Repeated Games with Many Players*

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Abstract

Motivated by the problem of sustaining cooperation in large communities with limited information, we analyze sequences of repeated games where the population size $N$, the discount factor $\delta$, and the monitoring structure (with channel capacity denoted by $C$) vary together. We show that if $(1 - \delta) N/C \rightarrow \infty$ then all Nash equilibrium payoffs are consistent with approximately myopic play. A folk theorem under a novel identifiability condition provides a near converse. For example, if $(1 - \delta) N \log (N)/C \rightarrow 0$ then a folk theorem holds under random auditing, where each player’s action is monitored with the same probability in every period. If attention is restricted to linear equilibria (a generalization of strongly symmetric equilibria), cooperation is possible only under much more severe parameter restrictions. Methodologically, we develop connections between repeated games and information theory.

Keywords: repeated games, large populations, information theory, Fisher information, channel capacity, random auditing, linear equilibrium

JEL codes: C72, C73

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1 Introduction

Large groups of individuals often have a remarkable capacity for cooperation, even in the absence of external contractual enforcement (Ostrom, 1990; Ellickson, 1991; Seabright, 2004). Cooperation in large groups usually seems to rely on high-quality monitoring of individual agents’ actions, together with sanctions that narrowly target deviators. These are key features of the community resource management settings documented by Ostrom (1990), as well as the local public goods provision environment studied by Miguel and Gugerty (2005), who in a development context found that parents who fell behind on their school fees and other voluntary contributions faced social sanctions. Large cartels seem to operate similarly. For example, the Federation of Quebec Maple Syrup Producers—a government-sanctioned cartel that organizes more than 7,000 producers, accounting for over 90% of Canadian maple syrup production—strictly monitors its members’ sales, and producers who violate its rules regularly have their sugar shacks searched and their syrup impounded, and can also face fines, legal action, and ultimately the seizure of their farms (Kuitenbrouwer, 2016; Edmiston and Hamilton, 2018). In contrast, we are not aware of any evidence that individual maple syrup producers—or the parents studied by Miguel and Gugerty, or the farmers, fishers, and herders studied by Ostrom—are incentivized by the fear of starting a price war or other general breakdown of cooperation.

The principle that large-group cooperation requires precise monitoring and personalized sanctions seems like common sense, but it is not reflected in current repeated game models. The standard analysis of repeated games with patient players (e.g., Fudenberg, Levine, and Maskin, 1994, henceforth FLM) fixes all parameters of the game except the discount factor $\delta$ and considers the limit as $\delta \to 1$. This approach does not capture situations where, while players are patient ($\delta \approx 1$), they are not necessarily patient compared to the population size $N$ (so $(1 - \delta)N$ may or may not be close to 0). In addition, since standard results are based on statistical identifiability conditions that hold generically regardless of the number of players, they also do not capture the possibility that more information may be required to support cooperation in larger groups. Finally, since there is typically a vast multiplicity of cooperative equilibria in the $\delta \to 1$ limit, standard results also say little about what kind of strategies must be used to support large-group cooperation, for example whether it is better to rely on individual sanctions (e.g., fines) or collective ones (e.g., price wars).
In this paper, we extend the standard analysis of repeated games with imperfect monitoring by considering sequences of games where the population size, discount factor, stage game, and monitoring structure all vary together. These sequences of games can be quite general: we assume only uniform upper bounds on the magnitude of the players’ stage-game payoffs and the number of actions available to each player, and a uniform lower bound on “noise,” formalized as a constraint that each player takes each available action with positive probability. Our results characterize sequences of $N$, $\delta$, and measures of the “informative-ness” of the monitoring structure for which, in the limit, cooperation is either impossible (i.e., play is almost myopic in every equilibrium) or possible (i.e., a folk theorem holds), for any stage games satisfying our conditions. We also show that cooperation is possible only under much more restrictive conditions if society exclusively relies on collective sanctions. In sum, we show that large-group cooperation requires a lot of patience and/or a lot of information, and cannot be based on collective sanctions for reasonable parameter values.

We now preview our main ideas and results. A useful measure of the informativeness of a monitoring structure is its channel capacity, $C$. This is a standard measure in information theory, which in the current context is defined as the maximum expected reduction in uncertainty (entropy) about the realized action profile that results from observing all players’ signals (which can be distinct, as monitoring need not be public), where the maximum is taken over strategy profiles. Channel capacity obeys the elementary inequality $C \leq \log |Y|$, where $Y$ is the set of possible signal realizations. Due to this inequality, our results based on channel capacity are more general than they would be if we simply measured informativeness by the cardinality of the set of signal realizations.

Our first result is that if $(1 - \delta) N/C \to \infty$ along a sequence of repeated games satisfying our assumptions, then in the limit payoffs in any Nash or communication equilibrium are consistent with almost-myopic play. This result is an immediate corollary of some general connections between repeated games and information theory that we develop. These connections take the form of two lemmas, the first of which shows that, for any communication equilibrium in any repeated game with a full-support monitoring structure, and any possible deviation by any player, we have

$$\text{deviation gain} \leq \sqrt{\frac{\delta}{1 - \delta}} \text{(detectability)} \text{(payoff variance)}.$$
In this inequality, a player’s deviation gain and payoff variance are both assessed at the equilibrium occupation measure, which is the discounted, time-averaged distribution over action profiles induced by the equilibrium; and a deviation’s detectability is defined as the expected square likelihood difference of the signal under the deviation, as compared to equilibrium play.\(^1\) The inequality is established by a variance decomposition approach. Our second lemma then shows that, in games with a uniform lower bound on noise, per-capita detectability is at most a constant multiple of per-capita channel capacity. This lemma is established by standard information theory methods, such as Pinsker’s inequality. Putting the lemmas together, we see that if \((1 - \delta) N/C \to \infty\) then the per-capita deviation gain at the occupation measure converges to 0, and hence payoffs are consistent with almost-myopic play.

Our second result provides a near-converse to the first, under some additional structure on monitoring. It shows that, for any sequence of repeated games with public, product-structure monitoring that satisfies an individual identifiability condition with \(\eta\) slack (where \(\eta\) varies along the sequence), and where \((1 - \delta) \log (N)/\eta \to 0\), a folk theorem holds in the limit.\(^2\) A simple example of public, product-structure monitoring that satisfies individual identifiability with \(\eta\) slack is random auditing, where in each period \(\eta N\) randomly chosen players have their realized actions revealed perfectly, and nothing is revealed about the other players’ actions. Since the channel capacity of random auditing is a constant multiple of \(\eta N\), this example describes a class of monitoring structures where a folk theorem holds whenever \((1 - \delta) N \log (N)/C \to 0\). This shows that the condition of our first theorem is tight up to \(\log (N)\) slack.

Our final result considers the implications of restricting society to “collective” sanctions and rewards. We formalize this restriction by focusing on linear equilibria, where all off-equilibrium path continuation payoff vectors lies on a line in \(\mathbb{R}^N\). Note that, when the stage game is symmetric and the line in question is the 45° line, linear equilibria reduce to strongly symmetric equilibria, which are a standard model of collusion through the threat of price wars (Green and Porter, 1984; Abreu, Pearce, and Stacchetti, 1986; Athey, Bagwell, and Sanchirico, 2004). We show that if there exists \(\rho > 0\) such that \((1 - \delta) \exp (N^{1-\rho}) \to \infty\)

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\(^1\) “Detectability” is thus a discrete version of the Fisher information of the signal, a standard concept in Bayesian statistics.

\(^2\) The definitions of public, product-structure monitoring and individual identifiability follow FLM. The required slack in identifiability is measured in terms of detectability/Fisher information.
along a sequence of repeated games, then equilibrium payoffs are almost-myopic in the limit. Since this condition holds even if $N \to \infty$ much slower than $\delta \to 1$, we interpret this result as an impossibility theorem for large-group cooperation in linear equilibria.\(^3\)

The paper is organized as follows: following a discussion of related literature, Section 2 presents the model, Section 3 presents our anti-folk theorem, Section 4 presents our folk theorem, Section 5 presents our stronger anti-folk theorem for linear equilibria, and Section 6 concludes. Proofs are in Appendix A, with some details deferred to (Online) Appendix B.

### 1.1 Related Literature

Prior research on repeated games has established folk theorems in the $\delta \to 1$ limit for fixed $N$, as well as anti-folk theorems in the $N \to \infty$ limit for fixed $\delta$, but has not considered the case where $N$ and $\delta$ vary together. As this is the case where monitoring precision is critical, our main results on the three-way interaction among $N$, $\delta$, and monitoring do not have close antecedents in the literature.

The most relevant folk theorem papers are FLM and Kandori and Matsushima (1998, henceforth KM). We build on their analysis (as well as that of Abreu, Pearce, and Stacchetti, 1990, henceforth APS), but as we explain in Section 4 their proofs of the folk theorem do not easily extend to the case where $N$ and $\delta$ vary together. We thus take a different approach, which is based on “block strategies” as in Matsushima (2004) and Hörner and Olszewski (2006), and involves a novel application of some large deviations bounds.

The most relevant anti-folk theorem papers are Fudenberg, Levine, and Pesendorfer (1998), Al-Najjar and Smorodinsky (2000, 2001), Pai, Roth, and Ullman (2014), and Awaya and Krishna (2016, 2019). Following earlier work by Green (1980) and Sabourian (1990), most of these papers establish conditions under which play becomes approximately myopic as $N \to \infty$ for fixed $\delta$.\(^4\) As we explain in Section 3, these papers can be adapted to give results that apply when $N$, $\delta$, and monitoring vary together, but these results are substantially weaker than ours, and in particular are not tight up to log terms. They key difference is that these papers do not use variance decomposition or information theory tools, which are

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\(^3\)It is well-known that strongly symmetric equilibria are typically less efficient than general public perfect equilibria in games with public monitoring. Our result is instead that the relationship between $N$ and $\delta$ required for any non-trivial incentive provision differs dramatically between strongly symmetric (more generally, linear) equilibria and general ones.

\(^4\)Awaya and Krishna instead establish conditions under which access to cheap talk is valuable.
not required for their goal of establishing convergence to myopic play as \( N \to \infty \) for fixed \( \delta \), but are needed for our goal of quantifying the interaction among \( N, \delta \), and monitoring.\(^5\)

Since the monitoring structure can vary with \( \delta \) in our model, we also relates to the literature on repeated games with frequent actions, where the monitoring structure varies with \( \delta \) in a particular, parametric manner (Abreu, Milgrom, and Pearce, 1991; Fudenberg and Levine, 2007, 2009; Sannikov and Skrzypacz, 2007, 2010; Rahman, 2014). The most related results here are Sannikov and Skrzypacz’s (2007) theorem on the impossibility of collusion in duopoly with frequent actions and Brownian noise, as well as a similar result by Fudenberg and Levine (2007). These results are related to our anti-folk theorem for linear equilibrium, as we explain in Section 5, but are not closely related to our general results.

We believe this paper is the first in the repeated games literature to measure monitoring precision by the mutual information between signals and actions, and to investigate maximal equilibrium payoffs subject to a constraint on mutual information.\(^6\) Our results thus concern optimal monitoring structure design in repeated games, although we consider only asymptotic results rather than exact optimality for fixed parameters. In static moral hazard problems, optimal monitoring design subject to information-theoretic constraints was recently studied by Georgiadis and Szentes (2020), Li and Yang (2020), and Hoffman, Inderst, and Opp (2021), while an earlier literature (Maskin and Riley, 1985; Khalil and Lawrence, 1995; Lewis and Sappington, 1995) studied the choice between monitoring inputs and outputs. Random auditing, which we find to be approximately optimal, also arises in static, costly state-verification models (Reinganum and Wilde, 1985; Border and Sobel, 1987; Mookherjee and Png, 1989).

Finally, in earlier work (Sugaya and Wolitzky, 2021a) we studied the relation among \( N, \delta \), and monitoring in a repeated random-matching game with private monitoring and incomplete information, where each player is “bad” with some probability. In that model, society has enough information to determine which players are bad after a single period

\(^5\)For fixed \( N \), Hörner and Takahashi (2016) consider the rate of convergence in \( \delta \) of the public perfect equilibrium payoff set to the limit set characterized by Fudenberg and Levine (1994) and FLM. While their analysis concerns rates of convergence in one parameter to a known limit payoff set, we ask how the relationship among different parameters determines the limit payoff set. Farther afield, there is also some work suggesting that cooperation in repeated games is harder to sustain in larger groups based on evolutionary models (Boyd and Richerson, 1988) and simulations (Bowles and Gintis, 2011; Chapter 5).

\(^6\)Entropy methods have previously been used in the repeated games literature to study automaton strategies (Neyman and Okada, 1999, 2000), communication (Gossner, Hernández, and Neyman, 2006), and reputation effects (Gossner, 2011; Ekmekci, Gossner, and Wilson, 2011; Faingold, 2020).
of play, but this information is decentralized, so the question is whether the players can aggregate information fast enough to ensure that it pays to be good. In contrast, in the current paper there is complete information and monitoring can be public, so the analysis concerns monitoring precision (the “amount” of information available to society) rather than the speed of information aggregation (the “distribution” of information).

2 Model

Stage Games. A stage game $G = (I, A, \hat{u})$ consists of a finite set of players $I = \{1, \ldots, N\}$, a finite action set $A_i$ for each player $i \in I$, and a payoff function $\hat{u}_i : A \rightarrow \mathbb{R}$ for each $i \in I$ (where $A = \prod_{i \in I} A_i$ and $\hat{u}(a) = (\hat{u}_i(a))_{i \in I}$ for $a \in A$). We assume that there is some independent noise in the execution of players’ actions: for each player $i$, there is a row-stochastic noise matrix $\pi^i \in [0, 1]^{A_i \times A_i}$ such that, for any $a_i, \hat{a}_i \in A_i$, when player $i$ chooses an intended action $a_i$, her realized action is $\hat{a}_i$ with probability $\pi^i_{a_i, \hat{a}_i}$. Let $\hat{\pi}^i = \min_{a_i, \hat{a}_i \in A} \pi^i_{a_i, \hat{a}_i}$ and assume that $\hat{\pi}^i > 0$ for each $i$. Define $\pi \in [0, 1]^{A \times A}$ by $\pi_{a, \hat{a}} = \prod_i \hat{\pi}^i_{a_i, \hat{a}_i}$ for all $a, \hat{a} \in A$. Given a stage game $G$ and noise $\pi$, define the expected payoff function $u_i : A \rightarrow \mathbb{R}$ by $u_i(a) = \sum_{\hat{a}} \pi_{a, \hat{a}} \hat{u}_i(\hat{a})$ for each $i$. Denote the range of player $i$’s expected payoffs by $\bar{u}_i = \max_a u_i(a) - \min_a u_i(a)$. As we will explain, the independent noise assumption is necessary for our anti-folk theorem (Theorem 1), but not for any of our other results, including our general bound on the strength of players’ incentives (Lemma 1).

Monitoring Structures and Channel Capacity. A monitoring structure $(Y, p)$ consists of a finite set of signals of the form $y = (y^i)_{i \in I}$ and a family of conditional probability distributions $p(y|\hat{a})$. The signal distribution thus depends only on the realized action profile. Given a monitoring structure $(Y, p)$, denote the probability of signal $y$ at intended action profile $a$ by $q(y|a) = \sum_{\hat{a}} \pi_{a, \hat{a}} p(y|\hat{a})$. Assume without loss that for each $y \in Y$, there exists $\hat{a} \in A$ such that $p(y|\hat{a}) > 0$. Together with the assumption that $\hat{\pi}^i > 0$ for each $i$, this implies that $q$ has full support: $q(y|a) > 0$ for all $y$ and $a$.

Given a distribution of realized action profiles $\hat{a} \in \Delta(A)$, a standard measure of the informativeness of a monitoring structure $(Y, p)$ about the realized action profile $\hat{a}$ is the
Mutual information between these random variables, defined as

\[ I(\hat{a}) = \sum_{y \in Y} \sum_{\hat{a} \in A} \hat{a}(\hat{a}) p(y|\hat{a}) \frac{\hat{a}(\hat{a}) p(y|\hat{a})}{\sum_{\hat{a}' \in A} \hat{a}(\hat{a}') p(y|\hat{a}')} . \]

Mutual information measures the expected reduction in uncertainty (entropy) about \( \hat{a} \) that results from observing \( y \). This is an endogenous object, as it depends on the prior distribution \( \hat{a} \) over realized action profiles. The channel capacity of \((Y, p)\) is defined as

\[ C = \max_{\hat{a} \in \Delta(A)} I(\hat{a}) . \]

This is an exogenous measure of the informativeness of \((Y, p)\) about \( \hat{a} \). Note that \( C \) is no greater than the entropy of the signal \( y \), which in turn is at most \( \log |Y| \) (Theorem 2.6.3 of Cover and Thomas, 2006, henceforth CT).

Our folk theorem (Theorem 2) will assume that monitoring is public and has a product structure. A monitoring structure \((Y, p)\) is public if all players observe the same signal: \( y^i = y^j \) for all \( i, j \in I, y \in Y \). In this case, in a slight abuse of notation, we simply denote the public signal by \( y \). A public monitoring structure \((Y, p)\) has a product structure if there exists sets \((Y^i)_i \in I\) and families of conditional distributions \((q_i(y^i|a^i))_{i \in I, y^i \in Y^i, a^i \in A^i}\) such that \( Y = \prod_i Y^i \) and \( q(y|a) = \prod_i q_i(y^i|a^i) \) for all \( y, a \); that is, the public signal \( y \) consists of conditionally independent signals of each player’s action.\(^8\) A sufficient condition for \((Y, p)\) to have a product structure is that there exist sets \((Y^i)_i \in I\) and families of conditional distributions \((p_i(y^i|\hat{a}^i))_{i \in I, y^i \in Y^i, \hat{a}^i \in A^i}\) such that \( Y = \prod_i Y^i \) and \( p(y|\hat{a}) = \prod_i p_i(y^i|\hat{a}^i) \) for all \( y, \hat{a} \).

**Repeated Games.** A repeated game is described by a stage game, a noise matrix, a discount factor \( \delta \in (0, 1) \), and a monitoring structure: that is, a tuple \( \Gamma = (G, \pi, \delta, Y, p) \). In each period \( t = 1, 2, \ldots \), (i) the players observe the outcome of a public randomizing device \( z_t \) drawn from the uniform distribution over \([0, 1]\), (ii) the players take actions \( a \), (iii) the realized action profile \( \hat{a} \) is drawn according to \( \pi_{a, \hat{a}} \), (iv) the signal \( y \) is drawn according to \( p(y|\hat{a}) \), and (v) each player \( i \) observes \( y^i \).\(^9\) A history \( h_t^i \) for player \( i \) at the beginning of period

\(^7\)In this paper, all logarithms are base \( e \).

\(^8\)Our notation is thus that \( Y^i \) denotes the set of possible signals observed by player \( i \) (for any monitoring structure), while \( Y^i \) denotes the set of public signals of player \( i \)’s action (for a public monitoring structure).

\(^9\)The analysis and results are unchanged if player \( i \) also observes her own realized action \( \hat{a}_i \).
thus takes the form \( h_i^t = ((z_t, a_i, y_t)_{t=1}^{t-1}, z_t) \), with \( h_i^1 = z_1 \), while a strategy \( \sigma_i \) for player \( i \) maps histories \( h_i^t \) to distributions of intended actions \( \Delta(A_i) \), for each \( t \). Players maximize discounted expected payoffs.

We consider sequences of repeated games \( \Gamma \) indexed by \( k \in \mathbb{N} \). Thus, \( I, A, \hat{u}, \pi, \delta, Y, \) and \( p \) (and hence also \( C \)) all implicitly depend on \( k \), although we usually suppress this dependence to simplify notation. Throughout, we restrict attention to sequences \( (\Gamma)^k \) with uniformly bounded payoffs and uniformly bounded noise: there exist \( \bar{u} > 0 \) and \( \bar{\pi} > 0 \) such that, for all \( k \in \mathbb{N} \) and \( i \in I^k \), we have \( \bar{u}^k_i \leq \bar{u} \) and \( \pi^k_i \geq \bar{\pi} \). Note that since \( \pi^k_i |A^k_i| \leq 1 \), the latter assumption implies that \( |A^k_i| \leq 1/\bar{\pi} \) for all \( k \) and \( i \).

