base game. If \( \sigma^* \) is a PCE in the base game, then \( \varepsilon(\sigma^*) \) is a PCE in the extended game.

In fact, starting from a PCE \( \sigma^* \) of the base game, we can construct more PCE of the extended game than \( \varepsilon(\sigma^*) \) by shifting around the probabilities assigned to different copies of the same base-game strategy, but all these profiles essentially correspond to the same outcome.

### 8 Concluding Discussion

PCE makes two key contributions. First, it generates new and sensible restrictions on equilibrium play by imposing cross-player restrictions on the relative probabilities that different
players assign to certain strategies — namely, those strategy pairs $s_i, s_j$ ranked by the compatibility relation $s_i \succeq s_j$. As we have shown through examples, these cross-player restrictions distinguish PCE from other refinement concepts and allows us to make comparative statics predictions in some games where other equilibrium refinements do not.

Second, PCE shows how restricted “trembles” can capture some of the implications of non-equilibrium learning. PCE’s cross-player restrictions arise endogenously for a general class of index learning rules, which includes both the standard model of Bayesian agents maximizing their expected discounted lifetime utility, and computationally tractable heuristics like Bayesian upper confidence bounds and weighted fictitious play. We conjecture that the result that $i$ is more likely to experiment with $s_i$ than $j$ is with $s_j$ when $s_i \succeq s_j$ applies in other natural models of learning or dynamic adjustment, such as those considered by Francetich and Kreps (2018), and that it may be possible to provide foundations for PCE in other and perhaps larger classes of games.

The strength of the PCE refinement depends on the completeness of the compatibility order $\succeq$, since $\epsilon$-PCE imposes restrictions on $i$ and $j$’s play only when the relation $s_i \succeq s_j$ holds. Our player compatibility definition supposes that player $i$ thinks all mixed strategies of other players are possible, as it considers the set of all totally mixed correlated strategies $\sigma_{-i} \in \Delta^\circ(S_{-i})$. If the players have some prior knowledge about their opponents’ utility functions, player $i$ might deduce a priori that the other players will only play strategies in some subset $\mathcal{A}_{-i}$ of $\Delta^\circ(S_{-i})$. As we show in Fudenberg and He (2020), in signaling games imposing this kind of prior knowledge leads to a more complete version of the compatibility order. It may similarly lead to a more refined version of PCE.

PCE is defined for general strategic forms. We have only provided learning foundations for player-compatible trembles in factorable games, but we view this as an improvement over the more typical situation in which refinements have no learning foundations at all.

References

Battigalli, P., S. Cerreia-Vioglio, F. Maccheroni, and M. Marinacci (2016): “Analysis of information feedback and selfconfirming equilibrium,” Journal of Mathemat-


Appendix

9 Proofs of Results Stated in the Main Text

9.1 Proof of Proposition 3

We first state an auxiliary lemma.

Lemma 4. If \( \sigma^0 \) is an \( \epsilon \)-PCE and \( s_i^* \succ s_j^* \), then

\[
\sigma_i^0(s_i^*) \geq \min \left[ \sigma_j^0(s_j^*), 1 - \sum_{s_i' \neq s_i^*} \epsilon(s_i'|i) \right].
\]

Proof. Suppose \( \epsilon \) is player-compatible and let \( \epsilon \)-equilibrium \( \sigma^0 \) be given. For \( s_i^* \succ s_j^* \), suppose \( \sigma_j^0(s_j^*) = \epsilon(s_j^* | j) \). Then \( \sigma_i^0(s_i^*) \geq \epsilon(s_i^* | i) \geq \epsilon(s_j^* | j) = \sigma_j^0(s_j^*) \), where the second inequality comes from \( \epsilon \) being player-compatible. On the other hand, suppose \( \sigma_j^0(s_j^*) > \epsilon(s_j^* | j) \). Since \( \sigma^0 \) is an \( \epsilon \)-equilibrium, the fact that \( j \) puts more than the minimum required weight on \( s_j^* \) implies \( s_j^* \) is at least a weak best response for \( j \) against \( \sigma^0 \), with \( \sigma^0 \) totally mixed due to the trembles. The definition of \( s_i^* \succ s_j^* \) then implies that \( s_i^* \) must be a strict best response for \( i \) against \( \sigma^0 \) as well. In the \( \epsilon \)-equilibrium, \( i \) must assign as much weight to \( s_i^* \) as possible, so that \( \sigma_i^0(s_i^*) = 1 - \sum_{s_i' \neq s_i^*} \epsilon(s_i'|i) \). Combining these two cases establishes the desired result. \( \square \)

Proposition 3: For any PCE \( \sigma^* \), player \( k \), and strategy \( \bar{s}_k \) such that \( \sigma_k^*(\bar{s}_k) > 0 \), there exists a sequence of totally mixed strategy profiles \( \sigma_{-k}^{(t)} \to \sigma_{-k}^* \) such that

(i) for every pair \( i, j \neq k \) with \( s_i^* \succ s_j^* \),

\[
\liminf_{t \to \infty} \frac{\sigma_i^{(t)}(s_i^*)}{\sigma_j^{(t)}(s_j^*)} \geq 1;
\]

and (ii) \( \bar{s}_k \) is a best response for \( k \) against every \( \sigma_{-k}^{(t)} \).
Proof. By Lemma 4, for every $e^{(t)}$-PCE we get

$$\frac{\sigma_i^{(t)}(s_i^*)}{\sigma_j^{(t)}(s_j^*)} \geq \min \left[ \frac{\sigma_j^{(t)}(s_j^*)}{\sigma_j^{(t)}(s_j^*)}, \frac{1 - \sum_{s_i \neq s_i^*} e^{(t)}(s_i'|i)}{\sigma_j^{(t)}(s_j^*)} \right]$$

$$= \min \left[ 1, \frac{1 - \sum_{s_i \neq s_i^*} e^{(t)}(s_i'|i)}{\sigma_j^{(t)}(s_j^*)} \right] \geq 1 - \sum_{s_i \neq s_i^*} e^{(t)}(s_i'|i).$$

This says

$$\inf_{t \geq T} \frac{\sigma_i^{(t)}(s_i^*)}{\sigma_j^{(t)}(s_j^*)} \geq 1 - \sup_{t \geq T} \sum_{s_i \neq s_i^*} e^{(t)}(s_i'|i).$$

For any sequence of trembles such that $e^{(t)} \to 0$,

$$\lim_{T \to \infty} \sup_{t \geq T} \sum_{s_i \neq s_i^*} e^{(t)}(s_i'|i) = 0,$$

so

$$\lim_{T \to \infty} \liminf_{t \to \infty} \frac{\sigma_i^{(t)}(s_i^*)}{\sigma_j^{(t)}(s_j^*)} = \lim_{T \to \infty} \left\{ \inf_{t \geq T} \frac{\sigma_i^{(t)}(s_i^*)}{\sigma_j^{(t)}(s_j^*)} \right\} \geq 1.$$

This shows that if we fix a PCE $\sigma^*$ and consider a sequence of player-compatible trembles $e^{(t)}$ and $e^{(t)}$-PCE $\sigma^{(t)} \to \sigma^*$, then each $\sigma_{-k}^{(t)}$ satisfies $\lim_{t \to \infty} \sigma_i^{(t)}(s_i^*)/\sigma_j^{(t)}(s_j^*) \geq 1$ whenever $i, j \neq k$ and $s_i^* \succeq s_j^*$. Furthermore, from $\sigma_k^*(\bar{s}_k) > 0$ and $\sigma_k^{(t)} \to \sigma_k^*$, we know there is some $T_1 \in \mathbb{N}$ so that $\sigma_k^{(t)}(\bar{s}_k) > \sigma_k^*(\bar{s}_k)/2$ for all $t \geq T_1$. We may also find $T_2 \in \mathbb{N}$ so that $e^{(t)}(\bar{s}_k|k) < \sigma_k^*(\bar{s}_k)/2$ for all $t \geq T_2$, since $e^{(t)} \to 0$. So when $t \geq \max(T_1, T_2)$, $\sigma_k^{(t)}$ places strictly more than the required weight on $\bar{s}_k$, so $\bar{s}_k$ is at least a weak best response for $k$ against $\sigma_{-k}^{(t)}$. Now the subsequence of opponent play $(\sigma_{-k}^{(t)})_{t \geq \max(T_1, T_2)}$ satisfies the requirement of this proposition.

\[ \square \]

9.2 Proof of Theorem 1

Theorem 1: PCE exists in every finite strategic-form game.

Proof. Consider a sequence of tremble profiles with the same lower bound on the probability of each strategy, that is $e^{(t)}(s_i|i) = e^{(t)}$ for all $i$ and $s_i$, and with $e^{(t)}$ decreasing monotonically to 0 in $t$. Each of these tremble profiles is player-compatible (regardless of the compatibility
structure \( \succcurlyeq \) and there is some finite \( T \) large enough that \( t \geq T \) implies an \( \text{e}^{(t)} \)-equilibrium exists, and some subsequence of these \( \text{e}^{(t)} \)-equilibria converges since the space of strategy profiles is compact. By definition these \( \text{e}^{(t)} \)-equilibria are also \( \text{e}^{(t)} \)-PCE, which establishes existence of PCE.

\[ \square \]

### 9.3 Proof of Proposition 4

**Proposition 4:** In a signaling game, every PCE \( \sigma^* \) is a Nash equilibrium satisfying the compatibility criterion, as defined in Fudenberg and He (2018).

**Proof.** Since every PCE is a trembling-hand perfect equilibrium and since this latter solution concept refines Nash, \( \sigma^* \) is a Nash equilibrium.

