Selective Memory Equilibrium*

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Abstract

We study agents who are more likely to remember some experiences than others, but who update their beliefs as if the experiences they remember are the only ones that occurred. To characterize their long-run behavior, we introduce the concept of selective memory equilibrium, where people choose actions that maximize their payoff given their distorted recollection of the outcome distribution. Selective memory equilibrium can explain why people are persistently overconfident, and can capture the long-run effects of “underinference,” where all experiences are remembered but some are given too little weight. When the expected number of recalled experiences is bounded, the long-run distribution of actions corresponds to a stochastic memory equilibrium. We use this to study the effect of “rehearsal,” where once an experience is recalled it is more likely to be recalled again. We also study the implications of agents who are only partially naïve about their selective memory.

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1 Introduction

We study agents who are more likely to recall some events than others. We assume that this selective memory is stochastic and exogenous. In most of the paper, we also assume that agents are unaware of their selective memory, so they update their beliefs as if the experiences they remember are the only ones that occurred. These assumptions fit evidence from both experimental and real-world settings.\(^1\)

Our focus is the long-run implications of selective memory. We show that if an agent’s behavior converges, their beliefs converge to a memory-weighted KL minimizer, i.e. a distribution that minimizes the KL divergence with respect to a version of the true outcome distribution that gives more weight to realizations that are more likely to be remembered. Moreover, the agent’s strategy then converges to a selective memory equilibrium, which is a strategy that myopically maximizes their expected payoff against a probability distribution over these minimizers. If all experiences are recalled with the same probability, then selective memory has no long-run effect. However, if agents are more likely to remember some experiences than others, selective memory can have a persistent effect. For example, an agent who is more likely to recall when they performed well in a task, relative to when they performed poorly, will underestimate the task’s difficulty and do it too often.

Selective memory equilibrium resembles Berk-Nash equilibrium (Esponda and Pouzo [2016]), which applies to agents with perfect memory but a misspecified prior. Indeed, we show that every uniformly strict Berk-Nash equilibrium (Fudenberg, Lanzani, and Strack [2021a]) is equivalent to a uniformly strict selective memory equilibrium for some memory function and a full support prior, and that every uniformly strict selective memory equilibrium is equivalent to a uniformly strict Berk-Nash equilibrium with the appropriate prior support. However, this equivalence fails for Berk-Nash equilibria that are not uniformly strict.\(^2\) In addition, unlike Berk-Nash equilibrium, in general selective memory equilibria do not reduce to self-confirming equilibria (Fudenberg and Levine [1993a]) when the agent is correctly specified. Importantly, the form of misspecification that would lead to the same behavior as a given form of selective memory depends on the environment. That is, par-

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1 Memory has been informally described as stochastic since the early stages of the psychology literature, as in James [1890], and recent neuroscience (e.g. Shadlen and Shohamy [2016]) supports this interpretation. Schacter [2008] and Kahana [2012] discuss evidence for memory biases that lead to some experiences being recalled more often, and e.g. Reder [2014], Zimmermann [2020], Gödker, Jiao, and Smeets [2021] provide evidence of partial or complete unawareness of memory biases.

2 A selective memory equilibrium is uniformly strict if it is the unique best reply to all the beliefs supported on the memory-weighted minimizers.
ticular forms of misspecification and selective memory that coincide under one information structure could lead to very different comparative statics with respect to changes of the information observed by the agent. We illustrate this point by showing that combining positive and negative feedback has qualitatively different effects on agents that have ego-boosting memory than on agents who are dogmatically overconfident.

Our framework is rich enough to include several commonly studied forms of memory bias, such as pleasant memory (Adler and Pansky [2020], Zimmermann [2020]),\textsuperscript{3} cognitive dissonance (Elkin and Leippe [1986], Chammat et al. [2017], Gödker, Jiao, and Smeets [2021]), associativeness (Thomson and Tulving [1970], Tulving and Schacter [1990], Enke, Schweter, and Zimmermann [2020]), confirmatory bias (Hastie and Park [1986], Snyder and Uranowitz [1978]), and the overweighting of extreme outcomes (Cruciani, Berardi, Cabib, and Conversi [2011]). We devote particular attention to pleasant memory bias and associativeness. Under the former, the agent is more likely to recall experiences that induced a larger utility. This is the endogenous memory counterpart of an exogenously overconfident misspecified agent, and we show that in a fixed environment pleasant memory bias leads to the same long-run behavior as exogenous overconfidence. However, we argue that the memory version is more susceptible to external manipulation. Under associativeness, it is easier to recall situations that are similar to the current decision problem. This is a “bias” if it leads the agent to underweight data relative to its true informativeness. The simplest version of associativeness, similarity weighting (Bordalo, Gennaioli, and Shleifer [2020]) does not alter the possible long-run outcomes for a correctly specified DM: we prove that all the selective memory equilibria are self-confirming.

We then extend our model to allow for the possibility that the expected number of recalled periods is bounded. Here typically the action process does not converge, but we can still show that whenever the action frequency converges the limit frequency is a (typically mixed) stochastic memory equilibrium. We use this to model the effect of “rehearsal,” where an experience that is recalled in one period is more likely to be recalled again. We also extend the model to allow for agents who are only partially naïve about their selective memory.

\textbf{Related Theoretical Work} Mullainathan [2002] studies selective memory where the probability of recalling an observation is the linear sum of a base rate, an “associativeness” term that measures the experience’s similarity to the current observation, and a “rehearsal” term that is an indicator for whether the experience was recalled in the previous period.\textsuperscript{3} Chew, Huang, and Zhao [2020] provide evidence that people selectively forget some negative events and create false memories of fictitious positive ones.
Like us, the paper assumes that agents are naïve about their selective memory. It also assumes that signals are normal, and are not influenced by the agent’s actions. Afrouzi, Kwon, Landier, Ma, and Thesmar [2020] also studies an agent who is forecasting the next realization of an AR(1) process. It assumes the agent knows the data generating process, and chooses which experiences to recall at a cost. Chauvin [2021] considers a model where agents distort their observations to reduce cognitive dissonance; this is an example of the “underinference” we discuss in Section 5.1. Bordalo, Conlon, Gennaioli, Kwon, and Shleifer [2021] considers an agent with a fixed sample size who assesses the relative likelihood of a set of hypotheses by sampling their memory with replacement. In addition to similarity-based sampling, the paper models the “interference” of some memories with others. None of these papers addresses our question of determining the agent’s long-run beliefs and actions.

Schwartzstein [2014] studies the long-run effects of a different, but related error: rather than not recalling some observations at a later date, the agent may misallocate attention and not observe them in the first place. As with selective memory and misspecified beliefs, this can lead the agent to make systematically biased forecasts. Battigalli, Generoso, et al. [2021] proposes a formalism to separate assumptions on the players’ information and memory. Further afield, Malmendier and Nagel [2016], Malmendier and Shen [2018], and Malmendier, Pouzo, and Vanasco [2020] consider models where agents apply a weight to events that depends on their age at the time the event happened.

2 Setup

We study a sequence of choices made by a single agent. In every period $t = 1, 2, \ldots$ the agent observes a signal $s$ from the finite set $S$ and then chooses an action $a$ from the finite set $A$. The realized signal $s$ and the chosen action $a$ induce an objective probability distribution $p_{a,s}^* \in \Delta(Y) \subset \mathbb{R}^Y$ over the finite set of possible outcomes $Y$. A (pure) strategy is a map $\sigma : S \rightarrow A$; $\Sigma$ denotes the set of all strategies. The agent’s flow payoff is given by $u : S \times A \times Y \rightarrow \mathbb{R}$.

As in Esponda and Pouzo [2016] and Fudenberg, Lanzani, and Strack [2021a], we assume that the agent knows there is a fixed and i.i.d. full-support distribution $\zeta \in \Delta(S)$ over signals. They also know that the map from actions and signals to probability distributions over outcomes is fixed and depends only on their current action and the realized signal, but

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4We denote objective distributions with a superscript $\ast$.

5This assumption lets us focus on our key points. When beliefs about the signal distribution are independent of beliefs about the contingent outcome distributions, the analysis of the paper is unchanged.
they are uncertain about the outcome distributions each signal-action pair induces. To model
this uncertainty, we suppose that the agent has a prior \( \mu_0 \) over data generating processes
\( p \in \Delta(Y)^{A \times S} \), where \( p_{a,s}(y) \) denotes the probability of outcome \( y \in Y \) when signal \( s \) is
observed and action \( a \) is played.\(^5\) The support of \( \mu_0 \) is \( \Theta \); its elements are the \( p \) that the
agent initially thinks are possible. The prior is \textit{correctly specified} if \( p^* \in \Theta \); if it is not, the
prior is \textit{misspecified}.

**Assumption 1.** For all \( p \in \Theta \), \( y \in Y \), \( a \in A \), and \( s \in S \), \( p^*_{a,s}(y) > 0 \) if and only if \( p_{a,s}(y) > 0 \).

This assumption means that the experiences that the agent initially thinks are impossible
are objectively impossible, and vice versa.