Our anti-folk theorem will apply not only for any Nash equilibrium, but also for any communication equilibrium with a mediator who observes \( y \) (but not \( a \) or \( \hat{a} \)). To define the relevant communication equilibrium concept, consider the mediated repeated game, where there is a mediator who at the end of each period observes \( y \), and at the beginning of each period sends a private message \( r_i \in R_i \) to each player \( i \), for some sets \( (R_i)_{i \in I} \).\(^{10}\) An outcome \( \mu \) of the game is a distribution over paths of actions and signals, \( (A \times Y)^\infty \). For any equilibrium concept, the set of equilibrium outcomes is larger in the mediated game than in the unmediated game, because the mediator can always babble. An equilibrium in the mediated game is canonical if, for each player \( i \), \( R_i = A_i \) and player \( i \)'s equilibrium strategy is to take \( a_i = r_i \) at every history. The mediator’s messages are thus interpreted as recommendations in a canonical equilibrium. Since \( q \) has full support, the set of sequential equilibrium outcomes in the mediated game equals the set of canonical Nash equilibrium outcomes in the mediated game.\(^{11}\) The elements of this set are the communication equilibria.\(^{12}\)

**Sets of Payoffs.** Given a stage game \( G \) with (expected) payoff functions \( (u_i)_i \), we define some relevant sets of payoff vectors. The feasible payoff set is \( V = \text{co} \{ u(a) \}_{a \in A} \), where \( \text{co} \) denotes convex hull. Let \( V^* \subseteq V \) denote the set of payoff vectors that Pareto-dominate a Nash equilibrium payoff in \( G \): that is, \( v \in V^* \) iff \( v \in V \) and there exists a static Nash equilibrium.
equilibrium $\alpha \in \prod_i \Delta (A_i)$ in $G$ such that $v \geq u (\alpha)$.\(^{13}\) For each $v \in \mathbb{R}^N$ and $\varepsilon > 0$, let $B_v (\varepsilon) = \prod_i [v_i - \varepsilon, v_i + \varepsilon]$, and let $B (\varepsilon) = \{ v \in \mathbb{R}^N : B_v (\varepsilon) \subseteq V^* \}$. That is, $B (\varepsilon)$ is the set of payoff vectors $v \in \mathbb{R}^N$ such that the cube with center $v$ and side-length $2\varepsilon$ lies entirely within $V^*$.

A *manipulation* for a player $i$ is a mapping $\tilde{\alpha}_i : A_i \to \Delta (A_i)$. The interpretation is that when player $i$ is recommended action $r_i$, she instead plays the mixed action $\tilde{\alpha}_i (r_i)$. Player $i$’s *deviation gain* from manipulation $\tilde{\alpha}_i$ at a (possibly correlated) action profile distribution $\alpha \in \Delta (A)$ is

$$D_i (\tilde{\alpha}_i | \alpha) = \sum_{r \in A} \alpha (r) \left( u_i (\tilde{\alpha}_i (r_i), r_{-i}) - u_i (r) \right),$$

and her *maximum deviation gain* at $\alpha \in \Delta (A)$ is

$$D_i (\alpha) = \max_{\tilde{\alpha}_i : A_i \to \Delta A_i} D_i (\tilde{\alpha}_i | \alpha).$$

The per-player average deviation gain at $\alpha$ is $\sum_i D_i (\alpha) / N$. For each $\varepsilon > 0$, the set of *action distributions consistent with $\varepsilon$-myopic play* is

$$A (\varepsilon) = \left\{ \alpha \in \Delta (A) : \frac{1}{N} \sum_i D_i (\alpha) \leq \varepsilon \right\},$$

and the set of *payoff vectors consistent with $\varepsilon$-myopic play* is

$$M (\varepsilon) = \{ v \in \mathbb{R}^N : v = u (\alpha) \text{ for some } \alpha \in A (\varepsilon) \}.$$

Note that a few players can have large deviation gains at an action distribution $\alpha \in A (\varepsilon)$. However, so long as the effect of each player’s action on the sum of her opponents’ payoffs is bounded, population payoffs at any $\alpha \in A (\varepsilon)$ must be close to those at an action distribution where all players’ deviation gains are less than $\varepsilon$. Note also that $D_i (\alpha)$ is convex as the maximum of affine functions, and hence $A (\varepsilon)$ and $M (\varepsilon)$ are convex sets.

Our anti-folk theorem will provide conditions under which all equilibrium payoff vectors are contained in $M (\varepsilon)$, while our folk theorem will provide conditions under which all payoff vectors in $B (\varepsilon)$ arise in equilibrium. As a check that the set $B (\varepsilon)$ is reasonably large for all $N$, in Appendix B.1 we consider a canonical public-goods game where each player chooses

\(^{13}\)Here and throughout, we linearly extend payoff functions to mixed actions.
Contribute or Don’t Contribute, and a player’s payoff is the fraction of players who contribute less a constant \( c \in (0, 1) \) (independent of \( N \)) if she contributes herself. In this game, we show that for every \( v \in (0, 1 - c) \) there exists \( \varepsilon > 0 \) such that the symmetric payoff vector where all players receive payoff \( v \) lies in \( B(\varepsilon) \) for all \( N \).

## 3 Necessary Conditions for Cooperation

### 3.1 Anti-Folk Theorem

Our first result is that whenever per-capita channel capacity is much smaller than the discount rate, payoffs are consistent with almost-myopic play.

**Theorem 1** Consider any sequence of repeated games \((\Gamma)^k\) such that \((1 - \delta) N/C \to \infty\) (where \( \delta, N, \) and \( C \) depend on \( k \)) and any corresponding sequence of communication equilibrium payoffs \((v)^k\). For any \( \varepsilon > 0 \), there exists \( \tilde{k} \) such that, for every \( k \geq \tilde{k} \), the payoff vector \( v^k \) is consistent with \( \varepsilon \)-myopic play.

Theorem 1 implies that cooperation in large groups requires a large amount of information, a high degree of patience, or both. Whether the implied necessary condition for cooperation—that \((1 - \delta) N/C \) is not too large—is highly restrictive for large \( N \) depends on the type of game under consideration. If the space of possible signal realizations \( Y \) is fixed independently of \( N \), then \( C \leq \log |Y| \) for all \( N \), so the necessary condition implies that \( \delta \) must converge to 1 at least as fast as \( N \to \infty \), which is a restrictive condition. This negative conclusion applies for traditional applications of repeated games with public monitoring where the signal space is fixed independently of \( N \), such as when the public signal is the market price facing Cournot competitors, the level of pollution in a common water source, or the output of team production.

However, in other types of games \( C \) naturally scales linearly with \( N \), so that \((1 - \delta) N/C \) is small whenever players are patient. In repeated games with random matching (Kandori, 1992; Ellison, 1994; Deb, Sugaya, and Wolitzky, 2020), players match in pairs each period and \( y_{it} = a_{m(i,t),t} \), where \( m(i,t) \in I \setminus \{i\} \) denotes player \( i \)'s period-\( t \) partner. In these games, \( C = N \log |A_i| \), so per-capita channel capacity is independent of \( N \). Intuitively, in random matching games each player gets a distinct piece of information, so the total amount
of information available to society is proportional to population size. Channel capacity may also scale linearly with \( N \) in public-monitoring games where the public information includes a distinct signal of each player’s action, as in the ratings systems used by websites like eBay and AirBnB. In general, \( C/N \) may be constant in games where each player is monitored “separately,” but it converges to 0 as \( N \to \infty \) in games where the players are jointly monitored through an aggregate statistic which is independent of \( N \).

**Remark 1** In applications like Cournot competition, pollution, or team production, the signal space may be modeled as a continuum, in which case the constraint \( C \leq \log |Y| \) is vacuous. However, our results extend to the case where \( Y \) is a compact metric space and there exists a compact metric space \( X \) and a function \( f^N : A^N \to X \) (which can vary with \( N \)) such that the signal distribution admits a conditional density of the form \( p_{Y|X}(y|x) \), where \( Y, X \), and \( p_{Y|X} \) are fixed independent of \( N \). (For example, in Cournot competition \( x \) is industry output and \( y \) is the market price; and the market price depends on \( x \) and noise, where the “amount of noise” is independent of \( N \).) In this case,

\[
C = \max_{\hat{a} \in \Delta(A)} \int_{y \in Y} \sum_{\hat{a} \in A} \hat{\alpha}(\hat{a}) p_{Y|X}(y|f^N(\hat{a})) \log \frac{\hat{\alpha}(\hat{a}) p_{Y|X}(y|f^N(\hat{a}))}{\sum_{\hat{a'} \in A} \hat{\alpha}(\hat{a'}) p_{Y|X}(y|f^N(\hat{a'}))},
\]

which is bounded by

\[
\bar{C} = \max_{p_X \in \Delta(X)} \int_{y \in Y} \int_{x \in X} p_X(x) p_{Y|X}(y|x) \log \frac{p_X(x) p_{Y|X}(y|x)}{\int_{x \in X} p_X(x) p_{Y|X}(y|x)}.\]

Since \( \bar{C} \) is independent of \( N \), it follows that \( C \) is bounded independently of \( N \).

The proof of Theorem 1 proceeds in two steps. First, we bound the strength of players’ incentives in terms of the *detectability* of manipulations (Lemma 1 in Section 3.2). This incentive bound holds generally for any repeated game and may have other applications. Second, we show that in our model with noisy action execution, detectability is bounded by per-capita channel capacity (Lemma 2 in Section 3.3). Combining the two bounds yields the theorem.
3.2 Bounding Incentives by Detectability

Note that if an action profile \( r \in A \) is recommended but player \( i \) manipulates according to \( \bar{\alpha}_i : A_i \rightarrow A_i \), the probability that signal \( y \) realizes equals

\[
q(y|r; \bar{\alpha}_i) := \sum_{a_i \in A_i} \bar{\alpha}_i(a_i|r_i) q(y|a_i, r_{-i}),
\]

where \( \bar{\alpha}_i(a_i|r_i) \) is the probability that \( \bar{\alpha}_i(r_i) \) assigns to action \( a_i \). The following is a key definition.

**Definition 1** The detectability of manipulation \( \bar{\alpha}_i \) at action profile \( r \) is

\[
F_i(\bar{\alpha}_i|r) := \sum_{y \in Y} q(y|r) \left( \frac{q(y|r; \bar{\alpha}_i) - q(y|r)}{q(y|r)} \right)^2.
\]

The maximum detectability of manipulation \( \bar{\alpha}_i \) is \( F_i(\bar{\alpha}_i) := \max_{r \in A} F_i(\bar{\alpha}_i|r) \).

That is, \( F_i(\bar{\alpha}_i|r) \) is the expected square likelihood difference when player \( i \) takes \( \bar{\alpha}_i(r_i) \) rather than \( r_i \) and her opponents take \( r_{-i} \). Note that \( F_i(\bar{\alpha}_i|r) \) (and hence \( F_i(\bar{\alpha}_i) \)) is well-defined because \( q \) has full support. For interpretation, note that \( F_i(\bar{\alpha}_i|r) \) is a discrete version of the *Fisher information* in Bayesian statistics, which in the current setting would equal

\[
\sum_{y \in Y} q(y|r) \left( \frac{\partial}{\partial r_i} q(y|r_i, r_{-i}) \right)^2.
\]

The Fisher information is the variance of the score, \( (\partial/\partial r_i) \log q(y|r_i, r_{-i}) \), which is the key information measure in moral hazard problems (Mirrlees, 1975; Holmström, 1979). Since the expectation of the score equals 0, the score is “often large” if the Fisher information is large, in which case the signal is useful for providing incentives.\(^{14}\)

Denote the variance of player \( i \)'s payoff under an action profile distribution \( \alpha \) by \( V_i(\alpha) = \text{Var}_{a \sim \alpha} (u_i(a)) \). For any subset of players \( J \subseteq I \), action profile distribution \( \alpha \in \Delta(A) \), and profile of manipulations \( \bar{\alpha}_J = (\bar{\alpha}_i)_{i \in J} \) for players \( i \in J \), we also define “group average”

\(^{14}\)It has previously been observed that the Fisher information plays an important role in moral hazard problems with quadratic utility (Jewitt, Kadan, and Swinkels, 2008; Hebert, 2018) or frequent actions (Sadzik and Stacchetti, 2015).
versions of the deviation gain $D_i$, detectability $F_i$, and payoff variance $V_i$, by

$$D_J(\tilde{\alpha}_J|\alpha) = \frac{1}{|J|} \sum_{i \in J} D_i(\tilde{\alpha}_i|\alpha), \quad D_J(\bar{\alpha}) = \frac{1}{|J|} \sum_{i \in J} D_i(\bar{\alpha}),$$

$$F_J(\tilde{\alpha}_J|r) = \frac{1}{|J|} \sum_{i \in J} F_i(\tilde{\alpha}_i|r), \quad F_J(\bar{\alpha}, r) = \frac{1}{|J|} \sum_{i \in J} F_i(\bar{\alpha}_i), \quad V_J(\alpha) = \frac{1}{|J|} \sum_{i \in J} V_i(\alpha).$$

Finally, given an outcome $\mu \in \Delta((A \times Y)^\infty)$, let $\alpha^\mu_t \in \Delta(A)$ denote the marginal of $\mu$ over period-$t$ action profiles, and define $\alpha^\mu \in \Delta(A)$, the occupation measure over action profiles induced by $\mu$, by

$$\alpha^\mu(a) = (1 - \delta) \sum_{t=1}^\infty \delta^{t-1} \alpha^\mu_t(a)$$

for each $a \in A$.

Note that the repeated-game payoffs under outcome $\mu$ equal

$$(1 - \delta) \sum_{t=1}^\infty \delta^{t-1} \sum_{a \in A} \alpha^\mu_t(a) u(a) = \sum_{a \in A} (1 - \delta) \sum_{t=1}^\infty \delta^{t-1} \alpha^\mu_t(a) u(a) = \sum_a \alpha^\mu(a) u(a) = u(\alpha^\mu).$$

(2)

We can now state our incentive bound.

**Lemma 1** For any communication equilibrium outcome $\mu$, any subset of players $J$, and any profile of manipulations $\tilde{\alpha}_J$, we have

$$D_J(\tilde{\alpha}_J|\alpha^\mu) \leq \sqrt{\frac{\delta}{1 - \delta} F_J(\tilde{\alpha}_J) V_J(\alpha^\mu)}. \quad (3)$$

In particular, any communication equilibrium payoff vector $v$ is consistent with $\varepsilon$-myopic play, where

$$\varepsilon = \sqrt{\frac{\delta}{1 - \delta} \left( \max_{\tilde{\alpha}_J} F_J(\tilde{\alpha}_J) \right)} \tilde{u}^2.$$
equilibria and are explicit only in the $\delta \to 1$ limit; and the incentive bounds of Fudenberg, Levine, and Pesendorfer (1998), al-Najjar and Smorodinsky (2000, 2001), Pai, Roth, and Ullman (2017), and Awaya and Krishna (2016, 2019) depend on the ratio of the total variation distance $\| q(y|r; \tilde{\alpha}_i) - q(y|r) \|_{TV} := (1/2) \sum_{y \in Y} |q(y|r; \tilde{\alpha}_i) - q(y|r)|$ and $(1 - \delta)$. Under our full support assumption, $\| q(y|r; \tilde{\alpha}_i) - q(y|r) \|_{TV}$ is greater than a constant multiple of $\sqrt{F_i(\tilde{\alpha}_i|r)}$, so these results give a looser comparison between information and discounting than Lemma 1 does, while the full force of the latter result is needed to prove Theorem 1. A relatively straightforward extension of the earlier results could be used to show that limit payoffs are almost-myopic if $(1 - \delta) \sqrt{N}/\log |Y| \to \infty$, but this hypothesis much more restrictive than that of Theorem 1, and in particular is not tight up to log terms. The key technical difference is that the earlier approaches place no restrictions on continuation payoffs beyond feasibility, while we use the law of total variance to recursively bound the variance of continuation payoffs.\footnote{See, e.g., Proposition 4.1 of Awaya and Krishna (2019), which actually involves $\| q_{-i}(y|r; \tilde{\alpha}_i) - q_{-i}(y|r) \|_{TV}$, where $q_{-i}$ denotes the marginal of $q$ on $Y^{-i}$. This modification leads to a tighter incentive bound for signals where the impact of a player’s action on the distribution of $y$ is much greater than its impact on the distribution of $y_{-i}$. The distinction between $q_{-i}$ and $q$ plays an important role in Awaya and Krishna’s analysis but is irrelevant in ours; our result and theirs are thus complementary.}

The proof of Lemma 1 (in Appendix A.1) shows that if deviating according to manipulation $\tilde{\alpha}_i$ is unprofitable in each period, then (3) must hold for $J = \{i\}$.\footnote{Notably, recursively decomposing the variance of continuation payoffs gives a useful bound on the strength of incentives, even though the equilibrium payoff set generally lacks a recursive structure (as is well known, cf. Kandori, 2002). This observation may be of independent interest.} There are three steps. First, if deviating according to $\tilde{\alpha}_i$ is unprofitable in period $t$, the conditional variance of player $i$’s period-$(t + 1)$ continuation payoff must be sufficiently high compared to the deviation gain from $\tilde{\alpha}_i$ in period $t$ and the (inverse of the) maximum detectability of $\tilde{\alpha}_i$ (equation (8) in Appendix A.1). This step follows from an application of Cauchy-Schwarz. Second, applying this lower bound on conditional variance recursively using the law of total variance, we show that a discounted sum of the variances of player $i$’s stage-game payoffs must exceed a discounted sum of the conditional variance bounds. Third, by Jensen’s inequality, this inequality relating a discounted sum of payoff variances and a discounted sum of bounds that depend on the deviation gain and detectability in each period implies a corresponding inequality relating the payoff variance and deviation gain evaluated at the equilibrium occupation measure and the maximum detectability (equation (9)), which simplifies to (3).\footnote{Once (3) is established for singleton $J$, the extension to non-singleton $J$ is immediate from Cauchy-Schwarz.}
Since Lemma 1 may be of independent interest, we illustrate it with an example.

**Example 1. Prisoner’s Dilemma with Public, Product-Structure Monitoring.**

Suppose the stage game is the 2-player prisoner’s dilemma:

\[
\begin{array}{cc}
C & D \\
C & 1, 1 & -1, 2 \\
D & 2, -1 & 0, 0
\end{array}
\]

Assume that monitoring is public and has a product structure, where 
\[ Y = \{C, D\} \times \{C, D\} \]
and \[ p_i(y_i = a_i|a_i) = \chi \in (1/2, 1) \] for each \( i \) and \( a_i \in \{C, D\} \).