To show that it satisfies the compatibility criterion, we need to show that \( \sigma^*_2 \) assigns probability 0 to plans in \( A^S \) that, for some \( s \in S \), do not best respond to the admissible beliefs at signal \( s \) under profile \( \sigma^* \), as defined in Fudenberg and He (2018) Definition 11. For any plan assigned positive probability under \( \sigma^*_2 \), by Proposition 3 we may find a sequence of totally mixed signal profiles \( \sigma^{(t)}_1 \) of the sender, so that whenever \( s_\theta \succcurlyeq s_{\theta'} \) we have \( \lim \inf_{t \to \infty} \sigma^{(t)}_1(s \mid \theta)/\sigma^{(t)}_1(s \mid \theta') \geq 1 \). Write \( q^{(t)}(\cdot \mid s) \) as the Bayesian posterior belief about sender’s type after signal \( s \) under \( \sigma^{(t)}_1 \), which is well defined because each \( \sigma^{(t)}_1 \) is totally mixed. Whenever \( s_\theta \succcurlyeq s_{\theta'} \), this sequence of posterior beliefs satisfies \( \lim \inf_{t \to \infty} q^{(t)}(\theta \mid s)/q^{(t)}(\theta' \mid s) \geq \lambda(\theta)/\lambda(\theta') \), so if the receiver’s plan best responds to every element in the sequence, it also best responds to an accumulation point \( (q^{\infty}(\cdot \mid s))_{s \in S} \) with \( q^{\infty}(\theta \mid s)/q^{\infty}(\theta' \mid s) \geq \lambda(\theta)/\lambda(\theta') \) whenever \( s_\theta \succcurlyeq s_{\theta'} \). Since the player compatibility definition used in this paper is slightly easier to satisfy than the type compatibility definition that the set \( P(s', \sigma^*) \) is based on, the plan best responds to \( P(s', \sigma^*) \) after every signal \( s' \). \[ \square \]

### 9.4 Proof of Proposition 5

Let \( N = \max_i |\mathbb{S}_i| \). We first show that \( i \)'s induced response against i.i.d. play drawn from \( \sigma_{-i} \) is the same as playing against a response path drawn from some distribution \( \eta \) at the start of \( i \)'s life. This \( \eta \) is the same for all agents and does not depend on their (possibly stochastic) learning rules.

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Lemma 5. In a factorable game, for each \( \sigma \in \times_k \Delta(S_k) \), there is a distribution \( \eta \) over response paths, so that for any player \( i \), any possibly random rule \( r_i : Y_i \to \Delta(S_i) \), and any strategy \( s_i \in S_i \), we have

\[
\phi_i(s_i; r_i, \sigma) = (1 - \gamma) \mathbb{E}_{\mathcal{S} \sim \eta} \left[ \sum_{t=1}^{\infty} \gamma^{t-1} \cdot (y_t^i(\mathcal{S}, r_i) = s_i) \right],
\]

where \( y_t^i(\mathcal{S}, r_i) \) refers to the \( t \)-th period history in \( y_t^i(\mathcal{S}, r_i) \).

Proof. In fact, we will prove a stronger statement: we will show there is such a distribution that induces the same distribution over period-\( t \) histories for every \( i \), every learning rule \( r_i \), and every \( t \).

Think of each response path \( \mathcal{S} \) as a two-dimensional array, \( \mathcal{S} = (\mathcal{S}_{t,n})_{t \in \mathbb{N}, 1 \leq n \leq N} \). For non-negative integers \( (m_n)_{n=1}^N \), each finite two-dimensional array of strategy profiles \( ((s_{t,n})_{t=1}^{m_n})_{n=1}^N \) with each \( s_{t,n} \in S \) defines a “cylinder set” of response paths with the form:

\[
\{ \mathcal{S} : \mathcal{S}_{t,n} = s_{t,n} \text{ for each } 1 \leq n \leq N, 1 \leq t \leq m_n \}. \]

That is, the cylinder set consists of those response paths whose first \( m_n \) elements for the \( n \)-th strategy match a given sequence of strategy profiles, \( (s_{t,n})_{t=1}^{m_n} \). (If \( m_n = 0 \), then there is no restriction on \( \mathcal{S}_{t,n} \) for any \( t \).) We specify the distribution \( \eta \) by specifying the probability it assigns to these cylinder sets:

\[
\eta \left\{ ((s_{t,n})_{t=1}^{m_n})_{n=1}^N \right\} = \prod_{n=1}^N \prod_{t=1}^{m_n} \sigma(s_{t,n}),
\]

where we have abused notation to write \( ((s_{t,n})_{t=1}^{m_n})_{n=1}^N \) for the cylinder set satisfying this profile of sequences, and we have used the convention that the empty product is defined to be 1.

We establish the claim by induction on \( t \) for period-\( t \) history. For \( t \geq 0 \), let \( Y_i[t] \subseteq Y_i \) be the set of possible period-\( t \) histories of \( i \), that is \( Y_i[t] := (S_i \times O_i)^t \). In the base case of \( t = 1 \), we show playing against a response path drawn according to \( \eta \) and playing against a pure strategy\(^{23}\) drawn from \( \sigma_{-i} \in \times_k \Delta(S_k) \) generate the same period-1 history. Fixing a learning

\(^{23}\)In the random matching model agents are facing a randomly drawn pure strategy profile each period.
rule \( r_i : Y_i \to S_i \) of \( i \), the probability of \( i \) having the period-1 history \((s_i^{(1)}, o^{(1)}) \in Y_i[1]\) in the random-matching model is \( 1(r_i(\emptyset) = s_i^{(1)}) \cdot \sigma(s : o_i(Z(s_i^{(1)}, s_{-i})) = o^{(1)}) \). That is, \( i \)'s rule must play \( s_i^{(1)} \) in the first period of \( i \)'s life. Then, \( i \) must encounter such a pure strategy that generates the required observation \( o^{(1)} \), and this has probability \( \sigma(s : o_i(Z(s_i^{(1)}, s_{-i})) = o^{(1)}) \).

The probability of this happening against a response path drawn from \( \eta \) is

\[
1(r_i(\emptyset) = s_i^{(1)}) \cdot \eta(\mathcal{S}:o_i(Z(s_i^{(1)}, s_{1,s_i^{(1)},-i})) = o^{(1)}) = 1(r_i(\emptyset) = s_i^{(1)}) \cdot \sigma(s : o_i(Z(s_i^{(1)}, s_{-i})) = o^{(1)}),
\]

where the second line comes from the probability \( \eta \) assigns to cylinder sets.

We now proceed with the inductive step. By induction, suppose random matching and the \( \eta \)-distributed response path induce the same distribution over the set of period-\( T \) histories, \( Y_i[T], \) where \( T \geq 1 \). Write this common distribution as \( \phi_{i,T}^{RM} = \phi_{i,T}^\eta = \phi_{i,T} \in \Delta(Y_i[T]) \). We prove that they also generate the same distribution over length \( T + 1 \) histories.

Suppose random matching generates distribution \( \phi_{i,T+1}^{RM} \in \Delta(Y_i[T + 1]) \) and the \( \eta \)-distributed response path generates distribution \( \phi_{i,T+1}^\eta \in \Delta(Y_i[T + 1]) \). Each length-\( T + 1 \) history \( y_{i}[T + 1] \in Y_i[T + 1] \) may be written as \((y_i[T], (s_i^{(T+1)}, o^{(T+1)}))\), where \( y_i[T] \) is a length-\( T \) history and \((s_i^{(T+1)}, o^{(T+1)})\) is a one-period history corresponding to what happens in period \( T + 1 \). Therefore, we may write for each \( y_i[T + 1] \),

\[
\phi_{i,T+1}^{RM}(y_i[T + 1]) = \phi_{i,T}^{RM}(y_i[T]) \cdot \phi_{i,T+1|T}^{RM}(s_i^{(T+1)}, o^{(T+1)}|y_i[T]),
\]

and

\[
\phi_{i,T+1}^\eta(y_i[T + 1]) = \phi_{i,T}^\eta(y_i[T]) \cdot \phi_{i,T+1|T}^\eta(s_i^{(T+1)}, o^{(T+1)}|y_i[T]),
\]

where \( \phi_{i,T+1|T}^{RM} \) and \( \phi_{i,T+1|T}^\eta \) are the conditional probabilities of the form “having history \((s_i^{(T+1)}, o^{(T+1)})\) in period \( T + 1 \), conditional on having history \( y_i[T] \in Y_i[T] \) in the first \( T \) periods.” If such conditional probabilities are always the same for the random-matching model and the \( \eta \)-distributed response path model, then from the hypothesis \( \phi_{i,T}^{RM} = \phi_{i,T}^\eta \), we (and not a fixed behavior strategy): they are matched with random opponents, who each play a pure strategy in the game as a function of their personal history. From Kuhn’s theorem, this is equivalent to facing a fixed profile of behavior strategies.
can conclude $\phi_{i,T+1}^{RM} = \phi_{i,T+1}^{0}$.

By argument exactly analogous to the base case, we have for the random-matching model

$$\phi_{i,T+1}^{RM}(s_{i}^{(T+1)}, o^{(T+1)})|y_{i}[T]) = 1(r_{i}(y_{i}(T)) = s_{i}^{(T+1)}) \cdot \sigma(s : o_{i}(Z(s_{i}^{(T+1)}, s_{-i})) = o^{(T+1)}),$$

since the matching is independent across periods.

But in the $\eta$-distributed response path model, since a single response path is drawn once and fixed, one must compute the conditional probability that the drawn $\mathcal{G}$ is such that the observation $o^{(T+1)}$ will be seen in period $T + 1$, given the history $y_{i}[T]$ (which is informative about which response path $i$ is facing).

For each $1 \leq n \leq N$, let the non-negative integer $m_{n}$ represent the number of times $i$ has used the $n$-th strategy in $S_{i}$ in the history $y_{i}[T]$. Let $(o_{t,n})_{1 \leq t \leq m_{n}}$ represent the sequence of observations seen after using the $n$-th strategy, in chronological order. Consider the following finite union of cylinder sets, $(s_{t,n} : o_{i}(Z(n, s_{t,n,-i})) = o_{t,n})_{1 \leq t \leq m_{n}, 1 \leq n \leq N}$. This is the set of response sequences consistent with the observations so far.

If $\mathcal{G}$ is to produce the observation $o^{(T+1)}$ from $i$’s next play of $s_{i}^{(T+1)}$, then $\mathcal{G}$ must belong to a more restrictive cylinder set that satisfies the additional restriction $(s_{m_{i}^{(T+1)}+1,s_{i}^{(T+1)}} : o_{i}(Z(s_{i}^{(T+1)}, s_{-i})) = o_{m_{i}^{(T+1)}+1,s_{i}^{(T+1)}}).$ The conditional probability of $\mathcal{G}$ belonging to this more restrictive cylinder set, given that it falls in $(s_{t,n} : o_{i}(Z(n, s_{t,n,-i})) = o_{t,n})_{1 \leq t \leq m_{n}, 1 \leq n \leq N},$ is then given by the ratio of $\eta$-probabilities of these unions of cylinder sets, which from the product structure of $\eta$ on cylinder sets, must be $\sigma(s : o_{i}(Z(s_{i}^{(T+1)}, s_{-i})) = o^{(T+1)})$. \hfill \Box

Thus, to prove that $\phi(s_{i}^{*}; r_{i}, \sigma_{-i}) \geq \phi(s_{j}^{*}; r_{j}, \sigma_{-j})$, it suffices to show that for every $\mathcal{G}$, the period where $s_{i}^{*}$ is played for the $k$-th time in induced history $y_{i}(\mathcal{G}, r_{i})$ happens earlier than the period where $s_{j}^{*}$ is played for the $k$-th time in history $y_{j}(\mathcal{G}, r_{j})$.

