Common priors, Duality, and No-Trade

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Motivation and goal

- **Belief-based** representation of information is now pervasive in models of information transmission (Bayesian persuasion and information design)

- **Single-receiver**: Bayes plausibility (Kamenica and Gentzkow 2011, KG) characterizes the set of feasible distributions over receiver’s posteriors

- **Multiple receivers**: a corresponding **general characterization** is missing:
  - Mathevet et al (2020) characterize (finite) distributions over hierarchies of beliefs rather implicitly
  - Arieli et al (2021) characterize distributions over first-order beliefs under binary states (no-trade)
  - Corrao (2021) characterizes distributions over first-order expectations with continuous states

- **This paper** provides an explicit characterization of feasible distributions of higher-order beliefs (and their "coarsenings") in terms of moment inequalities with no-trade interpretations
**Observation**: Characterizing feasible distributions over hierarchies of beliefs amounts to study the implications of Common Prior (CP) assumptions.


**Existing results**: Abstract state-space (usually finite), characterize existence of CP rather than the set of feasible distributions.

We provide a characterization for **CP-feasible distributions** over payoff-relevant states and higher-order beliefs with a no-arbitrage interpretation:

- A pair of priors over states and beliefs are CP-consistent iff they do not allow any arbitrage when seen as prices of independent bets that only depend respectively on states and beliefs.
As an illustration, we revisit the critical-path theorem of Kajii and Morris (1997)

Bounds on the probability of common-p belief in an event $E$ in terms of the prior probability of $E$

- No-trade interpretation of the critical bounds
- Tighter lower bound than KM97
- Extension to uncountable (but compact) spaces

**Information robustness**: smallest probability that both players invest in an investment game attains the implied bounds

**Information design**: If the designer’s objective depends on players’ hierarchies of beliefs and states then our characterization posits the problem as an (infinite-dimensional) linear program with moment constraints
However, equilibria in economic settings are often described by coarser features than the entire hierarchies of beliefs. Motivated by this, we introduce coarsened type spaces where the types of the agents correspond to these coarsened features (e.g., first-order beliefs, expectations, or actions). The beliefs of each coarsened type are not uniquely identified and only need to satisfy given restrictions (e.g., obedience when types correspond to actions).

We characterize the distributions over coarsened types that are CP-consistent: first-order beliefs that can arise under any information structure for a given CP, actions that can arise in any BCE, partitions induced by belief operators (robust info design).

Obtain moment restrictions on distributions over observable coarsenings that can falsify the CP assumption.

Simplify information design problems where designer’s objective depends on these coarsenings.
General incomplete information setting

- Finite set of agents \( N = \{1, \ldots, n\} \)
- Uncertain state of the world \( \theta \in \Theta \subseteq \mathbb{R}^m \) with \( \Theta \) compact (results extend to compact metric spaces)
- First-order beliefs of agent \( i: \ p^1_i \in P^1_i := \Delta(\Theta) \)
- Second-order beliefs of agent \( i: \ p^2_i \in P^2_i := \Delta(\Theta \times \prod_{j \neq i} P^1_j) \), so on and so forth...
- Universal types \( t_i \in T_i \) collect the entire (coherent) hierarchy of beliefs of agent \( i: \)
  \[
  t_i = (p^1_i, p^2_i, \ldots, p^k_i, \ldots) \in \prod_{k \in \mathbb{N}} P^k_i 
  \]
- Brandenburger and Dekel: \( T_i \) is a compact subset and there exists a (canonical) homeomorphism \( g_i : T_i \rightarrow \Delta(\Theta \times T_{\neg i}) \) mapping universal types to beliefs and viceversa
Common prior and distribution of beliefs

- A common prior (over states) is a probability measure $\mu \in \Delta(\Theta)$ shared by all agents.
- An information structure is a pair $\mathcal{I} = (S, \sigma)$ such that $S = \prod_{i \in N} S_i$ is the (product, measurable) signal space and
  \[ \sigma : \Theta \to \Delta(S) \]
  is a statistical experiment. Every agent $i$ only observes the private realization $s_i \in S_i$.
- A common prior $\mu \in \Delta(\Theta)$ and an information structure $\mathcal{I} = (S, \sigma)$ induce distributions $\pi_{\mu,\sigma} \in \Delta(\Theta \times T)$ and $\tau_{\mu,\sigma} \in \Delta(T)$ over universal types.
- We aim to characterize
  \[ \Delta_{CP}(\mu) = \left\{ \pi_{\mu,\sigma} \in \Delta(\Theta \times T) : \text{for some } \mathcal{I} = (S, \sigma) \right\}, \]
  \[ T_{CP}(\mu) = \left\{ \tau_{\mu,\sigma} \in \Delta(T) : \text{for some } \mathcal{I} = (S, \sigma) \right\} \]
as well as $\bigcup_{\mu \in \Delta(\Theta)} T_{CP}(\mu)$. 