Let \( J = \{1\} \), and consider the problem of finding \( \tilde{\alpha}_i \) to maximize \( D_i(\tilde{\alpha}_i|\alpha^\mu) \) subject to (3). For any (intended) action profile distribution \( \alpha = (\alpha_{CC}, \alpha_{CD}, \alpha_{DC}, \alpha_{DD}) \), we have \[ D_1(\tilde{\alpha}_1|\alpha) = (\alpha_{CC} + \alpha_{CD}) \tilde{\alpha}_1 (D|C) - 2(\alpha_{DC} + \alpha_{DD}) \tilde{\alpha}_1 (C|D) \]. Since \( D_1(\tilde{\alpha}_1|\alpha) \) is decreasing in \( \tilde{\alpha}_1 (C|D) \) and \( F_i(\tilde{\alpha}_i) \) is increasing in \( \tilde{\alpha}_1 (C|D) \) (as a manipulation that disobeys the recommendation with higher probability is weakly more detectable), the solution to this problem involves \( \tilde{\alpha}_1 (C|D) = 0 \). Next, when \( \tilde{\alpha}_1 (C|D) = 0 \), the detectability of manipulation \( \tilde{\alpha}_1 (F_1(\tilde{\alpha}_1|r)) \) is maximized at \( r_1 = C \) (and any value of \( r_2 \), since monitoring has a product structure). Letting \( \tilde{\alpha}_1 (D|C) = \rho \) to ease notation, we therefore have

\[
F_1(\tilde{\alpha}_1) = \chi \left( \frac{\rho (1-\chi) + (1-\rho) \chi - \chi}{\chi} \right)^2 + (1-\chi) \left( \frac{\rho \chi + (1-\rho) (1-\chi) - (1-\chi)}{1-\chi} \right)^2
\]

\[ = \frac{\rho^2 (2\chi - 1)^2}{\chi (1-\chi)}. \]

Note that \( d_1(\tilde{\alpha}_1|\alpha) \) and \( F_1(\tilde{\alpha}_1) \) are both linear in \( \rho \), so the manipulation that maximizes \( D_1(\tilde{\alpha}_1|\alpha^\mu) \) subject to (3) is given by \( (\tilde{\alpha}_1 (C|D) = 0, \tilde{\alpha}_1 (D|C) = 1) \). Therefore, (3) reduces to the requirement that, in any communication equilibrium, we have

\[
\alpha_{CC} + \alpha_{CD} \leq \sqrt{\frac{\delta}{1 - \delta} \frac{(2\chi - 1)^2}{\chi (1-\chi)} V_i(\alpha)},
\]

where \( V_i(\alpha) = \alpha_{CC} + 4\alpha_{DC} + \alpha_{CD} - (\alpha_{CC} + 2\alpha_{DC} - \alpha_{CD})^2 \). We thus obtain a simple upper bound on probability that the players cooperate, as a function of the discount factor \( \delta \) and the monitoring precision \( \chi \), which applies for any communication equilibrium.
3.3 Bounding Detectability by Channel Capacity

We now show that detectability can be bounded by per-capita channel capacity.

**Lemma 2** For any subset of players $J \subseteq I$ and any profile of manipulations $\tilde{\alpha}_J$, we have

$$F_J(\tilde{\alpha}_J) \leq \frac{4C}{\pi^2 |J|}.$$  

(4)

The proof of Lemma 2 (in Appendix A.2) follows from a relatively straightforward application of information theory results such as Pinsker’s inequality, as well as another application of Cauchy-Schwarz.

Theorem 1 follows immediately from Lemmas 1 and 2.

**Proof of Theorem 1.** By Lemmas 1 and 2, all communication equilibrium payoff vectors are consistent with $\varepsilon$-myopic play, where

$$\varepsilon = \sqrt{\frac{4\tilde{u}^2}{\pi^2} \times \delta C} \times \frac{1}{(1 - \delta) N}.$$  

Since $\tilde{u}$ and $\pi$ are fixed independently of $k$, it follows that if $\lim_k (1 - \delta) N/C = \infty$ then $\lim_k \varepsilon = 0$, completing the proof.

We can now explain why Theorem 1 requires noisy actions. Without noise, $\pi_{a,\tilde{a}} = 1 \{a = \tilde{a}\}$, and hence $q(y|a) = p(y|\tilde{a})$ and $\pi = 0$, so Lemma 2 is vacuous. Indeed, detectability cannot be bounded by channel capacity in the absence of noise. To see this, suppose that the stage game is an $N$-player prisoner’s dilemma with a binary public monitoring structure, where $y = 0$ if every player cooperates, and $y = 1$ if any player defects. Obviously, mutual cooperation is an equilibrium outcome in this game with a moderate discount factor, regardless of $N$: under grim trigger strategies where the signal $y = 1$ triggers permanent defection, each player’s incentives in this game are the same as in a 2-player prisoner’s dilemma with perfect monitoring. This observation is consistent with Lemma 1 because detectability is infinite in this example: when the other players cooperate, a deviation to defection is perfectly detectable. However, channel capacity in this example is only $\log 2$, so detectability is infinitely greater than channel capacity. Thus, in the absence of noise a monitoring structure can effectively detect deviations (and support strong incentives) even if it not very “informative” in terms of channel capacity. In contrast, Lemma 2 shows that when noise is present, only informative signals can effectively detect deviations.
4 Sufficient Conditions for Cooperation

4.1 Folk Theorem

We prove a folk theorem for unmediated repeated games with public, product-structure monitoring, which implies that the relationship among $N$, $\delta$, and $C$ in Theorem 1 is tight for these games (up to log $N$ slack). As we explain below, this result also implies a folk theorem for mediated repeated games with arbitrary (non-public) monitoring structures.\footnote{We also discuss possible extensions to non-product structure monitoring.}

Under public monitoring, the public history $h^t$ at the beginning of period $t$ takes the form $h^t = ((z_r, y_r)_{r=1}^{t-1}, z_t)$. A strategy $\sigma_i$ for player $i$ is public if it depends on player $i$’s history $h^t$ only through its public component $h^t$. A perfect public equilibrium (PPE) is a profile of public strategies that, beginning at any period $t$ and any public history $h^t$, forms a Nash equilibrium from that period on.\footnote{As usual, this definition allows players to consider deviations to arbitrary, non-public strategies; but such deviations are irrelevant because, whenever a player’s opponents use public strategies, she has a public strategy that is a best response.}

Denote the set of PPE payoff vectors by $E \subseteq \mathbb{R}^N$.

For any $\eta > 0$, we say that a public monitoring structure satisfies individual identifiability with $\eta$ slack if

$$\sum_{y_i: q_i(y_i|a_i) \geq \eta} q_i(y_i|a_i) \left( \frac{q_i(y_i|a_i) - q_i(y_i|\alpha_i)}{q_i(y_i|a_i)} \right)^2 \geq \eta \text{ for all } i \in I, a_i \in A_i, \alpha_i \in \Delta(A_i \setminus \{a_i\}).$$

This condition is a variant of FLM’s individual full rank condition and KM’s assumption (A2”). It says that the detectability (discrete Fisher information) of a deviation from $a_i$ to any mixed action $\alpha_i$ supported on $A_i \setminus \{a_i\}$ is at least $\eta$, from the perspective of an observer who ignores all signals that occur with probability less than $\eta$ under $a_i$. Intuitively, this requires that deviations from $a_i$ are detectable, and that in addition “detectability” does not come entirely from a few very rare signal realizations. This assumption will ensure that player $i$ can be incentivized through rewards with “magnitude” of order $(1 - \delta)/\eta$, when the magnitude of a reward is measured either by its maximum absolute value or its variance.\footnote{If (5) were weakened by taking the sum over all $y_i$ (rather than only $y_i$ such that $q_i(y_i|a_i) \geq \eta$), player $i$ could be incentivized by rewards with variance $O((1 - \delta)/\eta)$, but not necessarily with maximum absolute value $O((1 - \delta)/\eta)$. Our proof of the folk theorem requires controlling both the variance and absolute value of players’ rewards, so we need the stronger condition.}

For example, suppose that the monitoring structure is given by random auditing, where in
every period $\eta N$ players (i.e., fraction $\eta$ of the population) are selected uniformly at random, and the public signal reveals their identities and their realized actions: that is, $Y_i = A_i \cup \{\emptyset\}$ and

$$p_i (y_i|\hat{a}_i) = \begin{cases} 
\eta & \text{if } y_i = \hat{a}_i, \\
0 & \text{if } y_i \in A_i \setminus \{\hat{a}_i\}, \\
1 - \eta & \text{if } y_i = \emptyset,
\end{cases}$$

so that $q_i (y_i|a_i) = \begin{cases} 
\eta \pi_{a_i,y_i} & \text{if } y_i \in A_i, \\
1 - \eta & \text{if } y_i = \emptyset.
\end{cases}$

For simplicity, suppose also that $\pi_{a_i,y_i} = \pi < 1/(|A_i| + 1)$ for all $i, a_i \neq a'_i$. Since $q_i (y_i|a_i) \geq \eta \pi$ for all $a_i, y_i$, we then have

$$\sum_{y_i : q_i(y_i|a_i) \geq \eta \pi} q_i (y_i|a_i) \left( \frac{q_i (y_i|a_i) - q_i (y_i|a_i)}{q_i (y_i|a_i)} \right)^2 \geq \frac{(q_i (a_i|a_i) - \max_{a_i' \neq a_i} q_i (a_i'|a_i'))^2}{q_i (a_i|a_i)} = \frac{(\eta (1 - (|A_i| - 1) \pi) - \eta \pi)^2}{\eta (1 - (|A_i| - 1) \pi)} \geq \eta \pi^2,$$

so random auditing satisfies individual identifiability with $\eta \pi^2$ slack.

**Theorem 2** Consider any sequence of repeated games $(\Gamma)^k$ with public, product-structure monitoring, and any sequence of positive numbers $(\eta)_k$ such that $(1 - \delta) \log (N)/\eta \to 0$ (where $\delta$, $N$, and $\eta$ depend on $k$) and in game $\Gamma^k$ individual identifiability is satisfied with $\eta_k$ slack. For any $\varepsilon > 0$, there exists $\bar{k}$ such that, for every $k \geq \bar{k}$, in game $\Gamma^k$ we have $B(\varepsilon) \subseteq E$.

Theorem 2 implies that the relationship among $N$, $\delta$, and $C$ in Theorem 1 is tight up to $\log (N)$ slack. To see this, again consider random auditing, which satisfies individual identifiability with $\eta \pi^2$ slack. Since the number of possible realized actions for each player is $|A_i| \leq 1/\pi$, random auditing has a channel capacity of at most $\eta N \log (1/\pi)$. Therefore, under random auditing Theorem 1 implies that payoffs in any communication equilibrium are consistent with almost-myopic play if $(1 - \delta)/\eta \to \infty$, while Theorem 2 implies that a folk theorem holds in PPE if $(1 - \delta) \log (N)/\eta \to 0$.

Theorem 2 also implies a folk theorem in communication equilibrium for arbitrary (non-public) monitoring structures. To see this, given an arbitrary monitoring structure $\left( \hat{Y}, \hat{p} \right)$, call its publicization the public monitoring structure $(Y,p)$ where $Y^i = \hat{Y}$ for all $i$ and $p^i (\hat{y}|\hat{a}) = \hat{p} (\hat{y}|\hat{a})$ for all $i, \hat{y}, \hat{a}$, where $p^i$ denotes the marginal of $p$ on $Y^i$, that is, the signal.
profile \((\tilde{y}_i)_{i \in I}\) is drawn from the same family of conditional distributions \((p(\cdot|\hat{a}))_{\hat{a} \in A}\) under \((\tilde{Y}, \tilde{p})\) and \((Y, p)\), but under \((\tilde{Y}, \tilde{p})\) each player \(i\) observes only \(\tilde{y}_i\), while under \((Y, p)\) each player \(i\) observes the entire profile \((\tilde{y}_i)_{i \in I}\). Under our assumption that the mediator observes the signal profile, the Nash equilibrium payoff set in a mediated repeated game with an arbitrary monitoring structure \((\tilde{Y}, \tilde{p})\) (i.e., the communication equilibrium payoff set with monitoring structure \((\tilde{Y}, \tilde{p})\)) is larger than the Nash equilibrium payoff set in the mediated repeated game with the publicized monitoring structure \((Y, p)\), because the set of canonical strategy profiles is the same in both games, but the set of possible deviations is larger in the latter game. The Nash equilibrium payoff set in the mediated game with public monitoring structure \((Y, p)\) is also obviously larger than the PPE payoff set in the unmediated game with the same monitoring structure. Therefore, whenever the publicization of an arbitrary monitoring structure \((\tilde{Y}, \tilde{p})\) satisfies the conditions of Theorem 2, the conclusion of the theorem applies for the communication equilibrium payoff set with monitoring structure \((\tilde{Y}, \tilde{p})\).

Finally, note that Theorem 2 is a “Nash threat” folk theorem, recalling that \(V^*\) is defined as the set of payoffs that Pareto-dominate a static Nash equilibrium. To extend this result to a “minmax threat” theorem, players must be made indifferent among all actions in the support of a mixed strategy that minmaxes an opponent. This requires a stronger identifiability condition, similar to Kandori and Matsushima’s assumption (A1). \(^{22}\)

### 4.2 Remarks on the Proof

Theorem 2 is a folk theorem for PPE in games with public monitoring. The standard proof approach for such results, following FLM and KM, relies on transferring continuation payoffs among the players along hyperplanes that are tangent to the boundary of the PPE payoff set. Unfortunately, this approach encounters serious difficulties when \(N\) and \(\delta\) vary together. The basic problem is that when \(N\) is large, changing each player’s continuation payoff by a small amount can result in a large overall movement in the continuation payoff vector, which makes self generation difficult to satisfy. Mathematically, FLM’s proof relies the equivalence

\(^{21}\) As well as for the perfect communication equilibrium payoff set, defined by Tomala (2009).

\(^{22}\) Specifically, consider the \(|Y_i| \times |A_i|\) matrix \(Q_i\) whose \((y_i, a_i)\) entry is \(q_i(y_i|a_i)\). We assume this matrix has full column rank, and hence there exists a \(|A_i| \times |Y_i|\) matrix \(P_i^{-1}\) such that \(P_i^{-1}P_i\) is the identity matrix. A sufficient condition for the minmax threat folk theorem is that the absolute value of each entry of \(P_i^{-1}\) is no more than \(1/\eta\) (and \((1 - \delta)\log (N)/\eta \to 0\)).
of the $L^1$ norm and the Euclidean norm in $\mathbb{R}^N$. Since this equivalence is not uniform in $N$, their proof does not apply when $N$ and $\delta$ vary together.

To see the problem in more detail, note that when individual identifiability holds with $\eta$ slack, the per-period movement in each player’s continuation payoff required to provide incentives is of order $(1 - \delta)/\eta$, so the movement of the continuation payoff vector in $\mathbb{R}^N$ is $O\left(\sqrt{N} (1 - \delta)/\eta \right)$. Fix a ball $B$ contained in $V^*$, and consider the problem of generating the point $v = \arg\max_{w \in B} w_1$—the point in $B$ that maximizes player 1’s payoff—using continuation payoffs drawn from $B$. Since player 1’s continuation payoff must be within $O(1 - \delta)$ distance of $v$, and the greatest movement along a tangent hyperplane is $O\left(\sqrt{1 - \delta}\right)$. FLM’s proof approach thus requires $\sqrt{N} (1 - \delta)/\eta \ll \sqrt{1 - \delta}$, or $(1 - \delta) N/\eta^2 \ll 1$, while we assume only $(1 - \delta) \log(N)/\eta \ll 1$. So, while the conditions for Theorem 2 are tight up to $\log(N)$ slack, FLM’s approach requires slack $N$.

Our proof of Theorem 2 (in Appendix A.3) is instead based on the “block strategy” approach introduced by Matsushima (2004) and Hörner and Olszewski (2006) in the context of repeated games with private monitoring. We view the repeated game as a sequence of $T$-period blocks of periods, where $T$ is a number proportional to $1/(1 - \delta)$. At the beginning of each block, a target payoff vector is determined by public randomization, and with high probability the players take actions throughout the block that deliver the target payoff. Players accrue promised continuation payoff adjustments throughout the block based on the public signals of their actions, and the distribution of target payoffs in the next block is set so as to deliver the promised adjustments. Since individual identifiability is satisfied with $\eta$ slack, the required adjustment to each player’s continuation payoff in every period is $O(1/\eta)$. By the law of large numbers, when $T \gg 1/\eta$, with high probability the total adjustment that a given player accrues over a $T$-period block is of order less than $T$, and is thus small enough that it can be delivered by appropriately specifying the distribution of target payoffs at the start of the next block.

The main difficulty in the proof is caused by the low-probability event that a player accrues an unusually large total adjustment over a block, so that at some point the distribution of target payoffs for the next block cannot be modified any further. In this case, the player can no longer be incentivized to take a non-myopic best response, and all players’ behavior in the current block must change. Thus, if any player’s payoff adjustment is “abnormal,” all players’ payoffs in that block may be far from the target equilibrium payoffs.
To prove the theorem, we must ensure that rare payoff-adjustment abnormalities do not compromise either ex ante efficiency or the players’ incentives. Efficiency is preserved if the blocks are long enough that the probability that any player’s payoff adjustment is abnormal is small. Since the per-period payoff adjustment for each player is $O(1/\eta)$ and the length of a block is $O(1/(1 - \delta))$, standard concentration bounds imply that the probability that a given player’s payoff adjustment is abnormal is $\exp(-O(\eta/(1 - \delta)))$. Hence, by union bound, the probability that any player’s adjustment is abnormal is at most $N\exp(-O(\eta/(1 - \delta)))$, which converges to 0 when $(1 - \delta)\log(N)/\eta \to 0$. This step in the proof accounts for the log$(N)$ slack.

Finally, since all players’ payoffs are affected when any player’s payoff adjustment becomes abnormal, incentives would be threatened if one player’s action affected the probability that another player’s adjustment becomes abnormal. We avoid this problem by letting each player’s adjustment depend only on the signals of her own actions. This separation of payoff adjustments across players is possible because we assume product structure monitoring. We do not know if Theorem 2 can be extended to non-product structure monitoring.\footnote{As noted above, we conjecture that the approach of FLM and KM yields a folk theorem if $(1 - \delta)N/\eta^2 \to 0$. Since their approach requires only pairwise identifiability, we conjecture that product structure can be relaxed to pairwise identifiability if $(1 - \delta)N/\eta^2 \to 0$. We do not know if such a relaxation is possible under the more general hypothesis of Theorem 2.}

## 5 Comparison with Linear Equilibria

We say that a communication equilibrium is linear if the continuation payoff vectors at all mediator histories lie on a line: for each player $i \neq 1$, there exists a constant $b_i \in \mathbb{R}$ such that, for all on-path mediator histories $h, h'$, we have $w_i(h') - w_i(h) = b_i (w_1(h') - w_1(h))$, where $w_i(h)$ denotes player $i$’s equilibrium continuation payoff at mediator history $h$.\footnote{A mediator history $h$ takes the form $(z_t, y_t, r_t)_{t=1}^{t-1}, z_t)$ for some $t \in \mathbb{N}$. If $h$ is on path in a communication equilibrium then $(a_t)_{t=1}^{t-1} = (r_t)_{t=1}^{t-1}$, so the mediator history encodes past actions as well as signals and public randomization realizations.} Relabeling the players if necessary, we can take $|b_i| \leq 1$ for all $i$ without loss. Note that this notion of linear equilibrium generalizes that of a linear public perfect equilibrium in a game with public monitoring, where continuation payoff vectors at all public histories lie on a line, as well as that of a strongly symmetric equilibrium (SSE) in a symmetric game with public monitoring, where the line in question is additionally required to be the 45° line.
Linear equilibria are of interest because they capture collective incentive provision. If $b_i \geq 0$ for all $i$, all players have the same preferences over histories, and are thus all rewarded or punished together. If $b_i < 0$ for some $i$, the players can be divided into two groups, where each group is rewarded when the other is punished.

Our final result is that cooperation in a linear equilibrium is possible only under extremely restrictive conditions on $N$ and $\delta$. We view this result as essentially an impossibility theorem for large-group cooperation under collective incentives.

**Theorem 3** Consider any sequence of repeated games $(\Gamma^k)$ for which there exists $\rho > 0$ (independent of $k$) such that $(1 - \delta) \exp (N^{1-\rho}) \to \infty$, and consider any corresponding sequence of linear equilibrium payoffs $(v^k)$. For any $\varepsilon > 0$, there exists $k$ such that, for every $k \geq k$, the payoff vector $v^k$ is consistent with $\varepsilon$-myopic play.

Theorem 3 differs from Theorem 1 not only in the required relationship between $N$ and $\delta$, but also in that Theorem 3’s negative conclusion applies no matter how informative the monitoring structure is. Intuitively, this is because optimal linear equilibria have a bang-bang form even when the realized action profile is perfectly observed, so a binary signal that indicates which of two extreme continuation payoff vectors should be played is as effective as any more informative signal.