Now we turn to the proof of Proposition 5.

**Proof.** Let $0 \leq \gamma < 1$ and the social distribution $\sigma$ be fixed. Enumerate the strategy sets of $i$ and $j$ so that $s_{i}$ and $\varphi(s_{i})$ are assigned the same number for every $s_{i} \in S_{i}$. Consider the product distribution $\eta$ on the space of response paths, $((\mathcal{S})^{N})^{\infty}$, as in the proof of Lemma 5.

By Lemma 5, denote the period where $s_{i}^{*}$ appears in $y_{i}(\mathcal{G}, r_{i})$ for the $k$-th time as $T_{i}^{(k)}$,
the period where \( s_j^\ast \) appears in \( y_j(\mathfrak{S}, r_j) \) for the \( k \)-th time as \( T_j^{(k)} \). The quantities \( T_i^{(k)}, T_j^{(k)} \) are defined to be \( \infty \) if the corresponding strategies do not appear at least \( k \) times in the infinite histories. Write \( \#(s'_i; k) \in \mathbb{N} \cup \{\infty\} \) be the number of times \( s'_i \in \mathcal{S}_i \) is played in the history \( y_i(\mathfrak{S}, r_i) \) before \( T_i^{(k)} \). Similarly, \( \#(s'_j; k) \in \mathbb{N} \cup \{\infty\} \) denotes the number of times \( s'_j \in \mathcal{S}_j \) is played in the history \( y_j(\mathfrak{S}, r_j) \) before \( T_j^{(k)} \). Since \( \varphi \) establishes a bijection between \( \mathcal{S}_i \) and \( \mathcal{S}_j \), it suffices to show that for every \( k = 1, 2, 3, \ldots \) either \( T_j^{(k)} = \infty \) or for all \( s'_i \neq s_i^\ast \), 
\[ \#(s'_i; k) \leq \#(s'_j; k) \text{ where } s'_j = \varphi(s'_i). \]

We show this by induction on \( k \). First we establish the base case of \( k = 1 \).

Suppose \( T_j^{(1)} \neq \infty \), and, by way of contradiction, suppose there is some \( s'_i \neq s_i^\ast \) such that 
\[ \#(s'_i; 1) > \#(\varphi(s'_i); 1). \]
Find the subhistory \( y_i \) of \( y_i(\mathfrak{S}, r_i) \) that leads to \( s'_i \) being played for the \( (\#(\varphi(s'_i)); 1) + 1 \)-th time, and find the subhistory \( y_j \) of \( y_j(\mathfrak{S}, r_j) \) that leads to \( j \) playing \( s_j^\ast \) for the first time (\( y_j \) is well-defined because \( T_j^{(1)} \neq \infty \)). Note that \( y_i, s_i^\ast \sim y_j, s'_j \) vacuously, since \( i \) has never played \( s_i^\ast \) in \( y_i \) and \( j \) has never played \( s_j^\ast \) in \( y_j \).

Also, \( y_i, s'_i \sim y_j, s'_j \). To see this, note that \( i \) has played \( s'_i \) for \( \#(\varphi(s'_i); 1) \) times and \( j \) has played \( s'_j \) for the same number of times. The definition of response paths implies they faced the same sequence of opponent strategy profiles, and the definition of isomorphic learning problems implies they have gotten equivalent observations in all these periods.

Since \( r_j(y_j) = s_j^\ast \) and \( r_j \) is an index rule, \( s_j^\ast \) must have weakly the highest index at \( y_j \). Since \( r_i \) is more compatible with \( s_i^\ast \) than \( r_j \) is with \( s_j^\ast \), \( s'_i \) must not have the weakly highest index at \( y_i \). And yet \( r_i(y_i) = s'_i \) contradiction.

Now suppose this statement holds for all \( k \leq K \) for some \( K \geq 1 \). We show it also holds for \( k = K + 1 \). If \( T_j^{(K+1)} = \infty \) or \( T_j^{(K)} = \infty \), we are done. Otherwise, by way of contradiction, suppose there is some \( s'_i \neq s_i^\ast \) so that 
\[ \#(s'_i; K + 1) > \#(\varphi(s'_i); K + 1). \]
Find the subhistory \( y_i \) of \( y_i(\mathfrak{S}, r_i) \) that leads to \( s'_i \) being played for the \( (\#(\varphi(s'_i); K + 1) + 1 \)-th time. Since \( T_j^{(K)} \neq \infty \), from the inductive hypothesis \( T_i^{(K)} = \infty \) and \( \#(s'_i; K) \leq \#(\varphi(s'_i); K) \). That is, \( i \) must have played \( s'_i \) no more than \( \#(\varphi(s'_i); K) \) times before playing \( s_i^\ast \) for the \( K \)-th time. Since \( \#(\varphi(s'_i); K + 1) + 1 > \#(\varphi(s'_i); K) \), the subhistory \( y_i \) must extend beyond period \( T_i^{(K)} \), so it contains \( K \) instances of \( i \) playing \( s_i^\ast \).

Next, find the subhistory \( y_j \) of \( y_j(\mathfrak{S}, r_j) \) that leads to \( j \) playing \( s_j^\ast \) for the \( (K + 1) \)-th time. (This is well-defined because \( T_j^{(K+1)} \neq \infty \).) Note that \( y_i, s_i^\ast \sim y_j, s_j^\ast \), since \( i \) and \( j \) have played
s_i^*, s_j^* for K times each, and they were facing the same response paths. Also, y_{i,s_i} \sim y_{j,s_j} since i has played s_i^* for \#(\varphi(s_i^*); K + 1) times and j has played s_j^* for the same number of times. Since r_j(y_j) = s_j^* and r_j is an index rule, s_j^* must have weakly the highest index at y_j. Since r_i is more compatible with s_i^* than r_j is with s_j^*, s_i^* must not have the weakly highest index at y_i. And yet r_i(y_i) = s_i^* contradiction. □

9.5 Proof of Lemma 1

Proof. By way of contradiction, suppose there is some profile of moves by \(-i, (a_h)_{h \in H_{-i}}, \) so that h* is off the path of play in \((s_i, (a_h)_{h \in H_{-i}}) = (s_i, a_h^*, (a_h)_{h \in H_{-i} \setminus h^*}).\) Find a different action of j on h*, a'_h* \neq a_h^*. Since h* is off the path of play, both \((s_i, a_h^*, (a_h)_{h \in H_{-i} \setminus h^*})\) and \((s_i, a'_h^*, (a_h)_{h \in H_{-i} \setminus h^*})\) lead to the same payoff for i. But by Condition (1) in the definition of factorability and the fact that h* \in F_i[s_i], we will have found two \(-i\) action profiles s_{-i}, s'_{-i} in two different blocks of \(\Pi_i[s_i]\) with \(u_i(s_i, s_{-i}) = u_i(s_i, s'_i).\) This contradicts \(\Pi_i[s_i]\) being the coarsest partition of \(\mathbb{S}_{-i}\) that makes \(u_i(s_i, \cdot)\) measurable. □

9.6 Proof of Lemma 2

Proof. Since i’s payoff is not independent of h*, there exist actions a_h^* \neq a'_h* on h* and a profile a_{-h^*} of actions elsewhere in the game tree, so that \(u_i(a_h^*, a_{-h^*}) \neq u_i(a'_h^*, a_{-h^*}).\) Consider the strategy s_i for i that matches a_{-h^*} in terms of play on i’s information sets, so we may equivalently write

\[ u_i(s_i, a_h^*, (a_h)_{h \in H_{-i} \setminus h^*}) \neq u_i(s_i, a'_h^*, (a_h)_{h \in H_{-i} \setminus h^*}), \]

where \((a_h)_{h \in H_{-i} \setminus h^*}\) are the components of a_{-h^*} corresponding to information sets of \(-i. If h^* \notin F_i[s_i], then by Condition (1) of factorability, \((a_h^*, (a_h)_{h \in H_{-i} \setminus h^*})\) and \((a'_h^*, (a_h)_{h \in H_{-i} \setminus h^*})\) belong to the same block in \(\Pi_i[s_i]\). Yet, they give different payoffs to i, which contradicts that i’s payoff after s_i must be measurable with respect to \(\Pi_i[s_i]\). □
9.7 Proof of Proposition 6

Proof. Combining Lemmas 1 and 2 implies there is an extensive-form strategy \( s_i \in \mathcal{S}_i \) such that \( h^* \) is on the path of play whenever \( i \) chooses \( s_i \). Consider some strategy profile \((s_i^0, s_{-i}^0)\) where \( h^* \) is off the path. Then \( i \) can unilaterally deviate to \( s_i \), and \( h^* \) is on the path of \((s_i, s_{-i}^0)\). Furthermore, \( i \)'s play differs on the new path relative to the old path on exactly one information set, since \( i \) plays at most once on any path. So instead of deviating to \( s_i \), \( i \) can deviate to \( s'_i \) that matches \( s_i \) in terms of this information set where \( i \)'s play is modified, but otherwise is the same as \( s_i^0 \). So \( h^* \) is also on the path of play for \((s'_is_{-i}^0)\), where \( s'_i \) differs from \( s_i^0 \) only on one information set. \( \Box \)

10 Compatibility for Index Rules

In this section, we show that weighted fictitious play is index-compatible under the conditions of Proposition 8, the Heuristic Gittins index is index-compatible under the conditions of Proposition 7, and the UCB index is index-compatible under the conditions of Theorem 2.

10.1 Weighted Fictitious Play

We give the proof of Proposition 8.