**Recalled Histories** We assume that the agent always recalls the signal they just observed,
and that this signal may also act as a cue for previous periods. We suppose that the agent’s
memory of past periods is distorted by a collection of signal-dependent \textit{memory functions}
\( m_{s'} : (S \times A \times Y) \to [0, 1] \), where \( m_{s'}(s, a, y) \) specifies the probability with which an agent
remembers a past realization of the signal, action, outcome triplet \( (s, a, y) \) when they observe
signal \( s' \). We call these triplets \textit{experiences}, and we assume that for each \( s \), \( s' \), and \( a \) there
is at least one \( y \) with \( m_{s'}(s, a, y) > 0 \).\(^7\)

Let \( H_t = (S \times A \times Y)^t \) denote the set of all histories of length \( t \), and \( H = \cup_t H_t \) the
set of all histories. After history \( h_t = (s_\tau, a_\tau, y_\tau)_{\tau=1}^t \) and signal \( s_{t+1} \), the \textit{recalled periods}
\( r_t \) are a random subset of \( \{1, \ldots, t\} \). Period \( \tau \) with experience \( (s_\tau, a_\tau, y_\tau) \) is remembered
with probability \( m_{s_{t+1}}(s_\tau, a_\tau, y_\tau) \) independent of which other periods are remembered.\(^8\) The \textit{recalled history} is the subsequence of recalled experiences \( h_{r_t} = (s_\tau, a_\tau, y_\tau)_{\tau \in r_t} \).

\(^6\)For every \( X \subseteq \mathbb{R}^k \), we let \( \Delta(X) \) denote the set of Borel probability distributions on \( X \) endowed with
the topology of weak convergence and the associated Prokhorov metric \( d_p \).

\(^7\)As a simplification, we assume that the agent either remembers \( (s, a, y) \) perfectly or not at all, as
opposed to remembering only certain aspects of it, such as one of two components, or whether \( y \) was positive
or negative. We also suppress the possible dependence of the agent’s memory function on an exogenous state,
though that would be easy to add, and allow us to capture e.g. the effect of an exogenous endowment
on beliefs in Hartzmark, Hirshman, and Imas [2019].

\(^8\)Thus, the distribution over subsets \( B \subseteq \{1, \ldots, t\} \) is given by
\( P[r_t = B| h_t = (s_\tau, a_\tau, y_\tau)_{\tau=1}^t, s_{t+1}] = \prod_{\tau=1}^t (1_{B \cap \{s_\tau, a_\tau, y_\tau\}}(s_\tau, a_\tau, y_\tau) + 1_{B^c \cap \{s_\tau, a_\tau, y_\tau\}}) \). In the Online Appendix, we show how to relax
this to allow the probability of recalling the \( n \)-th instance of the same experience to depend on \( n \).
Beliefs  We assume the agent is unaware of their selective memory and naïvely updates their beliefs as if the experiences they remember are the only ones that occurred,\(^9\) so that

\[
\mu(C|h_t(r_t)) = \frac{\int_{p \in C} \prod_{\tau \in r_t} p_{a_{r_{-1}}, (y_{r_{-1}})}(y_{r_{-1}}) d\mu(p)}{\int_{\Theta} \prod_{\tau \in r_t} p_{a_{r_{-1}}, (y_{r_{-1}})}(y_{r_{-1}}) d\mu(p)} \quad \forall C \subseteq \Theta. \tag{1}
\]

Best Responses and Optimal Policies  The agent’s belief \(\mu\) determines the subjective expected utility of each action. We denote by \(BR(s, \mu)\) the actions that maximize expected utility when signal \(s\) is observed: 

\[
BR(s, \mu) = \arg\max_{a \in A} \int_{\Theta} \sum_{y \in Y} u(s, a, y)p_{a, s}(y) d\mu(p).
\]

A policy \(\pi : H \rightarrow \Sigma\) specifies a strategy for every recalled history. We assume that the agent is myopic and uses an optimal policy, i.e., a map \(\pi : H \rightarrow \Sigma\) such that for all \(s \in S\) and recalled histories \(h_t \in H\), \(\pi(h_t)(s) \in BR(s, \mu(\cdot|h_t))\). (Note that this requires the agent to choose a pure strategy.)

2.1 Examples

We illustrate our model with five commonly studied examples of memory bias.

Example 1 (Utility-Dependent Memory). Suppose that the agent is more likely to remember pleasant experiences. This corresponds to \(m_{s'}(s, a, y) = \Phi(u(s, a, y))\) for all \(s \in S\) and for some increasing function \(\Phi : \mathbb{R} \rightarrow [0, 1]\). Similarly, a single dipped function \(\Phi\) captures the fact that extreme utility realizations are more easily remembered. ▲

Example 2 (Positive Memory Bias). Positive memory bias is the tendency to over-remember experiences that reflect positively on oneself, such as a high test score (Adler and Pansky [2020] for a survey of the evidence for positive memory bias). To model this we can let one dimension \(y_1\) of the outcome \(y = (y_1, y_2)\) reflect the self-image consequences of the experience, and make the recall probability an increasing function depending on that dimension \(m_{s'}(s, a, y) = \Phi(y_1)\) for some increasing \(\Phi\). ▲

Example 3 (Cognitive Dissonance and Ex-post Regret). Cognitive dissonance is a memory bias where the pleasantness or unpleasantness of a memory is relative instead of absolute: the probability of recalling an experience depends on how well the chosen alternative performed compared to the counterfactual payoff the agent would have received under

\(^9\)See, e.g., Reder [2014] for evidence supporting this assumption. In particular agents do not make inferences about their forgotten observations from the actions they remember taking. Because our agents do not correctly perceive the process that generates their memories, there is a sense in which they are “misspecified” even when their prior has full support.
the ex-post optimal choice (Elkin and Leippe 1986). This corresponds to $m_{s'}(s, a, y) = f(\max_{s' \in A} u(s, a', y) - u(s, a, y))$ where $f : \mathbb{R}_+ \rightarrow [0, 1]$ is decreasing. ▲

**Example 4** (Associative Memory). We can model associative memory (Thomson and Tulving 1970) as

$$\forall s, s' \in S, y \in Y, a \in A \quad \frac{m_s(s, a, y)}{m_{s'}(s', a, y)} > \frac{m_{s'}(s, a, y)}{m_s(s', a, y)},$$

so that a signal is more likely to trigger memories of experiences where the signal was the same. In general, signals represent the condition under which the choice is made. For example, when in a particular mood, agents tend to recall more situations in which they were in the same mood (Matt, Vázquez, and Campbell [1992], Mayer, McCormick, and Strong [1995].)\(^{10}\) A leading special case is similarity-weighted memory, where the probability of recalling a past experience only depends on the context in which the choice is taken: Here there is a metric $d$ on the space of signals, and $m_{s'}(s, a, y) = f(d(s, s'))$ for some decreasing function $f : \mathbb{R}_+ \rightarrow [0, 1]$. ▲

**Example 5** (Confirmatory Memory Bias). Suppose that $\Theta = \{p^0, p^1\}$, and that $\mu(p^0) > \mu(p^1)$. The agent has confirmatory memory bias (Hastie and Park 1986) if they are more likely to remember experiences that the prior views as more likely:

$$\frac{p_{a,s}^0(y)}{p_{a,s}^1(y)} \geq (>) \frac{p_{a,s}^0(y)}{p_{a,s}^1(y)} \implies m_{s'}(s, a, y) \geq (>) m_{s'}(s, a, y').$$

▲

### 3 Long Run Outcomes

The objective action-contingent probability distribution $p^*$ and the agent’s policy $\pi$ induce a probability measure $\mathbb{P}_\pi$ on $H$; we use this measure to analyze the agent’s long-run behavior.

**Definition 1.** Strategy $\sigma$ is a limit strategy if there is an optimal policy $\pi$ such that $\mathbb{P}_\pi [\sup \{t: a_t \neq \sigma(s_t)\} < \infty] > 0$.

In words, a strategy is a limit strategy if there is positive probability that it will be played in every period after some random but finite time. This section gives some general results about limit strategies. These results make no assumptions about the memory function and

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\(^{10}\)See Koszegi, Loewenstein, and Murooka [2021] for an approach that makes mood endogenous.
thus can be used to study the effects of different memory biases. Section 4 then discusses the consequences of specific memory biases.

### 3.1 Selective Memory Equilibrium

To characterize the strategies that can arise as limit behavior, we define for each strategy \( \sigma \) the set of memory-weighted KL minimizers after signal \( s' \):

\[
KL^{\Theta,m}_{s'}(\sigma) = \arg\min_{p \in \Theta} \sum_{s \in S} \sum_{y \in Y} m_{s'}(s, \sigma(s), y)p_{\sigma(s),s}(y) \log p_{\sigma(s),s}(y).
\]  

These are the elements of \( \Theta \) that maximize the log-likelihood of the memory-weighted outcome distribution induced by \( \sigma \). Note that only the relative sizes of the weights \( m \) matter for determining \( KL^{\Theta,m}_{s'}(\sigma) \): if \( \hat{m}(\cdot) = \lambda m(\cdot) \) for some \( \lambda > 0 \) then \( \hat{m} \) and \( m \) have the same memory-weighted KL minimizers.