Theorem 3 is proved in Appendix A.4. To see the main idea, consider the case where the game is symmetric and $b_i = 1$ for all $i$, so linear equilibria are SSE. Suppose we wish to enforce a symmetric pure action profile $\bar{a}_0 = (a_0, \ldots, a_0)$, where $D_i (\bar{a}_0) = \eta$, and suppose for simplicity that $|A_i| = 2$ and $\pi_{a_i, \bar{a}_i} = 1 - \pi$ whenever $a_i = \bar{a}_i$, and $\pi_{a_i, \bar{a}_i} = \pi$ otherwise. By standard arguments, an optimal SSE takes the form of a “tail test,” where the players are all punished if the number of players $n$ who take action $a_0$ falls below a threshold $n^*$. Due to the i.i.d. noise $\pi$, the distribution of $n$ is approximately normal when $N$ is large, with mean $(1 - \pi) N$ and standard deviation $\sqrt{\pi (1 - \pi) N}$. Denote the threshold z-score of the tail test by $z^* = (n^* - (1 - \pi) N) / \sqrt{\pi (1 - \pi) N}$, let $\phi$ and $\Phi$ denote the standard normal pdf and cdf, and let $x \in [0, \bar{u} / (1 - \delta)]$ denote the size of the penalty when the tail test is failed. We then must have

$$\frac{\phi(z^*)}{\sqrt{\pi (1 - \pi) N}} x \geq \eta \quad \text{and} \quad \Phi(z^*) x \leq \bar{u},$$
where the first inequality is incentive compatibility, and the second inequality says that the expected penalty cannot exceed the maximum difference between any two stage game payoffs. Dividing the first inequality by the second, we obtain
\[
\frac{\phi(z^*)}{\Phi(z^*)} \geq \frac{\eta \sqrt{\pi (1 - \pi)} N}{\bar{u}}.
\]
The left-hand size of this inequality is the statistical score of a normal tail test, which is approximately equal to \(-z^*\) for \(z^* < 0\). Hence, \(z^*\) must decrease at least linearly with \(\sqrt{N}\). But since \(\phi(z^*)\) decreases exponentially with \(z^*\), and hence exponentially with \(N\), Theorem 3 now follows from incentive compatibility, which implies that the product of \(\phi(z^*)/\sqrt{\pi (1 - \pi)} N\) and \(\bar{u}/(1 - \delta)\) must exceed \(\eta\).\(^{25}\)

Theorem 3 is related to Proposition 1 of Sannikov and Skrzypacz (2007), which is an anti-folk theorem for SSE in a two-player repeated game where actions are observed with additive, normally distributed noise, with variance proportional to \(1/(1 - \delta)\). (The interpretation is that players change their actions every \(\Delta\) units of time, where \(\delta = e^{-r\Delta}\) for fixed \(r > 0\) and variance is inversely proportional to \(\Delta\), for example as a consequence of observing the average increments of a Brownian process.) As a tail test is also optimal in their setting, the reasoning just given implies that incentives can be provided only if \(1/(1 - \delta)\) increases exponentially with the variance of the noise. Since in their model \(1/(1 - \delta)\) increases with the variance only linearly, they likewise obtain an anti-folk theorem. Also, Proposition 2 of Fudenberg and Levine (2007) is an anti-folk theorem in a game with one long-run player and series of short-run players, where the long-run player’s action is observed with additive, normal noise with variance proportional to \(1/(1 - \delta)^{\rho}\) for some \(\rho > 0\); and their Proposition 3 is a folk theorem when the variance is constant in \(\delta\). Theorem 3 suggests that their anti-folk theorem extends whenever the variance asymptotically dominates \((\log 1/(1 - \delta))^{1/(1 - \rho)}\) for some \(\rho > 0\), while their folk theorem extends whenever the variance is asymptotically dominated by \((\log 1/(1 - \delta))^{1/(1 + \rho)}\) for some \(\rho > 0\).\(^{26}\)

\(^{25}\)Conversely, if \(\pi_{a_i, a_i}\) is sufficiently large for each \(a_i\) and \((1 - \delta) \exp (N^{1+\rho}) \rightarrow 0\) for some \(\rho > 0\), then a folk theorem holds for linear equilibria. Intuitively, a target action profile \(a\) can now be enforced by a tail test where the players are all punished only if \(\tilde{a}_i \neq a_i\) for all \(i\), so that no one took the prescribed action. An earlier version of this paper contains a formal statement of such a result.

\(^{26}\)The analysis of tail tests as optimal incentive contracts under normal noise goes back to Mirrlees (1975). The logic of Theorem 3 shows that the size of the penalty in a Mirrleesian tail test must increase exponentially with the variance of the noise. We are not aware of references to this point in the literature.
6 Conclusion

This paper has developed a theory of large-group cooperation by considering sequences of repeated games where the population size, discount factor, stage game, and monitoring structure all vary together in a flexible manner. We derived general necessary and sufficient conditions for cooperation to be possible in the limit of such a sequence. For a specific class of monitoring structures (random auditing), these conditions coincide up to \( \log(N) \) slack. We also showed that cooperation in a linear equilibrium is possible only under much more stringent conditions. Methodologically, our results rely on some new connections between repeated games and information theory.

We hope our results raise questions for future theoretical and applied research. On the theory side, we believe that the connections between repeated games, mechanism design, and information theory remain under-explored. Our earlier work (Sugaya and Wolitzky, 2021a) exploited these connections in the context of repeated random-matching games with incomplete information; other promising topics for such an approach include repeated games with frequent actions and static moral hazard in teams problems with large teams. As for applied work, it would be interesting to see more systematic empirical or experimental evidence on the determinants of large-group cooperation, for example on the relative efficacy of individual and collective sanctions in populations of different sizes.

A Appendix

A.1 Proof of Lemma 1

First, note that the first sentence of the lemma implies the second, because if (3) holds for \( J = I \) then \( \alpha^0 \in A(\varepsilon) \) for \( \varepsilon = (\delta / (1 - \delta)) \left( \max_{\tilde{\alpha}_i} F_I(\tilde{\alpha}_I) \right) \bar{u}^2 \), and hence, by (2),

\[
(1 - \delta) \sum_{t \in \mathbb{N}} \delta^{t-1} \sum_{a \in A} \alpha_t^\mu(a) u(a) = u(\alpha^\mu) \in M(\varepsilon).
\]

Next, note that it suffices to establish (3) for singleton \( J \), because if \( D_i(\tilde{\alpha}_i|\alpha^\mu) \leq \)
\[\sqrt{(\delta/(1-\delta))} \, F_i(\tilde{\alpha}_i) \, V_i(\alpha^\mu) \] for each \(i \in I\), then by Cauchy-Schwarz, for any \(J \subseteq I\),

\[
D_J(\tilde{\alpha}_J|\alpha^\mu) = \frac{1}{|J|} \sum_{i \in J} D_i(\tilde{\alpha}_i|\alpha^\mu) \leq \frac{1}{|J|} \sum_{i \in J} \sqrt{\frac{\delta}{1-\delta}} \, F_i(\tilde{\alpha}_i) \, V_i(\alpha^\mu) \\
\leq \frac{1}{|J|} \sqrt{\frac{\delta}{1-\delta}} \sum_{i \in J} F_i(\tilde{\alpha}_i) \sum_{i \in J} V_i(\alpha^\mu) = \sqrt{\frac{\delta}{1-\delta}} \, F_J(\tilde{\alpha}_J) \, V_i(\alpha^\mu).
\]

The remainder of the proof thus fixes \(i \in I\) and establishes (3) for \(J = \{i\}\). Given a path of action profiles \((a_t)_{t=1}^\infty\), let \(u_{i,t} = u_i(a_t)\), and denote player \(i\)'s continuation payoff at the beginning of period \(t\) by

\[w_{i,t} = (1-\delta) \sum_{t'=t}^\infty \delta^{t'-t} u_{i,t}.
\]

Denote a history of actions and signals at the beginning of period \(t\) by \(h^t = (a^t, y^t)\).

Fix a communication equilibrium outcome \(\mu\) and a manipulation \(\tilde{\alpha}_i\). Let \(H^t\) denote the set of period-\(t\) histories \(h^t\) that are reached with positive probability under \(\mu\), and define a \(H^t\)-measurable random variable \(W_{i,t} : H^t \to \mathbb{R}\) by \(W_{i,t}(h^t) = \mathbb{E}[w_{i,t}|h^t]\) for all \(h^t \in H^t\). By the law of total variance, we have

\[
\text{Var}(W_{i,t+1}) = \text{Var}\left(\mathbb{E}[W_{i,t+1}|h^t]\right) + \mathbb{E}\left[\text{Var}(W_{i,t+1}|h^t)\right].
\]

Similarly, define \(U_{i,t} : H^t \to \mathbb{R}\) by \(U_{i,t}(h^t) = \mathbb{E}[u_{i,t}|h^t]\) for all \(h^t \in H^t\). Let \(\mu(h^t, r)\) denote the probability that history \(h^t\) is reached in period \(t\) and then action profile \(r\) is played.

**Claim 1** For each period \(t\), we have

\[
\text{Var}\left(\mathbb{E}[W_{i,t+1}|h^t]\right) \geq \frac{1}{\delta} \text{Var}(W_{i,t}) - \frac{1-\delta}{\delta} \text{Var}(U_{i,t}) \quad \text{and} \\
\mathbb{E}\left[\text{Var}(W_{i,t+1}|h^t)\right] \geq \left(\frac{1-\delta}{\delta}\right)^2 \frac{D_i(\tilde{\alpha}_i|\alpha^\mu)^2}{F_i(\tilde{\alpha}_i)}.
\]

**Proof.** The derivation of equation (7) is elementary and is deferred to Appendix B.2. For equation (8), since \(\mu\) is an equilibrium outcome, we have

\[
\frac{1-\delta}{\delta} D_i(\tilde{\alpha}_i) \leq \sum_{h^t, r_t, y_t} \mu(h^t, r_t) (q(y_t|r_t) - q(y_t|r_t; \tilde{\alpha}_i)) W_{i,t+1}(h^t, r_t, y_t).
\]

This holds because, if she manipulates according to \(\tilde{\alpha}_i\) in period \(t\), player \(i\) can guarantee an
expected continuation payoff of \( \sum_{h', r_t, y_t} \mu(h', r_t) q(y_t| r_t; \tilde{\alpha}_i) W_{i, t+1}(h', r_t, y_t) \) by following the mediator’s recommendations from period \( t + 1 \) onwards. (In other words, in the continuation game player \( i \) plays as if her period-\( t \) action were \( r_{i,t} \) rather than \( \tilde{\alpha}_i(r_{i,t}) \). This continuation play may not be optimal, but we are only giving a necessary condition.) Therefore,

\[
\frac{1 - \delta}{\delta} D_i (\tilde{\alpha}_i | \alpha_i^u) \leq \sum_{h', r_t, y_t} \mu(h', r_t) (q(y_t|r_t) - q(y_t| r_t; \tilde{\alpha}_i)) W_{i, t+1}(h', r_t, y_t)
\]

\[
= \sum_{h', r_t, y_t} \mu(h', r_t) q(y_t|r_t) \left( \frac{q(y_t|r_t) - q(y_t| r_t; \tilde{\alpha}_i)}{q(y_t|r_t)} \right) (W_{i, t+1}(h', r_t, y_t) - \mathbb{E}[W_{i, t+1}| h^t])
\]

\[
\leq \sqrt{\sum_{h', r_t, y_t} \mu(h', r_t) q(y_t|r_t) \left( \frac{q(y_t|r_t) - q(y_t| r_t; \tilde{\alpha}_i)}{q(y_t|r_t)} \right)^2} \times \sqrt{\sum_{h', r_t, y_t} \mu(h', r_t) q(y_t|r_t) (W_{i, t+1}(h', r_t, y_t) - \mathbb{E}[W_{i, t+1}| h^t])^2}
\]

\[
\leq \sqrt{F_i(\tilde{\alpha}_i)} \sqrt{\mathbb{E}[\text{Var}(W_{i, t+1}| h^t)]},
\]

where the second inequality follows from Cauchy-Schwarz, and the third follows from the definition of \( F_i(\tilde{\alpha}_i) \). Squaring both sides and rearranging yields (8). □

By (6), (7), and (8), for each period \( t \), we have

\[
\text{Var}(W_{i, t+1}) \geq \frac{1}{\delta} \text{Var}(W_{i, t}) - \frac{1 - \delta}{\delta} \text{Var}(U_{i, t}) + \left( \frac{1 - \delta}{\delta} \right)^2 \frac{D_i(\tilde{\alpha}_i| \alpha_i^u)^2}{F_i(\tilde{\alpha}_i)}.
\]

Since \( \text{Var}(W_{i, 1}) = 0 \), recursively applying this inequality, for each \( T \in \mathbb{N} \) we have

\[
\delta^T \text{Var}(W_{i, T+1}) \geq (1 - \delta) \sum_{t=1}^{T} \delta^{t-1} \left( \frac{1 - \delta}{\delta} \frac{D_i(\tilde{\alpha}_i| \alpha_i^u)^2}{F_i(\tilde{\alpha}_i)} - \text{Var}(U_{i, t}) \right).
\]

As payoffs are bounded, the left-hand side of this inequality converges to 0 as \( T \to \infty \), while the right-hand side converges to \((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \left( ((1 - \delta)/\delta) \left( D_i(\tilde{\alpha}_i| \alpha_i^u)^2 / F_i(\tilde{\alpha}_i) \right) - \text{Var}(U_{i, t}) \right)\), which therefore must be non-­positive. Hence, we have

\[
(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \text{Var}(U_{i, t}) \geq (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \frac{1 - \delta}{\delta} \frac{D_i(\tilde{\alpha}_i| \alpha_i^u)^2}{F_i(\tilde{\alpha}_i)} \geq \frac{1 - \delta}{\delta} \frac{(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} D_i(\tilde{\alpha}_i| \alpha_i^u)^2}{F_i(\tilde{\alpha}_i)},
\]

(9)
where the second inequality follows from Jensen. Moreover, we have

\[ V_i(\alpha^\mu) = \sum_a \alpha^\mu(a) (u_i(a) - u_i(\alpha^\mu))^2 \]

\[ = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \sum_a \alpha^\mu_t(a) (u_i(a) - u_i(\alpha^\mu))^2 \]

\[ \geq (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \sum_a \alpha^\mu_t(a) (u_i(a) - u_i(\alpha^\mu))^2 \]

\[ = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \text{Var}(u_i^t) \geq (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \text{Var}(U_i^t), \quad (10) \]

where the first inequality follows because \( \mathbb{E}[(X - x)^2] \geq \mathbb{E}[(X - \mathbb{E}[X])^2] \) for all \( x \in \mathbb{R} \), and the second follows by the law of total variance; and

\[ D_i(\tilde{\alpha}_i|\alpha^\mu) = \sum_a \alpha^\mu(a) (u_i(\tilde{\alpha}_i(a_i), a_{-i}) - u_i(a)) \]

\[ = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \sum_a \alpha^\mu_t(a) (u_i(\tilde{\alpha}_i(a_i), a_{-i}) - u_i(a)) = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} D_i(\tilde{\alpha}_i|\alpha^\mu_t) \quad (11) \]

By (9), (10), and (11), we have

\[ V_i(\alpha^\mu) \geq \frac{1 - \delta}{\delta} \frac{D_i(\tilde{\alpha}_i|\alpha^\mu)^2}{F_i(\tilde{\alpha}_i)}. \]

Rearranging and taking square roots yields (3).

**A.2 Proof of Lemma 2**

For any \( r \in A \), let \( \Pr(\cdot|r) \) denote the probability distribution over realized action profiles \( \hat{a} \) and outcomes \( y \) that results under intended action profile \( r \). Throughout the proof, \( \hat{a}_i \) denotes a realized action for player \( i \). Note that, for all \( \hat{a}_i \in A_i, y \in Y, \) and \( r \in A \), we have

\[ \Pr(\hat{a}_i, y|r) = \pi_{r_i, \hat{a}_i} \Pr(y|r, \hat{a}_i) = q(y|r) \Pr(\hat{a}_i|r, y), \]

and therefore, since \( \pi_{r_i, \hat{a}_i} \geq \pi \),

\[ (\Pr(y|r, \hat{a}_i) - q(y|r))^2 = \left( \frac{q(y|r)}{\pi_{r_i, \hat{a}_i}} (\Pr(\hat{a}_i|r, y) - \pi_{r_i, \hat{a}_i}) \right)^2 \leq \left( \frac{q(y|r)}{\pi} (\Pr(\hat{a}_i|r, y) - \pi_{r_i, \hat{a}_i}) \right)^2. \quad (12) \]
For any $r \in A$ and $a_i \in A_i$, we thus have

$$
\sum_y \frac{1}{q(y|r)} (q(y|a_i, r_{-i}) - q(y|r))^2 = \sum_y \frac{1}{q(y|r)} \left( \sum_{\hat{a}_i} (\pi_{a_i, \hat{a}_i} - \pi_{r, \hat{a}_i}) \Pr(y|r, \hat{a}_i) \right)^2 \\
= \sum_y \frac{1}{q(y|r)} \left( \sum_{\hat{a}_i} (\pi_{a_i, \hat{a}_i} - \pi_{r, \hat{a}_i}) (\Pr(y|r, \hat{a}_i) - q(y|r)) \right)^2 \\
\leq \sum_{\hat{a}_i} (\pi_{a_i, \hat{a}_i} - \pi_{r, \hat{a}_i})^2 \sum_y \frac{1}{q(y|r)} \sum_{\hat{a}_i} (\Pr(y|r, \hat{a}_i) - q(y|r))^2 \\
\leq \frac{2}{\pi^2} \sum_y q(y|r) \sum_{\hat{a}_i} (\Pr(\hat{a}_i|r, y) - \pi_{r, \hat{a}_i})^2 \\
\leq \frac{2}{\pi^2} \sum_y q(y|r) \left( \sum_{\hat{a}_i} |\Pr(\hat{a}_i|r, y) - \pi_{r, \hat{a}_i}| \right)^2 \\
\leq \frac{4}{\pi^2} \sum_y q(y|r) \sum_{\hat{a}_i} \Pr(\hat{a}_i|r, y) \log \frac{\Pr(\hat{a}_i|r, y)}{\pi_{r, \hat{a}_i}} \\
= \frac{4}{\pi^2} \sum_{\hat{a}_i, y} \Pr(\hat{a}_i, y|r) \log \frac{\Pr(\hat{a}_i|r, y)}{\pi_{r, \hat{a}_i}}.
$$

where the first inequality follows by Cauchy-Schwarz, the second follows by (12) and $\sum_{\hat{a}_i} (\pi_{a_i, \hat{a}_i} - \pi_{r, \hat{a}_i})^2 \leq 2$, the third is immediate, and the fourth follows by Pinsker’s inequality (CT, Lemma 11.6.1). Note that the last line equals $(4/\pi^2) I(\hat{a}_i; y|r)$, where $I(\cdot;\cdot|r)$ denotes mutual information conditional on $r$.

Hence, for any subset of players $J \subseteq I$, any profile of manipulations $\hat{a}_J$, and any action profile $r \in A$, we have

$$
F_J(\hat{a}_J|r) = \frac{1}{|J|} \sum_{i \in J} \sum_y \frac{(q(y|\hat{a}_i(r_i, r_{-i}) - q(y|r))^2}{q(y|r)} \leq \frac{4}{\pi^2 |J|} \sum_{i \in J} I(\hat{a}_i; y|r) = \frac{4}{\pi^2 |J|} I(\hat{a}_J; y|r),
$$

where the last equality follows because $(\hat{a}_i)_{i \in J}$ are independent conditional on $r$.