Corrao and Morris (MIT)
First-step: getting rid of information structures

Lemma

π is CP-consistent, that is π ∈ ∪_μ∈Δ(Θ) Δ_CP (μ), if and only if, for every i ∈ N, g_i : T_i → Δ (Θ × T_\_i) is a version of the conditional probability of π given t_i ∈ T_i.

**Immediate implication:** the following are equivalent

(i) τ ∈ Δ (T) is consistent with the common prior assumption (resp. with μ ∈ Δ (Θ))

(ii) There exists π ∈ Δ (Θ × T) that admits (g_i)_{i ∈ N} as versions of its conditional probabilities and marg_T π = τ (resp. also marg_Θ π = μ)
Trades

- State- and beliefs-contingent trades are profile of *continuous* functions
  
  \[ h = (h_i)_{i \in N} \in H := C(\Theta \times T)^N \]

- Continuity needed for *countable additivity* of CPs (extension to *finitely additive* CPs with *bounded and measurable trades*).

- Consider a dummy agent \( i_0 \) with no information in the interim stage.

- All the agents’ preferences are *linear in money*. 
No-trade

- For trades \((h_{i_0}, (h_i)_{i \in N})\) define:

  - **Feasibility:** \(-h_{i_0}(\theta, t) \geq \sum_{i \in N} h_i(\theta, t)\) for every \((\theta, t) \in \Theta \times T\)
  
  - **Acceptability:** \(\int_{\Theta \times T} h_i(\theta, t) \, dg(t_i)(\theta, t_{-i}) \geq 0\) for every \(t_i \in T_i\) and \(i \in N\)

**Definition**

\(\pi \in \Delta(\Theta \times T)\) satisfies **no-trade** if there does not exists a feasible and acceptable profile of trades \((h_{i_0}, (h_i)_{i \in N})\) such that

\[
\int_{\Theta \times T} h_j(\theta, t) \, d\pi(\theta, t) \geq 0 \quad \text{for all } j \in N \cup \{i_0\},
\]

\[
\int_{\Theta \times T} h_j(\theta, t) \, d\pi(\theta, t) > 0 \quad \text{for some } j \in N \cup \{i_0\}.
\]
Zero-value trades

- **Alternative definition**: get rid of the dummy trader $i_0$ and replace it with an external trader who is still uniformed in the interim

- External trader offers $h = (h_i)_{i \in N} \in H$ to the agents who then choose whether to accept or not in the interim stage

### Definition

A trade $h \in H$ is **zero-value** if, for every $t_i \in T_i$ and $i \in N$,

$$\int_{\Theta \times T_{-i}} h_i(\theta, t) \, dg(t_i)(\theta, t_{-i}) = 0.$$  

Let $H_0 \subseteq H$ denote the set of zero-value trades.

- Every type of every agent is indifferent between accepting or rejecting a zero-value trade
Money pumps

Definition

π ∈ Δ(Θ × T) satisfies **no-money-pump** if, for every h ∈ H₀,

\[ \int_{\Theta \times T} \sum_{i \in N} h_i(\theta, t) \, d\pi(\theta, t) \geq 0 \]

**Interpretation:** If \( \int_{\Theta \times T} \sum_{i \in N} h_i(\theta, t) \, d\pi(\theta, t) < 0 \), then an external trader with beliefs π can make a strictly positive expected profit by offering h to the agents.
Theorem

The following are equivalent:

(i) $\pi$ is CP-consistent, that is, $\pi \in \bigcup_{\mu \in \Delta(\Theta)} \Delta_{CP}(\mu)$

(ii) $\pi$ satisfies no-money pump

(iii) $\pi$ satisfies no-trade

Add the requirement that $\text{marg}_{\Theta}\pi = \mu$ to (ii) and (iii) to obtain a characterization of $\Delta_{CP}(\mu)$
Supereplicating independent bets

- Next, focus only on marginals over higher-order beliefs (extension of Bayes-plausibility condition)
- Define the set
  \[ S = \left\{ (\phi, \psi) \in C(\Theta) \times C(T) : \exists h \in H_0, \phi + \psi \geq \sum_{i \in N} h_i \right\} \]

- **Interpretation**: Suppose that the external trader has access to "independent" trades \((\phi, \psi) \in C(\Theta) \times C(T)\) that only depend either on the state \(\theta\) or on the beliefs of the agents \(t\)

- The elements of \(S\) are those independent trades that *supereplicate* a portfolio of acceptable trades \(h \in H_0\)
Main characterization

Theorem

Fix $\mu \in \Delta (\Theta)$ and $\tau \in \Delta (T)$. The following are equivalent:

(i) There exists an information structure $(S, \sigma)$ such that $\tau = \tau_{\mu, \sigma}$, that is, $\tau \in \mathcal{T}_{CP}(\mu)$

(ii) For every $(\phi, \psi) \in S$,

$$\int_{\Theta} \phi(\theta) \, d\mu(\theta) + \int_{T} \psi(t) \, d\tau(t) \geq 0$$  \hspace{1cm} (1)
No-arbitrage interpretation

- We interpret condition (ii) as a no-arbitrage condition: suppose that $\mu$ and $\tau$ are the (linear) price functionals for independent trades $\phi \in C(\Theta)$ and $\psi \in C(T)$.