**Definition 2.** A strategy \( \sigma \in A^S \) is a

1. **Selective memory equilibrium** if for all \( s \in S \) there is \( \nu_s \in \Delta(KL^{\Theta,m}_{s'}(\sigma)) \) such that \( \sigma(s) \in BR(s, \nu_s) \).

2. **Uniformly strict selective memory equilibrium** if for all \( s \in S \) and all \( \nu \in \Delta(KL^{\Theta,m}_{s'}(\sigma)) \), \( \{\sigma(s)\} = BR(s, \nu) \).

In a selective memory equilibrium \( \sigma \), the action played after each signal \( s \) is a best reply to some belief over memory-weighted KL minimizers from the distribution of experiences generated by \( \sigma \). The uniformly strict version adds the restriction that for each of these minimizers there is the same unique best reply. Note that both concepts allow the actions played in response to different signals to be justified by different beliefs, because which memories are triggered depends on the current realization of the signal.

**Theorem 1.** Every limit strategy is a selective memory equilibrium.

The idea of this proposition is that when the agent plays a fixed strategy, the empirical distribution converges, and the distribution of recalled experiences after each signal converges to a deterministic limit where the best fitting models after each signal \( s \) are the memory-weighted KL minimizers \( KL^{\Theta,m}_{s'}(\sigma) \).

Theorem 1 lets us provide a learning foundation for some equilibrium concepts that have been used in past work. For example, Koszegi, Loewenstein, and Murooka [2021] propose
an equilibrium concept where the agent is more likely to remember successes than failures if they are in a good mood, and the agent’s mood is determined by their self-esteem, which is a function of the number of past successes she remembers. This is a special case of our model where that the agent’s mood is an action chosen to match her perceived probability of succeeding at a task (i.e. ability). Our equilibrium concept then coincides with Koszegi, Loewenstein, and Murooka [2021]’s “self-esteem personal equilibrium,” and Theorem 1 shows that any long-run learning outcome must be such an equilibrium.

Theorem 1 shows that any long-run outcome is a selective memory equilibrium. Proposition 10 in the Appendix establishes a partial converse: every uniformly strict selective memory equilibrium is stable, meaning that play converges to it with arbitrarily high probability for an open set of beliefs. This result parallels the fact that every uniformly strict Berk-Nash equilibrium is stable. In Section 5.2 we establish a tight connection between these two equilibrium concepts.

4 Specific Forms of Selective Memory

4.1 Similarity-Weighted Memory and Self-Confirming Equilibrium

Proposition 1. For a correctly specified agent with similarity-weighted memory, a strategy is a selective memory equilibrium if and only if it is a self-confirming equilibrium.

This result follows from the fact that when the agent is correctly specified and has a large sample, since $m_s(s, a, y)$ does not depend on $a$ or $y$, the true distribution is the best fit for every signal, so the weight assigned to each signal does not matter. However, similarity weighting can change the set of selective memory equilibria when the agent is misspecified, i.e. when $p^* \notin \Theta$. Whether this occurs depends on how the agent thinks the outcome distribution varies with signals. For example, if the agent thinks that the outcome distributions associated with different signals are independent, similarity is irrelevant whether or not the agent is correctly specified: The set of selective memory equilibria coincides with the set of Berk-Nash equilibria with or without this bias. At the other extreme, the agent might think that the distribution of outcomes is the same for all signals. In this case similarity weighting can change the set of selective memory equilibria, as shown by Example 10 in the Online Appendix.\textsuperscript{11}

\textsuperscript{11}Also, even when there is a unique selective memory equilibrium and it is objectively optimal, the speed of convergence to the equilibrium can be influenced by similarity weighting. This is similar to kernel density
4.2 Ego-Boosting Memory Bias and Overconfidence

Many people are more likely to recall situations that reflect positively on themselves.\footnote{See, e.g. Mischel, Ebbesen, and Zeiss [1976] for an early experiment. This resembles situations where people choose beliefs that reflect positively on themselves as in Bénabou and Tirole [2002], with the difference that the overly positive self is not consciously chosen but produced by a memory bias.} This can lead to a particular kind of pleasant memory bias: they are more likely to remember experiences that boost their self-assessment than those that give negative signals.

Consider a situation where the agent observes i.i.d. outcomes $y_t \in Y \subset \mathbb{R}$ that reveal information about an ego-relevant characteristic such as IQ or the ability to be a successful investor. We assume that there are no signals, and that the agent (correctly) believes that their action does not affect the realized outcome. The next proposition shows that that in this case, a larger bias leads to a more positive limit belief and higher limit action.

**Proposition 2.** Suppose that $m$ and $m'$ and $p^*$ are constant in $a$, $m'(a, y) = f(y)m(a, y)$ for some increasing function $f$, $u(a, y)$ is supermodular, and that $\Theta = \Delta(\Delta(Y))$. The agent’s long-run belief with memory $m'$ concentrates on a distribution of outcomes that is higher in first-order stochastic dominance than the distribution long-run belief with memory $m$, and the limit action with memory $m'$ will be higher than the limit action with memory $m$.

In particular, comparing an increasing $m'$ with the constant memory function $m = 1$ shows that agent’s long-run belief with memory $m'$ concentrates on a distribution of outcomes that is higher in first-order stochastic dominance than the true distribution, and that their limit action will be higher than the objective optimum.

**Example 6.** Suppose that the agent repeatedly observes the outcome of an IQ test, which is either pass, $y = 1$, or fail, $y = 0$. The agent passes the test with i.i.d. probability $p^*$. Each period the agent takes an action $a \in \{0, 1\}$, with $u(a, y) = a(y - z)$, $z \in (0, 1)$, so $a = 1$ is optimal if and only if the probability of passing the test exceeds $z$. The agent always recalls passed IQ tests; they recall failed tests with probability $\phi$:

$$m(a, y) = \begin{cases} 1 & \text{if } y = 1 \\ \phi & \text{if } y = 0 \end{cases}.$$
In the long run the agent overestimates the probability of passing an IQ test to be

\[ p = p^* + \frac{p^*(1 - p^*)}{\phi(1 - \phi) + p^*}. \]

For example, if the true probability \( p^* \) is .5, and the agent remembers failing an IQ test with probability .8, in the long run believe they believe that they pass the test with probability .556. As a consequence, the agent will behave like an agent who is exogenously misspecified and dogmatically believes their ability of passing is at least .556.

This example relates to an experiment by Zimmermann [2020] in which subjects took an IQ test and received three noisy observations of how well they performed relative to other subjects. Zimmermann [2020] finds that all subjects are able to recall the signals immediately after observing them, but subjects who received negative feedback were less likely to recall the feedback a month later than subjects who received positive feedback: subjects are roughly 20% more likely to state that they “cannot recall” the result of the IQ test if the feedback was negative, even though that answer is payoff dominated in the experiment, and there were only three things for subjects to try to remember. Thus here selective memory is a better explanation for long-run overconfidence than selective attention.

Example 6 and Proposition 2 also relate to the literature on overconfidence and financial decision making. Walters and Fernbach [2021] finds that investors are 10% less likely to recall an investment that led to a loss as compared to an investment that led to a gain. It also finds that selective forgetting is a significant predictor of investor overconfidence, and that overconfidence is reduced when investors rely less on their memory. In an incentivized experiment, Gödker, Jiao, and Smeets [2021] finds that subjects over-remember good investment outcomes and under-remember bad investment outcomes. In line with the prediction of Proposition 2, it shows that this leads subjects to have overly optimistic beliefs about their investments and to hold bad investments longer.

**Ego Boosting Bias and Misattribution** Now we show how an agent with ego-boosting bias can misinterpret data about about other aspects of the world.

**Example 7.** Suppose that besides taking an IQ test, the agent is involved in a joint project with a coworker. The possible outcome distributions \((p_1, q) \in [0, 1]^2\) and outcome \((y_1, y_2) \in \ldots\)

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\(^{13}\) Zimmermann [2020] finds that “negative feedback is indeed recalled with significantly lower accuracy, compared to positive feedback.” Here lower accuracy means both that the agents are more likely to report that they not recall the experience, and that they misreport the experience.
{$0,1}^2$ are two dimensional, where the first component denotes whether or not the agent passed an IQ test and the second component denotes whether a group project succeeded. The probability that the agent passes the IQ test is $p_1$ and the probability with which the group project succeeds equals $p_2 = \alpha p_1 + (1 - \alpha)q$. Intuitively, the group project depends on the ability of the agent’s $p_1$ and their coworker’s ability $q$. The agent always remembers periods with positive IQ test results, and remembers periods with negative test results with probability $\phi \in (0,1)$. If ($p_1^*, q^*$) denote the true ability of the agent and coworker then the agent’s long-run beliefs will satisfy

$$p_1 = p^* + \frac{p^*(1 - p^*)}{\phi/(1 - \phi) + p^*} \quad q = q^* - \frac{\alpha}{1 - \alpha} \frac{p^*(1 - p^*)}{\phi/(1 - \phi) + p^*}.$$  

The agent thus underestimates the coworker’s ability as a consequence of selective memory. The induced long-run beliefs are similar to the attribution bias derived in Heidhues, Kőszegi, and Strack [2018] for exogenously overconfident agents, where low outcomes are attributed to an exogenous state. ▲

To generalize this example, we consider a two dimensional outcome space $Y = Z \times Z \subset \mathbb{R}^2$, where $y_1$ corresponds to an ego-relevant characteristic, and is distributed according to $p_1^*$. The second component $y_2$ is independently drawn, with $p_2^*(y_2) = \alpha p_1^*(y_2) + (1 - \alpha)q^*(y_2)$ for some $\alpha \in (0,1)$. The agent knows that the outcomes are independently drawn each period according to a $p^*$ satisfying these conditions, but does not know $p_1^*$ or $q^*$, and their prior belief has full support over $(p_1, q)$.