Now note that

$$
I(\hat{a}_J; y|r) = I(\hat{a}; y|r) - I(\hat{a}_{\hat{J}}; y|r, \hat{a}_J) \leq I(\hat{a}; y|r) = I(\hat{a}; r; y) - I(r; y) \\
= I(\hat{a}; y) + I(r; y; \hat{a}) - I(r; y) = I(\hat{a}; y) + 0 - I(r; y) \leq I(\hat{a}; y) \leq C,
$$

where the first equality follows by the chain rule for mutual information (CT, Theorem
2.2.1), the first inequality follows because mutual information is non-negative, the second and third equalities again follow by the chain rule, the fourth equality follows because \( r \) and \( y \) are independent conditional on \( \hat{a} \), the second inequality again follows by non-negativity, and the last inequality follows from the definition of channel capacity. Hence,

\[
F_J (\alpha_J | r) \leq \frac{4C}{\pi^2 |J|} \quad \text{for all } r \in A,
\]

and (4) follows.

### A.3 Proof of Theorem 2

#### A.3.1 Preliminaries

Fix any \( \varepsilon > 0 \). If \( \varepsilon \geq \bar{u}/2 \) then \( B(\varepsilon) = \emptyset \) and the conclusion of the theorem is trivial, so assume without loss that \( \varepsilon < \bar{u}/2 \). We begin with two preliminary lemmas. First, for each \( i \in I \) and \( r_i \in A_i \), we define a function \( f_{i,r_i} : Y_i \to \mathbb{R} \) that will later be used to specify player \( i \)'s continuation payoff as a function of \( y_i \).

**Lemma 3** If individual identifiability holds with \( \eta \) slack, then for each \( i \in I \) and \( r_i \in A_i \) there exists a function \( f_{i,r_i} : Y_i \to \mathbb{R} \) such that

\[
\mathbb{E} [f_{i,r_i} (y_i) | r_i] - \mathbb{E} [f_{i,r_i} (y_i) | a'_i] \geq \bar{u} \text{ for all } a'_i \neq r_i, \quad (13)
\]

\[
\mathbb{E} [f_{i,r_i} (y_i) | r_i] = 0, \quad (14)
\]

\[
\text{Var} (f_{i,r_i} (y_i) | r_i) \leq \bar{u}^2/\eta, \quad (15)
\]

\[
|f_{i,r_i} (y_i)| \leq 2\bar{u}/\eta \text{ for all } y_i. \quad (16)
\]

**Proof.** Fix \( i \) and \( r_i \). Let \( Y_i^* = \{ y_i : q_i (y_i | r_i) \geq \eta \} \), and let

\[
Q_i (r_i; Y_i^*) = \left( \frac{q_i (y_i | r_i)}{\sqrt{q_i (y_i | r_i)}} \right)_{y_i \in Y_i^*} \quad \text{and } \quad \tilde{Q}_i (r_i; Y_i^*) = \bigcup_{a'_i \neq r_i} \left( \frac{q_i (y_i | a'_i)}{\sqrt{q_i (y_i | r_i)}} \right)_{y_i \in Y_i^*}.
\]

Note that (5) is equivalent to

\[
d \left( Q_i (r_i; Y_i^*), \text{co} (\tilde{Q}_i (r_i; Y_i^*)) \right) \geq \sqrt{\eta} \text{ for all } i \in I, r_i \in A_i,
\]

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where \(d(\cdot, \cdot)\) denotes Euclidean distance in \(\mathbb{R}^{V_i^*}\). Hence, by the separating hyperplane theorem, there exists \(x = (x(y_i))_{y_i \in Y_i^*} \in \mathbb{R}^{V_i^*}\) such that \(\|x\| = 1\) and \((Q_i(r_i; Y_i^*) - q) \cdot x \geq \sqrt{\eta}\) for all \(q \in \tilde{Q}_i(r_i; Y_i^*)\). By definition of \(Q_i\) and \(\tilde{Q}_i\), this implies that \(\sum_{y_i \in Y_i^*} (q_i(y_i|r_i) - q_i(y_i|a'_i)) x(y_i) \geq \sqrt{q_i(y_i|r_i) \eta}\) for all \(a'_i \neq r_i\). Defining \(f_{i,r_i}(y_i) = \bar{u} \left( x(y_i) - \sum_{\bar{y}_i \in Y_i^*} q(\bar{y}_i|r_i) x(\bar{y}_i) \right) / \sqrt{q_i(y_i|r_i) \eta}\) for all \(y_i \in Y_i^*\) and \(f_{i,r_i}(y_i) = 0\) for all \(y_i \notin Y_i^*\), conditions (13) and (14) hold. Moreover, since \(\mathbb{E}[f_{i,r_i}(y_i)|r_i] = 0\) and the term \(\sum_{\bar{y}_i \in Y_i^*} q(\bar{y}_i|r_i) x(\bar{y}_i)\) is independent of \(y_i\), we have

\[
\text{Var}(f_{i,r_i}(y_i)|r_i) = \mathbb{E} \left[ \frac{\bar{u}^2 x(y_i)^2}{q_i(y_i|r_i) \eta} \right] - \mathbb{E} \left[ \frac{\bar{u} x_i(y_i)}{\sqrt{q_i(y_i|r_i) \eta}} \right] \leq \sum_{y_i \in Y_i^*} \frac{\bar{u}^2 x(y_i)^2}{\eta} \leq \frac{\bar{u}^2}{\eta},
\]

and hence (15) holds. Finally, (16) holds since, for each \(y_i \in Y_i^*\),

\[
|f_{i,r_i}(y_i)| \leq \bar{u} \left( |x(y_i)| + \sum_{\bar{y}_i \in Y_i^*} q(\bar{y}_i|r_i) |x_i(\bar{y}_i)| \right) / \sqrt{q_i(y_i|r_i) \eta} \leq \bar{u} \left( 1 + \sum_{\bar{y}_i \in Y_i^*} q(\bar{y}_i|r_i) \right) / \eta \leq 2\bar{u} / \eta.
\]

Now fix \(i \in I\) and \(r_i \in A_i\), and suppose that \(y_{i,t} \sim q_i(\cdot|r_i)\) for each period \(t \in \mathbb{N}\), independently across periods. By (15), for any \(T \in \mathbb{N}\), we have

\[
\text{Var} \left( \sum_{t=1}^{T} \delta^{t-1} f_{i,r_i}(y_{i,t}) \right) = \sum_{t=1}^{T} \delta^{2(t-1)} \text{Var}(f_{i,r_i}(y_{i,t})) \leq \frac{1 - \delta^{2T}}{1 - \delta^2} \frac{\bar{u}^2}{\eta}.
\]

Together with (14) and (16), Bernstein's inequality (Boucheron, Lugosi, and Massart, 2013, Theorem 2.10) now implies that, for any \(T \in \mathbb{N}\) and \(\bar{f} \in \mathbb{R}_+\), we have

\[
\Pr \left( \sum_{t=1}^{T} \delta^{t-1} f_{i,r_i}(y_{i,t}) \geq \bar{f} \right) \leq \exp \left( -\frac{\bar{f}^2 \eta}{2 \left( \frac{1 - \delta^{2T}}{1 - \delta^2} \bar{u}^2 + \frac{2}{3} \bar{f} \bar{u} \right)} \right). \quad (17)
\]

Our second lemma defines \(T\) and \(\bar{f}\) (as a function of \(k\)) so that the bound in (17) is sufficiently small, and some other conditions used in the proof also hold. This lemma is where we use the assumption that \((1 - \delta) \log (N) / \eta \to 0\).

**Lemma 4** There exists \(\bar{k}\) such that, for every \(k \geq \bar{k}\), there exist \(T \in \mathbb{N}\) and \(\bar{f} \in \mathbb{R}\) that...
satisfy the following three inequalities:

\[
60\bar{u}N \exp \left( -\frac{\left( \frac{f}{3} \right)^2 \eta}{2 \left( \frac{1-\delta^{2T}}{1-\delta} \bar{u}^2 + \frac{2}{3} \frac{\bar{u}}{u} \right)} \right) \leq \varepsilon, \quad (18)
\]

\[
8 \frac{1 - \delta}{1 - \delta^T} \left( \bar{f} + \frac{2\bar{u}}{\eta} \right) \leq \varepsilon, \quad (19)
\]

\[
4\bar{u} \frac{1 - \delta^T}{\delta^T} + \frac{1 - \delta}{\delta^T} \left( \bar{f} + \frac{2\bar{u}}{\eta} \right) \leq \varepsilon. \quad (20)
\]

**Proof.** For each \( k \), let \( T \) be the largest integer such that \( 8\bar{u} \left( 1 - \delta^T \right) /\delta^T \leq \varepsilon \), and let

\[
\bar{f} = \sqrt{\frac{36 \log \left( \frac{60\bar{u}}{\varepsilon} \right) \log (N) \frac{1 - \delta^T}{1 - \delta} \frac{\bar{u}^2}{\eta}}. \]

Note that \( 1 - \delta^T \rightarrow \varepsilon \left( \varepsilon + 8\bar{u} \right) \) (a constant independent of \( k \)) as \( k \rightarrow \infty \). Since \( (1 - \delta) \log (N) /\eta \rightarrow 0 \), it follows that \( (1 - \delta) \log (N) / (\eta (1 - \delta^T)) \rightarrow 0 \). Therefore, there exists \( \bar{k} \) such that, for every \( k \geq \bar{k} \), we have

\[
\frac{4}{9} \sqrt{36 \frac{\log \left( \frac{60\bar{u}}{\varepsilon} \right) \log (N) \frac{1 - \delta}{1 - \delta^T} \frac{1}{\eta}} \leq 1 \quad \text{and} \quad (21)
\]

\[
8\bar{u} \left( \sqrt{36 \log \left( \frac{60\bar{u}}{\varepsilon} \right) \log (N) \frac{1 - \delta}{1 - \delta^T} \frac{1}{\eta} + \frac{1 - \delta}{\eta}} \frac{2}{1 - \delta^T} \right) \leq \varepsilon. \quad (22)
\]

It now follows from straightforward algebra (provided in Appendix B.3) that (18)–(20) hold for every \( k \geq \bar{k} \).  

**A.3.2 Equilibrium Construction**

Fix any \( k, T \), and \( \bar{f} \) that satisfy (18)–(20), as well any \( v \in B(\varepsilon) \). For each extreme point \( v^* \) of \( B_v(\varepsilon/2) \), we construct a PPE in a \( T \)-period, finitely repeated game augmented with continuation values drawn from \( B_v(\varepsilon/2) \) that generates payoff vector \( v^* \). By standard arguments, this implies that \( B_v(\varepsilon/2) \subseteq E(\Gamma) \), and hence that \( v \in E(\Gamma) \).\(^{27}\) Since \( v \in B(\varepsilon) \) was chosen arbitrarily, it follows that \( B(\varepsilon) \subseteq E(\Gamma) \).

\(^{27}\)Specifically, at each history \( h^{T+1} \) that marks the end of a block, public randomization can be used to select an extreme point \( v^* \) to be targeted in the following block, with probabilities chosen so that the expected payoff \( E[v^*] \) equals the promised continuation value \( w(h^{T+1}) \).
Specifically, for each $\zeta \in \{-1, 1\}^N$ and $v^* = \arg\max_{v \in B_v(\varepsilon/2)} \zeta \cdot v$, we construct a public strategy profile $\sigma$ in a $T$-period, finitely repeated game (which we call a \textit{block strategy profile}) together with a continuation value function $w : H^{T+1} \to \mathbb{R}^N$ that satisfy

\textbf{Promise Keeping.} $v_i^* = \mathbb{E}^\sigma \left[ (1 - \delta) \sum_{t=1}^T \delta^{t-1} u_{i,t} + \delta^T w_i (h^{T+1}) \right]$ for all $i \in I$.

\textbf{Incentive Compatibility.} $\sigma_i \in \arg\max_{\tilde{\sigma}_i} \mathbb{E}^{\tilde{\sigma}_i, \sigma_{-i}} \left[ (1 - \delta) \sum_{t=1}^T \delta^{t-1} u_{i,t} + \delta^T w_i (h^{T+1}) \right]$ for all $i \in I$.

\textbf{Self Generation.} $w(h^{T+1}) \in B_v(\varepsilon/2)$ for all $h^{T+1}$. (Note that, since $B_v(\varepsilon/2)$ is cube with side-length $\varepsilon$ and $v^* = \arg\max_{v \in B_v(\varepsilon/2)} \zeta \cdot v$, this is equivalent to $\zeta_i \left( w_i(h^{T+1}) - v_i^* \right) \in [-\varepsilon, 0]$ for all $i$ and $h^{T+1}$.)

Defining $\pi_i(h^{T+1}) = (\delta^T / (1 - \delta)) \left( w_i(h^{T+1}) - v_i^* \right)$, these conditions can be rewritten as

\textbf{Promise Keeping.}

$$v_i^* = \frac{1 - \delta}{1 - \delta^T} \mathbb{E}^\sigma \left[ \sum_{t=1}^T \delta^{t-1} u_{i,t} + \pi_i(h^{T+1}) \right] \text{ for all } i. \tag{23}$$

\textbf{Incentive Compatibility.}

$$\sigma_i \in \arg\max_{\tilde{\sigma}_i} \mathbb{E}^{\tilde{\sigma}_i, \sigma_{-i}} \left[ \sum_{t=1}^T \delta^{t-1} u_{i,t} + \pi_i(h^{T+1}) \right] \text{ for all } i. \tag{24}$$

\textbf{Self Generation.}

$$\zeta_i \frac{1 - \delta}{\delta^T - \pi_i(h^{T+1})} \in [-\varepsilon, 0] \text{ for all } i \text{ and } h^{T+1}. \tag{25}$$

Fix $\zeta \in \{-1, 1\}^N$ and $v^* = \arg\max_{v \in B_v(\varepsilon/2)} \zeta \cdot v$. We construct a block strategy profile $\sigma$ and continuation value function $\pi$ which, in the next subsection, we show satisfy these three conditions. This will complete the proof of the theorem.

First, fix a correlated action profile $\tilde{\alpha} \in \Delta(A)$ such that

$$u_i(\tilde{\alpha}) = v_i^* + \zeta_i \varepsilon/2 \text{ for all } i, \tag{26}$$

and fix a static Nash equilibrium $\alpha^{NE} \in \prod_i \Delta(A_i)$ such that $u_i(\alpha^{NE}) \leq v_i^* - \varepsilon/2$ for all $i$. Such $\tilde{\alpha}$ and $\alpha^{NE}$ exist because $v^* \in B_v(\varepsilon/2)$ and $B_v(\varepsilon) \subseteq F^*$. 

32
We now construct the block strategy profile $\sigma$. For each player $i \in I$ and period $t \in \{1, \ldots, T\}$, we define a state $F_{i,t} \in \{0, 1\}$ for player $i$ in period $t$, which will determine player $i$’s prescribed equilibrium action in period $t$. The states are determined by the public history, and so are common knowledge among the players. We first specify players’ prescribed actions as a function of the state, and then specify the state as a function of the public history.

**Prescribed Equilibrium Actions:** For each period $t$, let $r_t \in A$ be a pure action profile which is drawn by public randomization at the start of period $t$ from the distribution $\bar{\alpha} \in \Delta(A)$ fixed in (26). The prescribed equilibrium actions are defined as follows.

1. If $F_{i,t} = 0$ for all $i \in I$, the players take $a_t = r_t$.

2. If there is a unique player $i$ such that $F_{i,t} = 1$, the players take $a_t = (r'_i, r_{-i,t})$ for some $r'_i \in BR_i(r_{-i})$ if $\zeta_i = 1$, and they take $\alpha^{NE}$ if $\zeta_i = -1$, where $BR_i(r_{-i}) = \arg\max_{a_i \in A_i} u_i(a_i, r_{-i})$ is the set of $i$’s best responses to $r_{-i}$.

3. If there is more than one player $i$ such that $F_{i,t} = 1$, the players take $\alpha^{NE}$.

Let $\alpha^*_i \in \prod_i \Delta(A_i)$ denote the distribution of prescribed equilibrium actions, prior to public randomization $z_t$.

(It may be helpful to informally summarize the prescribed actions. So long as $F_{i,t} = 0$ for all players, the players take actions drawn from the target action distribution $\bar{\alpha}$. If $F_{i,t} = 1$ for multiple players, the Pareto-dominated Nash equilibrium $\alpha^{NE}$ is played. The most subtle case is when there is a unique player $i$ such that $F_{i,t} = 1$. Intuitively, this case will correspond to situations where the signals of player $i$’s actions are “abnormal,” which later in the proof will imply that her continuation payoffs cannot be adjusted further without violating the self-generation constraint. In this case, player $i$ starts taking static best responses. Moreover, if $\zeta_i = -1$—so that player $i$’s continuation payoff is already “low”—$\alpha^{NE}$ is played.)

It will be useful to introduce the following additional state variable $S_{i,t}$, which summarizes player $i$’s prescribed action as a function of $(F_{j,t})_{j \in I}$:

---

28Technically, the public randomization device $Z_t$ is always a uniform $[0, 1]$ random variable. Throughout the proof, whenever we say that a certain variable is “drawn by public randomization,” we mean that its realization is determined by public randomization, independently of the other variables in the construction. Since we define only a finite number $B$ of such variables, this can be done by, for example, specifying that if $n = b \mod B$ then the $n^{th}$ digit of $z$ is used to encode the realization of the $b^{th}$ such variable we define.
1. \( S_{i,t} = 0 \) if \( F_{j,t} = 0 \) for all \( j \in I \), or if there exists a unique player \( j \neq i \) such that \( F_{j,t} = 1 \), and for this player we have \( \zeta_j = 1 \). In this case, player \( i \) is prescribed to take \( a_{i,t} = r_{i,t} \).

2. \( S_{i,t} = NE \) if \( F_{i,t} = 0 \) and either (i) there exists a unique player \( j \) such that \( F_{j,t} = 1 \), and for this player we have \( \zeta_j = -1 \), or (ii) there are two distinct players \( j, j' \) such that \( F_{j,t} = F_{j',t} = 1 \). In this case, player \( i \) is prescribed to take \( \alpha_i^{NE} \).

3. \( S_{i,t} = BR \) if \( F_{i,t} = 1 \). In this case, player \( i \) is prescribed to best respond to her opponents’ actions (which equal either \( r_{-i,t} \) or \( \alpha_{-i}^{NE} \), depending on \( (F_{j,t})_{j \neq i} \)).

**States:** At the start of each period \( t \), conditional on the public randomization draw of \( r_t \in A \) described above, an additional random variable \( \tilde{y}_t \in Y \) is also drawn by public randomization, with distribution \( q(\tilde{y}_t | r_t) \). That is, the distribution of the public randomization draw \( \tilde{y}_t \) conditional on the draw \( r_t \) is the same as the distribution of the realized public signal profile \( \tilde{y}_t \) when the profile of the players’ intended actions is \( r_t \); however, the distribution of \( \tilde{y}_t \) depends only on the public randomization draw \( r_t \), and not on the players’ actions. For each player \( i \) and period \( t \), let \( f_{i,r_i,t} : Y_i \to \mathbb{R} \) be defined as in Lemma 3, and let

\[
f_{i,t} = \begin{cases} 
  f_{i,r_i,t}(y_{i,t}) & \text{if } S_{i,t} = 0, \\
  f_{i,r_i,t}(\tilde{y}_{i,t}) & \text{if } S_{i,t} = NE, \\
  0 & \text{if } S_{i,t} = BR.
\end{cases}
\]

Thus, the value of \( f_{i,t} \) depends on the state \( (F_{n,t})_{n \in t} \), the target action profile \( r_t \) (which is drawn from distribution \( \alpha \) as described above), the public signal \( y_{t_i} \), and the additional variable \( \tilde{y}_t \).\(^29\) Later in the proof, \( f_{i,t} \) will be a component of the “reward” earned by player \( i \) in period \( t \), which will be reflected in player \( i \)’s end-of-block continuation payoff function \( \pi : H^{T+1} \to \mathbb{R} \).

We can finally define \( F_{i,t} \) as

\[
F_{i,t} = 1 \left\{ \exists t' \leq t : \left| \sum_{t''=1}^{t'-1} \delta^{t''-1} f_{i,t''} \right| \geq \tilde{f} \right\}.
\]

\(^29\)Intuitively, introducing the variable \( \tilde{y}_t \), rather than simply using \( y_{i,t} \) everywhere in (27), ensures that the distribution of \( f_{i,t} \) does not depend on player \( i \)’s opponents’ strategies.
That is, \( F_{i,t} \) is the indicator function for the event that the magnitude of the component of player \( i \)'s reward captured by \( (f_{i,i'}^{t-1})_{i'=1}^{t} \) exceeds \( \bar{f} \) at any time \( t' \leq t \).