- These prices correspond to the marginal distribution over states and beliefs due to fair pricing.

- Suppose that there exist $(\phi, \psi) \in S$ for some $h \in H_0$ such that (1) is not satisfied.

- The external trader can then buy these two assets to obtain

$$- \left( \int_{\Theta} \phi(\theta) \, d\mu(\theta) + \int_T \psi(t) \, d\tau(t) \right) > 0$$

and offer the profile of acceptable trades $(h_i)_{i \in N}$ to the agents.

- Since $(\phi, \psi)$ supereplicate $(h_i)_{i \in N}$ pointwise, the external trader obtains a strictly positive profit.
When \( N = \{i\} \), we have \( T = \Delta(\Theta) \)

- Fix any \( \phi \in C(\Theta) \) and define
  \[
  \psi(t) = \mathbb{E}_t[\phi] \quad \forall t \in \Delta(\Theta)
  \]

- We have
  \[
  \mathbb{E}_t[\phi - \psi(t)] = 0 \quad \forall t \in \Delta(\Theta)
  \]
  so that \( h(\theta, t) = \phi(\theta) - \psi(t) \) is a zero-value trade for the unique agent

- Our result gives
  \[
  \int_\Theta \phi(\theta) \, d\mu(\theta) - \int_T \psi(t) \, d\tau(t) = 0 \iff \int \phi \, d\mu = \int \mathbb{E}_t[\phi] \, d\tau(t)
  \]
  that is, Bayes plausibility
Define the set
\[ \Delta (\mu, \tau) = \{ \pi \in \Delta (\Theta \times T) : \text{marg}_\Theta \pi = \mu, \text{marg}_T \pi = \tau \} \]

Form previous Theorem, \( \tau \in \mathcal{T}_{CP} (\mu) \) if and only if there exists \( \pi \in \Delta (\mu, \tau) \) such that
\[
\int_{\Theta \times T} \sum_{i \in N} h_i (\theta, t) \, d\pi (\theta, t) \geq 0 \quad \forall h \in H_0,
\]
that is
\[
\sup_{\pi \in \Delta (\mu, \tau)} \inf_{h \in H_0} \left\{ \int_{\Theta \times T} \sum_{i \in N} h_i (\theta, t) \, d\pi (\theta, t) \right\} \geq 0
\]

\( \Delta (\mu, \tau) \) weakly compact and convex, \( H_0 \) convex, objective function doubly linear \( \implies \) Apply Sion Maxmin Theorem
Formal Proof: Kantorovich Duality

- We then have $\tau \in \mathcal{T}_{\text{CP}}(\mu)$ if and only if

\[
\inf_{h \in H_0} \sup_{\pi \in \Delta(\mu, \tau)} \left\{ \int_{\Theta \times T} \sum_{i \in N} h_i(\theta, t) \, d\pi(\theta, t) \right\} \geq 0
\]

- Fix $h \in H_0$ and focus on inner maximization: optimal transport problem with marginals $(\mu, \tau)$ and cost $c = -\sum h_i$

- Apply Kantorovich Duality to obtain

\[
\sup_{\pi \in \Delta(\mu, \tau)} \left\{ \int_{\Theta \times T} \sum_{i \in N} h_i(\theta, t) \, d\pi(\theta, t) \right\} = \inf_{(\phi, \psi) \in C(\Theta) \times C(T) : \phi + \psi \geq \sum h_i} \left\{ \int_{\Theta} \phi \, d\mu + \int_{T} \psi \, d\tau \right\}
\]

- The result follows
Consider only two players $N = \{a, b\}$ and define simple $\Theta$-events as

$$E \times T \subseteq \Theta \times T$$

for some event $E \subseteq \Theta$

The belief operator of $i \in N$ is

$$B^p_i (E) = \{t_i \in T_i : g_i (E \mid t_i) \geq p\} \quad \forall p \in [0, 1]$$

As usual we define

$$B^p,1_*(E) = B^p_* (E) = B^p_a (E) \times B^p_b (E)$$

and for all $n \in \mathbb{N}$

$$B^p,n+1_* (E) = B^p_* (B^p,n_* (E))$$

The common-$p$ belief operator is

$$C^p (E) = \bigcap_{n \in \mathbb{N}} B^p,n_* (E) \subseteq T$$
Corollary

Fix $\mu \in \Delta(\Theta)$, a closed set $E \subseteq \Theta$, and $p \in (0, 1/2)$. For every $\tau \in \mathcal{T}_{CP}(\mu)$, we have

$$\frac{\mu(E)(1+p) - 3p}{1-2p} \leq \frac{\mu(E) - 2p}{1-2p} \leq \tau[C^P(E)] \leq \frac{1}{p} \mu(E),$$

where the lower bound is tight (if we consider finitely additive measures).