**Proposition 3.** Suppose that $m$ and $p^*$ are constant in $a$ and $y_2$ and are increasing in $y_1$. Then the agent’s long-run belief about $y_1$ concentrates on a distribution that is higher in first-order stochastic dominance than $p_1^*$ and the agent’s long-run belief about $y_2$ concentrates on a distribution that is lower.

In Section 5.2 we will see that the long-run belief induced by selective memory can be replicated by exogenous misspecification in any fixed environment, and vice versa. However, the two models can lead to very different comparative statics. Suppose for example that negative feedback is delivered along with positive feedback about an unrelated trait of the agent. Combining positive and negative information in this way is known as a “feedback sandwich;” it is suggested as a way of making feedback more impactful in the management and psychology literature. If the positive feedback makes the periods where the agent

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14 Procházka, Ovcari, and Durinik [2020] describes an experiment where bundling negative feedback with positive feedback about an unrelated domain helps agents perform better.
failed the IQ test less unpleasant, an agent with positive memory bias would be more likely to remember them, and their long-run belief about their own ability would move closer to their true ability. So they would become less biased about their coworker’s ability. This is in contrast to exogenous misspecification, where positive feedback about an unrelated state would not affect the agent’s beliefs about their own ability or their coworker’s.

4.3 Selective Memory and Risk Attitudes

This section shows that for choices over lotteries, memory distortions can generate the same behavior as a distorted risk preference. We again simplify by supposing there are no signals, and let the outcome $y \in \mathbb{R}$ be the amount of money received by the agent, with $u(s, a, y) = v(y)$ for some concave $v$.

**Extreme Experience Bias** Suppose the agent chooses between a safe action $a = 0$ they know always induces outcome $y_0$ and a lottery, $a = 1$, for which the agent thinks that all outcome distributions are possible.\(^{15}\) We say that the agent has an extreme experience bias if the probability of remembering an experience $m$ is an increasing function of the distance of the outcome $y$ from its expected value

$$m(s, a, y) = h(|y - \mathbb{E}_{p^*_1}(y)|).$$

Our next result shows extreme experience bias makes the agent less likely to accept lotteries that are symmetric around the reference point.

**Proposition 4.** Suppose $p^*_1(\mathbb{E}_{p^*_1}(y) + y) = p^*_1(\mathbb{E}_{p^*_1}(y) - y)$. If choosing the lottery is not a selective memory equilibrium with perfect memory, it is not a selective memory equilibrium with extreme experience bias.

Intuitively, because the agent over-remembers extreme experiences, the environment seems more risky than it truly is, so in the long run they do not take the risky action if it would not be optimal for an agent without extreme experience bias.\(^{16}\) By making the tail realizations relatively more memorable, extreme bias can make a risk-averse agent act as if they were even more risk averse. This may help explain why the risk aversion needed to match the real-world investment choices is unrealistically high: the agents can be attracted

\(^{15}\)Thus the support of the agent’s prior is $\Theta = \{p \in \Delta(Y) \times \Delta(Y) : p_0(y_0) = 1\}$.

\(^{16}\)This behavior can also be induced by a misspecified belief with overly fat tails.
by safe alternatives because they are moderately risk averse and their memory exaggerates the riskiness of the uncertain alternatives.

**Rare Experience Bias**  Similarly, some forms of selective memory are equivalent to preferences that arise from distorting outcome probabilities. Suppose that the agent is more likely to remember experiences that happen more rarely, i.e. there is a decreasing function $h : [0, 1] \rightarrow [0, 1]$ such that $m(s, a, y) = h(p^*_i(y))$. In this case, in the long run the decision maker believes that the outcome distribution for the risky action is

$$\frac{h(p^*_i(y))}{\sum_{z \in Y} h(p^*_i(z))}.$$ 

They will thus act as if they have probability weighting, as in prospect theory (Kahneman and Tversky [1979]).

## 5 Alternative Models

This section compares our selective memory model with *underinference* and *misspecification*, which are two other ways to model similar effects.

### 5.1 Underinference

The phenomenon of underinference (Phillips and Edwards [1966]) is distinct from selective memory but has similar long-run implications. Here agents remember (or are presented with) a record of past observations, and so memory is not an issue, and the agent's beliefs are a deterministic function of the sequence of observations. However, they underweight a given observation $(s, a, y)$ when applying Bayes rule. In particular, they use the deterministic updating rule

$$\mu^U(C|(s_i, a_i, y_i)_{i=1}^t) = \frac{\int_{p \in C} \prod_{i=1}^t (p_{a_i, s_i}(y_i))^{m(s_i, a_i, y_i)} d\mu(p)}{\int_{\Theta} \prod_{i=1}^t (p'_{a_i, s_i}(y_i))^{m(s_i, a_i, y_i)} d\mu(p)},$$

(4)

where $m(s, a, y) \in [0, 1]$ is the *underinference distortion* applied to experience $(s, a, y)$.

As with selective memory, this memory distortion leads beliefs to concentrate on a set of weighted KL minimizers, and as as the next result shows the underinference distortion maps directly to a selective memory function.
Proposition 5. If $\sigma$ is a limit strategy with underinference distortion $m$, it is a selective memory equilibrium with memory function $m$.

A leading special case is uniform underinference where $m(s, a, y) = c < 1$ and the agent discounts all observations by the same amount. In this case, Propositions 1 and 5 imply that the limit strategy for a correctly specified agent must be a self-confirming equilibrium. A natural question is whether the two theories can be distinguished from observable data. Most of the current evidence on overconfidence only regards agent beliefs, and not the recalled histories, so it cannot tell the two distortions apart (see Benjamin [2019]).

If signals are absent and actions are real-valued, the way actions respond to outcomes can be used to distinguish underinference and selective memory. Under overconfidence, the realization of $y_t$ is sufficient to predict whether $a_{t+1}$ is more or less than $a_t$. Under selective memory, the set of past experiences retrieved at time $t + 1$ may differ from those at time $t$, so in general the previous period’s outcome and action are not sufficient to predict the way that actions change. Moreover, the action sequence features a sort regression to the mean: after an action that is particularly high, it is likely the next action will be lower.

In general, with an exogenous data generating process the agent’s beliefs will converge to the same limit as with underinference, so their limit action will be the same. If instead the data generating process is endogenous, random memory realizations can induce switches in actions, so actions are less likely to converge. The next example illustrates this possibility.

Example 8. Suppose $A = \{a, b\}$, $Y = \{0, 1\}$, $S$ is null, and $u(a, y) = y$, and that for some $c \in (0, 1)$, $p_u(1) = c$ for all $p \in \Theta$. The effect of action $b$ is unknown, but the agent’s initial belief is that it is larger than that of action $a$, so $BR(\mu_0) = b$, although there is $p' \in \Theta$ with $p'_b(1) < c$. The truth is that $p^*(b) > c$, so action $b$ is optimal, but if the memory function is strictly positive, both $a$ and $b$ are selective memory equilibria. In the underinference model both $a$ and $b$ have positive probability of being limit actions, and moreover if $a_t = a$ then $a_{\tau} = a$ almost surely for all $\tau > t$, while if $m(b, 1) \geq m(b, 0)$ with the selective memory model we have $\lim_{t \to \infty} a_t = b$ almost surely. ▲

5.2 Selective Memory and Misspecification

Using selective memory equilibrium, we can relate the long-run implications of selective memory to those of misspecification in the sense of the statistics literature, where the true model is not in the support of the agent’s prior.

\[17\] This resembles fictitious play with incomplete sampling, as in Kaniovski and Young [1995].
Let
\[ KL^{\Theta,1}(\sigma) = \arg\min_{p \in \Theta} \sum_{s \in S} \zeta(s) \sum_{y \in Y} p^s_{\sigma(s),s}(y) \log p^s_{\sigma(s),s}(y). \tag{5} \]
be the set of KL minimizers for \( \Theta \) when the function \( m \) is identically equal to 1, so the agent does not have selective memory.

**Definition 3.**

1. Strategy \( \sigma \) is a *Berk-Nash equilibrium* if for all \( s \in S \), there exists \( \nu \in \Delta(KL^{\Theta,1}(\sigma)) \) such that \( \sigma(s) \in BR(s, \nu) \).