Proof. Let histories \( y_i, y_j \) and strategy \( s'_j \neq s^*_j \) be given with \( y_{i,s'_i} \sim y_{j,s'_j}^*, y_{i,s'_i} \sim y_{j,\varphi(s'_i)} \), and \( s^*_j \) having weakly the highest index for \( j \). Construct two lists of strategy profiles, \((\tilde{\sigma}[s_i])_{s_i \in \mathcal{S}_i}, (\sigma^{[s_j]}))_{s_j \in \mathcal{S}_j}\), so that for each \( s_j \in \mathcal{S}_j \), \( \sigma^{[s_j]}(h) = \alpha_h(\cdot; y_j) \) for all \( h \in F_j[s_j] \). For each \( s_i \in \{s_i^*, s_i'\} \), \( \tilde{\sigma}[s_i](h) = \alpha_h(\cdot; y_i) \) for all \( h \in F_i[s_i] \). Finally, for \( s_i \in \mathcal{S}_i \setminus \{s_i^*, s_i'\} \), let \( \tilde{\sigma}[s_i](h) = \sigma^{[\varphi(s_i)]}(h) \) for all \( h \in F_i[s_i] \cap F_j[\varphi(s_i)] \). So either by construction or from the definition of equivalent histories, \( \tilde{\sigma}[s_i]|_{F_i[s_i] \cap F_j[\varphi(s_i)]} = \sigma^{[\varphi(s_i)]}|_{F_i[s_i] \cap F_j[\varphi(s_i)]} \) for all \( s_i \in \mathcal{S}_i \). Also, since \( j \)'s payoff from each \( s_j \) only depends on \( -j \)'s play on \( F_j[s_j] \), \( \sum_{s_j \in \mathcal{S}_j} u_j(s'_j, s_{-j}) \cdot \sigma^{[s_j]}(s) \) equals to the index that the weighted fictitious play agent assigns to \( s'_j \) after history \( y_j \). Since \( s^*_j \) has the weakly highest index, \( \sum_{s_j \in \mathcal{S}_j} u_j(s'_j, s_{-j}) \cdot \sigma^{[s_j]}(s) = \max_{s'_j \in \mathcal{S}_j} \sum_{s_j \in \mathcal{S}_j} u_j(s'_j, s_{-j}) \cdot \sigma^{[s_j]}(s) \). From the definition of strong player compatibility, \( \sum_{s_i \in \mathcal{S}_i} u_i(s^*_i, s_{-i}) \cdot \tilde{\sigma}[s_i](s) > \sum_{s_i \in \mathcal{S}_i} u_i(s'_i, s_{-i}) \cdot \tilde{\sigma}[s_i](s) \), which says \( s'_i \) does not have the weakly highest index for \( i \) after \( y_i \). \( \Box \)
10.2 The Heuristic Gittins Index

Write $V(\tau; s_i, \nu_{s_i})$ for the value of the above auxiliary problem under the (not necessarily optimal) stopping time $\tau$ in the definition of the Gittins index. The Gittins index of $s_i$ is $\sup_{\tau>0} V(\tau; s_i, \nu_{s_i})$. We begin by linking $V(\tau; s_i, \nu_{s_i})$ to $i$’s stage-game payoff from playing $s_i$. From belief $\nu_{s_i}$ and stopping time $\tau$, we will construct the correlated profile $\alpha(\nu_{s_i}, \tau) \in \Delta^{\nu}(\times_{h \in F_i[s_i]} A_h)$, so that $V(\tau; s_i, \nu_{s_i})$ is equal to $i$’s expected payoff when playing $s_i$ while opponents play according to this correlated profile on the $s_i$-relevant information sets.

**Definition 21.** A full-support belief $\nu_{s_i} \in \times_{h \in F_i[s_i]} \Delta(\Delta(A_h))$ for player $i$ together with a (possibly random) stopping rule $\tau > 0$ together induce a stochastic process $(\tilde{a}_{(-i),t})_{t \geq 1}$ over the space $\times_{h \in F_i[s_i]} A_h \cup \{\emptyset\}$, where $\tilde{a}_{(-i),t} \in \times_{h \in F_i[s_i]} A_h$ represents the opponents’ actions observed in period $t$ if $\tau \geq t$, and $\tilde{a}_{(-i), t} = \emptyset$ if $\tau < t$. We call $\tilde{a}_{(-i), t}$ player $i$’s internal history at period $t$ and write $\mathbb{P}_{(-i)}$ for the distribution over internal histories that the stochastic process induces.

Internal histories live in the same space as player $i$’s actual experience in the learning problem, represented as a history in $O_i$. The process over internal histories is $i$’s prediction about what would happen in the auxiliary problem (which is an artificial device for computing the Gittins index) if he were to use $\tau$.

Enumerate all possible profiles of moves at information sets $F_i[s_i]$ as $\times_{h \in F_i[s_i]} A_h = \{a_{(-i)}^{(1)}, \ldots, a_{(-i)}^{(K)}\}$, let $p_{t,k} := \mathbb{P}_{(-i)}[\tilde{a}_{(-i), t} = a_{(-i)}^{(k)}]$ for $1 \leq k \leq K$ be the probability under $\nu_{s_i}$ of seeing the profile of actions $a_{(-i)}^{(k)}$ in period $t$ of the stochastic process over internal histories, $(\tilde{a}_{(-i), t})_{t \geq 0}$, and let $p_{t,0} := \mathbb{P}_{(-i)}[\tilde{a}_{(-i), t} = \emptyset]$ be the probability of having stopped before period $t$.

**Definition 22.** The synthetic correlated profile at information sets in $F_i[s_i]$ is the element of $\Delta^{\nu}(\times_{h \in F_i[s_i]} A_h)$ (i.e. a correlated random action) that assigns probability $\frac{\sum_{t=1}^{\infty} \beta^{t-1} p_{t,k}}{\sum_{t=1}^{\infty} \beta^{t-1}(1-p_{t,0})}$ to the profile of actions $a_{(-i)}^{(k)}$. Denote this profile by $\alpha(\nu_{s_i}, \tau)$.

Note that the synthetic correlated profile depends on the belief $\nu_{s_i}$, stopping rule $\tau$, and effective discount factor $\beta$. Since the belief $\nu_{s_i}$ has full support, there is always a positive probability assigned to observing every possible profile of actions on $F_i[s_i]$ in the first period,
so the synthetic correlated profile is totally mixed. The significance of the synthetic correlated profile is that it gives an alternative expression for the value of the auxiliary problem under stopping rule \( \tau \).

**Lemma 6.**

\[
V(\tau; s_i, \nu_{s_i}) = u_i(s_i, \alpha(\nu_{s_i}, \tau))
\]

The proof is the same as in Fudenberg and He (2018) and is omitted.\(^{24}\)

Consider now the situation where \( i \) and \( j \) share the same beliefs about play of \(-ij\) on the common information sets \( F_i[s_i] \cap F_j[s_j] \subseteq \mathcal{H}_{-ij} \). For any pure-strategy stopping time \( \tau_j \) of \( j \), we define a random stopping rule of \( i \), the *mimicking stopping time* for \( \tau_j \). Lemma 7 will establish that the mimicking stopping time generates a synthetic correlated profile that matches the corresponding profile of \( \tau_j \) on \( F_i[s_i] \cap F_j[s_j] \).

The key issue in this construction is that \( \tau_j \) maps \( j \)'s internal histories to stopping decisions, which does not live in the same space as \( i \)'s internal histories. In particular, \( \tau_j \) could make use of \( i \)'s play to decide whether to stop. To mimic such a rule, \( i \) makes use of external histories, which include both the common component of \( i \)'s internal history on \( F_i[s_i] \cap F_j[s_j] \), as well as simulated histories on \( F_j[s_j] \setminus (F_i[s_i] \cap F_j[s_j]) \).

For a given isomorphism \( \varphi \) between \( i \) and \( j \) with \( \varphi(s_i) = s_j \) and \( F_i, F_j \), we may write \( F_i[s_i] = F^C \cup \overline{F}^{-i} \) with \( F^C \subseteq \mathcal{H}_{-ij} \) and \( \overline{F}^{-i} \subseteq \mathcal{H}_{-i} \). Similarly, we may write \( F_j[s_j] = F^C \cup \overline{F}^{-j} \) with \( \overline{F}^{-j} \subseteq \mathcal{H}_{-j} \). (So, \( F^C \) is the common information sets that are observed after both \( s_i \) and \( s_j \).) Whenever \( j \) plays \( s_j \), he observes some \((a_{(C)}, a_{(-j)}) \in (x_{h \in F^CA_h}) \times (x_{h \in \overline{F}^{-j}A_h})\), where \( a_{(C)} \) is a profile of actions at information sets in \( F^C \) and \( a_{(-j)} \) is a profile of actions at information sets in \( \overline{F}^{-j} \). So, a pure-strategy stopping rule in the auxiliary problem defining \( j \)'s Gittins index for \( s_j \) is a function \( \tau_j : \cup_{t \geq 1}[(x_{h \in F^CA_h}) \times (x_{h \in \overline{F}^{-j}A_h})]^t \rightarrow \{0, 1\} \) that maps finite histories in \( O_j \) to stopping decisions, where “0” means continue and “1” means stop.

**Definition 23.** Player \( i \)'s *mimicking stopping rule* for \( \tau_j \) draws \( \alpha^{-j} \in x_{h \in \overline{F}^{-j}} \Delta(A_h) \) from \( j \)'s belief \( \nu_{s_j} \) on \( \overline{F}^{-j} \), and then draws \((a_{(-j), t})_{t \geq 1} \) by independently generating \( a_{(-j), t} \) from \( \alpha^{-j} \).

\(^{24}\)Notice that even though \( i \) starts with the belief that opponents randomize independently at different information sets, and also holds an independent prior belief, \( V(\tau; s_i, \nu_{s_i}) \) may not be the payoff of playing \( s_i \) against a independent randomizations by the opponent because of the endogenous correlation that we discussed in the text.
each period. Conditional on \((a_{(-j),t})\), \(i\) stops according to the rule 
\[ \tau_i|(a_{(-j),t})(a_{(C),t}, a_{(-i),t})_{t=1}^\infty := \tau_j((a_{(C),t}, a_{(-j),t})_{t=1}^\infty). \]

That is, the mimicking stopping rule involves ex-ante randomization across a family of pure-strategy stopping rules \(\tau_i|(a_{(-j),t})_{t=1}^\infty\), indexed by \((a_{(-j),t})_{t=1}^\infty\). First, \(i\) draws a behavior strategy on the information sets \(\bar{F}^{-j}\) according to \(j\)'s belief about \(-j\)'s play there. Then, \(i\) simulates an infinite sequence \((a_{(-j),t})_{t=1}^\infty\) of \(i\)'s play using this drawn behavior strategy and follows the pure-strategy stopping rule \(\tau_i|(a_{(-j),t})_{t=1}^\infty\).