This completes the definition of the equilibrium block strategy profile \( \sigma \). Before proceeding further, we note that a unilateral deviation from \( \sigma \) by any player \( i \) does not affect the distribution of the state vector \( (F_{j,t})_{j \neq i}^{T} \). (However, such a deviation does affect the distribution of \( (F_{i,t})_{t=1}^{T} \).)

**Lemma 5** For any player \( i \) and block strategy \( \bar{\sigma}_i \), the distribution of the random vector \( (F_{j,t})_{j \neq i}^{T} \) is the same under block strategy profile \( (\bar{\sigma}_i, \sigma_{-i}) \) as under block strategy profile \( \sigma \).

**Proof.** Since \( F_{j,t} = 1 \) implies \( F_{j,t+1} = 1 \), it suffices to show that, for each \( t \), each \( J \subseteq I \setminus \{i\} \), each \( h^t \) such that \( J = \{j \in I \setminus \{i\} : F_{j,t} = 0\} \), and each \( z_t \), the probability \( \Pr \left( (F_{j,t+1})_{j \in J} | h^t, z_t, a_{i,t} \right) \) is independent of \( a_{i,t} \). Since \( F_{j,t+1} \) is determined by \( h^t \) and \( f_{j,t} \), it is enough to show that \( \Pr \left( (f_{j,t})_{j \in J} | h^t, z_t, a_{i,t} \right) \) is independent of \( a_{i,t} \).

Recall that \( S_{j,t} \) is determined by \( h^t \), and that if \( j \in J \) (that is, \( F_{j,t} = 0 \)) then \( S_{j,t} \in \{0, NE\} \). If \( S_{j,t} = 0 \) then player \( j \) takes \( r_{j,t} \), which is determined by \( z_t \), \( y_{j,t} \) is distributed according to \( p_j(y_{j,t}|r_{j,t}) \), and \( f_{j,t} \) is determined by \( y_{j,t} \), independently across players conditional on \( z_t \). If \( S_{j,t} = NE \) then \( \tilde{y}_{j,t} \) is distributed according to \( p_j(\tilde{y}_{j,t}|r_{j,t}) \), where \( r_{j,t} \) is determined by \( z_t \), and \( f_{j,t} \) is determined by \( \tilde{y}_{j,t} \), independently across players conditional on \( z_t \). Thus, \( \Pr \left( (f_{j,t})_{j \in J} | h^t, z_t, a_{i,t} \right) = \prod_{j \neq i} \Pr \left( f_{j,t} | S_{j,t}, r_{j,t} \right) \), which is independent of \( a_{i,t} \) as desired.

**Continuation Value Function:** We now construct the continuation value function \( \pi : H^{T+1} \rightarrow \mathbb{R}^N \). For each player \( i \) and end-of-block history \( h^{T+1} \), player \( i \)'s continuation value \( \pi_i(h^{T+1}) \) will be defined as the sum of \( T \) “rewards” \( \pi_{i,t} \), where \( t = 1, \ldots, T \), and a constant term \( c_i \) that does not depend on \( h^{T+1} \).

The rewards \( \pi_{i,t} \) are defined as follows:

1. If \( F_{j,t} = 0 \) for all \( j \in I \), then
   \[
   \pi_{i,t} = \delta^{t-1} \left( v^*_i + \zeta_i \varepsilon / 4 - u_i(a^*_i) + f_{i,r_i}(y_{i,t}) \right). \tag{29}
   \]

2. If \( F_{i,t} = 1 \) and \( F_{j,t} = 0 \) for all \( j \neq i \),
   \[
   \pi_{i,t} = \delta^{t-1} \left( v^*_i + \zeta_i \varepsilon / 4 - u_i(a^*_i) \right). \tag{30}
   \]
3. Otherwise, 
\[ \pi_{i,t} = \delta^{t-1} \left( -\zeta_i \bar{u} - u_i^* + 1 \{ S_{i,t} = 0 \} f_{i,r_i,t} (y_{i,t}) \right). \]  

The constant \( c_i \) is defined as 
\[ c_i = -\mathbb{E} \left[ \sum_{t=1}^{T} \delta^{t-1} \left( \max_{j \neq i} \max_{F_{j,t} = 0} (v_i^* + \zeta_i \varepsilon/4) - 1 \{ \max_{j \neq i} F_{j,t} = 1 \} \zeta_i \bar{u} \right) + \frac{1 - \delta^T}{1 - \delta} v_i^* \right]. \]  

Note that, since \( v_i^* + \zeta_i \varepsilon/4 \) and \( v_i^* \) are both feasible payoffs, we have 
\[ |c_i| \leq 2 \bar{u} \frac{1 - \delta^T}{1 - \delta}. \]  

Finally, for each \( i \) and \( h^{T+1} \), player \( i \)‘s continuation value at end-of-block history \( h^{T+1} \) is defined as 
\[ \pi_i (h^{T+1}) = c_i + \sum_{t=1}^{T} \pi_{i,t}. \]  

A.3.3 Verification of the Equilibrium Conditions

We now verify that \( \sigma \) and \( \pi \) satisfy promise keeping, incentive compatibility, and self generation. We first show that \( F_{i,t} = 0 \) for all \( i \) and \( t \) with high probability, and then verify the three desired conditions in turn.

**Lemma 6** We have 
\[ \Pr \left( \max_{i \in I, t \in \{1, \ldots, T\}} F_{i,t} = 0 \right) \geq 1 - \frac{\varepsilon}{20 \bar{u}}. \]  

**Proof.** By union bound, it suffices to show that, for each \( i \), \( \Pr (\max_{t \in \{1, \ldots, T\}} F_{i,t} = 1) \leq \varepsilon/20 \bar{u} N \), or equivalently 
\[ \Pr \left( \max_{t \in \{1, \ldots, T\}} \left| \sum_{t'=1}^{t} \delta^{t'-1} f_{i,t'} \right| \geq \tilde{f} \right) \leq \frac{\varepsilon}{20 \bar{u} N}. \]  

To see this, let \( \tilde{f}_{i,t} = f_{i,r_i,t} (\bar{y}_{i,t}) \). Note that the variables \( \left( \tilde{f}_{i,t} \right)_{t=1}^{T} \) are independent (unlike the variables \( (f_{i,t})_{t=1}^{T} \)). Since \( \left( \tilde{f}_{i,t'} \right)_{t'=1}^{t} \) and \( (f_{i,t'})_{t'=1}^{t} \) have the same distribution if \( S_{i,t} \neq BR \),
while $f_{i,t} = 0$ if $S_{i,t} = BR$, we have
\[
\Pr \left( \max_{t \in \{1, \ldots, T\}} \left| \sum_{t' = 1}^{t} \delta^{t'-1} f_{i,t'} \right| \geq \bar{f} \right) \leq \Pr \left( \max_{t \in \{1, \ldots, T\}} \left| \sum_{t' = 1}^{T} \delta^{t'-1} \bar{f}_{i,t'} \right| \geq \bar{f} \right).
\] (37)

Since $\left( \bar{f}_{i,t} \right)_{t=1}^{T}$ are independent, Etemadi’s inequality (Billingsley, 1995; Theorem 22.5) implies that
\[
\Pr \left( \max_{t \in \{1, \ldots, T\}} \left| \sum_{t' = 1}^{t} \delta^{t'-1} \bar{f}_{i,t'} \right| \geq \bar{f} \right) \leq 3 \max_{t \in \{1, \ldots, T\}} \Pr \left( \sum_{t' = 1}^{t} \delta^{t'-1} \bar{f}_{i,t'} \geq \bar{f}/3 \right).
\] (38)

Letting $x_{i,t} = \delta^{t'-1} \bar{f}_{i,t}$, note that $|x_{i,t}| \leq 2\bar{u}/\eta$ with probability 1 by (16), $\mathbb{E}[x_{i,t}] = 0$ by (14), and
\[
\text{Var} \left( \sum_{t' = 1}^{t} x_{i,t'} \right) = \sum_{t' = 1}^{t} \text{Var} (x_{i,t'}) \leq \sum_{t' = 1}^{T} \text{Var} (x_{i,t'}) = \frac{1 - \delta T \bar{u}^2}{1 - \delta} \eta \text{ by (15).}
\]

Therefore, by Bernstein’s inequality ((17), which again applies because $\left( \bar{f}_{i,t} \right)_{t=1}^{T}$ are independent) and (18), we have, for each $t \leq T$,
\[
\Pr \left( \left| \sum_{t' = 1}^{t} \delta^{t'-1} \bar{f}_{i,t'} \right| \geq \bar{f}/3 \right) \leq \frac{\varepsilon}{60\bar{u}N}.
\] (39)

Finally, (37), (38), and (39) together imply (36). □

**Incentive Compatibility:** We use the following lemma (proof in Appendix B.4).

**Lemma 7** For each player $i$ and block strategy profile $\sigma$, incentive compatibility holds (i.e., (24) is satisfied) if and only if
\[
\text{supp } \sigma_i (h^t) \subseteq \arg\max_{a_{i,t} \in A_i} \mathbb{E}^{\sigma_{-i}} \left[ \delta^{t-1} u_{i,t} + \pi_{i,t} | h^t, a_{i,t} \right] \text{ for all } t \text{ and } h^t.
\] (40)

In addition, for all $t \leq t'$ and $h^t$, we have
\[
\mathbb{E}^{\sigma} \left[ \delta^{t-1} u_{i,t} + \pi_{i,t'} | h^t \right] = \mathbb{E}^{\sigma} \left[ \delta^{t'-1} \left( 1 \left\{ \max_{j \neq i} F_{j,t'} = 0 \right\} (v_i^* + \zeta_i \varepsilon / 4) - 1 \left\{ \max_{j \neq i} F_{j,t'} = 1 \right\} \zeta_i \bar{u} \right) | h^t \right].
\] (41)

We now verify (40). Fix a player $i$, period $t$, and history $h^t$. We consider several cases, which parallel the definition of the reward $\pi_{i,t}$.
1. If $F_{j,t} = 0$ for all $j \in I$, recall that the equilibrium action profile is $r_t$ that is prescribed by public randomization $z_t$. For each action $a_i \neq r_{i,t}$, by (13) and (29), and recalling that $u \geq \max_a u_i (a) - \min_a u_i (a)$, we have

$$\mathbb{E}^{\sigma^{i_t}} \left[ \delta^{t-1} u_{i,t} + \pi_i | h^t, z_t, a_{i,t} = r_{i,t} \right] - \mathbb{E}^{\sigma^{i_t}} \left[ \delta^{t-1} u_{i,t} + \pi_i | h^t, z_t, a_{i,t} = a_i \right]$$

$$= \delta^{t-1} \left( \mathbb{E} \left[ u_i (r_t) + f_{i,r_{i,t}} (y_{i,t}) | a_{i,t} = r_{i,t} \right] - \mathbb{E} \left[ u_i (a_i, r_{i,t}) + f_{i,r_{i,t}} (y_{i,t}) | a_{i,t} = a_i \right] \right)$$

$$\leq 0,$$ so (40) holds.

2. If $F_{i,t} = 1$ and $F_{j,t} = 0$ for all $j \neq i$, then the reward $\pi_{i,t}$ specified by (30) does not depend on $y_{i,t}$. Hence, (40) reduces to the condition that every action in $\text{supp} \sigma_i (h^t)$ is a static best responses to $\sigma_{-i} (h^t)$. This conditions holds for the prescribed action profile, $(r_i' \in BR_i (r_{-i,t}), r_{-i,t})$ or $\alpha^{NE}$.

3. Otherwise,

(a) If $S_{i,t} = 0$, then (40) holds because it holds in Case 1 above and (29) and (31) differ only by a constant independent of $y_{i,t}$.

(b) If $S_{i,t} \neq 0$, then either $F_{j,t} = F_{j',t} = 1$ for distinct players $j, j'$, or there exists a unique player $j \neq i$ with $F_{j,t} = 1$, and for this player we have $\zeta_j = -1$. In both cases, $\alpha^{NE}$ is prescribed. Since the reward $\pi_{i,t}$ specified by (31) does not depend on $y_{i,t}$, (40) reduces to the condition that every action in $\text{supp} \sigma_i (h^t)$ is a static best responses to $\sigma_{-i} (h^t)$, which holds for the prescribed action profile $\alpha^{NE}$.

**Promise Keeping:** This essentially holds by construction: we have

$$\frac{1 - \delta}{1 - \delta^T} \mathbb{E}^\sigma \left[ \sum_{t=1}^T \delta^{t-1} u_{i,t} + \pi_i (h^{T+1}) \right]$$

$$= \frac{1 - \delta}{1 - \delta^T} \left( \mathbb{E}^\sigma \left[ \sum_{t=1}^T \delta^{t-1} u_{i,t} + \pi_i (h^t) \right] + c_i \right) \quad \text{(by (34))}$$

$$= \frac{1 - \delta}{1 - \delta^T} \sum_{t=1}^T \delta^{t-1} \left( 1 \left\{ \max_{j \neq i} F_{j,t} = 0 \right\} (v_i^* + \zeta_i \varepsilon/4) - 1 \left\{ \max_{j \neq i} F_{j,t} = 1 \right\} \zeta_i \bar{u} \right) + c_i \quad \text{(by (41))}$$

$$= v_i^* \quad \text{(by (32))},$$ so (23) holds.

**Self Generation:** We use the following lemma (proof in Appendix B.5).
Lemma 8 For every end-of-block history \( h^{T+1} \), we have

\[
\zeta_i \sum_{t=1}^{T} \pi_{i,t} \leq \bar{f} + \frac{2\bar{u}}{\eta} \text{ and } \quad \sum_{t=1}^{T} \pi_{i,t} \leq \bar{f} + \frac{2\bar{u}}{\eta} + 2\bar{u} \frac{1 - \delta^T}{1 - \delta}. \tag{42}
\]

In addition,

\[
\zeta_i c_i \leq -\frac{1 - \delta^T \varepsilon}{1 - \delta \cdot 8} \tag{44}
\]

To establish self generation ((25)), it suffices to show that, for each \( h^{T+1}, \) \( \zeta_i \pi_i (h^{T+1}) \leq 0 \) and \( \left( (1 - \delta) / \delta^T \right) \pi_i (h^{T+1}) \leq \varepsilon. \) This now follows because

\[
\zeta_i \pi_i (h^{T+1}) = \zeta_i \left( c_i + \sum_{t=1}^{T} \pi_{i,t} \right) \leq -\frac{1 - \delta^T \varepsilon}{1 - \delta \cdot 8} + \bar{f} + \frac{2\bar{u}}{\eta} \quad \text{(by (42) and (44))}
\]

\[
\leq \frac{1 - \delta^T}{8(1 - \delta)} \left( \varepsilon - 8 \left( \frac{1 - \delta}{1 - \delta^T} \right) \left( \bar{f} + \frac{2\bar{u}}{\eta} \right) \right) \leq 0 \quad \text{(by (19)), and}
\]

\[
\frac{1 - \delta}{\delta^T} \pi_i (h^{T+1}) \leq \frac{1 - \delta}{\delta^T} \left| c_i + \sum_{t=1}^{T} \pi_{i,t} \right| \leq \frac{1 - \delta}{\delta^T} \left( 4\bar{u} \frac{1 - \delta^T}{1 - \delta} + \bar{f} + \frac{2\bar{u}}{\eta} \right) \quad \text{(by (33) and (43))}
\]

\[
= \frac{1 - \delta}{\delta^T} - 4\bar{u} + \frac{1 - \delta}{\delta^T} \left( \bar{f} + \frac{2\bar{u}}{\eta} \right) \leq \varepsilon \quad \text{(by (20))},
\]

which completes the proof.

A.4 Proof of Theorem 3

Fix a linear equilibrium with weights \( b = (1, b_2, \ldots, b_N) \), where \( |b_i| \leq 1 \) for all \( i \). Let \( I^+ = \{i : b_i \geq 0\} \) and \( I^- = \{i : b_i \leq 0\} \). Define

\[
\underline{v}_i = \begin{cases} 
\inf_h u_i (h) & \text{if } i \in I^+, \\
\sup_h u_i (h) & \text{if } i \in I^-
\end{cases}
\]

and \( \bar{v}_i = \begin{cases} 
\sup_h u_i (h) & \text{if } i \in I^+, \\
\inf_h u_i (h) & \text{if } i \in I^-
\end{cases} \).

By standard arguments, for every \( \beta \in [0, 1] \), there exists a linear equilibrium with the same weights \( b \) and expected payoff \( v = (1 - \beta) \underline{v} + \beta \bar{v} \) such that the set \( \{v : \exists h \text{ s.t. } v = u (h)\} \) is
closed and there exist histories $h$ and $h'$ such that $u(h) = \underline{v}$ and $u(h') = \bar{v}$. Since $M(\varepsilon)$ is convex, it thus suffices to show that $\underline{v}, \bar{v} \in M(\varepsilon)$.

In the following lemma, given $\alpha \in \Delta(A)$ and a function $f : A \times Y \to \mathbb{R}$, $\mathbb{E}^\alpha [f(r,y)]$ denotes expectation where $r \sim \alpha$ and then $y \sim q(\cdot|r)$, and $\mathbb{E}^{\alpha \alpha'}[f(r,y)]$ denotes expectation where $r \sim \alpha$ and then $y \sim q(\cdot|\alpha', r)$). The proof is deferred to Appendix B.6.

**Lemma 9** There exist $\alpha \in \Delta(A)$ and $x : A \times Y \to \mathbb{R}$ such that

$$
\bar{v} = \mathbb{E}^\alpha [u(r) - bx(r,y)],
$$
$$
\mathbb{E}^\alpha [u_i(r) - b_ix(r,y) | r_i = a_i] \geq \mathbb{E}^{\alpha \alpha'}[u_i(\alpha', r_{-i}) - b_ix(r,y) | r_i = a_i] \text{ for all } i, a_i \in \text{supp } \alpha, \alpha' \in A,
$$
$$
x(r,y) \in \left[0, \frac{\delta}{1-\delta} \bar{u}\right] \text{ for all } r, y.
$$

If the constraint $x(r,y) \in [0, (\delta/(1-\delta)) \bar{u}]$ is replaced with $x(r,y) \in [-((\delta/(1-\delta)) \bar{u}, 0]$, then the same statement holds with $\underline{v}$ in place of $\bar{v}$.

Taking $\alpha$ and $x$ as in Lemma 9, we have, for any player $i$ and manipulation $\tilde{\alpha}_i$,

$$
D_i(\tilde{\alpha}_i|\alpha) \leq \sum_{a_i} \alpha_i(a_i) \left( \mathbb{E}^{\alpha \tilde{\alpha}_i}(a_i) [b_i x(r,y) | r_i = a_i] - \mathbb{E}^\alpha [b_i x(r,y) | r_i = a_i] \right)
$$
$$
\leq \sum_{a_i} \alpha_i(a_i) \max_{a_i} \left| \mathbb{E}^{\alpha \alpha'}[x(r,y) | r_i = a_i] - \mathbb{E}^\alpha [x(r,y) | r_i = a_i] \right|
$$
$$
\leq \sum_r \alpha(r) \max_{a_i} \left| \mathbb{E}[x(r,y) | r, a_i] - \mathbb{E}[x(r,y) | r] \right|
$$

where the second inequality uses $|b_i| \leq 1$. Hence,

$$
\frac{1}{N} \sum_i D_i(\alpha) \leq \frac{1}{N} \sum_i \sum_r \alpha(r) \max_{a_i} \left| \mathbb{E}[x(r,y) | r, a_i] - \mathbb{E}[x(r,y) | r] \right|
$$
$$
\leq \max_{r,a} \frac{1}{N} \sum_i \left| \mathbb{E}[x(y) | a_i, r_{-i}] - \mathbb{E}[x(y) | r] \right|.
$$
We conclude that \( \sum_i D_i(\alpha)/N \) is bounded by the solution to the program

\[
\max_{(Y,p),r,a,x} \frac{1}{N} \sum_i \left[ \mathbb{E}[x(y)|a_i,r_{-i}] - \mathbb{E}[x(y)|r] \right] \\
\text{s.t.} \\
x(y) \in \left[ 0, \frac{\delta}{1-\delta} \bar{u} \right] \text{ for all } y, \\
\mathbb{E}[x(y)|r] \leq \bar{u},
\]

where the last constraint follows because \( \mathbb{E}[x(y)|r] = u_1(r) - \bar{v}_1 \leq \bar{u} \). The remainder of the proof shows that the value of this program converges to 0 along any sequence of equilibria satisfying the theorem’s hypothesis.