2. A Berk-Nash equilibrium \( \sigma \) is a *self-confirming equilibrium* if there is \( \nu \in \Delta(\Theta) \) such that for all \( s \in S \) and \( p \in \Theta \), \( p_{\sigma(s),s} = p^s_{\sigma(s),s} \) and \( \sigma(s) \in BR(s, \nu) \).

3. Strategy \( \sigma \) is a *uniformly strict Berk-Nash equilibrium* if for all \( \nu \in \Delta(KL^{\Theta,1}(\sigma)) \) and all \( s \in S \), \( \{\sigma(s)\} = BR(s, \nu) \).

Esponda and Pouzo [2016] shows that Berk-Nash equilibrium is a necessary condition for a strategy to be the long-run outcome of a possibly misspecified learning process. Fudenberg and Levine [1993b] shows that self-confirming equilibrium corresponds to the steady states of a learning model with long-lived but myopic agents; it requires that agents have correct beliefs about the consequences of their equilibrium action but allows them to have incorrect beliefs about actions they do not use. Fudenberg, Lanzani, and Strack [2021a] shows that uniformly strict Berk-Nash equilibrium is a necessary condition for a strategy to be the long-run outcome with probability near 1.

Berk-Nash equilibrium is the special case of selective memory equilibrium where memory is perfect as opposed to selective. Moreover, even beyond this case there is a close relationship between the uniformly strict versions of these equilibrium concepts: For a given prior support \( \Theta \) and objective distribution \( p^s \), every uniformly strict Berk-Nash equilibria is equivalent to a selective memory equilibrium with full support prior for some memory function, and every uniformly strict selective memory equilibria is equivalent to a Berk-Nash equilibrium for some support. To formalize this idea we will need to consider two different priors \( \mu_0 \) and \( \mu'_0 \) with possibly different supports \( \Theta \) and \( \Theta' \).

**Definition 4.** A Berk-Nash equilibrium \( \sigma \) with support \( \Theta \) and a selective memory equilibrium \( \sigma' \) with support \( \Theta' \) are *belief equivalent* if \( \sigma = \sigma' \), and for all \( s \in S \) there exists a belief \( \nu \in \Delta(KL^{\Theta,1}(\sigma) \cap KL^{\Theta',m}_{\sigma'}(\sigma)) \) such that \( \sigma(s) \in BR(s, \nu) \).
Two equilibria are belief equivalent if they prescribe the same strategies, and behavior after each signal can be justified by the same belief. Note that in a selective memory equilibrium the signal can influence the agent’s memory and the set of KL minimizers, while in a Berk-Nash equilibrium the KL minimizers are the same regardless of the signal.

**Proposition 6.**

1. Every uniformly strict Berk-Nash equilibrium with support $\Theta$ is belief equivalent to a selective memory equilibrium with support $\Theta' = \Delta(Y)^A \times S$ for some memory function.

2. Every uniformly strict selective memory equilibrium with support $\Theta$ is belief equivalent to a uniformly strict Berk-Nash equilibrium for some $\Theta'$.

The idea behind the first part of the proposition is that if we start from a KL minimizer $p$ with perfect memory but incomplete support, we can choose a memory function that rescales the probability of an outcome by $p_{\sigma(s),s}(y)/p_{\sigma(s),s}^*(y)$, making the recalled frequency equal to $p$. This makes $p$ a weighted-memory minimizer, so $\sigma$ is a best reply.\(^\text{18}\) The second part of the proposition is trivial: to construct a strict Berk-Nash equilibrium that leads to the same beliefs and behavior as in the selective memory equilibrium we can endow the agent with a degenerate belief that equals the long-run belief in the selective memory equilibrium.\(^\text{19}\)

**Remark.** As we prove in Appendix A, the uniform strictness conditions of Proposition 6 are needed:

1. There are Berk-Nash equilibria that are not belief equivalent to any selective memory equilibrium with support $\Theta' = \Delta(Y)^A \times S$.

2. There are selective memory equilibria that are not belief equivalent to any Berk-Nash equilibrium.

3. Moreover, unlike Berk-Nash equilibria, selective memory equilibria do not reduce to self-confirming equilibria when the agent is correctly specified. In particular, selective memory equilibria need not be objectively optimal when the distribution of outcomes is independent of the agent’s action.

\(^\text{18}\)There may be other minimizers, and $\sigma$ may not be a best response to some of them, so it need not be a uniformly strict selective memory equilibrium.

\(^\text{19}\)Formally, we take $\Theta'$ to be a singleton $p$ with with $p_{a,s}(y) = p'_{a,s}(y)$ where $p' \in KL^\Theta_{s,m}(\sigma)$. 

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To illustrate the equivalence result, consider a buyer who submits an offer for a good in double-blind two-sided auction where the price $z$ is set at the buyer’s bid, so the seller’s dominant strategy is to bid their value. Suppose that the buyer has correlation neglect and believes that the price sellers ask is independent of the quality of the good they are selling. If the buyer’s value of the good is $x + v + \varepsilon$ where $x$ is the value for the seller, $v$ measures the gains from trade, and $\varepsilon$ is a noise term, and if after every interaction the buyer observes their payoff and the ask price $x$, then in equilibrium they submit a bid that is too low, as in Esponda [2008].\textsuperscript{20} Proposition 6 shows that memory distortions can lead the agent to believe that value and bid are independent, and thus have the same behavior and beliefs as in a selective memory equilibrium. This is obtained with a memory function that gives just enough additional weight to experiences with a larger gap between values and ask prices.\textsuperscript{21}

While Proposition 6 implies that selective memory and misspecification will have similar long-run implications in a fixed environment, in Section 4.2 we have shown that the two models have different comparative statics with respect to changes in the environment.\textsuperscript{22} Therefore, empirical work that distinguishes between the two models based on variation in information is needed.

6 Partial naïvete

So far we assumed that the agent is unaware that they are forgetful, and treats the experiences they remember as if these were the only ones that happened. This section generalizes the model to agents who are partially aware of their memory limitations.\textsuperscript{23} We suppose throughout this section that there are no signals, and that actions have no effect on the outcome distribution. We also assume that the agent either does not remember their own actions or mistakenly believes their actions convey no information. so that they do not draw inferences about the experiences they forgot from their actions. Finally we suppose that the

\begin{footnotesize}
\footnotesize
\begin{enumerate}
  \item Here $(a,y) = (z,(x,\mathbf{1}_{z>x}(x + v + \varepsilon - z)))$, and $u(a,y) = \mathbf{1}_{z>x}(x + v + \varepsilon - z)$.
  \item Here we use Assumption 1. The result fails without it, as we do not allow the memory function to create memories of things that didn’t happen. Note also that since memory cannot affect the perceived relative likelihoods of experiences that are not observed, and only some distributions for the counterfactual justify the equilibrium strategy, the resulting equilibrium may not be uniformly strict.
  \item There is a sense in which selective memory can also be viewed as a form of misspecification as the agent is not aware of their memory limitations. From that perspective, our results show that the classic misspecification studied in Bayesian statistics is closely related to a psychologically founded form of misspecification derived from selective memory.
  \item To focus on the learning implications of selective memory, we are focusing on myopic decision-makers. Without myopia, one has to take a stance on the commitment power of the agent and on how they try to affect their future memory, as in Piccione and Rubinstein [1997].
\end{enumerate}
\end{footnotesize}
agent knows the current period and so knows how many observations they have made.\textsuperscript{24}

Suppose the agent believes that they remember an outcome with probability \( \hat{m}(y) \), instead of the true probability \( m(y) \). The subjective likelihood of a remembered history \( h_{rt} \) under the data generating process \( p \) is then

\[
\left( \frac{t}{|r_t|} \right) \left[ \sum_{z \in Y} p(z)(1 - \hat{m}(z)) \right] t - |r_t| \prod_{i \in r_t} p(y_i) \hat{m}(y_i). 
\]

Thus up to a constant the subjective log-likelihood of outcome frequency \( f \) equals

\[
\left( 1 - \frac{|r_t|}{t} \right) \log \left[ \sum_{z \in Y} p(z)(1 - \hat{m}(z)) \right] + \frac{|r_t|}{t} \sum_{y \in Y} f_t(y, r_t) \log(p(y)\hat{m}(y)) \tag{6}
\]

where \( |r_t| \) is the number of events the agent remembers. (Note that the first term does not appear when the agent believes they remember everything, which corresponds to \( |r_t|/t = 1 \).)

Because the expected value of \( |r_t|/t \) is \( 1 - \sum_{y \in Y} p(y)m(y) \), (6) suggests the following generalization of (2):

\[
KL^{\Theta,m,\hat{m}}(\sigma) = \arg\min_{\nu \in \Theta} \left( 1 - \sum_{y \in Y} m(y)p^*(y) \right) \log \left( 1 - \sum_{y \in Y} p(y)\hat{m}(y) \right) + \sum_{y \in Y} m(y)p^*(y) \log(\hat{m}(y)p(y))
\]

Notice that when \( \hat{m} = 1 \), the second term of the equation is equal to the minimized function in (2), and the first term does no longer depends on \( p \), becoming irrelevant for the minimization.