As in the definition of internal histories, the mimicking strategy and \(i\)'s belief \(\nu_{s_i}\) generates a stochastic process \((\tilde{a}_{(-i),t}, \tilde{a}_{(C),t})_{t=1}^\infty\) of internal histories for \(i\) (representing actions on \(F_i[s_i]\) that \(i\) anticipates seeing when he plays \(s_i\)). It also induces a stochastic process \((\tilde{e}_{(-j),t}, \tilde{e}_{(C),t})_{t=1}^\infty\) of “external histories” defined in the following way:

**Definition 24.** The stochastic process of external histories \((\tilde{e}_{(-j),t}, \tilde{e}_{(C),t})_{t=1}^\infty\) is defined from the process of internal histories \((\tilde{a}_{(-i),t}, \tilde{a}_{(C),t})_{t=1}^\infty\) that \(i\) generates and given by: (i) if \(\tau_i < t\), then \((\tilde{e}_{(-j),t}, \tilde{e}_{(C),t}) = \emptyset\); (ii) otherwise, \(\tilde{e}_{(C),t} = \tilde{a}_{(C),t}\), and \(\tilde{e}_{(-j),t}\) is the \(t\)-th element of the infinite sequence \((a_{(-j),t})_{t=1}^\infty\) that \(i\) simulated before the first period of the auxiliary problem.

Write \(P_\nu\) for the distribution over the sequence of of external histories generated by \(i\)'s mimicking stopping time for \(\tau_j\), which is a function of \(\tau_j, \nu_{s_j},\) and \(\nu_{s_i}\).

To understand the distinction between internal and external histories, note that the probability of \(i\)'s first-period internal history satisfying \((\tilde{a}_{(-i),1}, \tilde{a}_{(C),1}) = (\tilde{a}_{(-i),1}, \tilde{a}_{(C)})\) for some fixed values \((\tilde{a}_{(-i),1}, \tilde{a}_{(C)}) \in \times_h \in F_i[s_i] A_h\) is given by the probability that a mixed play \(\alpha_{-i}\) on \(F_i[s_i]\), drawn according to \(i\)'s belief \(\nu_{s_i}\), would generate the profile of actions \((\tilde{a}_{(-i),1}, \tilde{a}_{(C)})\). On the other hand, the probability of \(i\)'s first-period external history satisfying \((\tilde{e}_{(-j),1}, \tilde{e}_{(C),1}) = (\tilde{a}_{(-j),1}, \tilde{a}_{(C)})\) for some fixed values \((\tilde{a}_{(-j),1}, \tilde{a}_{(C)}) \in \times_h \in F_i[s_i] A_h\) also depends on \(j\)'s belief \(\nu_{s_j}\), for this belief determines the distribution over \((a_{(-j),t})_{t=1}^\infty\) drawn before the start of the auxiliary problem.

When using the mimicking stopping time for \(\tau_j\) in the auxiliary problem, \(i\) expects to see the same distribution of \(-ij\)'s play before stopping as \(j\) does when using \(\tau_j\), on the information sets in \(F_i[s_i] \cap F_j[s_j]\). This is formalized in the next lemma.

---

\[25\text{Note this is a valid (stochastic) stopping time, as the event } \{\tau_i \leq T\} \text{ only depends on } i\text{'s observations in } \mathcal{O}_i \text{ in the first } T \text{ periods, plus some private randomizations of } i.\]
Lemma 7. Fix $F_i, F_j$ and an isomorphism $\varphi$ between $S_i$ and $S_j$ with $\varphi(s_i) = s_j$, and suppose $i$ holds belief $\nu_{s_i}$ over play in $F_i[s_i]$ and $j$ holds belief $\nu_{s_j}$ over play in $F_j[s_j]$, such that $\nu_{s_i} | F_i[s_i] \cap F_j[s_j] = \nu_{s_j} | F_i[s_i] \cap F_j[s_j]$, that is the two sets of beliefs match when marginalized to the common information sets in $\mathcal{H}_{ij}$. Let $\tau_i$ be $i$’s mimicking stopping rule for $s_i$. Proposition 10. Fix $F_i, F_j$ and an isomorphism $\varphi$ between $S_i$ and $S_j$ with $\varphi(s_i) = s_j$, $\varphi(s'_i) = s'_j$, where $s'_i \neq s'_j$. Suppose $i$ is strongly more player-compatible with $s'_i$ than $j$ is with $s'_j$, with respect to $\varphi, F_i, F_j$. Suppose $i$ holds belief $\nu_{s_i} \in \times_{h \in F_i[s_i]} \Delta(\Delta(A_h))$ about opponents’ play after each $s_i$ and $j$ holds belief $\nu_{s_j} \in \times_{h \in F_j[s_j]} \Delta(\Delta(A_h))$ about opponents’ play after each $s_j$, such that $\nu_{s'_i} | F_i[s'_i] \cap F_j[s'_j] = \nu_{s'_j} | F_i[s'_i] \cap F_j[s'_j]$ and $\nu_{s'_j} | F_i[s'_i] \cap F_j[s'_j] = \nu_{s'_j} | F_i[s'_i] \cap F_j[s'_j]$. If $s'_j$ has the weakly highest Gittins index for $j$ under effective discount factor $0 \leq \delta \gamma < 1$, then $s'_i$ does not have the weakly highest Gittins index for $i$ under the same effective discount factor.

Proof. We begin by defining a collection of totally mixed correlated profiles $(\alpha_{s_i})_{s_i \in S_i}$ where $\alpha_{s_i} \in \Delta^0(\times_{h \in F_i[s_i]} A_h)$. For each $s_j \neq s'_j$ the profile $\alpha_{s_j}$ is the synthetic correlated profile $\alpha(\nu_{s_j}, \tau^*_{s_j})$, where $\tau^*_{s_j}$ is an optimal pure-strategy stopping time in $j$’s auxiliary stopping problem involving $s_j$. For $s_j = s'_j$, the correlated profile $\alpha_{s'_j}$ is instead the synthetic correlated profile associated with the mimicking stopping rule for $\tau^*_{s'_i}$, i.e. mimicking agent $i$’s pure-strategy optimal stopping time in $i$’s auxiliary problem for $s'_i$.

Next, define a profile of totally mixed correlated actions $(\alpha_{F_i[s_i]})_{s_i \in S_i}$ for $i$’s opponents on information sets $(F_i[s_i])_{s_i \in S_i}$. For each $s_i \notin \{s'_i, s'_j\}$, just use the marginal distribution of $\alpha_{\varphi(s_i)}$ constructed before on $F_i[s_i] \cap F_j[\varphi(s_i)]$, then arbitrarily specify play in $F_i[s_i] \setminus F_j[\varphi(s_i)]$, if any. For $s'_i$ the correlated profile is $\alpha(\nu_{s'_i}, \tau^*_{s'_i})$, i.e. the synthetic move associated with $i$’s optimal stopping rule for $s'_i$. Finally, for $s'_j$, the correlated profile $\alpha_{s'_j}$ is the synthetic correlated profile associated with the mimicking stopping rule for $\tau^*_{s'_j}$.

From Lemma 7, for every $s_i$, the profiles of correlated actions $\alpha_{s_i}$ and $\alpha_{\varphi(s_i)}$ agree when marginalized to the information sets $F_i[s_i] \cap F_j[\varphi(s_i)]$. Therefore, they can be completed into two lists of totally mixed correlated strategy profiles, $(\tilde{\sigma}_{s_i})_{s_i \in S_i}$ and $(\sigma_{s_j})_{s_j \in S_j}$, such that $\tilde{\sigma}_{s_i} | F_i[s_i] \cap F_j[\varphi(s_i)] = \sigma_{\varphi(s_i)} | F_i[s_i] \cap F_j[\varphi(s_i)]$ for every $s_i$. For each $s_j \neq s'_j$, the Gittins index of $s_j$
for $j$ is $u_j(s_j, \sigma_{s_j})$. Also, since $\alpha_{[s_j']}$ is the mixed profile associated with the suboptimal mimicking stopping time, $u_j(s_j', \sigma_{s_j'})$ is no larger than the Gittins index of $s_j'$ for $j$. By the hypothesis that $s_j'$ has the weakly highest Gittins index for $j$, $u_j(s_j', \sigma_{s_j'}) \geq \max_{s_j \neq s_j'} u_j(s_j, \sigma_{s_j})$. By the definition of strong player compatibility, we must also have $u_i(s_i^*, \sigma_{s_i^*}) > \max_{s_i \neq s_i^*} u_i(s_i, \sigma_{s_i})$, so in particular $u_i(s_i^*, \sigma) > u_i(s_i', \sigma_{s_i'})$. But $u_i(s_i^*, \sigma_{s_i^*})$ is no larger than the Gittins index of $s_i^*$, for $\alpha_{[s_i^*]}$ is the synthetic strategy associated with a suboptimal mimicking stopping time. As $u_i(s_i' \sigma_{s_i'})$ is equal to the Gittins index of $s_i'$ this shows $s_i'$ cannot have even weakly the highest Gittins index at this belief, for $s_i^*$ already has a strictly higher Gittins index than $s_i'$ does. 

Finally, we provide the proof of Proposition 7.

Proof. The construction of the prior is such that $i$ holds independent beliefs about opponents’ play at different information sets and after different $s_i$, and same holds for $j$. Further, for every $s_i \in S_i$ and $F \in F_i[s_i] \cap F_j[\varphi(s_i)]$, $i$ and $j$ assign the same prior belief to play on $F$. So, the same must hold for $s_i \in \{s_i^*, s_i'\}$ after histories $y_i, y_j$ with $y_i, s_i \sim y_j, s_j$ and $y_i, s_i' \sim y_j, \varphi(s_i')$, since equivalence is defined in terms of having same observations on the common information sets $F_i[s_i] \cap F_j[\varphi(s_i)]$. After such histories, if $s_i^*$ has weakly the highest Gittins index for $j$, we use the hypothesis of strong player compatibility and Proposition 10 to conclude the proof.

Note that $HG_i$ and the rational policy coincide in factorable games, so this also establishes the first statement of Theorem 2.