- Upper bound is simple. For lower bound we let $\psi = I_{CP(E)}$, $\phi = -\frac{I_{E-2p}}{1-2p}$, and find $h_0 \in H_0$ such that

$$I_{CP(E)}(t) - \frac{I_E(\theta) - 2p}{1-2p} = \sum_{i \in N} h_i(t, \theta) \quad \forall (\theta, t) \in \Theta \times T$$

- The lower bound then follows from the characterization theorem by approximating $\psi$ and $\phi$ with continuous functions from above
Construction of the critical trade

- Define coarsenings $\kappa_i : T_i \to \mathbb{N} \cup \{\infty\}$ as

$$
\kappa_i(t_i) = \begin{cases} 
0, & \text{if } t_i \notin B^p_i(E) \\
 k, & \text{if } t_i \in [B^p_i]^k(E) \text{ but } t_i \notin [B^p_i]^{k+1}(E) \\
\infty, & \text{if } t_i \in B^p_i(C^p(E))
\end{cases}
$$

- The trade $h_i(\theta, t)$ is $\kappa_i, \kappa_j$-measurable and pays
  - 0 if $\kappa_i = \infty$
  - $p$ if $\kappa_i < \infty$ and either ($\neg E$) or ($E$ and $\kappa_j < \kappa_i$)
  - $-(1-p)$ if $E$ and $\kappa_i < \kappa_j$
  - $p - 1/2$ if $E$ and $\kappa_i = \kappa_j$

- **Intuition**: betting on rank beliefs about $E$
Information design: characterization

- General information design problem (cf. Mathevet et al. 2020) given common prior $\mu \in \Delta(\Theta)$

$$
\mathcal{P} = \sup \left\{ \int_{\Theta \times T} V(\theta, t) \, d\pi(\theta, t) : \pi = \pi_{\mu, \sigma} \text{ for some } \mathcal{I} = (S, \sigma) \right\}
$$

for some continuous objective function $V : \Theta \times T \to \mathbb{R}$

- Our characterization simplifies the problem to

$$
\mathcal{P} = \sup \left\{ \int_{\Theta \times T} V(\theta, t) \, d\pi(\theta, t) \right\}
$$

such that

$$
\mathbb{E}_{\pi} \left[ \phi + \sum_i h_i \right] \geq 0
$$

for all $\phi \in C(\Theta)$ and $h \in H_0$. 
Information design: duality

The dual information design problem is

\[ D = \inf_{\phi \in C(\Theta), h_0 \in H_0} \left\{ \int_\Theta \phi(\theta) d\mu(\theta) : \forall (\theta, t), \phi(\theta) \geq V(\theta, t) + \sum_{i \in N} h_i(\theta, t) \right\} \]

Get rid of \( \phi \) by defining \( \phi(\theta) = \sup_{t \in T} \{ V(\theta, t) + \sum_{i \in N} h_i(\theta, t) \} \)

**Theorem**

We have:

1. **No Duality Gap**: \( \mathcal{P} = D \)
2. **The pair** \((\pi, h)\) **solve the primal and the dual problems if and only if**

\[ t \in \arg \max_{\tilde{t} \in T} \left\{ V(\theta, \tilde{t}) + \sum_{i \in N} h_i(\theta, \tilde{t}) \right\} \quad \text{for } \pi\text{-almost all } (\theta, t) \]

- Corrao, Wolitzky, and Kolotilin (2021): same duality approach to solve single-receiver persuasion
Motivation for coarsened types

- Sometimes analyst interested in coarsened description of the hierarchies of beliefs (e.g., sufficient to describe equilibria)

- Coarsened types $x = (x_i)_{i \in N} \in X$ can be description of the agents’ beliefs or behavior

- Distributions $\nu \in \Delta(X)$ over coarsened types $x \in X$ are potentially observable

- **Goal**: characterize $\pi \in \Delta(\Theta \times X)$ and $\nu \in \Delta(X)$ that are consistent with common prior assumption in terms of falsifiable implications
Generalization: coarsened type-spaces

- A coarsened type space is a structure \((X_i, \Delta_i)_{i \in N}\) where, for every \(i \in N\),
  - \(X_i\) is a compact metric space of coarsened types
  - \(\Delta_i : X_i \rightrightarrows \Delta(\Theta \times X_{-i})\) is a \textit{closed and convex-valued correspondence} mapping types \(x_i\) to possible beliefs

Examples:

1. **Standard type space**: \(X_i = T_i\) and \(\Delta_i(t_i) = \{g_i(t_i)\}\) for all \(t_i \in T_i\)
2. **First-order beliefs**: \(X_i = P^1_i = \Delta(\Theta)\) and
   \[
   \Delta_i \left( p^1_i \right) = \left\{ \gamma \in \Delta(\Theta \times P^1_{-i}) : \text{marg}_{\Theta} \gamma = p^1_i \right\}
   \]
3. **Kth-order beliefs**: \(X_i = P^k_i\)
   \[
   \Delta_i \left( p^k_i \right) = \left\{ \gamma \in \Delta(\Theta \times P^k_{-i}) : \forall l \leq k, \text{marg}_{P^l_{-i}} \gamma = \text{marg}_{P^l_{-i}} p^k_i \right\}
   \]
Coarsened type space: other examples