**Definition 5.** A \textit{selective memory equilibrium for a partially naïve agent} is a strategy \( \sigma \) such that there exists a belief \( \nu \in KL^{\Theta,m,\hat{m}}(\sigma) \) with \( \sigma \in BR(\nu) \).

For an agent who is aware of their own forgetfulness, but not aware that their memory is selective, i.e. who believes that their memory function \( m \) is constant, \( KL^{\Theta,m,\hat{m}} = KL^{\Theta,m} \) and the selective memory equilibria of a partially naïve and fully naïve agent coincide.\textsuperscript{25} This shows that for our results it is not important that the agent is unaware of their forgetfulness, but it is important that they are unaware of the selective nature of their memory. At the

\textsuperscript{24}Otherwise we would need to model how the agent make inferences about how many observations they have forgotten.

\textsuperscript{25}To see this, note that for \( \hat{m} \) constant the minimized function is \( \sum_{y \in Y} m(y)p^*(y) \log(p(y)) + \left[ \sum_{y \in Y} p^*(y)m(y) \log(\hat{m}) \right] + \left[ \sum_{y \in Y} p^*(y)(1 - m(y)) \right] \log(1 - \hat{m}). \) The last two terms are independent of \( p \) so \( KL^{\Theta,m,\hat{m}} = KL^{\Theta,m} \).
other extreme, if agents are fully aware of their memory function then $\delta_{p^*} \in KL^{\Theta,m,m}$ and thus any action that is optimal for the true data generating process is always a selective memory equilibrium.\footnote{That $\delta_{p^*} \in KL^{\Theta,m,m}$ follows directly from the Gibbs inequality.}

The next result, whose proof is omitted, follows from an argument analogous to the proof of Theorem 1.

**Proposition 7.** When the agent is partially naïve, every limit strategy is a selective memory equilibrium.

Moreover, as with notions of partial naïveté in cursed equilibrium and quasi-hyperbolic discounting, one can define a parametric notion of partial naïveté by assuming that $\hat{m}(y) = (1 - \alpha) + \alpha m(y)$. For $\alpha = 0$ the agent is fully naïve and unaware of their memory limitations. For $\alpha = 1$ the agent is sophisticated and understands their memory limitations which in consequence implies that the agent will have correct long-run beliefs.

As the next proposition illustrate, the degree of naïvete can amplify existing memory biases. Consider again the setting of Section 4.2 which studied positive memory bias by assuming that $y$ is a scalar and $m(y)$ is increasing in $y$.

**Proposition 8.** Suppose $m$ and $p^*$ are constant in $a$ and $m$ is increasing in $y$, that $\hat{m}(y) = (1 - \alpha) + \alpha m(y)$, $\Theta = \Delta(\Delta(Y))$, and the utility function is supermodular. Then the agent’s long-run belief concentrates on a distribution of outcomes that is increasing in first-order stochastic dominance in $\alpha$, i.e. the naïvete of the agent.

### 7 Finite Expected Memory

#### 7.1 Stationary Bounded Memory

In our base model, the number of recalled experiences goes to infinity as the agent’s sample size increases. This section modifies the model so that even when the actual sample size goes to infinity, the expected number of instances recalled by the agent remains bounded. This provides an explanation of why time-changing behavior may persist in apparently stationary environments that complements explanations based on recency bias (see Foster and Young [2003], Fudenberg and Levine [2014], Fudenberg and Peysakhovich [2016], and Erev and Haruvy [2016]).
We suppress the signals for simplicity, and assume that the agent’s memory at time $t$ is distorted through a memory function that now depends on calendar time $t$: For some fixed integer $k$,

$$m_t(a, y) = \min\{1, k/t\} m(a, y).$$

(7)

With this specification, as the number of experiences $t$ grows to infinity, the probability that any specific experience is recalled decreases at rate $k/t$. Thus asymptotically the expected number of recalled experiences is no more than $k$, so even when the action process converges, the recalled outcome frequency can have a non-degenerate distribution.

After history $h_t = (a_i, y_i)_{i=1}^t$, the recalled periods $r_t$ are a random subset of $\{1, \ldots, t\}$ with probability distribution

$$\mathbb{P}[r_t = B|h_t] = \prod_{i=1}^t \left(1 - \mathbbm{1}_{i\notin B} m_t(a_i, y_i) + \mathbbm{1}_{i\notin B} (1 - m_t(a_i, y_i))\right) \quad \forall B \subseteq \{1, \ldots, t\}.$$  

The recalled history is the subsequence of action-outcome pairs $h_{r_1} = (a_i, y_i)_{i \in B}$ of the recalled periods, and $\mu_{t+1}$ is the random period-$t + 1$ belief $\mu(\cdot|h_{r_1})$ induced by the recalled history $h_{r_1}$.\(^\text{27}\)

Let $\eta_\alpha$ be the distribution on the numbers of times each experience is recalled, defined as follows: The distribution of occurrences of each action-outcome pair $(a, y)$ is Poisson with parameter $\alpha(a)p^*_a(y)km(a, y)$ independent of the number of occurrences for the other pairs. We are going to show that $\eta_\alpha$ is the limit distribution of the recalled experiences if the action frequencies converge to $\alpha$. Intuitively, the expected number of times a pair $(a, y)$ is recalled is proportional to the frequency of the action $\alpha(a)$, the probability of the outcome given the action $p^*_a(y)$, and how memorable that experience is, i.e. $m(a, y)$. Let $F_\alpha$ be the distribution of beliefs induced by $\eta_\alpha$, i.e., $F_\alpha(B) = \eta_\alpha(\{h : \mu(\cdot|h) \in B\})$ for all $B \subseteq \Delta(\Theta)$.\(^\text{28}\)

**Definition 6.** A stochastic memory equilibrium is a mixed strategy $\alpha$ for which there is a Markovian policy $\rho : \Delta(\Theta) \rightarrow A$ such that

(i) at every belief $\mu$ the action is optimal, i.e. $\rho(\mu) \in BR(\mu)$, and

(ii) $\alpha$ equals the action frequencies induced by $\rho$, i.e. $\alpha(a) = F_\alpha(\{\mu : a = \rho(\mu)\})$ for all

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\(^{27}\)Note that because the $m_t$ are independent, for $t > k$ there is positive probability that no periods are recalled. In this case $\mu$ is simply the agent’s prior.

\(^{28}\)Note a small abuse of notation: $\eta_\alpha$ defines a probability measure over the number of time each experience is recalled. This is not equal to the set of histories, as two histories composed by the same experiences in a different order are distinct. However, as such two histories induce the same posterior, the definition of $F_\alpha$ is unambiguous.
For every $t \in \mathbb{N}$, define the action frequency at time $t$ by

$$\alpha_t(h_t)(a) = \frac{1}{t} \sum_{\tau=1}^{t} I_a(a_{\tau})$$

for all $h_t \in H_t$ and for all $a \in A$.

**Theorem 2.** Suppose that $\alpha_t \rightarrow \alpha$ with positive probability. Then $\alpha$ is a stochastic memory equilibrium.

Intuitively, if play converged to a limit that is not a stochastic memory equilibrium, the distribution of actions would not correspond to the distribution of best responses generated by the agent’s beliefs. To prove this, we first show that when the $\alpha_t$ converge to $\alpha$, the distribution of recalled experiences has the Poisson form given above, which follows from the Poisson limit theorem on the sum of binomials. We then show that beliefs converges to $F_\alpha$, and then we use stochastic approximation results of Benaim, Hofbauer, and Sorin [2005] to show that if $\alpha$ is not a stochastic memory equilibrium, the agent’s best response to their beliefs would lead the distribution of actions to move away from $\alpha$.

In a stochastic memory equilibrium, the agent will sometimes rely on a small number of past instances to make decisions. This can induce long-run underreaction of beliefs and insensitivity to sample size, a form of representativeness bias first documented by Kahneman and Tversky [1972]. We can calibrate the memory capacity parameter $k$ to match various aspects of the evidence. For example, the probability that the agent decides only on the basis of their prior decreases exponentially in $k$, and is equal to:

$$\prod_{(a,y) \in A \times Y} \exp(-\alpha(a)p_a^*(y) km(a, y)).$$

There is a similar formula for the probability of making a decision based on at most seven experiences, which is claimed to be the “magical” number for working memory (Miller [1956]), so we can calibrate the model by choosing $k$ that makes the probability of choosing based on 7 or fewer memories close to 1. Moreover, a falsifiable implication of the model is that the decrease in the probability of recalling an additional $t$-th experience decreases factorially regardless of the value of $k$.

While Theorem 2 shows that the every limit point of the empirical frequency must be a stochastic memory equilibrium the same is not true for the expectation of the empirical
frequency). We illustrate with Example 9 in the OLA.