10.3 Bayes-UCB

We start with a lemma that shows the Bayes-UCB index for a strategy $s_i$ is equal to $i$’s payoff from playing $s_i$ against a certain profile of mixed actions on $F_i[s_i]$, where this profile depends on $i$’s belief about actions on $F_i[s_i]$, the quantile $q$, and how $u_{s_i, h}$ ranks mixed actions in $\Delta(A_h)$ for each $h \in F_i[s_i]$.

Lemma 8. Let $n_{s_i}$ be the number of times $i$ has played $s_i$ in history $y_i$ and let $q_{s_i} = q(n_{s_i}) \in (0, 1)$. Then the Bayes-UCB index for $s_i$ and given quantile-choice function $q$ after history
Proof. For each $h \in F_s[s_i]$, the random variable $\tilde{u}_{s_i,h}(y_{i,h})$ only depends on $y_{i,h}$ through the posterior $g_i(\cdot | y_{i,h})$. Furthermore, $Q(\tilde{u}_{s_i,h}(y_{i,h}); q_{s_i})$ is strictly between the highest and lowest possible values of $u_{s_i,h}(\cdot)$, each of which can be attained by some pure action on $A_h$, so there is a totally mixed $\tilde{\alpha}_h \in \Delta^0(A_h)$ so that $Q(\tilde{u}_{s_i,h}(y_{i,h}); q_{s_i}) = u_{s_i,h}(\tilde{\alpha}_h)$. Moreover, if $u_{s_i,h}$ and $u'_{s_i,h}$ rank mixed strategies on $\Delta(A_h)$ in the same way, there are $a \in \mathbb{R}$ and $b > 0$ so that $u'_{s_i,h} = a + bu_{s_i,h}$. Then $Q(\tilde{u}'_{s_i,h}(y_{i,h}); q_{s_i}) = a + bQ(\tilde{u}_{s_i,h}(y_{i,h}); q_{s_i})$, so $\tilde{\alpha}_h$ still works for $u'_{s_i,h}$.

The second statement of Theorem 2 follows as a corollary.

Corollary 1. If $s_i^* \gtrsim s_j^*$, and the hypotheses of Theorem 2 are satisfied, then UCB$_i$ is more index-compatible with $s_i^*$ than UCB$_j$ is with $s_j^*$.

Proof. When $i$ and $j$ have matching beliefs, by Lemma 8 we may calculate their Bayes-UCB indices for different strategies as their myopic expected payoff of using these strategies against some common opponents’ play, as in the similar argument for the Gittins index in Lemma 8. Applying the definition of compatibility, we can deduce that when $s_i^* \gtrsim s_j^*$ and $\varphi(s_i^*) = s_j^*$, if $s_j^*$ has the highest Bayes-UCB index for $j$ then $s_i^*$ must have the highest Bayes-UCB index for $i$.

10.4 Proof of Lemma 7

Proof. Let $(\tilde{a}_{(-i),t}, \tilde{a}_{(C),t})_{t \geq 1}$ and $(\tilde{e}_{(-j),t}, \tilde{e}_{(C),t})_{t \geq 1}$ be the stochastic processes of internal and external histories for $\tau_i$, with distributions $\mathbb{P}_{-i}$ and $\mathbb{P}_e$. Enumerate possible profiles of actions on $F^C$ as $\times_{h \in F^C} A_h = \{a^{(1)}_{(C)}, \ldots, a^{(K)}_{(C)}\}$, possible profiles of actions on $\bar{F}^{-i}$ as $\times_{h \in \bar{F}^{-i}} A_h = \{a^{(1)}_{(-i)}, \ldots, a^{(K)}_{(-i)}\}$, and possible profiles of actions on $\bar{F}^{-j}$ as $\times_{h \in \bar{F}^{-j}} A_h = \{a^{(1)}_{(-j)}, \ldots, a^{(K)}_{(-j)}\}$.

Write $p_{t,(k_{-i},k_C)} := \mathbb{P}_{-i}[(\tilde{a}_{(-i),t}, \tilde{a}_{(C),t}) = (a^{(k_{-i})}_{(-i)}, a^{(k_C)})]$ for $k_{-i} \in \{1, \ldots, K_{-i}\}$ and $k_C \in \{1, \ldots, K_C\}$. Also write $q_{t,(k_{-j},k_C)} := \mathbb{P}_e[(\tilde{e}_{(-j),t}, \tilde{e}_{(C),t}) = (a^{(k_{-j})}_{(-j)}, a^{(k_C)})]$ for $k_{-j} \in \{1, \ldots, K_{-j}\}$.
and \( k_C \in \{1, ..., K_C\} \). Let \( p_{t,(0,0)} = q_{t,(0,0)} := \mathbb{P}_{-i}[\tau_i < t] = \mathbb{P}_e[\tau_i < t] \) be the probability of having stopped before period \( t \).

The distribution of external histories that \( i \) expects to observe before stopping under belief \( \nu_{s_i} \) when using the mimicking stopping rule \( \tau_i \) is the same as the distribution of internal histories that \( j \) expects to observe when using stopping rule \( \tau_j \) under belief \( \nu_{s_j} \), because \( i \) simulates the data-generating process on \( \tilde{F}^{-j} \) by drawing a mixed action \( \alpha^{-j} \) according to \( j \)'s belief \( \nu_{s_j|\tilde{F}^{-j}} \) and \( \nu_{s_j|F_C} = \nu_{s_j|\tilde{F}^{-j}} \). Thus for every \( k_{-j} \in \{1, ..., K_{-j}\} \) and every \( k_C \in \{1, ..., K_C\} \),

\[
\frac{\sum_{t=1}^{\infty} (\delta \gamma)^{t-1} q_{t,(k_{-j},k_C)}}{\sum_{t=1}^{\infty} (\delta \gamma)^{t-1}(1 - q_{t,(0,0)})} = \alpha(\nu_{s_j}, \tau_j)(\alpha^{(k_{-j})}_j, \alpha^{(k_C)}_C).
\]

For a fixed \( k_C \in \{1, ..., K_C\} \), summing across \( k_{-j} \) gives

\[
\frac{\sum_{t=1}^{\infty} (\delta \gamma)^{t-1} \sum_{k_{-j}=1}^{K_{-j}} q_{t,(k_{-j},k_C)}}{\sum_{t=1}^{\infty} (\delta \gamma)^{t-1}(1 - q_{t,(0,0)})} = \alpha(\nu_{s_j}, \tau_j)(\alpha^{(k_C)}_C).
\]

By definition, the processes \((\tilde{a}_{(-i),t}, \tilde{a}_{(C),t})_{t \geq 0} \) and \((e_{(-j),t}, e_{(C),t})_{t \geq 0} \) have the same marginal distribution on the second dimension:

\[
\sum_{k_{-j}=1}^{K_{-j}} q_{t,(k_{-j},k_C)} = \mathbb{P}_{-i}[\tilde{a}_{(C),t} = \tilde{a}^{(k_C)}_{(C)}] = \sum_{k_{-j}=1}^{K_{-j}} p_{t,(k_{-j},k_C)}.
\]

Making this substitution and using the fact that \( p_{t,(0,0)} = q_{t,(0,0)} \),

\[
\frac{\sum_{t=1}^{\infty} (\delta \gamma)^{t-1} \sum_{k_{-j}=1}^{K_{-j}} p_{t,(k_{-j},k_C)}}{\sum_{t=1}^{\infty} (\delta \gamma)^{t-1}(1 - p_{t,(0,0)})} = \alpha(\nu_{s_j}, \tau_j)(\tilde{a}^{(k_C)}_C).
\]

But by the definition of synthetic correlated profile, the LHS is \( \sum_{k_{-j}=1}^{K_{-j}} \alpha(\nu_{s_i}, \tau_i)(\alpha^{(k_{-j})}_i, \tilde{a}^{(k_C)}_C) = \alpha(\nu_{s_i}, \tau_i)(\alpha^{(k_C)}_C) \).

Since the choice of \( \tilde{a}^{(k_C)}_C \in \times_{h \in F_C} A_h \) was arbitrary, we have shown that the synthetic profile \( \alpha(\nu_{s_j}, \tau_j) \) of the original stopping rule \( \tau_j \) and the one associated with the mimicking strategy of \( i \), \( \alpha(\nu_{s_i}, \tau_i) \), coincide on \( F_C \). \( \square \)
10.5 Proof of Corollary 1

**Corollary 1:** The Bayes-UCB rule $r_{i,\text{UCB}}$ and $r_{j,\text{UCB}}$ satisfy the hypotheses of Proposition 5 when $s_i^* \succeq s_j^*$, provided the hypotheses of Theorem 2 are satisfied.

**Proof.** Consider histories $y_i, y_j$ with $y_i.s_i^* \sim y_j.s_j^*$ and $y_i.s_i' \sim y_j.s_j'$. By Lemma 8, there exist $\tilde{\alpha}_h^{-i} \in \Delta^\circ(A_h)$ for every $h \in \cup_{s_i \in S_i} F_i[s_i]$ and $\tilde{\alpha}_h^{-j} \in \Delta^\circ(A_h)$ for every $h \in \cup_{s_i \in S_i} F_j[s_j]$ so that $\nu_{i,s_i}(y_i) = u_i(s_i, (\alpha_h^{-i})_{h \in F_i[s_i]})$ and $\nu_{j,s_j}(y_j) = u_j(s_j, (\alpha_h^{-j})_{h \in F_j[s_j]})$ for all $s_i, s_j$, where $\nu_{i,s_i}(y_i)$ is the Bayes-UCB index for $s_i$ after history $y_i$ and $\nu_{j,s_j}(y_j)$ is the Bayes-UCB index for $s_j$ after history $y_j$.

Because $y_i.s_i^* \sim y_j.s_j^*$ and $y_i.s_i' \sim y_j.s_j'$, $y_i$ contains the same number of $s_i^*$ experiments as $y_j$ contains $s_j^*$, and $y_i$ contains the same number of $s_i'$ experiments as $y_j$ contains $s_j'$. Also by third-party equivalence and the fact that $i$ and $j$ start with the same beliefs on common relevant information sets, they have the same posterior beliefs $g_i(\cdot \mid y_i,t), g_j(\cdot \mid y_j,t)$ for any $h \in F_i[s_i^*] \cap F_j[s_j^*]$ and $h \in F_i[s_i'] \cap F_j[s_j']$. Finally, the hypotheses of Theorem 2 say that on any $h \in F_i[s_i^*] \cap F_j[s_j^*]$, $u_{s_i^*,h}$ and $u_{s_i',h}$ have the same ranking of mixed actions, while on any $h \in F_i[s_i'] \cap F_j[s_j']$, $u_{s_i',h}$ and $u_{s_j',h}$ have the same ranking of mixed actions. So, by Lemma 8, we may take $\tilde{\alpha}_h^{-i} = \tilde{\alpha}_h^{-j}$ for all $h \in F_i[s_i^*] \cap F_j[s_j^*]$ and $h \in F_i[s_i'] \cap F_j[s_j']$.