4. **Belief operators**: induce a partition of the universal type space

5. **Best-responses**: Consider a game with incomplete info \((A_i, U_i(a, \theta))_{i \in N}\) and let \(X_i = A_i\) and

\[
\Delta_i (a_i) = \left\{ \gamma \in \Delta(\Theta \times A_{-i}) : a_i \in \arg \max_{\tilde{a_i} \in A_i} \left\{ \mathbb{E}_\gamma [U_i(\tilde{a_i}, \cdot)] \right\} \right\} \quad \forall a_i \in A_i
\]

6. **Strategic type space**: coarsened types correspond to the sequences of action sets \(x_i = \left\{ A_{i}^n \right\}_{n \in \mathbb{N}}\) resulting from Interim Correlated Rationalizability
Common priors over coarsened types

Definition

We say that $\pi \in \Delta (\Theta \times X)$ is CP-$X$-consistent if, for every $i \in N$, there exists a version of the conditional probability $(\pi_{x_i})_{x_i \in X_i}$ such that

$$\pi_{x_i} \in \Delta_i (x_i) \quad \text{for } \pi-\text{almost all } x_i.$$ 

We say that $\nu \in \Delta (X)$ is CP-consistent if there exists a CP-consistent $\pi$ such that $\text{marg}_X \pi = \nu$.

- As before, we can also require consistency with a fixed prior over states $\mu \in \Delta (\Theta)$
- **First-order-belief coarsening**: the CP-consistent distributions $\nu \in \Delta (X)$ correspond to those that can be induced by an information structure (cf. Arieli et al 2021)
- **Best-response coarsening**: the CP-consistent distributions $\pi \in \Delta (\Theta \times A)$ correspond to Bayes correlated equilibria (BCE) of the underlying game (cf. Bergemann and Morris 2016, 2017)
Cautiously zero-value trades

- X-measurable trades \( h = (h_i)_{i \in N} \in H_X = C(\Theta \times X)^N \) are trades that only depend on the state and the coarsened types.
- Type \( x_i \in X_i \) of agent \( i \) can evaluate \( h_i \) according to multiple beliefs \( \gamma \in \Delta (x_i) \).
- Consider the worst possible evaluation: for every \( h_i \in C(\Theta \times X) \), define

\[
\xi_i(h_i)(x_i) = \inf \{ \mathbb{E}_\gamma[h_i(x_i, \cdot)] : \gamma \in \Delta_i(x_i) \} \quad \forall x_i \in X_i
\]

Definition

A trade \( h \in H_X \) is **cautiously zero-value** if, for every \( x_i \in X_i \) and \( i \in N \),

\[
\xi_i(h_i)(x_i) = 0
\]

**Interpretation**: Every type \( x_i \) is indifferent between accepting or rejecting the trade under the worst possible belief, hence they will always weakly prefer to accept for every \( \gamma \in \Delta_i(x_i) \).

Let \( H_{X,0} \) denote the set of cautiously zero-value for coarsening \( X \).
Main characterization

Definition

\( \pi \in \Delta (\Theta \times X) \) satisfies **no cautious money pump** if, for every \( h \in H_X,0 \),

\[
\int_{\Theta \times X} \sum_{i \in N} h_i (\theta, x) \, d\pi (\theta, x) \geq 0
\]

- **Interpretation:** If \( \int_{\Theta \times X} \sum_{i \in N} h_i (\theta, x) \, d\pi (\theta, x) < 0 \) for some \( h \in H_X,0 \), then an external trader with beliefs \( \pi \) can make a strictly positive expected profit by offering \( h \) to the agents.

Theorem

*Fix \( \pi \in \Delta (\Theta \times X) \). The following are equivalent:*

(i) \( \pi \) is CP-X-consistent

(ii) \( \pi \) satisfies no cautious money pump

- **Remark:** novel characterization for BCE in incomplete info games
Sketch of the proof: Strassen 65

(i) $\implies$ (ii) If $\pi$ is CP-consistent $\pi$, then there exists a regular conditional probability $(\pi_{x_i})_{x_i \in X_i}$ of $\pi$ such that $\pi_{x_i} \in \Delta (x_i)$ for $\pi$-almost all $x_i$

For every $i \in N$, we then have

$$\int_{\Theta \times X} h_i (\theta, x) \, d\pi (\theta, x) \geq \inf \{ \mathbb{E}_\gamma [h_i (x_i, \cdot)]: \gamma \in \Delta_i (x_i) \}$$