We first consider the sub-program where \((Y,p)\) is fixed, so maximization is over \((r,a,x)\). Recall that \( q(y|a) = \sum_{\hat{a}} \pi_{a,\hat{a}} P(y|\hat{a}) \). Note that the value of the sub-program with signal distribution \( q \) is greater than that with signal distribution \( \hat{q} \), if \( \hat{q} \) is a garbling of \( q \). (That is, there exists a Markov matrix \( M \) such that \( \hat{q} = Mq \).) To see this, fix any \((r,a,\hat{x})\) that are feasible with signal distribution \( \hat{q} \), and define \( x(r,y) = \sum_y M(\hat{y}|y) \hat{x}(r,\hat{y}) \). Then

\[
\sum_y q(y|a) x(r,y) = \sum_y q(y|a) \sum_{\hat{y}} M(\hat{y}|y) \hat{x}(r,\hat{y}) = \sum_{\hat{y}} \hat{q}(\hat{y}|a) \hat{x}(r,\hat{y}) \text{ for all } a,
\]

so \((r,a,x)\) is feasible with signal distribution \( q \) and yields the same value in the sub-program.

Consequently, it is without loss to let \( Y = A \) and \( p(y|\hat{a}) = \hat{a} \) for all \( y, \hat{a} \), so that \( q(\hat{a}|a) = \pi_{a,\hat{a}} \) for all \( a, \hat{a} \). Now fix \( r,a \in A \), and for each \( i \), define

\[
\pi^i_{a_i,\hat{a}_i} = \begin{cases} 
1 - \bar{\pi} & \text{if } \hat{a}_i = a_i, \\
\bar{\pi} & \text{if } \hat{a}_i = r_i, \\
0 & \text{otherwise}
\end{cases} \\
\pi_{r_i,\hat{a}_i} = \begin{cases} 
1 - \bar{\pi} & \text{if } \hat{a}_i = r_i, \\
\bar{\pi} & \text{if } \hat{a}_i = a_i, \\
0 & \text{otherwise}
\end{cases} \\
\pi^i_{\hat{a}_i,\hat{a}_i} = \begin{cases} 
1 \{ \hat{a}_i = \hat{a}_i \} & \text{for } \hat{a}_i \notin \{ a_i, r_i \}.
\end{cases}
\]

Then define \( \bar{\pi}_{\hat{a},\hat{a}} = \prod_i \pi^i_{\hat{a}_i,\hat{a}_i} \) for all \( \hat{a}, \hat{a} \). The following lemma (proved in Appendix B.7) implies that the value of the program is upper-bounded by that with \( \pi = \bar{\pi} \).

**Lemma 10** \( \pi \) is a garbling of \( \bar{\pi} \). 

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We conclude that \( \sum D_i(\alpha)/N \) is bounded by the solution to the program

\[
\max_{r,\alpha,x} \frac{1}{N} \sum_i \left[ \mathbb{E} \left[ x(\hat{\alpha}) \right] | a_i, r_{-i} \right] - \mathbb{E} \left[ x(\hat{\alpha}) \right] | r \right] \quad \text{s.t.} \quad (45)
\]

\[
x(\hat{\alpha}) \in \left[ 0, \frac{\delta}{1-\delta} \bar{u} \right] \quad \text{for all} \ \hat{\alpha},
\]

\[
\mathbb{E} \left[ x(\hat{\alpha}) \right] | r \leq \bar{u}, \quad (47)
\]

where \( \hat{\alpha} \) is distributed \( \pi_{\hat{\alpha},\hat{\alpha}} \). Note that, for \( \hat{\alpha} = r \) or \( \hat{\alpha} = (a_i, r_{-i}) \) for some \( i, \pi_{\hat{\alpha},\hat{\alpha}} > 0 \) iff \( \hat{\alpha} \in \prod_i \{a_i, r_i\} \). Note also that it is without loss to take \( a_i \neq r_i \) for all \( i \). For if \( a_i = r_i \) then the program becomes

\[
\max_{a_{-i}, r_{-i}, x_{-i}:A_{-i} \in \mathbb{R}} \frac{1}{N} \sum_{j \neq i} \left[ \mathbb{E} \left[ x_{-i}(\hat{a}_{-i}) \right] | a_j, r_{-j} \right] - \mathbb{E} \left[ x_{-i}(\hat{a}_{-i}) \right] | r \right] \quad \text{s.t.} \quad (46), (47).
\]

Any feasible triple \( (a_{-i}, r_{-i}, x_{-i}) \) in this reduced program can be extended to a feasible triple \( (a, r, x) \) with \( a_i \neq r_i \) in the original program which gives the same value, by defining \( x(\hat{\alpha}) = x_{-i}(\hat{a}_{-i}) \) for all \( \hat{\alpha} \). We thus assume that \( a_i \neq r_i \) for all \( i \).

We now show that the value of program (45)–(47) converges to 0, which will complete the proof. Note that this value is less than the sum of the values of the two programs

\[
\max_{r,\alpha,x} \frac{1}{N} \sum_i \left( \mathbb{E} \left[ x(\hat{\alpha}) \right] | a_i, r_{-i} \right] - \mathbb{E} \left[ x(\hat{\alpha}) \right] | r \right]_+ \quad \text{s.t.} \quad (46), (47), \quad \text{and}
\]

\[
\max_{r,\alpha,x} \frac{1}{N} \sum_i \left( \mathbb{E} \left[ x(\hat{\alpha}) \right] | r \right] - \mathbb{E} \left[ x(\hat{\alpha}) \right] | a_i, r_{-i} \right]_+ \quad \text{s.t.} \quad (46), (47).
\]

We show that the value of the first of these programs converges to 0. A symmetric argument shows that the value of the second program also converges to 0, which implies that the value of program (45)–(47) converges to 0 as well, as desired.

Letting \( \lambda \geq 0 \) denote the multiplier on (47), it is immediate that the solution to the first program takes the form

\[
x(\hat{\alpha}) = \left\{ \begin{array}{ll}
\delta \frac{\bar{u}}{1-\delta} & \text{if } \frac{\sum_i \pi(a_i,r_{-i}),\hat{\alpha}}{\pi_{r,\hat{\alpha}}}-\frac{\pi_{r,\hat{\alpha}}}{\pi_{r,\hat{\alpha}}} > \lambda, \\
0 & \text{if } \frac{\pi(a_i,r_{-i}),\hat{\alpha}}{\pi_{r,\hat{\alpha}}}-\frac{\pi_{r,\hat{\alpha}}}{\pi_{r,\hat{\alpha}}} < \lambda
\end{array} \right.
\]

\[
\quad \quad \quad \lambda = \left\{ \begin{array}{ll}
\frac{\delta}{1-\delta} \bar{u} & \text{if } \frac{1}{N} \sum_i \pi(a_i,r_{-i}),\hat{\alpha}\pi_{r,\hat{\alpha}} > \lambda + 1, \\
0 & \text{if } \frac{1}{N} \sum_i \pi(a_i,r_{-i}),\hat{\alpha}\pi_{r,\hat{\alpha}} < \lambda + 1
\end{array} \right.
\]
For all $\hat{a} \in \prod_i \{a_i, r_i\}$,

$$\pi_{(a_i, r_{-i}), \hat{a}} = \begin{cases} 
\frac{1-\pi}{\pi} & \text{if } \hat{a}_i = a_i, \\
\frac{\pi}{1-\pi} & \text{if } \hat{a}_i = r_i.
\end{cases}$$

Since $(1 - \pi)/\pi > \pi/(1 - \pi)$ (as $\pi < 1/2$), it follows that there exists $n^* \in \{0, 1, \ldots, N\}$ and $\beta \in [0, 1]$ such that

$$x(\hat{a}) = \begin{cases} \frac{\delta}{1-\delta} \bar{u} & \text{if } \{i : \hat{a}_i = a_i\} > n^*, \\
\beta \frac{\delta}{1-\delta} \bar{u} & \text{if } \{i : \hat{a}_i = a_i\} = n^*, \\
0 & \text{if } \{i : \hat{a}_i = a_i\} < n^*
\end{cases}.$$

Let $n = |\{i : \hat{a}_i = a_i\}|$ and let $n_{-i} = |\{j \neq i : \hat{a}_j = a_j\}|$. Note that, for any $n^*$,

$$\Pr(n = n^*|a_i, r_{-i}) = (1 - \pi) \Pr(n_{-i} = n^* - 1|r_{-i}) + \pi \Pr(n_{-i} = n^*|r_{-i}) \quad \text{and} \quad \Pr(n = n^*|r) = \pi \Pr(n_{-i} = n^* - 1|r_{-i}) + (1 - \pi) \Pr(n_{-i} = n^*|r_{-i}),$$

and hence $\Pr(n \geq n^*|a_i, r_{-i}) - \Pr(n \geq n^*|r_{-i}) = (1 - 2\pi) \Pr(n_{-i} = n^* - 1|r_{-i})$. Therefore, the program becomes

$$\max_{n^* \in \{0, 1, \ldots, N\}, \beta \in [0, 1]} \frac{\delta}{1-\delta} \bar{u} (1 - 2\pi) \left(\beta \Pr(n_{-i} = n^* - 1|r_{-i}) + (1 - \beta) \Pr(n_{-i} = n^*|r_{-i})\right) \quad \text{(48)}$$

subject to

$$\beta \Pr(n = n^*|r) + \Pr(n \geq n^* + 1|r) \leq \frac{1 - \delta}{\delta}, \quad \text{(49)}$$

where

$$\Pr(n_{-i} = n^*|r_{-i}) = \binom{N-1}{n^*} \pi^{n^*} (1 - \pi)^{N-1-n^*} \quad \text{and} \quad \Pr(n = n^*|r) = \binom{N}{n^*} \pi^{n^*} (1 - \pi)^{N-n^*}.$$

Fix a sequence, indexed by $k$, of games with $(1 - \delta) \exp(N^{1-\rho}) \to \infty$ for some $\rho > 0$ and pairs $(n^*, \beta)$ that satisfy the constraint (49). Fix $\varepsilon > 0$, and suppose towards a contradiction that, for every $\bar{k}$, there is some $k \geq \bar{k}$ such that the value of the objective (48) exceeds $\varepsilon$. Taking a subsequence and relabeling $\bar{k}$ if necessary, this implies that there exists $\bar{k}$ such that, for every $k \geq \bar{k}$, the value of the objective (48) exceeds $\varepsilon$.

We consider two cases, and derive a contradiction in each of them.

First, suppose that there exists $c > 0$ such that, for every $\bar{k}$, there is some $k \geq \bar{k}$ satisfying
\[ |\pi - (n^* - 1)/(N - 1)| > c. \] By Hoeffding’s inequality,

\[
\Pr(n-i \geq n^* - 1|r_{-i}) \leq \exp\left(-2\left(\frac{\pi - n^* - 1}{N - 1}\right)^2(N - 1)\right).
\]

Hence, for every \( \tilde{k} \), there is some \( k \geq \tilde{k} \) such that the value of (48) is at most

\[
\frac{\delta}{1 - \delta} \bar{u} (1 - 2\pi) \exp\left(-2\left(\frac{\pi - n^* - 1}{N - 1}\right)^2(N - 1)\right) \leq \frac{\delta}{1 - \delta} \bar{u} (1 - 2\pi) \exp(-2c^2(N - 1)).
\]

Since \((1 - \delta) \exp(N^{1-\rho}) \to \infty\), we have \(\exp(-c^2N) / (1 - \delta) \to 0\) for all \( c > 0 \), and hence (48) is less than \( \varepsilon \) for sufficiently large \( k \), a contradiction.

Second, suppose that for any \( c > 0 \), there exists \( \tilde{k} \) such that, for every \( k \geq \tilde{k} \), we have

\[
\left|\frac{\pi - n^* - 1}{N - 1}\right| \leq c.
\]

(50)

Fix \( c > 0 \) and take \( k \) sufficiently large that (50) holds. For any \( m \in \mathbb{N} \), we have

\[
\frac{\Pr(n \geq n^* + 1|r)}{\Pr(n-i = n^*|r_{-i})} = \sum_{n=n^*+1}^{N} \frac{N(1 - \pi)(N - n^*)!n^!}{N - n^*} \left(\frac{\pi}{1 - \pi}\right)^{n-n^*}
\]

\[
\geq \sum_{n=n^*+1}^{N} \frac{N(1 - c)}{N - 1} \left(\frac{N - n^*}{n}\right)^{n-n^*} \left(\frac{n^* - 1 - c(N - 1)}{N - n^* + c(N - 1)}\right)^{n-n^*}
\]

\[
\geq \sum_{n=n^*+1}^{n^*+m} (1 - c) \left(\frac{N - n^*}{n^* + m} \times \frac{n^* - 1 - c(N - 1)}{N - n^* + c(N - 1)}\right)^{m}
\]

\[
= m(1 - c) \left(\frac{N - n^*}{n^* + m} \times \frac{n^* - 1 - c(N - 1)}{N - n^* + c(N - 1)}\right)^{m}.
\]

By (50), for any \( \gamma' > 0 \), for sufficiently large \( k \) we have \((n^* - 1)/(n^* + m) \geq 1 - \gamma'\), and hence

\[
\frac{N - n^*}{n^* + m} \times \frac{n^* - 1 - c(N - 1)}{N - n^* + c(N - 1)} \geq (1 - \gamma') \frac{N - n^*}{n^* - 1} \times \frac{n^* - 1 - c(N - 1)}{N - n^* + c(N - 1)}
\]

\[
= (1 - \gamma') \frac{1 - c}{1 + c N - n^*}
\]

\[
\geq (1 - \gamma') \frac{1 - c}{1 + \frac{c}{1 - \pi}} = \frac{(1 - \gamma')(\pi - 2c)(\pi - c)}{(\pi - c)(1 - \pi)}.
\]
which converges to $1 - \gamma'$ as $c \to 0$. Hence, for any $\gamma > 0$, there exists $\tilde{k}$ sufficiently large such that for every $k \geq \tilde{k}$,

$$\frac{\Pr (n \geq n^* + 1| r)}{\Pr (n_{-i} = n^*| r_{-i})} \geq m(1 - c) \left( \frac{(1 - \gamma')(\pi - 2c)(1 - \pi - c)}{(\pi - c)(1 - \pi)} \right)^m \geq m(1 - \gamma).$$

We therefore have

$$\frac{\beta \Pr (n = n^*| r) + \Pr (n \geq n^* + 1| r)}{\Pr (n_{-i} = n^*| r_{-i})} \geq \frac{\Pr (n \geq n^* + 1| r)}{\Pr (n_{-i} = n^*| r_{-i})} \geq m(1 - \gamma).$$

Similarly, for each $m$ and $\gamma > 0$, there exists $\tilde{k}$ such that for every $k \geq \tilde{k}$, we have

$$\frac{\beta \Pr (n = n^*| r) + \Pr (n \geq n^* + 1| r)}{\Pr (n_{-i} = n^* - 1| r_{-i})} \geq m(1 - \gamma).$$

Thus, for each $m$ and $\gamma > 0$, there exists $\tilde{k}$ such that for every $k \geq \tilde{k}$, we have

$$\frac{\beta \Pr (n = n^*| r) + \Pr (n \geq n^* + 1| r)}{\beta \Pr (n_{-i} = n^* - 1| r_{-i}) + (1 - \beta) \Pr (n_{-i} = n^*| r_{-i})} \geq m(1 - \gamma).$$

The value of (48) thus satisfies

$$\frac{\delta}{1 - \delta} \bar{u} (1 - 2\pi) \left( \beta \Pr (n_{-i} = n^* - 1| r_{-i}) + (1 - \beta) \Pr (n_{-i} = n^*| r_{-i}) \right) \leq \bar{u} (1 - 2\pi) \frac{\beta \Pr (n_{-i} = n^* - 1| r_{-i}) + (1 - \beta) \Pr (n_{-i} = n^*| r_{-i})}{\beta \Pr (n = n^*| r) + \Pr (n \geq n^* + 1| r)} \leq \frac{\bar{u} (1 - 2\pi)}{m(1 - \gamma)}.\text{ (by (49))}$$

Taking $m$ and $\gamma$ such that $\bar{u} (1 - 2\pi) / (m(1 - \gamma)) < \varepsilon$ gives the desired contradiction.

**References**


B Online Appendix

B.1 The Set \( B(\varepsilon) \) in A Public-Goods Game

Consider the public-goods game where each player chooses \textit{Contribute} or \textit{Don’t Contribute}, and a player’s payoff is the fraction of players who contribute less a constant \( c \in (0,1) \) (independent of \( N \)) if she contributes herself. Fix any \( v \in (0,1-c) \), let \( v = (v,\ldots,v) \in \mathbb{R}^N \), and let \( \varepsilon = cv (1-c-v) / 4 > 0 \). We show that \( B_v(\varepsilon) \subseteq V \) for all \( N \). Since no one contributing is a Nash equilibrium with 0 payoffs, this implies that \( B_v(\varepsilon) \subseteq V^* \), and hence \( v \in B(\varepsilon) \), for all \( N \).

To see this, fix any \( N \). Since the game is symmetric, to show that \( B_v(\varepsilon) \subseteq V \) it suffices to show that, for any number \( n \in \{0,\ldots,N\} \), there exists a feasible payoff vector where \( n \) “favored” players receive payoffs no less than \( v + \varepsilon \), and the remaining \( N - n \) “disfavored” players receive payoffs no more than \( v - \varepsilon \). Fix such an \( n \), and let \( x = n/N \).

Consider the mixed action profile \( \alpha^1 \) where favored players \textit{Contribute} with probability \( (v + \varepsilon) / (1-c) \in (0,1) \) and disfavored players always \textit{Contribute}. At this profile, favored players receive payoff

\[
 f(x) := x \left( \frac{v + \varepsilon}{1 - c} + (1 - x) (1) - c \right) \frac{v + \varepsilon}{1 - c},
\]

while disfavored players receive payoff

\[
 g(x) := x \left( \frac{v + \varepsilon}{1 - c} + (1 - x) (1) - c \right).
\]

Note that \( f'(x) < 0 \), so \( f(x) \geq f(1) = v + \varepsilon \).

Now, with \( f(x) \) so defined, consider the mixed action profile \( \alpha^2 \) where favored players \textit{Contribute} with probability \( (v + \varepsilon)^2 / ((1-c)f(x)) \in (0,1) \) and disfavored players \textit{Contribute} with probability \( (v + \varepsilon) / f(x) \in (0,1) \). Note that each player’s payoff at profile \( \alpha^2 \) equals her payoff at profile \( \alpha^1 \) multiplied by \( (v + \varepsilon) / f(x) \). Therefore, at profile \( \alpha^2 \), favored players receive payoff

\[
 f(x) \frac{v + \varepsilon}{f(x)} = v + \varepsilon,
\]

while disfavored players receive payoff

\[
 g(x) \frac{v + \varepsilon}{f(x)} = \left( f(x) - \left( \frac{1 - v + \varepsilon}{1 - c} \right) c \frac{v + \varepsilon}{f(x)} \right) \frac{v + \varepsilon}{f(x)} \\
 \leq v + \varepsilon - \left( \frac{1 - v + \varepsilon}{1 - c} \right) c (v + \varepsilon) \quad \text{(since } f(x) \leq 1) \\
 \leq v - \varepsilon,
\]
where the last inequality follows from $\varepsilon = cv (1 - c - v) / 4$ and straightforward algebra.