Find some $\sigma_{-j} = (\sigma_{-ij}, \sigma_i) \in \times_{k \neq j} \Delta^\circ(S_k)$ so that $\sigma_{-j}$ generates the random actions $(\tilde{\alpha}_h^{-j})$ on every $h \in \cup_{s_j \in S_j} F_j[s_j]$. Then we have $\nu_{j,s_j}(y_j) = u_j(s_j, \sigma_{-j})$ for every $s_j \in S_j$. The fact that $s_j^*$ has weakly the highest index means $s_j^*$ is weakly optimal against $\sigma_{-j}$. Now take $\sigma_{-i} = (\sigma_{-ij}, \sigma_j)$ where $\sigma_j \in \Delta^\circ(S_j)$ is such that it generates the random actions $(\tilde{\alpha}_h^{-i})$ on $F_i[s_i^*] \cap \mathcal{H}_j$ and $F_i[s_i'] \cap \mathcal{H}_j$. But since $\tilde{\alpha}_h^{-i} = \tilde{\alpha}_h^{-j}$ for all $h \in F_i[s_i^*] \cap F_j[s_j^*]$ and $h \in F_i[s_i'] \cap F_j[s_j']$, $\sigma_{-i}$ generates the random actions $(\tilde{\alpha}_h^{-i})$ on all of $F_i[s_i^*]$ and $F_i[s_i']$, meaning $\nu_{i,s_i}(y_i) = u_i(s_i^*, \sigma_{-i})$ and $\nu_{i,s_i'}(y_i) = u_i(s_i', \sigma_{-i})$. The definition of compatibility implies $u_i(s_i^*, \sigma_{-i}) > u_i(s_i', \sigma_{-i})$, so $\nu_{i,s_i}(y_i) > \nu_{i,s_i'}(y_i)$. This shows $s_i'$ does not have weakly the highest Bayes-UCB index, since $s_i^*$ has a strictly higher one. \qed
11 Proofs Omitted from the Appendix

11.1 Proof of Proposition 1

Proof. Suppose \( s^*_k \) is weakly optimal for \( k \) against some totally mixed correlated profile \( \sigma^{(k)} \). We show that \( s^*_i \) is strictly optimal for \( i \) against any totally mixed and correlated \( \sigma^{(i)} \) with the property that \( \text{marg}_{-ik}(\sigma^{(k)}) = \text{marg}_{-ik}(\sigma^{(i)}) \).

To do this, we first modify \( \sigma^{(i)} \) into a new totally profile by copying how the action of \( i \) correlates with the actions of \(-ik\) in \( \sigma^{(k)} \). For each \( s_{-ik} \in S_{-ik} \) and \( s_i \in S_i \), \( \sigma^{(k)}(s_i, s_{-ik}) > 0 \) since \( \text{marg}_{-k}(\sigma^{(k)}) \in \Delta^o(S_k) \). So write \( p(s_i \mid s_{-ik}) := \frac{\sigma^{(k)}(s_i, s_{-ik})}{\sum_{s_i' \in S_i} \sigma^{(k)}(s_i', s_{-ik})} > 0 \) as the conditional probability that \( i \) plays \( s_i \) given \(-ik\) play \( s_{-ik} \), in the profile \( \sigma^{(k)} \). Now construct the profile \( \hat{\sigma} \in \Delta^o(S) \), where

\[
\hat{\sigma}(s_i, s_{-ik}, s_k) := p(s_i \mid s_{-ik}) \cdot \sigma^{(i)}(s_{-ik}, s_k).
\]

Profile \( \hat{\sigma} \) has the property that \( \text{marg}_{-jk}(\hat{\sigma}) = \text{marg}_{-jk}(\sigma^{(k)}) \). To see this, note first that because \( \hat{\sigma} \) and \( \sigma^{(k)} \) agree on the \(-ijk\) marginal \( \text{marg}_{-ik}(\sigma^{(k)}) = \text{marg}_{-ik}(\sigma^{(i)}) \). Also, by construction, the conditional distribution of \( i \)'s action given profile of \((-ijk)\)’s actions is the same.

From the hypothesis that \( s^*_j \succeq s^*_k \), we get \( j \) finds \( s^*_j \) strictly optimal against \( \hat{\sigma} \).

But at the same time, \( \text{marg}_{-i}(\hat{\sigma}) = \text{marg}_{-i}(\sigma^{(i)}) \) by construction, so this implies also \( \text{marg}_{-ij}(\hat{\sigma}) = \text{marg}_{-ij}(\sigma^{(i)}) \). From \( s^*_i \succeq s^*_j \), and the conclusion that \( j \) finds \( s^*_j \) strictly optimal against \( \hat{\sigma} \) just obtained, we get \( i \) finds \( s^*_i \) strictly optimal against \( \sigma^{(i)} \) as desired.  

11.2 Proof of Proposition 2

Proof. Suppose that \( s^*_i \succeq s^*_j \) and that neither (ii) nor (iii) holds. We show that these assumptions imply \( s^*_j \nsubseteq s^*_i \).

Partition the set \( \Delta^o(S) \) into three subsets, \( \Pi^+ \cup \Pi^0 \cup \Pi^- \), with \( \Pi^+ \) consisting of \( \sigma \in \Delta^o(S) \) that make \( s^*_j \) strictly better than the best alternative pure strategy, \( \Pi^0 \) the elements of \( \Delta^o(S) \) that make \( s^*_j \) indifferent to the best alternative, and \( \Pi^- \) the elements that make
\(s'_j\) strictly worse. (These sets are well defined because \(|S_j| \geq 2\), so \(j\) has at least one alternative pure strategy to \(s'_j\).) If \(\Pi^0\) is non-empty, then there is some \(\sigma \in \Pi^0\) such that 
\[
\sum_{s \in S} u_j(s_j^*, s_{-j})\sigma(s) = \max_{s'_j \in S \setminus \{s_j^*\}} \sum_{s \in S} u_j(s'_j, s_{-j})\sigma(s).
\]
Because \(s_j^* \succeq s_j^*\), \(\sum_{s \in S} u_i(s_i^*, s_{-i})\bar{\sigma}(s) > \max_{s'_i \in S \setminus \{s_i^*\}} \sum_{s \in S} u_i(s'_i, s_{-i})\bar{\sigma}(s)\) for every \(\bar{\sigma} \in \Delta^\circ(S)\) such that \(\text{marg}_{s_{-i}}(\sigma) = \text{marg}_{s_{-i}}(\bar{\sigma})\).

Since at least one such \(\bar{\sigma}\) exists, we do not have \(s_j^* \succeq s_i^*\).

Also, if both \(\Pi^+\) and \(\Pi^-\) are non-empty, then \(\Pi^0\) is non-empty. This is because both \(\sigma \mapsto \sum_{s \in S} u_j(s_j^*, s_{-j})\sigma(s)\) and \(\sigma \mapsto \max_{s'_j \in S \setminus \{s_j^*\}} \sum_{s \in S} u_j(s'_j, s_{-j})\sigma(s)\) are continuous functions. If
\[
\sum_{s \in S} u_j(s_j^*, s_{-j})\sigma(s) - \max_{s'_j \in S \setminus \{s_j^*\}} \sum_{s \in S} u_j(s'_j, s_{-j})\sigma(s) > 0
\]
and also \(\sum_{s \in S} u_j(s_j^*, s_{-j})\bar{\sigma}(s) - \max_{s'_j \in S \setminus \{s_j^*\}} \sum_{s \in S} u_j(s'_j, s_{-j})\bar{\sigma}(s) < 0\), then some mixture between \(\sigma\) and \(\bar{\sigma}\) must belong to \(\Pi^0\).

So we have shown that if either \(\Pi^0\) is non-empty or both \(\Pi^+\) and \(\Pi^-\) are non-empty, then \(s_j^* \not\preceq s_i^*\).

If only \(\Pi^+\) is non-empty, then \(s_j^*\) is strictly interior dominant for \(j\). Together with \(s_i^* \succeq s_j^*\), this would imply that \(s_i^*\) is strictly interior dominant for \(i\), contradicting the assumption that (iii) does not hold.

Finally suppose that only \(\Pi^-\) is non-empty, so that for every \(\sigma \in \Delta^\circ(S)\) there exists a strictly better pure response than \(s_j^*\) against \(\sigma_{-j}\). Then, from Lemma 4 of Pearce (1984), there is a mixed strategy \(\sigma_j\) for \(j\) that weakly dominates \(s_j^*\) against all correlated strategy distributions. This \(\sigma_j\) strictly dominates \(s_j^*\) against strategy distributions in \(\Delta^\circ(S_{-j})\), so \(s_j^*\) is strictly interior dominated for \(j\). Since (ii) does not hold, there is a \(\sigma_{-i} \in \Delta^\circ(S_{-i})\) against which \(s_i^*\) is a weak best response. Then, the fact that \(s_j^*\) is not a strict best response against any \(\sigma_{-j} \in \Delta^\circ(S_{-j})\) means \(s_j^* \not\preceq s_i^*\).

\[\square\]

### 11.3 Proof of Lemma 3

**Proof.** We prove the first statement by contraposition. If \(C(\bar{\sigma})\) is not an \(\mathcal{C}(\bar{\epsilon})\)-equilibrium in the base game, then some \(i\) assigns more than the required weight to some \(s'_i \in S_i\) that does not best respond to \(C(\bar{\sigma})_{-i}\). This means no \((s'_i, n_i) \in \bar{S}_i\) best responds to \(\bar{\sigma}_{-i}\), since all copies of a strategy are payoff equivalent. Since \(C(\bar{\sigma})\) and \(\mathcal{C}(\bar{\epsilon})\) are defined by adding up the respective extended-game probabilities, \(C(\bar{\sigma})(s'_i) > \mathcal{C}(\bar{\epsilon})(s'_i | i)\) means \(\sum_{n_i} \bar{\sigma}_i(s'_i, n_i) > \sum_{n_i} \bar{\epsilon}(s'_i, n_i | i)\). So for at least one \(n'_i\), \(\bar{\sigma}_i(s'_i, n'_i) > \bar{\epsilon}(s'_i, n'_i | i)\), that is \(\bar{\sigma}_i\) assigns more than
required weight to the non best response \((s'_i, n'_i) \in \tilde{S}_i\). We conclude \(\tilde{\sigma}\) is not an \(\epsilon\)-equilibrium, as desired.