Next, fix $h \in H_0$ and observe that

$$\sum_{i \in N} \int_{\Theta \times X} h_i (\theta, x) \, d\pi (\theta, x) \geq \sum_{i \in N} \int_{X_i} \xi_i (h_i) (x_i) \, d\text{marg}_{\chi_i} \pi (x_i) = 0$$

proving the implication
Sketch of the proof: Strassen 65

(ii) $\implies$ (i) Fix $h_i \in C(\Theta \times X)$ and define

$$\hat{h}_i(\theta, x) = h_i(\theta, x) - \xi_i(h_i)(x_i) \quad \forall i \in N$$

Next argue that $\hat{h} \in H_0$ and apply no cautious money pump to conclude that

$$\int_{\Theta \times X} h_i(\theta, x) \ d\pi(\theta, x) \geq \int_{X_i} \xi_i(h_i)(x_i) \ d\text{marg}_{X_i}\pi(x_i) \quad \forall i \in N$$

Finally, Theorem 3 in Strassen 65 implies that, for every $i \in \mathbb{N}$, there exists a regular conditional probability $\pi_{x_i} \in \Delta(x_i)$ for $\pi$-almost all $x_i$
CP-consistent marginals

- Define

\[ S_X = \left\{ (\phi, \psi) \in C(\Theta) \times C(X) : \exists h \in H_{X,0}, \phi + \psi \geq \sum_{i \in N} h_i \right\} \]

**Corollary**

*Fix \( \mu \in \Delta(\Theta) \) and \( \nu \in \Delta(X) \). The following are equivalent:

(i) There exists a CP-consistent \( \pi \in \Delta(\Theta \times X) \) such that \( \text{marg}_\Theta \pi = \mu \) and \( \text{marg}_X \pi = \nu \)

(ii) For every \( (\phi, \psi) \in S_X \), we have

\[ \int_{\Theta} \phi(\theta) \, d\mu(\theta) + \int_{T} \psi(x) \, d\nu(x) \geq 0 \]

- No-arbitrage interpretation as before
Next, we do not fix $\mu \in \Delta (\Theta)$ to capture the general implications of the CP assumptions for coarsenings.

**Corollary**

$v \in \Delta (X)$ is CP-consistent for some $\mu \in \Delta (\Theta)$ if and only if

$$\int_X \max_{\theta \in \Theta} \left[ \sum_{i \in N} h_i (\theta, x) \right] \, d\nu (x) \geq 0$$

Generalizes the main result in Arieli et al. (2021) to continuous states and arbitrary (coarsened) type spaces.
Illustration: first-order expectations

- Let $\Theta = X_i = [0, 1]$ and consider the first-order expectation coarsening with

  $$\Delta_i (x_i) = \{ \gamma \in \Delta (\Theta \times X_{-i}) : E_\gamma [\tilde{\theta}] = x_i \}$$

- A sufficient class of cautiously zero-value trades is given by

  $$h_i (\theta, x) = q_i (x_i) (\theta - x_i) \quad q_i \in C (A_i)$$

- Obtain result in Arieli et al (2021): $\nu \in \Delta (X)$ is CP-consistent for some $\mu \in \Delta (\Theta)$ if and only if

  $$\int_X \left\{ \sum_{i \in N} q_i (x_i) x_i - \left[ \sum_{i \in N} q_i (x_i) \right]^+ \right\} d\nu (x) \leq 0$$

  for all $q_i \in C ([0, 1])$ and $i \in N$
Consider an incomplete information game with $\Theta = [0, 1]$, $A_i = [0, 1]$ and payoff functions $U_i$ are smooth and strictly concave in $a_i$.

The belief-correspondence is

$$\Delta_i (a_i) = \left\{ \gamma \in \Delta (\Theta \times A_{-i}) : E_{\gamma} \left[ \frac{\partial}{\partial a_i} U_i (a_i, \cdot) \right] = 0 \right\}$$

A sufficient class of cautiously zero-value trades is given by

$$h_i (\theta, a) = q_i (a_i) \frac{\partial}{\partial a_i} U_i (\theta, a) \quad q_i \in C (A_i)$$

Interpretation: trades are proportional to the marginal utility of the players (cf. Nau and McCardle 90 characterization of correlated equilibrium).
Illustration: smooth incomplete information games

Corollary

The distribution over actions $\nu \in \Delta (A)$ is a BCE for some common prior $\mu \in \Delta (\Theta)$ if and only if

$$\int_A \max_{\theta \in \Theta} \left\{ \sum_{i \in N} q_i (a_i) \frac{\partial}{\partial a_i} U_i (\theta, a) \right\} d\nu (a) \geq 0$$

for every $(q_i)_{i \in N} \in \prod_{i \in N} C (A_i)$.

