Selling Impressions: Efficiency vs Competition

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Efficiency Versus Competition in Digital Advertising

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  - A little, to maximize competition?
  - Or something in between?
consider classic problem of second price auction of single object to buyers with symmetric independent private values.....
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- would the seller prefer full information (buyers know their values perfectly), no information (buyers know nothing about their values), or something in between?

with full information: efficient allocation but information rents - revenue is expectation of second highest value

with no information: inefficiency but no information rent - revenue is common ex ante expected value
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intuition: competition is lowest when there is a high winning value

this is our main theoretical result and first main contribution
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- So the publisher *can* control the information that the advertiser has about the value of the impression (by controlling the advertiser’s access to information about attributes).
- Advertisers values’ might well be correlated, but will be independent as long as advertiser / viewer variation is "horizontal", i.e., attributes and preferences are symmetric across viewers and bidders.
Selling Impressions

- So (I claim) our result applies to the market for impressions
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- You can trust me on this, or....
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A model of the market for impressions with two sided information.
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- So (I claim) our result applies to the market for impressions
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  - A model of the market for impressions with two sided information
  - We describe when this model reduces statistically and strategically to our first model and result
Hour Long Talk

1 Main Result
2 Market for Impressions
   1 Institutional Details
   2 Stylized Model of Market for Impressions with Two-Sided Information
   3 Statistical and Strategic Analysis
Part I: Main Result
Setting for Main Result

- $N$ bidders
Setting for Main Result

- $N$ bidders
- Private values symmetrically and independently distributed according to $F$

Blackwell/Strassen/Rothscild-Stiglitz show: there exists a signal $s$ that induces a distribution of expected valuations $G$ from $F$ if and only if $F$ is a mean preserving spread of $G$. We write $F \preceq G$. 

If $F$ is a mean preserving spread of $G$, we have: 

$$
\int_{0}^{1} \int \frac{v}{dF}(t) = \int_{0}^{1} \int \frac{v}{dG}(t),
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- if $F$ is a mean preserving spread of $G$ we write $F \prec G$
Revenue

- second-order statistic $w_{(2)}$ of $N$ symmetrically and independently distributed random variables is

$$\mathbb{P}(w_{(2)} \leq t) = N G^{N-1}(t)(1 - G(t)) + G^N(t)$$
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- non-linear problem in optimization variable $G$
- neither convex nor concave program
denote by $q_i$ a random variable that is uniformly distributed in $[0, 1]$ and

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Key Quantile Change of Variables

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- quantile distribution $S$ is independent of the underlying distribution $F$ or $G$

- just as quantile of any random variable is uniformly distributed, the quantile of second-order statistic of $N$ random variables is distributed according to $S$ for every distribution
Quantile Representation of Revenue

- revenue is expectation over quantiles using measure $S(q)$
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- $F \prec G$ if and only if $G^{-1} \prec F^{-1}$
- objective is linear in $G^{-1}$
Proposition (Optimal Information Structure)

Suppose that $F$ is absolutely continuous, then the unique optimal symmetric information structure is given by:

$$s(v_i) = \begin{cases} v_j & \text{if } q_i(v_i) \leq q^* \\ \mathbb{E}[v_j | F(v_j) \geq q] & \text{if } q_i(v_i) \geq q^* \end{cases}$$

where $q^* \in [0, 1)$ is independent of $F$.

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- otherwise reveal no information beyond the fact that the valuation is above the threshold
Optimal Information Structure

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- reveal the valuation of all those bidders who have a valuation lower than some threshold determined by a fixed quantile $q^*$
- otherwise reveal no information beyond the fact that the valuation is above the threshold
- with change of variables, "upper censorship"
optimal information structure supports competition at the top of the distribution at the expense of an efficient allocation
Competition through Information

- optimal information structure supports competition at the top of the distribution at the expense of an efficient allocation
- bundles for every bidder all valuations above the threshold $F^{-1}(q^*)$ into a single mass point
optimal information structure supports competition at the top of the distribution at the expense of an efficient allocation

bundles for every bidder all valuations above the threshold $F^{-1}(q^*)$ into a single mass point

information rent of the winning bidder is depressed considerably with a corresponding gain in the revenue for the seller
Intuitive Proof Step 1: Integrate by Parts

- if $\bar{v} = G^{-1}(1)$ is the upper bound on expected value, by integration by parts, revenue is:

$$\int_0^1 S'(q)G^{-1}(q)\,dq = \bar{v} - \int_0^1 S(q)dG^{-1}(q)$$

so we have minimization problem

$$\min_{G^{-1}} \int_0^1 S(q)dG^{-1}(q)$$
subject to $G^{-1} \prec F^{-1}$

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so we have minimization problem

$$
\min_{G^{-1}} \int_0^1 S(q) dG^{-1}(q)
$$
subject to $G^{-1} < F^{-1}$

- HINT: if $\bar{v} = 1$, $G^{-1}$ is itself a distribution function.
Step 2: Convexification of Second Order Statistic