Again by contraposition, suppose \(\mathcal{E}(\sigma)\) is not an \(\mathcal{E}(\epsilon)\)-equilibrium in the extended game. This means some \(i\) assigns more than the required weight to some \((s'_i, n'_i) \in \tilde{S}_i\) that does not best respond to \(\mathcal{E}(\sigma)_{-i}\). This implies \(s'_i\) does not best respond to \(\sigma_{-i}\). By the definition of \(\mathcal{E}(\epsilon)\) and \(\mathcal{E}(\sigma)\), if \(\mathcal{E}(\sigma)(s'_i, n'_i) > \mathcal{E}(\epsilon)((s'_i, n'_i) | i)\), then also \(\mathcal{E}(\sigma)(s'_i, n_i) > \mathcal{E}(\epsilon)((s'_i, n_i) | i)\) for every \(n_i\) such that \((s'_i, n_i) \in \tilde{S}_i\). Therefore, we also have \(\sigma_i(s'_i) > \epsilon(s'_i | i)\), so \(\sigma\) is not an \(\epsilon\)-equilibrium in the base game as desired.

\[\square\]

11.4 Proof of Proposition 9

Proof. Suppose \(\tilde{\sigma}^*\) is a PCE in the extended game. So, we have \(\tilde{\sigma}^{(t)} \to \tilde{\sigma}^*\) where each \(\tilde{\sigma}^{(t)}\) is an \(\tilde{\mathcal{E}}^{(t)}\)-PCE, and each \(\tilde{\mathcal{E}}^{(t)}\) is player-compatible (in the extended game sense). This means each \(\mathcal{C}(\tilde{\mathcal{E}}^{(t)})\) is player compatible in the base game sense, and furthermore each \(\mathcal{C}(\tilde{\sigma}^{(t)})\) is an \(\mathcal{C}(\tilde{\mathcal{E}}^{(t)})\)-equilibrium (by Lemma 3), hence an \(\mathcal{C}(\tilde{\mathcal{E}}^{(t)})\)-PCE. Since \(\tilde{\mathcal{E}}^{(t)} \to 0\), \(\mathcal{C}(\tilde{\mathcal{E}}^{(t)}) \to 0\) as well. Since \(\tilde{\sigma}^{(t)} \to \tilde{\sigma}^*\), \(\mathcal{C}(\tilde{\sigma}^{(t)}) \to \mathcal{C}(\tilde{\sigma}^*)\). We have shown \(\mathcal{C}(\tilde{\sigma}^*)\) is a PCE in the base game. The proof of the other statement is exactly analogous. \(\square\)

12 Refinements in the Link-Formation Game

Proposition 11. Each of the following refinements selects the same subset of pure Nash equilibria when applied to the anti-monotonic and co-monotonic versions of the link-formation game: extended proper equilibrium, proper equilibrium, trembling-hand perfect equilibrium, \(p\)-dominance, Pareto efficiency, and strategic stability. Pairwise stability does not apply to the link-formation game. Finally, the link-formation game is not a potential game.

Proof. Step 1. Extended proper equilibrium, proper equilibrium, and trembling-hand perfect equilibrium allow the “no links” equilibrium in both versions of the game. For \((q_i)\) anti-monotonic with \((c_i)\), for each \(\epsilon > 0\) let N1 and S1 play Active with probability \(\epsilon^2\), N2 and S2 play Active with probability \(\epsilon\). For small enough \(\epsilon\), the expected payoff of Active for player \(i\) is approximately \((10 - c_i)\epsilon\) since terms with higher order \(\epsilon\)
are negligible. It is clear that this payoff is negative for small $\epsilon$ for every player $i$, and that under the utility re-scalings $\beta_{N1} = \beta_{S1} = 10$, $\beta_{N2} = \beta_{S2} = 1$, the loss to playing Active smaller for N2 and S2 than for N1 and S1. So this strategy profile is a $(\beta, \epsilon)$-extended proper equilibrium. Taking $\epsilon \to 0$, we arrive at the equilibrium where each player chooses Inactive with probability 1.

For the version with $(q_i)$ co-monotonic with $(c_i)$, consider the same strategies without re-scalings, i.e. $\beta = 1$. Then already the loss to playing Active smaller for N2 and S2 than for N1 and S1, making the strategy profile a $(1, \epsilon)$-extended proper equilibrium.

These arguments show that the “no links” equilibrium is an extended proper equilibrium in both versions of the game. Every extended proper equilibrium is also proper and trembling-hand perfect, which completes the step.

Step 2. $p$-dominance eliminates the “no links” equilibrium in both versions of the game. Regardless of whether $(q_i)$ are co-monotonic or anti-monotonic with $(c_i)$, under the belief that all other players choose Active with probability $p$ for $p \in (0, 1)$, the expected payoff of playing Active (due to additivity across links) is $(1 - p) \cdot 0 + p \cdot (10 + 30 - 2c_i) > 0$ for any $c_i \in \{14, 19\}$.

Step 3. Pareto eliminates the “no links” equilibrium in both versions of the game. It is immediate that the no-links equilibrium outcome is Pareto dominated by the all-links equilibrium outcome under both parameter specifications, so Pareto efficiency would rule it out whether $(c_i)$ is anti-monotonic or co-monotonic with $(q_i)$.

Step 4. Strategic stability (Kohlberg and Mertens, 1986) eliminates the “no links” equilibrium in both versions of the game. First suppose the $(c_i)$ are anti-monotonic with $(q_i)$. Let $\eta = 1/100$ and let $\epsilon' > 0$ be given. Define $\epsilon_{N1}(\text{Active}) = \epsilon_{S1}(\text{Active}) = 2\epsilon'$, $\epsilon_{N2}(\text{Active}) = \epsilon_{S2}(\text{Active}) = \epsilon'$ and $\epsilon_i(\text{Inactive}) = \epsilon'$ for all players $i$. When each $i$ is constrained to play $s_i$ with probability at least $\epsilon_i(s_i)$, the only Nash equilibrium is for each player to choose Active with probability $1 - \epsilon'$. (To see this, consider N2’s play in any such equilibrium $\sigma$. If N2 weakly prefers Active, then N1 must strictly prefer it, so $\sigma_{N1}(\text{Active}) = 1 - \epsilon' \geq \sigma_{N2}(\text{Active})$. On the other hand, if N2 strictly prefers Inactive, then $\sigma_{N2}(\text{Active}) = \epsilon' < 2\epsilon' \leq \sigma_{N1}(\text{Active})$. In either case, $\sigma_{N1}(\text{Active}) \geq \sigma_{N2}(\text{Active})$.)

When both North players choose Active with probability $1 - \epsilon'$, each South player has
Active as their strict best response, so $\sigma_{S_1}(\text{Active}) = \sigma_{S_2}(\text{Active}) = 1 - \epsilon'$. Against such a profile of South players, each North player has Active as their strict best response, so $\sigma_{N_1}(\text{Active}) = \sigma_{N_2}(\text{Active}) = 1 - \epsilon'$.

Now suppose the $(c_i)$ are co-monotonic with $(q_i)$. Again let $\eta = 1/100$ and let $\epsilon' > 0$ be given. Define $\epsilon_{N_1}(\text{Active}) = \epsilon_{S_1}(\text{Active}) = \epsilon'$, $\epsilon_{N_2}(\text{Active}) = \epsilon'/1000$, $\epsilon_{S_2}(\text{Active}) = \epsilon'$ and $\epsilon_i(\text{Inactive}) = \epsilon'$ for all players $i$. Suppose by way of contradiction there is a Nash equilibrium $\sigma$ of the constrained game which is $\eta$-close to the Inactive equilibrium. In such an equilibrium, N2 must strictly prefer Inactive, otherwise N1 strictly prefers Active so $\sigma$ could not be $\eta$-close to the Inactive equilibrium. Similar argument shows that S2 must strictly prefer Inactive. This shows N2 and S2 must play Active with the minimum possible probability, that is $\sigma_{N_2}(\text{Active}) = \epsilon'/1000$ and $\sigma_{S_2}(\text{Active}) = \epsilon'$. This implies that, even if $\sigma_{N_1}(\text{Active})$ were at its minimum possible level of $\epsilon'$, S1 would still strictly prefer playing Inactive because S1 is 1000 times as likely to link with the low-quality opponent as the high-quality opponent. This shows $\sigma_{S_1}(\text{Active}) = \epsilon'$. But when $\sigma_{S_1}(\text{Active}) = \sigma_{S_2}(\text{Active}) = \epsilon'$, N1 strictly prefers playing Active, so $\sigma_{N_1}(\text{Active}) = 1 - \epsilon'$. This contradicts $\sigma$ being $\eta$-close to the no-links equilibrium.

**Step 5. Pairwise stability** (Jackson and Wolinsky, 1996) **does not apply to this game.** This is because each player chooses between either linking with every player on the opposite side who plays Active, or linking with no one. A player cannot selectively cut off one of their links while preserving the other.

**Step 6. The game does not have an ordinal potential, so refinements of potential games** (Monderer and Shapley, 1996) **do not apply.** To see that this is not a potential game, consider the anti-monotonic parametrization. Suppose a potential $P$ of the form $P(a_{N_1}, a_{N_2}, a_{S_1}, a_{S_2})$ exists, where $a_i = 1$ corresponds to $i$ choosing Active, $a_i = 0$ corresponds to $i$ choosing Inactive. We must have

$$P(0, 0, 0, 0) = P(1, 0, 0, 0) = P(0, 0, 0, 1),$$

since a unilateral deviation by one player from the Inactive equilibrium does not change any player’s payoffs. But notice that $u_{N_1}(1, 0, 0, 1) - u_{N_1}(0, 0, 0, 1) = 10 - 14 = -4$, while
$u_S(1,0,0,1) - u_S(1,0,0,0) = 30 - 19 = 11$. If the game has an ordinal potential, then both of these expressions must have the same sign as $P(1,0,0,1) - P(1,0,0,0) = P(1,0,0,1) - P(0,0,0,1)$, which is not true. A similar argument shows the co-monotonic parametrization does not have a potential either. 

□