- If the marginal utility of every $i$ is affine $\frac{\partial}{\partial a_i} U_i (\theta, a) = \theta - \beta_i (a)$, then the previous condition becomes

$$\int_A \left\{ \sum_{i \in N} q_i (a_i) \beta_i (a) - \left[ \sum_{i \in N} q_i (a_i) \right]^+ \right\} d\nu (a) \geq 0$$

generalizing Arieli et al (2021) to incomplete information games.
Conclusion and future research

- Provided a no-trade characterization of feasible distributions over higher-order beliefs under CP
- Introduced language of coarsened type space and characterized CP-implications
- This allowed to unify and revisit several scattered results in information design and information economics
- Propose a dual approach to implementation and optimal design of information

**Future research:** The (simple) math trick was to express conditional moments conditions in terms of unconditional ones (Econometricians know better)

- Same trick can be used to characterize other conditional moment conditions:
  - Truthful reporting in communication equilibria and mechanism design
  - Inscrutability principle in mechanism design with informed principal
  - REE and Self-confirming equilibrium
Explicit trade illustration

For every $i \in N$, consider the trade

$$h_i(\theta, t) = \begin{cases} 
  p & \text{if } \kappa_i(t_i) < \infty \text{ and } \theta \notin E \\
  p - \frac{1}{2} & \text{if } \kappa_i(t_i) < \infty, \theta \in E \text{ and } \kappa_j(t_j) < \kappa_i(t_i) \\
  -(1-p) & \text{if } \kappa_i(t_i) < \infty, \theta \in E \text{ and } \kappa_j(t_j) = \kappa_i(t_i) \\
  0 & \text{if } \kappa_i(t_i) < \infty, \theta \in E \text{ and } \kappa_i(t_i) < \kappa_j(t_j) \\
  0 & \text{if } \kappa_i(t_i) = \infty 
\end{cases}$$
Construction of the critical trade

- This trade is acceptable for both $i$ and all $t_i$ by construction.

- Now let

$$F = \{(t_a, t_b) \in T : \kappa_a(t_a) < \infty \text{ and } \kappa_b(t_b) < \infty\}$$
$$S = \{(t_a, t_b) \in T : \kappa_i(t_i) = \infty \text{ and } \kappa_j(t_j) < \infty \text{ for some } i\}$$

- Observe that

$$h_a(\theta, t) + h_b(\theta, t) = \begin{cases} 
2 & \text{if } \theta \notin E \text{ and } t \in F \\
1 & \text{if } \theta \notin E \text{ and } t \in S \\
0 & \text{if } \theta \notin E \text{ and } t \in C_P(E) \\
1 - 2p & \text{if } \theta \in E \text{ and } t \in F \\
p - 1 & \text{if } \theta \in E \text{ and } t \in S \\
0 & \text{if } \theta \in E \text{ and } t \in C_P(E) 
\end{cases}$$

and verify that

$$I_{C_P(E)}(t) - \frac{I_E(\theta) - 2p}{1 - 2p} = h_a(\theta, t) + h_b(\theta, t) \text{ for all } (\theta, t).$$
Simpler characterization for linear coarsenings

- We say that $\Delta_i$ is **linear** if the set $\Delta_i(x_i) \subseteq \Delta(\Theta \times X_{-i})$ is described by (potentially infinite) linear inequalities (half-spaces in the finite case).
- $E \subseteq \Theta \times X$ is an $i$-event if $E = E_i \times E_{-i}$ for some $E_i \subseteq X_i$ and $E_{-i} \subseteq \Theta \times E_{-i}$

**Theorem**

Let $(\Delta_i)_{i \in I}$ be linear and fix $\pi \in \Delta(\Theta \times X)$. The following are equivalent:

(i) $\pi$ is CP-consistent

(ii) For every $i \in N$ and every $i$-event $E = E_i \times E_{-i}$, we have

$$\pi(E_i \times E_{-i}) \geq \int_{E_i} \min \{ \gamma(E_{-i}) : \gamma \in \Delta_i(x_i) \} \, d\pi_i(x_i)$$
Sketch of the proof

- **Sketch**: The proof is similar to the previous one by replacing Theorem 4 of Strassen to his Theorem 3.

- The non-trivial part is to show that, for every $x_i \in X_i$, the set-function

  $$E_{-i} \mapsto \min \{ \gamma (E_{-i}) : \gamma \in \Delta_i (x_i) \}$$

  is supermodular in the inclusion order, which is necessary to invoke Strassen’s result.
Appendix

CP-consistent supports

- What coarsened types (e.g. actions) are consistent with the common prior assumption?

**Corollary**

Fix a compact $S \subseteq \Theta \times X$. The following are equivalent:

(i) There exists a CP-consistent $\pi \in \Delta (\Theta \times X)$ such that $\pi (S) = 1$

(ii) For every $h \in H_{X,0}$, we have

$$\sup_{(\theta, x) \in S} \sum_{i \in N} h_i (\theta, x) \geq 0$$

**Sketch:** Point (i) can be expressed as a maxmin problem, then use Sion (compactness of $S$) to obtain result
Implications for the single-receiver case

Assume that $N = \{i\}$. For every $f \in C(\Theta)$, define the $U$-concavification of $f$ as

$$f^U(a) = \max_{\lambda,\bar{\theta},\theta: \lambda \frac{\partial}{\partial a} U(\bar{\theta}, a) + (1-\lambda) \frac{\partial}{\partial a} U(\theta, a)} \{ \lambda f(\bar{\theta}) + (1-\lambda) f(\theta) \} \quad \forall a \in A$$