The Role of the Prior under Finite and Infinite Expected Memory  One important difference between finite and infinite expected memory is the role of the prior. To see this in the starkest form, we compare limit behavior with a selective memory function $\hat{m}$ and that in a stochastic memory equilibrium with memory $\hat{m}_t = \min\{1, k/t\}m$. We first consider the case of an exogenous data generating process where the action does not influence the outcome distribution.

**Corollary 1.** Fix $\Theta \subseteq \Delta(Y)^{A \times S}$ and $m : A \times Y \rightarrow [0, 1]$. If the data generating process and the memory function are exogenous and there exists a unique uniformly strict selective memory equilibrium:

1. Under infinite expected memory, the limit action will be the unique selective memory equilibrium independent of the prior.

2. For every action $a$ that is not weakly dominated in $\Theta$ and expected memory capacity $k \in \mathbb{N}$, there exists a prior belief such that $\alpha(a) > 0$ for every stochastic memory equilibrium $\alpha$.

When the data generating process is exogenous, the empirical distribution of recalled outcomes converges almost surely, and the agent ends up playing the best reply to this distribution. Instead, with finite memory there is a positive fraction of periods in which the number of instances recalled by the agent is so low that they best reply to the prior. Since a (weakly) undominated action is always a best reply to some prior, the result follows. More generally, when the action does influence the distribution of outcomes, the prior may influence the probability of converging to a specific selective memory equilibrium, but the set of selective memory equilibria is the same for every prior. This is not the case with stochastic memory, because the prior influences the chosen action when the number of recalled experiences is small.

### 7.2 Rehearsal

We expand the model to incorporate the effect of rehearsal: If an experience is recalled in one period, it is more likely to be recalled in subsequent periods, see Kandel et al. [2000] and the references therein. To model this phenomenon we assume that the agent’s memory
at time $t$ is distorted through a \textit{rehearsal memory function} that can depend on the history recalled in the previous period

$$m_t(a, y|(r_{t-1}, (a_{t-1}, y_{t-1})) = \min\{1, k/t\}[m(a, y) + r\mathbb{1}_{t_{t-1}}(a, y)], \tag{8}$$

where the experiences $\tilde{r}_{t-1} = r_{t-1} \cup \{(a_{t-1}, y_{t-1})\}$ that were recalled or experienced last period have an additional probability of $r$ of being recalled.\footnote{Rehearsal is a key feature of Mullainathan [2002]'s model of memory, which analyzed the period-ahead impact of rehearsal but not its long-run implications. To make it simpler to compare our results, we maintain the same assumption that only what was recalled yesterday affects what is recalled today.}

After history $h_t = (a_i, y_i)_{i=1}^t$, the recalled periods $r_t$ are a random subset of $\{1, \ldots, t\}$ with probability distribution

$$\mathbb{P}[r_t = B|h_t, r_{t-1}] = \prod_{i=1}^t \left(1_{i \in B} m_t(a_i, y_i|(r_{t-1}, (a_{t-1}, y_{t-1}))) + 1_{i \notin B}(1 - m_t(a_i, y_i|(r_{t-1}, (a_{t-1}, y_{t-1}))))\right)$$

for all $B \subseteq \{1, \ldots, t\}$. We call the experiences the agent recalls the \textit{recalled history}.

We now define a key Markov chain over histories.

\textbf{Definition 7.} For every $t \in \mathbb{N}$ and $h' = (a_i, y_i)_{i=1}^t$, let $\eta_{\alpha, h'}$ be the distribution where the number of occurrences of each action-outcome pair $(a, y)$ has a Poisson distribution with parameter

$$\begin{align*}
\alpha(a)p_a^*(y)k [m(a, y) + r] & \quad \text{if } \exists i \in \{1, \ldots, t\}: (a, y) = (a_i, y_i) \\
\alpha(a)p_a^*(y)k [m(a, y) + rp_a^*(y))] & \quad \text{if } \exists i \in \{1, \ldots, t\}: (a, y) = (a_i, y_i) \text{ and } (a = \pi(h')) \\
\alpha(a)p_a^*(y)k m(a, y) & \quad \text{otherwise}
\end{align*}$$

independent of the number of the instances for the other pairs. The \textit{induced Markov chain} $\eta_\alpha$ has state space $H$ and Markov kernel $\eta_{\alpha, h'}(h)$.

At any time every subset of $H$ has positive probability of being the recalled history, so the chain is irreducible on the subsets of histories that can be reached with positive probability starting from the empty history. A calculation in the appendix shows it is positive recurrent, which yields the following lemma.

\textbf{Lemma 1.} $\eta_\alpha$ admits a unique stationary distribution $\mathcal{H}_\alpha \in \Delta(H)$.

We will show that the unique stationary distribution of this Markov chain is the limit time average distribution over recalled histories. Intuitively, the expected number of times
a pair \((a, y)\) is recalled is proportional to the frequency of the action \(\alpha(a)\), the probability of the outcome given the action \(p^{a}(y)\), how memorable that realization is, \(m(a, y)\), and whether it occurred or was recalled in the last period.

Let \(F_{\alpha,h'}\) be the distribution of beliefs induced by the \(\eta_{\alpha,h'}\), and let \(\chi_{\alpha,h'}\) denote the distributions over actions induced by an optimal Markovian policy \(\rho\) and random beliefs \(\mu\):

\[
\chi_{\alpha,h'} = \{\alpha' \in \Delta(A) : \exists \rho \in \Pi_o \text{ such that for all } a \in A, \alpha'(a) = F_{\alpha,h'}(\{\mu : a = \rho(\mu)\})\}.
\]

**Definition 8.** An *ergodic memory equilibrium* is a mixed strategy \(\alpha\) such that there exists \((\alpha^h)_{h \in H} \in \Delta(A)^H\) with \(\alpha^h \in \chi_{\alpha,h}\) for all \(h \in H\) and \(\alpha = \mathbb{E}_{\mathcal{H}_\alpha}[\alpha^h]\).

**Theorem 3.** Suppose that \(\alpha_t \rightarrow \alpha\) with positive probability. Then \(\alpha\) is an ergodic memory equilibrium.

### 7.3 Income Forecasts and Asset Pricing

Our model of finite expected memory and rehearsal lets us generalize the findings of Mullainathan [2002] about income forecasts beyond the specific parametric structure it assumed. It also lets us provide a novel memory-based explanation of the equity-premium and equity-volatility puzzles. We suppose that the outcome \(y_t\) is i.i.d. \(y_t = \theta + \epsilon_t\), independent of the action of the agent, where the \(\epsilon_t\) are mean-0 shocks.\(^{30}\) As elsewhere in this section, we assume there are no signals, so the agent only observes the outcome.

**Memory Rehearsal and correlated prediction errors** The rehearsal memory function of equation (8) generates the same predictions about one-period correlations as Mullainathan [2002], without assuming associativeness. First, a high outcome last period triggers memories of equally high past realizations, so that the forecasting error will be negatively correlated with the most recent information.\(^{31}\) Second, when the baseline probability of remembering an event is low, and the rehearsal effect is strong, the forecast errors in successive periods will be positively correlated for the same reason as in as Mullainathan [2002]: memories that are forgotten are more likely to be forgotten again.

\(^{30}\)Both Mullainathan [2002] and Weitzman [2007] assumed that the outcomes follow an AR1 process, so that a standard Bayesian would always place non-vanishing weight on the most recent outcome; our assumption of finite expected memory has the same implication.

\(^{31}\)Mullainathan [2002] supposes that \(y\) has a positive density on the real line, so that some form of associativeness is needed for rehearsal to have any effect.
**Asset pricing**  Suppose that the agent can choose between purchasing a safe asset $a_0$ and purchasing a (representative) equity portfolio $a_1$ at prices $p_{0,t}$ and $p_{1,t}$. If held one period, the safe asset has net return $i$, while the risky asset provides net return $1 + i + \theta$, where $\theta$ is a random variable whose distribution is unknown. In this setting, the *equity premium puzzle* is that, if the distribution of $\theta$ were known and equal to that observed in the data, a very large amount of risk aversion would be needed to justify the observed difference in asset prices.

Weitzman [2007] explains this with the combination of an overly pessimistic prior and incomplete learning. With a small sample of observations, initially pessimistic agents will be willing to pay a premium price for the safe asset. To match the persistent risk premium in face of an arbitrarily long sequence of observations, Weitzman [2007] posits that the agent believes in a time-evolving distribution for $\theta$, and so they discard old observations.

Ergodic memory equilibrium predicts the same effect even with a perceived constant risk premium. Theorem 3 guarantees that even in the long run the agent will rely on a limited number of observation, so that the pessimistic prior is able to sustain the premium. Moreover, if we allow for signals together with the widely documented extreme experience bias and similarity-weighted memory, our model predicts the excess volatility with respect to fundamentals featured by the equity-volatility puzzle.