$N q^{N-1} (1-q) + q^N$

Graph of $S(q)$ for $N = 3$
Step 2: Convexification of Second Order Statistic

Graph of $S(q)$ for $N = 3$

unique inflection point for all $N$
Convex Hull of Quantile Function

\[ N q^{N-1} (1-q) + q^N \]

- find largest convex function below the original one
Convex Hull of Quantile Function

- find largest convex function below the original one
- problem reduces to finding \( q \) such that:

\[
S(q) + S''(q)(1 - q) = S'(1) = 1
\]
we take the mass of $F^{-1}$ to the extremes of the affine segment
we take the mass of $F^{-1}$ to the extremes of the affine segment
the mass at each extreme must keep the expected mean of quantile constant
Step 3: Back to Value Distribution

We draw the quantile function for $F(v) = p^v$. 

- map back to value distribution of bidder $i$
Step 3: Back to Value Distribution

- map back to value distribution of bidder $i$
- we draw the quantile function for $F(v) = \sqrt{v}$
From Quantile to Convexified Quantile

- the mass is moved to the end points
the mass is moved to the end points
while keeping expectation of quantile constant
we have been working with the quantile function
we have been working with the quantile function
to recover the distribution we rotate
we now have the distribution $Q$
we now have the distribution $Q$
there is one step in distribution of expected value
Verification

- this is an example of a problem of characterizing extreme points of monotone functions subject to majorization constraints (Kleiner et al. 2021)

Proposition (Kleiner et al. Proposition 2)

Let $G^{-1}$ be such that for some countable collection of intervals $\{[x_i, \bar{x}_i] \mid i \in I\}$,

$$G^{-1}(q) = \begin{cases} F^{-1}(q) & q \notin \bigcup_{i \in I} [x_i, \bar{x}_i] \\ \frac{\int_{x_i}^{\bar{x}_i} F^{-1}(t)dt}{\bar{x}_i - x_i} & q \in [x_i, \bar{x}_i] \end{cases}$$

If $\text{conv } S$ is affine on $[x_i, x_i]$ for each $i \in I$ and if $\text{conv } S = S$ otherwise, then $G$ solves the maximization problem. Moreover, if $F$ is strictly increasing the converse holds.
What is the Critical Quantile?

Proposition (Critical Quantile)

The quantile $q^*(N) \in [0, 1)$ that determines the optimal information structure is 0 if $N = 2$, is increasing in $N$ and approaches 1 as $N \to \infty$; for $N \geq 3$, it is implicitly defined as the solution of:

$$S'(q)(1 - q) = 1 - S(q)$$

- this is an $N$th degree polynomial in $q$
Variational Intuition

- suppose we initially have quantile threshold $q$ and write
  
  \[ \bar{v} = F^{-1}(q) \text{ and } \overline{v} = \mathbb{E}_F [v | v \geq \bar{v}] \]
Variational Intuition

- suppose we initially have quantile threshold $q$ and write $\underline{v} = F^{-1}(q)$ and $\overline{v} = \mathbb{E}_F [v | v \geq v]$
- conditional on $v \geq \underline{v}$
Variational Intuition

- suppose we initially have quantile threshold $q$ and write $v = F^{-1}(q)$ and $\overline{v} = \mathbb{E}_F [v | v \geq v]

- conditional on $v \geq v$

  - expected gain in approximately:

$$\frac{1 - S(q)}{1 - q} \times \mathbb{E} [\overline{v} - v]$$

  - prob high payment
  - increase in payment
Variational Intuition

- suppose we initially have quantile threshold $q$ and write $\nu = F^{-1}(q)$ and $\bar{v} = \mathbb{E}_F[v | v \geq v]$
- conditional on $v \geq v$
  - expected gain in approximately:
    $$\frac{1 - S(q)}{1 - q} \times \varepsilon [\bar{v} - v]$$
  - expected loss is approximately
    $$\varepsilon S'(q) \times \bar{v} - v$$
Critical Quantiles

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Part II: Market for Impressions
Market for Impressions: Qualitative Features

- private information in digital advertising takes a particular distributed form....
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- viewer is object of auction and has many attributes (demographics, past browsing behavior, past purchase behavior, etc.)
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- publisher as seller has private information about attributes of viewer
- advertiser as bidder has private information about their preference (willingness to pay) for attributes of viewer
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- publisher as seller has private information about attributes of viewer
- advertiser as bidder has private information about their preference (willingness to pay) for attributes of viewer

value of the match or impression between advertiser and viewer is jointly determined by these different sources of private information
Market for Impressions: Selling Mechanism