**B.2 Derivation of Equation (7)**

Since $w_{i,t} = (1 - \delta) u_{i,t} + \delta w_{i,t+1}$, for every history $h^t \in H^t$ we have

\[ W_{i,t} (h^t) = (1 - \delta) U_{i,t} (h^t) + \delta \mathbb{E} [W_{i,t+1} | h^t] \iff \mathbb{E} [W_{i,t+1} | h^t] = \frac{1}{\delta} W_{i,t} (h^t) - \frac{1 - \delta}{\delta} U_{i,t} (h^t). \]

Therefore,

\[
\text{Var} \left( \mathbb{E} [W_{i,t+1} | h^t] \right) = \text{Var} \left( \frac{1}{\delta} W_{i,t} - \frac{1 - \delta}{\delta} U_{i,t} \right) \\
= \frac{1}{\delta^2} \text{Var} (W_{i,t}) + \left( \frac{1 - \delta}{\delta} \right)^2 \text{Var} (U_{i,t}) - 2 \frac{1 - \delta}{\delta^2} \text{Cov} (U_{i,t}, W_{i,t}) \\
\geq \frac{1}{\delta} \text{Var} (W_{i,t}) + \left( \frac{1 - \delta}{\delta} \right)^2 \text{Var} (U_{i,t}) - \frac{1 - \delta}{\delta^2} \text{Var} (W_{i,t}) \\
= \frac{1}{\delta} \text{Var} (W_{i,t}) - \frac{1 - \delta}{\delta} \text{Var} (U_{i,t}).
\]

**B.3 Omitted Details for the Proof of Lemma 4**

We show that, with the stated definitions of $T$ and $\bar{f}$, (21) and (22) imply (18)–(20). First, note that

\[
\frac{1 - \delta^2}{1 - \delta^{2T}} = \frac{(1 + \delta) (1 - \delta)}{(1 + \delta^T) (1 - \delta^T)} < 2 \frac{1 - \delta}{1 - \delta^T}.
\]

Hence,

\[
\frac{2 \bar{f} (1 - \delta^2)}{9 \bar{u} (1 - \delta^{2T})} < \frac{4}{9} \frac{1 - \delta}{1 - \delta^T} \sqrt{36 \log \left( \frac{60 \bar{u}}{\varepsilon} \right) \log (N) \frac{1 - \delta^T}{1 - \delta} \frac{\bar{u}^2}{\eta}} \\
= \frac{4}{9} \sqrt{36 \log \left( \frac{60 \bar{u}}{\varepsilon} \right) \log (N) \frac{1 - \delta}{1 - \delta^T} \frac{1}{\eta}} \leq 1 \quad \text{(by (21)).}
\]

Therefore,

\[
60 \bar{u} N \exp \left( - \frac{\bar{f}^2}{2} \frac{\eta}{(1 - \delta^T \bar{u}^2 + 2 \frac{\bar{f}^2}{3} \bar{u})} \right) \leq 60 \bar{u} N \exp \left( - \frac{\bar{f}^2}{2} \frac{\eta}{(1 - \delta^T \bar{u}^2 + 1 - \delta^T \bar{u}^2)} \right) = 60 \bar{u} N \exp \left( - \frac{\bar{f}^2 \eta}{36 \frac{1 - \delta^T \bar{u}^2}{1 - \delta^T} \bar{u}^2} \right).
\]

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Moreover,
\[
\frac{f^2 \eta}{36 \frac{1-\delta^T u^2}{1-\delta^T}} = \frac{36 \log \left( \frac{60u}{\varepsilon} \right) \log (N) \frac{1-\delta^T}{1-\delta^T}}{36 \frac{1-\delta^T u^2}{1-\delta^T}} = \frac{1 + \delta}{1 + \delta^T} \log \left( \frac{60u}{\varepsilon} \right) \log (N) \geq \log \left( \frac{60u}{\varepsilon} \right) \log (N).
\]

Hence, we have
\[
60u \exp \left( - \left( \frac{1}{3} \right)^2 \frac{\eta}{2} \frac{1-\delta^T u^2}{1-\delta^T} \right) \leq 60u \exp \left( - \log \left( \frac{60u}{\varepsilon} \right) \log (N) \right) = \varepsilon.
\]

This establishes (18).

Next, we have
\[
8 \frac{1 - \delta}{1 - \delta^T} \left( \frac{f + 2u}{\eta} \right) = 8u \sqrt{36 \log \left( \frac{60u}{\varepsilon} \right) \log (N) \frac{1-\delta}{1-\delta^T} \eta + \frac{1 - \delta}{1-\delta^T} \frac{2}{3} \frac{f + 2u}{\eta}} \leq \varepsilon \quad \text{(by (22)).}
\]

(51)

This establishes (19).

Finally, by (51) and \(8 \frac{1 - \delta^T}{\delta^T} / \delta^T \leq \varepsilon\), we have
\[
4u \frac{1 - \delta^T}{\delta^T} + \frac{1 - \delta}{\delta^T} \left( \frac{f + 2u}{\eta} \right) = 4u \frac{1 - \delta^T}{\delta^T} + \frac{1 - \delta}{\delta^T} \frac{1 - \delta}{1 - \delta^T} \left( \frac{f + 2u}{\eta} \right) \leq 4 \frac{\varepsilon}{8} + \frac{\varepsilon}{8} \leq \varepsilon.
\]

This establishes (20).

### B.4 Proof of Lemma 7

We show that player \(i\) has a profitable one-shot deviation from \(\sigma_i\) at some history \(h^t\) if and only if (40) is violated at \(h^t\). To see this, we first calculate player \(i\)'s continuation payoff under \(\sigma\) from period \(t + 1\) onwards (net of the constant \(c_i\) and the rewards already accrued \(\sum_{t'=1}^{t'} \pi_{i,t'}\)). For each \(t' \geq t + 1\), there are several cases to consider.

1. If \(F_{j,t'} = 0\) for all \(j\), then by (14) and (29) we have
   \[
   \mathbb{E}_\sigma \left[ \delta^{t'-1} u_{i,t'} + \pi_{i,t'} | h^t \right] = \delta^{t'-1} \left( u_i(\alpha_{\delta}^*) + v_i^* + \zeta_i\varepsilon/4 - u_i(\alpha_{\delta}^*) + \mathbb{E} \left[ f_{i,r_{i,t'}} (y_{i,t'}) | r_{i,t'} \right] \right) = \delta^{t'-1} (v_i^* + \zeta_i\varepsilon/4).
   \]

2. If \(F_{i,t'} = 1\) and \(F_{j,t'} = 0\) for all \(j \neq i\), then by (30) we have
   \[
   \mathbb{E}_\sigma \left[ \delta^{t'-1} u_{i,t'} + \pi_{i,t'} | h^t \right] = \delta^{t'-1} \left( u_i(\alpha_{\delta}^*) + v_i^* + \zeta_i\varepsilon/4 - u_i(\alpha_{\delta}^*) \right) = \delta^{t'-1} (v_i^* + \zeta_i\varepsilon/4).
   \]

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3. Otherwise,

(a) If $S_{i,t'} = 0$, then by (14) and (31) (and recalling that player $i$’s equilibrium action is $r_{i,t}$ when $S_{i,t'} = 0$) we have

\[
\mathbb{E}^\sigma \left[ \delta^{t'-1} u_{i,t'} + \pi_{i,t'} \mid h^{t'} \right] = \delta^{t'-1} \left( u_i(\alpha^*_t) - \zeta_i \bar{u} - u(\alpha^*_t) \right) + \mathbb{E} \left[ f_{i,r_{i,t'}}(y_{i,t'}) \mid r_{i,t'} \right] = \delta^{t'-1} (-\zeta_i \bar{u}).
\]

(b) If $S_{i,t'} \neq 0$, then by (31) we have

\[
\mathbb{E}^\sigma \left[ \delta^{t'-1} u_{i,t'} + \pi_{i,t'} \mid h^{t'} \right] = \delta^{t'-1} \left( u_i(\alpha^*_t) - \zeta_i \bar{u} - u(\alpha^*_t) \right) = \delta^{t'-1} (-\zeta_i \bar{u})
\]

In total, (41) holds, and player $i$’s continuation payoff under $\sigma$ from period $t + 1$ onwards equals

\[
\mathbb{E}^\sigma \left[ \sum_{t'=t+1}^T \delta^{t'-1} \left( 1 \left\{ \max_{j \neq i} F_{j,t'} = 0 \right\} \left( v_i^* + \zeta_i \bar{u} / 4 \right) - 1 \left\{ \max_{j \neq i} F_{j,t'} = 1 \right\} \zeta_i \bar{u} \right) \mid h^t \right].
\]

By Lemma 5, the distribution of $\left( (F_{n,t'})_{n \neq i} \right)_{t'=t+1}^T$ does not depend on player $i$’s period-$t$ action, and hence neither does player $i$’s continuation payoff under $\sigma$ from period $t + 1$ onwards. Therefore, player $i$’s period-$t$ action $a_{i,t}$ maximizes her continuation payoff from period $t$ onwards if and only if it maximizes $\mathbb{E}^\sigma \left[ \delta^{t'-1} u_{i,t'} + \pi_{i,t'} \mid h^{t'}, a_{i,t} \right]$.

**B.5 Proof of Lemma 8**

Define

\[
\pi_{i,t}^v = \begin{cases} 
\delta^{t-1} \left( v_i^* + \zeta_i \bar{u} / 4 - u_i(\alpha^*_t) \right) & \text{if } F_{j,t} = 0 \text{ for all } j \neq i, \\
\delta^{t-1} (-\zeta_i \bar{u} - u_i(\alpha^*_t)) & \text{otherwise}
\end{cases}
\]

and

\[
\pi_{i,t}^f = \begin{cases} 
\delta^{t-1} f_{i,a_{i,t}}(y_{i,t}) & \text{if either } F_{j,t} = 0 \text{ for all } j \text{ or } S_{i,t} = 0, \\
0 & \text{otherwise}.
\end{cases}
\]
Note that, by (29)–(31), we can write \( \pi_{i,t} = \pi_{i,t}^v + \pi_{i,t}^f \). We show that, for every end-of-block history \( h^{T+1} \), we have

\[
\zeta_i \sum_{t=1}^{T} \pi_{i,t}^v \in \left[ -2\bar{u} \frac{1 - \delta^T}{1 - \delta}, 0 \right] \quad \text{and} \\
\left| \zeta_i \sum_{t=1}^{T} \pi_{i,t}^f \right| \leq \bar{f} + 2\bar{u}/\eta. \tag{52}
\]

Since \( \pi_{i,t} = \pi_{i,t}^v + \pi_{i,t}^f \), (52) and (53) imply (42) and (43), which proves the first part of the lemma.

For (52), note that, by definition of the prescribed equilibrium actions, if \( F_{j,t} = 0 \) for all \( j \neq i \), then (i) if \( \zeta_i = 1 \), we have \( u_i(\alpha_i^*) \geq \sum_a \alpha(a) \min \left\{ u_i(a), \max_{a_i'} u_i(a_i', a_{-i}) \right\} \geq u_i(\bar{a}) = v_i^* + \varepsilon/2 \), by (26); and (ii) if \( \zeta_i = -1 \), we have \( u_i(\alpha_i^*) \leq \max \left\{ u_i(\bar{a}), u_i(\alpha_i^{NE}) \right\} = u_i(\bar{a}) = v_i^* - \varepsilon/2 \), again by (26). In total, we have \( \zeta_i (v_i^* + \zeta_i \varepsilon/4 - u_i(\alpha_i^*)) \leq -\varepsilon/4 \). Since obviously \( \zeta_i (v_i^* + \zeta_i \varepsilon/4 - u_i(\alpha_i^*)) \geq -2\bar{u} \) and \(-\bar{u} - \zeta_i u_i(\alpha_i^*) \geq -2\bar{u} \), we have

\[
\zeta_i \pi_{i,t}^v = \begin{cases} \\
\delta^{t-1} \zeta_i (v_i^* + \zeta_i \varepsilon/4 - u_i(\alpha_i^*)) & \text{if } F_{j,t} = 0 \text{ for all } j \neq i \\
\delta^{t-1} (-\bar{u} - \zeta_i u_i(\alpha_i^*)) & \text{otherwise} \\
\end{cases} \in [-2\bar{u} \delta^{t-1}, 0].
\]

For (53), note that \( S_{i,t} = 0 \) implies \( F_{i,t} = 0 \), and hence

\[
\left| \zeta_i \sum_{t=1}^{T} \pi_{i,t}^f \right| \leq \left| \zeta_i \sum_{t=1}^{T} 1 \{ F_{i,t} = 0 \} \delta^{t-1} f_{i,a_{i,t}}(y_{i,t}) \right|.
\]

Since \( F_{i,t+1} = 1 \) whenever \( \sum_{t'=1}^{t} \delta^{t-1} f_{i,a_{i,t}}(y_{i,t}) \geq \bar{f} \), and in addition \( |f_{i,a_{i,t}}(y_{i,t})| \leq 2\bar{u}/\eta \) by (16), this inequality implies (53).
For the second part of the lemma, by (32), we have

\[
\zeta_i c_i = \zeta_i \left( -\mathbb{E} \left[ \sum_{t=1}^{T} \delta^{t-1} \left( 1 \left\{ \max_{j \neq i} F_{j,t} = 0 \right\} (v_i^* + \zeta_i \varepsilon/4) - 1 \left\{ \max_{j \neq i} F_{j,t} = 1 \right\} (\zeta_i \bar{u}) \right) + \frac{1 - \delta^T}{1 - \delta} v_i^* \right] \right)
\]

\[
= \mathbb{E} \left[ \sum_{t=1}^{T} \delta^{t-1} \left( 1 \left\{ \max_{j \neq i} F_{j,t} = 0 \right\} (-\varepsilon/4) + 1 \left\{ \max_{j \neq i} F_{j,t} = 1 \right\} (\bar{u} + \zeta_i v_i^*) \right) \right] \bigg| \in [0, 2\alpha]
\]

\[
\leq \mathbb{E} \left[ \sum_{t=1}^{T} \delta^{t-1} \left( 1 \left\{ \max_{j \neq i} F_{j,t} = 0 \right\} (-\varepsilon/4) + 1 \left\{ \max_{j \neq i} F_{j,t} = 1 \right\} 2\bar{u} \right) \right]
\]

\[
\leq -\frac{1 - \delta^T}{1 - \delta} \left( 1 - \frac{\varepsilon}{20\bar{u}} \right) \varepsilon/4 + \left( \frac{\varepsilon}{20\bar{u}} \right) 2\bar{u} \quad \text{(by (35))}
\]

\[
\leq -\frac{1 - \delta^T}{1 - \delta} \varepsilon/8 \quad \text{(as } \varepsilon < \bar{u}/2)\]

**B.6 Proof of Lemma 9**

Let \( E = \{(1 - \beta) v + \beta \bar{v} : \beta \in [0, 1]\}. By standard arguments (e.g., APS), \( E \) is self-generating: for any \( v \in E \), there exist \( \alpha \in \Delta(A) \) and \( w : A \times Y \to E \) such that

\[
v = \mathbb{E}^\alpha [u(r) + \delta w(r, y)] \quad \text{and} \quad \mathbb{E}^\alpha [u_i(r) + \delta w_i(r, y) | r_i = a_i] \geq \mathbb{E}^{\alpha, a'_i} [u_i(a'_i, r_{-i}) + \delta w_i(r, y) | r_i = a_i] \quad \text{for all } i, a_i \in \text{supp} \alpha_i, a'_i \in A_i.
\]

Since \( v \in E \) and \( w(r, y) \in E \) for all \( r, y \), we have

\[
v_i - w_i(r, y) = b_i (v_i - w_i(r, y)) \quad \text{for all } i \neq 1, r \in A, y \in Y.
\]

Since \( \bar{v}_1 \geq v_1 \) for all \( v \in E \), if \( v = \bar{v} \) then \( w_1(r, y) \leq v_1 \) for all \( r, y \). Hence, taking \( v = \bar{v} = (1 - \delta) u(\alpha) + \delta \mathbb{E} [w(r, y) | \alpha] \) and defining \( x(r, y) = (\delta/ (1 - \delta)) (\bar{v}_1 - w_1(r, y)) \in [0, (\delta/ (1 - \delta)) \bar{u}] \) for all \( r, y \), and letting \( \mathbb{E} [\cdot] \) denote expectation where \( y \sim q(\cdot | a) \), we have, for all \( a, r, \)

\[
u(a) - b \mathbb{E} [x(r, y)] = u(a) - b \mathbb{E} \left[ \frac{\delta}{1 - \delta} (\bar{v}_1 - w_1(r, y)) \right] = u(a) - \mathbb{E} \left[ \frac{\delta}{1 - \delta} (\bar{v} - w(r, y)) \right] = (1 - \delta) u(a) + \delta \mathbb{E} [w(r, y)],
\]

and the result follows. Similarly, if \( v = v \) then \( w_1(r, y) \geq v_1 \) for all \( r, y \), and the symmetric argument applies.
B.7 Proof of Lemma 10

Since \((\hat{a}_i)_{i \in I}\) are independent conditional on \((a_i)_{i \in I}\), it suffices to show that \(\pi^i\) is a garbling of \(\hat{\pi}^i\) for each \(i\). Since \(\pi < 1/2\), the matrix \(\hat{\pi}^i\) is invertible, with inverse matrix \(\hat{\pi}^i\) given by

\[
\hat{\pi}^i_{\hat{a}_i, a_i} = \begin{cases}
\frac{1-\pi}{\pi} \text{ if } \hat{a}_i = a_i, \\
-\frac{\pi}{1-2\pi} \text{ if } \hat{a}_i = r_i, \\
0 \text{ otherwise}
\end{cases}
\]

\[
\hat{\pi}^i_{a_i, \hat{a}_i} = \begin{cases}
\frac{1-\pi}{\pi} \text{ if } \hat{a}_i = a_i, \\
-\frac{\pi}{1-2\pi} \text{ if } \hat{a}_i = r_i, \\
0 \text{ otherwise}
\end{cases}
\]

The matrix \(M^i := \pi^i \hat{\pi}^i\) is easily calculated as

\[
M^i_{\hat{a}_i, a_i} = \begin{cases}
\pi^i_{\hat{a}_i, a_i} \frac{1-\pi}{\pi} - \left(1 - \pi^i_{\hat{a}_i, a_i}\right) \frac{\pi}{1-2\pi} & \text{if } a_i, a_i' \in \{a_i, r_i\}, \\
1 \{\hat{a}_i = \tilde{a}_i\} & \text{otherwise}
\end{cases}
\]

Note that, for \(\tilde{a}_i \in \{a_i, r_i\},\)

\[
\sum_{\hat{a}_i} M^i_{\hat{a}_i, \tilde{a}_i} = \frac{|A_i| - 1 - \pi}{|A_i| - 1 - |A_i| \pi} - (|A_i| - 1) \frac{\pi}{|A_i| - 1 - |A_i| \pi} = 1,
\]

and clearly \(\sum_{\hat{a}_i} M^i_{\hat{a}_i, \tilde{a}_i} = 1\) for \(\tilde{a}_i \notin \{a_i, r_i\}\). In addition, since \(\pi^i_{\hat{a}_i, a_i} \geq \pi\) for all \(\hat{a}_i, a_i'\), we have

\[
\frac{\pi^i_{\hat{a}_i, a_i} (1 - \pi) - (1 - \pi^i_{\hat{a}_i, a_i}) \pi}{1 - 2\pi} \geq \frac{\pi (1 - \pi) - (1 - \pi) \pi}{|A_i| - 1 - |A_i| \pi} = 0,
\]

and clearly \(M^i_{a_i, \hat{a}_i} \leq 1\) for all \(a_i, a_i'\). So \(M^i\) is a Markov matrix and \(\pi^i = M^i \hat{\pi}^i\), completing the proof.