Corollary

Fix $\mu \in \Delta(\Theta)$. The distribution over actions $\nu \in \Delta(A)$ is implementable by an information structure if and only if

$$\int_A f^U(a) \, d\nu(a) \geq \int_\Theta f(\theta) \, d\mu(\theta) \quad \forall f \in C(\Theta)$$

For $\frac{\partial}{\partial a} U(\theta, a) = (\theta - a)$ this reduces to standard convex ordering $\mu \succeq_{cvx} \nu$
A differential characterization

- Define the cost function $R : \Delta(\Theta) \times \Delta(X) \rightarrow \overline{\mathbb{R}}_+$

$$R(\mu, \nu) = -\inf_{\mu_0 \in H_{X, 0}} \sup_{\pi \in \Delta(\mu, \nu)} \left\{ \int_{\Theta \times X} \sum_{i \in N} h_i(\theta, x) \, d\pi(\theta, x) \right\}$$

- **Interpretation**: Capture a measure of "distance" between $\mu$ and $\nu$ with respect to the cautiously zero-value trades $h \in H_{X, 0}$

- Define the operators

$$I_\mu(\psi) = \min_{\nu \in \Delta(X)} \left\{ \int_X \psi(x) \, d\nu(x) + R(\mu, \nu) \right\} \quad \forall \psi \in C(X)$$

and

$$I_\nu(\phi) = \min_{\mu \in \Delta(\Theta)} \left\{ \int_{\Theta} \phi(\theta) \, d\mu(\theta) + R(\mu, \nu) \right\} \quad \forall \phi \in C(\Theta)$$
A differential characterization

- These operators evaluate independent bets $\psi$ and $\phi$ under the worst possible distributions with higher penalization for those that are "distant" from $\mu$ and $\nu$
- Decision theory under uncertainty: $I_\mu$ and $I_\nu$ represent variational preferences
- These operators are concave and 1-Lipschitz continuous

**Corollary**

We have:

$$\{\nu \in \Delta (X) : \text{o}utcomes \ \nu \ \text{consistent with common prior } \mu\} = \partial I_\mu (0)$$

and

$$\{\mu \in \Delta (\Theta) : \text{common priors } \mu \ \text{consistent with outcome } \nu\} = \partial I_\nu (0)$$
Extreme points

Let $\Delta_{CP}^X(\mu)$ denote the set of CP-consistent $\pi \in \Delta(\Theta \times X)$ such that $\text{marg}_\Theta \pi = \mu$.

**Theorem**

Fix $\mu \in \Delta(\Theta)$ and define

$$\hat{H}_{X,0} = \left\{ \phi + \sum_{i \in \mathbb{N}} h_i \in C(\Theta \times X) : \phi \in C(\Theta), h \in H_{X,0} \right\}.$$ 

The following are equivalent:

(i) $\pi$ is an extreme point of $\Delta_{CP}^X(\mu)$

(ii) $\text{marg}_\Theta \pi = \mu$ and $\hat{H}_{X,0}$ is dense in $L_1(\pi)$
Version of conditional probability

We say that $g_i : T_i \rightarrow \Delta (\Theta \times T_{-i})$ is a version of the conditional probability of $\pi \in \Delta (\Theta \times T)$ given $T_i$ if

$$
\int_{\Theta \times T} h(\theta, t) \, d\pi(\theta, t) = \int_{T_i} \left[ \int_{\Theta \times T_{-i}} h(\theta, t) \, dg(t_i)(\theta, t_{-i}) \right] \, dm_{\text{marg}}(\pi)(t_i)
$$

for all $i \in N$ and all $h \in C(\Theta \times T)$.  

(2)
Moment conditions and information design

- Consider a (possibly multidimensional) bounded objective function $V : \Theta \times T \to \mathbb{R}$
- Recall that $\Delta_{CP} (\mu) = \{ \pi \in \Delta (\Theta \times T) : \text{ for some } I = (S, \sigma) \}$
- We aim to characterize the set of feasible moments:

$$V_{CP} (\mu) = \{ \mathbb{E}_\mu [V] \in \mathbb{R} : \pi \in \Delta_{CP} (\mu) \} \subseteq \mathbb{R}$$

**Theorem**

Fix $v \in \mathbb{R}^m$. The following are both equivalent to $v \in V_{CP} (\mu)$:

1. For every $h \in H_0$, there exist $\lambda \in [0, 1]$ and $(\theta_0, t_0), (\theta_1, t_1) \in \Theta \times T$ such that the collections of vectors $V (\theta_0, t_0) \neq V (\theta_1, t_1)$ and

$$\lambda V (\theta_0, t_0) + (1 - \lambda) V (\theta_1, t_1) = v,$$

$$\lambda h (\theta_0, t_0) + (1 - \lambda) h (\theta_1, t_1) \geq 0.$$