- auction, e.g., second price auction
Market for Impressions: Selling Mechanism

- auction, e.g., second price auction
- publisher generates information for advertisers by combining reported preferences with own attribute information

Two variants:

1. **Auto-bidding**
   - Advertisers report preferences to publisher and publisher commits to submitting advertiser optimal bids conditional on reported preferences
   - Our focus: reduces to our main result

2. **Manual bidding**
   - Advertisers select bids after receiving information from publisher
   - Extension: motivates modifications of our main result
Market for Impressions: Selling Mechanism

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- There is an induced distribution $F$ over value $v_i$. 
Statistical Assumptions

- An advertiser’s preference tells them nothing about their or others’ valuation of the object (without knowing the attribute)

\[(x, v_1, \ldots, v_N) \text{ and } (y, v_1, \ldots, v_N)\]

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- as $J \to \infty$, can induce any distribution of values $F$
A Model of Auto-Bidding

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3. Preferences and attributes are realized, preferences are reported to the advertiser, signals and bids are realized and the impression is allocated to the highest bidder at the second highest price.
Auto-Bidding

Proposition (Truthful Reporting)

Advertisers have an incentive to truthfully report their preferences in the auto-bidding mechanism.

Corollary: With those commitment powers, publisher’s problem reduces to our main result
Comment on Manual Bidding

- suppose that the advertiser chooses his bid after receiver signal from publisher
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- however, there is an information structure where seller pools high and low values and reveals values in between, which attains close to first best
Easy and Hard Extensions

1. Could replace second price auction with standard auction without reserve (easy)

2. With reserve, will also have pooling at reserve price (easy)

3. Can't solve asymmetric case, no counterexample (hard)

4. We abstracted from adverse selection via independence assumption, would be nice to add in (hard)

5. As mentioned, manual bidding or other mechanisms (hard)

6. Relaxing statistical assumptions, allow vertical differentiation of bidders and viewers (hard)
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- Bergemann, Brooks and Morris (2017): seller can do even better if he can reveal information to buyers about other buyers’ valuations
Literature II: Mechanics of Bidding in the Market for Impressions

- automated versus manual bidding, Aggarwal et al. (2019), Deng et al. (2020)
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- automated versus manual bidding, Aggarwal et al. (2019), Deng et al. (2020)
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Key Takeaways

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- publishers can and do control the amount of information reflected in bids by limiting bidding language, releasing information
- a key incentive to pool premium impressions while allowing information about non-premium impressions
Fixed $N$ doesn’t seem very empirically relevant/interpretable. We solve two interesting cases with a large number of bidders....

1. Entry: With probability $p$, bidder has valuation 0 ("non-entrant"); with probability $1 - p$, bidder is entrant with valuation distributed according to $F$.  

2. Consider $N \rightarrow \infty$ and $p \rightarrow 1$ limit, where $N(1 - p)$ (expected number of entrants) is constant.

3. If $N \rightarrow \infty$, no information to entrants

4. If $N \rightarrow \infty$, pool bidders at quantiles above $p$.

5. Power law tails gain from pooling is strictly positive even as $N \rightarrow \infty$ and $p \rightarrow 1$. 

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- gain from pooling is strictly positive even as $N \to \infty$ and $q^* \to 1$
Large Markets

- large number of (possible) bidders is arguably the prevailing condition in digital advertising how does information respond to random participation of bidders
- revenue performance of auction with optimal information structure when the actual number of participating bidders grows large.
Random Number of Bidder

- with probability $p$, valuation is equal zero
- with probability $1 - p$, valuation is distributed with $F$
- limit as $N \to \infty$ and $p \to 1$ while expected number of bidders with positive values constant at:

$$\lambda \triangleq N(1 - p)$$

- critical number $\rho$ of expected bidders

$$\rho \triangleq N(1 - q^*)$$ \hspace{1cm} (1)

- as $N \to \infty$, (1) converges in terms of $\rho$:

$$\rho^2 e^{-\rho} = 1 - e^{-\rho} - \rho e^{-\rho} \iff \rho \approx 1.793$$
Proposition

As $N \to \infty$, $p \to 1$, the optimal information structure is:

1. If $\lambda \leq \rho$, then bidders observe binary signals and expected value is either 0 or $\mathbb{E}[v_i] \frac{\lambda}{\rho}$.

2. If $\lambda > \rho$, bidder $v_i$ with $F(v_i) \leq (\lambda - \rho)/\lambda$ learns value, and bidder $v_i \in [F^{-1}((\lambda - \rho)/\lambda), 1]$ is bundled.

- bundle zero values with positive values ("broad search")
- increase number of bidders even at cost of decreasing expected valuations
- with sufficiently many bidders, we have pooling of high-valuation bidders
Arnosti, Beck and Milgrom (2016) argued heavy tails distribution prevail in digital advertising.

$F$ has regularly varying tails with index $\alpha$, if

$$\lim_{t \to \infty} \frac{1 - F(kt)}{1 - F(t)} = k^\alpha,$$

we assume $\alpha < 0$, and with $\alpha < -1$, we guarantee finite mean

for example Pareto distribution satisfies this assumption
Revenue Comparison with Heavy Tails

- expected revenue in second price auction with complete disclosure of information, \( R_c \):

\[
R_c \triangleq \mathbb{E}[v(2)].
\]

- compare revenue of optimal information structure, \( R \) with revenue of complete disclosure, \( R_c \) for large \( N \)

Proposition (Revenue Ratio with Many Bidders)

As \( N \to \infty \), there exists \( z \in (1, \infty) \) s.th.:

\[
\lim_{N \to \infty} \frac{R}{R_c} = z.
\]

Furthermore, in the limit \( \alpha \to -1 \), \( z \to \infty \).
Revenue Gains

- gains from optimal information structure do not vanish when the distribution has fat tails, or $\alpha < 0$

\[ \mathbb{E}[v_1] - \mathbb{E}[v_2] \to \infty, \text{ as } N \to \infty. \]

- optimal information structure thickens the market at the tail of the distribution

- thus provide a revenue improvement even as the numbers of bidders becomes arbitrarily large
Honesty and Obedience
Eliciting Advertisers’ Preferences

- examine advertisers’ incentives to truthfully report their preferences.
- a reporting strategy for bidder $i$ is denoted by:
  $$\tilde{y}_i : \{-1, 1\}^J \rightarrow \Delta\{-1, 1\}^J.$$
- given reported preferences, the seller discloses to the bidder a signal $s(\tilde{v}_i)$, where
  $$\tilde{v}_i \triangleq u\left(\frac{1}{\sqrt{J}} \sum_{j=1}^{J} \tilde{y}_{ij}(y_{ij})x_j\right)$$
- since preferences and attributes are symmetrically distributed, a sufficient statistic for the bidder’s strategy is the fraction of preferences truthfully reported:
  $$t_i \triangleq \sum_{j=1}^{J} \frac{\tilde{y}_i y_i}{J}$$
Critical Reporting Strategies

- with preferences and attributes symmetrically distributed, a sufficient statistic is:

\[
    t_i \triangleq \sum_{j=1}^{J} \frac{\tilde{y}_i y_i}{J}
\]

- in other words, for any reporting strategy \( \tilde{y}_i, \tilde{y}'_i \) satisfying \( t_i = t'_i \), the induced distribution of expected valuations will be the same: \( \tilde{G}_i = \tilde{G}'_i \)

- if \( t = 1 \) then preferences have been correctly reported; if \( t = 0 \) then half of all preference components have been misreported; if \( t = -1 \) then every preference component has been incorrectly reported
Honesty and Informativeness

The following lemma establishes that the only relevant incentive constraints are those induced by reporting the exact opposite preference.

**Lemma (Informativeness of Signals)**

Let $s$ be the optimal information structure. For every $t \in [0, 1)$, the generated signal is less informative than the signal generated when reporting truthfully. For every $t \in (-1, 0]$, the generated signal is less informative than the signal generated when reporting the exact opposite preference (i.e., $t = -1$).
Truthful Reporting Under Auto Bidding

- informative lemma helps to establish:

Proposition (Honesty)

Under auto bidding and the optimal information structure, it is a dominant strategy for the advertiser to report his preference truthfully.

- misreporting leads to automated bids different from the expected value given limited information
- truhttelling guarantees that bid always equals expected value
Manual Bidding

- truthtelling is not an equilibrium for every $N, u$
- there is a class of information structures balancing revenue-maximization and incentive compatibility with large $N$
- consider the two-sided pooling structure:

\[
s(v_i) = \begin{cases} 
\mathbb{E}[v_j | F(v_j) \leq 1 - q] & \text{if } F(v_j) \leq 1 - q^* \\
v_j & \text{if } 1 - q^* \leq F(v_j) \leq q^* \\
\mathbb{E}[v_j | F(v_j) \geq q] & \text{if } F(v_j) \geq q^*
\end{cases}
\]

- above information structure adds pooling at the bottom to pooling at the top
Truthful Reporting Under Manual Bidding

Proposition (Honesty and Obedience)

Under manual bidding, it is a dominant strategy for the advertiser to report his preference truthfully in the two-sided pooling structure.

Proposition (Approximate Optimality)

Under the two-sided pooling information structure the revenue converges to the one under the optimal information structure when the number of bidders grows large:

$$\lim_{N \to \infty} \left( \mathbb{E}[w(2)] - R \right) = 0.$$ 

- revenue under two-sided pooling is given by $w(2)$
Discussion and Conclusion

- correlated values and adverse selection
- vertical differentiation of attributes
- auction format
- reserve price and optimal auction
- asymmetric information structure