To see this in a simple example, suppose the signal $s$ is a mean-zero shift to the outcome distribution, and the similarity-weighted memory takes the form $d(s, s') = 1$ if $s'$ and $s$ have the same sign and 0 otherwise, and that the extreme experience bias has the form assumed in (3), with reference point the return of the low asset, i.e., $r = 1 + i$. Similarity-weighted memory implies that after a negative signal the agent is more likely to sample negative realizations, and extreme experience bias implies that situations worse than among these negative realizations the worse ones will be exaggerated, inducing an excessively low willingness to pay for the asset. While this form of similarity-based memory is very stark, the same qualitative feature holds if the similarity between negative signals is high compared to the correlation of different-sign signals. Here the prediction of excess volatility is not obtained from recency alone: If memory is limited but not selective, the prediction is underreaction to signals.
8 Discussion

This is the first paper to explore the long-run implications of selective memory. We develop our analysis under the assumption that the agent is naïve about their selective memory.

There are several natural extensions for our work. One extension generalizes our assumption of i.i.d. signals to a Markov process. This extension would let us capture some other relevant biases; for example, the gambler’s fallacy (see Rabin and Vayanos [2010] and He [2021]) would arise if a signal is more memorable when it is different than the signal in the previous period.32 Or it might be much easier for agents to recall whether an experience happened at all than whether it happened five or six times; we could capture this by supposing that less frequent experiences are more likely to be recalled. Another generalization would be to memory functions with recency bias, such as \( m_{s',t}(s_\tau, a_\tau, y_\tau) = m_s(s_\tau, a_\tau, y_\tau) f(t - \tau) \) where \( f \) is a decreasing function. As with associative memory, when the outcomes are exogenous this bias only leads to slower learning, but when actions are endogenous it can prevent the agent from locking on to the optimal learning. We leave these extensions for future work.

A Appendix

For every \( h_t \in H \) let \( f(h_t) \in \Delta(S \times A \times Y) \) denote the empirical distribution over signals, actions, and outcomes in history \( h_t \), and for every nonempty recalled history \( r_t \) let

\[
\hat{f}(h_t, r_t)(s, a, y) = \frac{1}{|r_t|} \sum_{i=1}^{t} \mathbb{1}_{(s_i, a_i, y_i) = (s, a, y)} \mathbb{1}_{i \in r_t}
\]

denote the recalled empirical distribution in history \( h_t \) when the recalled periods are \( r_t \). For all \( \tau \geq t \) we write \( h_\tau \succeq h_t \) if the history \( h_\tau \) is consistent with \( h_t \), i.e. \( h_\tau = (h_t, (s_i, a_i, y_i)_{i=t+1}^{\tau}) \).

32At a technical level this extension can be achieved using the belief concentration result for misspecified agents with Markov models developed in Fudenberg, Lanzani, and Strack [2021b].
Observe that for all $C \subseteq \Theta$ equation (1) can be rewritten as

$$
\mu(C|h_{t_i}) = \frac{\int_{p \in C} \prod_{(s,a,y)} (p_{a,s}(y)) \sum_{i=1}^{t} 1_{(s_i,a_i,y_i) = (s,a,y)} \, dp}{\int_{p \in \Theta} \prod_{(s,a,y)} (p_{a,s}(y)) \sum_{i=1}^{t} 1_{(s_i,a_i,y_i) = (s,a,y)} \, dp}
= \frac{\int_{p \in C} \prod_{(s,a,y)} (p_{a,s}(y)) |r_t| \hat{f}(h_{t_i}, r_{i})(s,a,y) \, dp}{\int_{p \in \Theta} \prod_{(s,a,y)} (p_{a,s}(y)) |r_t| \hat{f}(h_{t_i}, r_{i})(s,a,y) \, dp}
= \frac{\int_{p \in C} \exp \left(|r_t| \sum_{(s,a,y)} \log(p_{a,s}(y)) \hat{f}(h_{t_i}, r_{i})(s,a,y) \right) \, dp}{\int_{p \in \Theta} \exp \left(|r_t| \sum_{(s,a,y)} \log(p_{a,s}(y)) \hat{f}(h_{t_i}, r_{i})(s,a,y) \right) \, dp}.
$$

(9)

Also, for every $\gamma \in \Delta(S \times A \times Y)$, and $p \in \Delta(Y)^{A \times S}$ let $D(\gamma||p) = - \sum_{(s,a,y)} \gamma(s,a,y) \log(p_{a,s}(y))$.

**Lemma A.1.** For all Borel measurable $C, C' \subseteq \Delta(Y)^{A \times S}$, $t \in \mathbb{N}$ $h_{t} \in H_t$ and $r_{t} \subseteq \{1, ..., t\}$ we have

$$
\frac{\mu(C|h_{t_i})}{1 - \mu(C'|h_{t_i})} \geq \frac{\mu(C)}{1 - \mu(C')} \exp \left(|r_t| \left[ \inf_{p \in C} D(\hat{f}(h_{t_i}, r_{i})||p) - \sup_{p \in \Theta \setminus C'} D(\hat{f}(h_{t_i}, r_{i})||p) \right] \right).
$$

**Proof.** Using (9) we have that

$$
\frac{\mu(C|h_{t_i})}{1 - \mu(C'|h_{t_i})} = \frac{\int_{p \in C} \exp \left(|r_t| \sum_{(s,a,y)} \log(p_{a,s}(y)) \hat{f}(h_{t_i}, r_{i})(s,a,y) \right) \, dp}{\int_{p \in \Theta \setminus C'} \exp \left(|r_t| \sum_{(s,a,y)} \log(p_{a,s}(y)) \hat{f}(h_{t_i}, r_{i})(s,a,y) \right) \, dp}
= \frac{\int_{p \in C} \exp \left(- |r_t| D(\hat{f}(h_{t_i}, r_{i})||p) \right) \, dp}{\int_{p \in \Theta \setminus C'} \exp \left(- |r_t| D(\hat{f}(h_{t_i}, r_{i})||p) \right) \, dp}
\geq \frac{\mu(C) \exp \left(- |r_t| \sup_{p \in C} D(\hat{f}(h_{t_i}, r_{i})||p) \right)}{1 - \mu(C')} \exp \left(- |r_t| \inf_{p \in \Theta \setminus C'} D(\hat{f}(h_{t_i}, r_{i})||p) \right).
$$

Let $KL_{\sigma}^{\Theta,m}(\sigma, \varepsilon) = \{p \in \Theta : \exists q \in KL_{\sigma}^{\Theta,m}(\sigma), ||p - q||_{\infty} \leq \varepsilon\}$.

**Lemma A.2.** For every $\sigma \in A^S$, $s \in S$, and $\varepsilon > 0$, $KL_{\sigma}^{\Theta,m}(\sigma)$ and $KL_{\sigma}^{\Theta,m}(\sigma, \varepsilon)$ are nonempty and compact.

**Proof.** By Assumption 1, the function

$$
p \mapsto \sum_{s' \in S} \zeta(s') \sum_{y \in Y} m_s(s', \sigma(s), y) p_{s'}^*(s', s, y) \log p_{\sigma(s')}(s', y)
$$
is finite-valued and continuous on the compact set $\Theta$. Therefore, $KL^\Theta_{s^m}(\sigma)$ is nonempty and compact by Theorem 2.43 in Aliprantis and Border [2013]. The result for $KL^\Theta_{s^m}(\sigma, \epsilon)$ is an immediate consequence given the continuity of $|| \cdot ||_\infty$. □

**Lemma A.3.** For any strategy $\sigma$ and sequence of actions $a^t$ let $\pi^{\sigma,a^t}$ be the map that prescribes action $a_t$ at period $t$ and action $\sigma(s_\tau)$ at all periods $\tau > t$. If, for every $t$, every sequence of actions $a^t$, and every optimal policy $\tilde{\pi}$, $P_{\pi^{\sigma,a^t}}[\sigma(s_\tau) = \tilde{\pi}(h_\tau(r_\tau), s_\tau) \text{ for all } \tau \geq t | t] = 0$ then $\sigma$ is not a limit strategy.

**Proof.** Fix an arbitrary optimal policy $\tilde{\pi}$ and a history $(s^t, a^t, y^t)$ with $P_{\tilde{\pi}}(s^t, a^t, y^t) > 0$. Let $\tau = \min \{ t' > t : \sigma(s_{t'}) \neq \tilde{\pi}((s^{t'}, a^{t'}, y^{t'}(r_{t'}))(s_{t'+1}) \} \}$ be the first time after $(s^t, a^t, y^t)$ when $\tilde{\pi}$ does not prescribe $\sigma$. Note that since $\tilde{\pi}(a^{t'}, y^{t'}(s_{t'})(s_{t'+1}) = \pi^{b,a^t}(s_{t'+1}, a^{t'}, y^{t'})$ for all $t' \in [t, \tau - 1]$, the agent’s belief until period $\tau$ is the same under $\pi^{b,a^t}$ and $\tilde{\pi}$. As $P_{\tilde{\pi}}(s^{t'+1}, a^{t'}, y^{t'}, r_{t'}) > 0$ implies $P_{\pi^{b,a^t}}(s^{t'+1}, a^{t'}, y^{t'}, r_{t'}) > 0$, the probability that the agent uses strategy $\sigma$ forever (i.e. $\tau = \infty$) after history $(s^t, a^t, y^t)$ equals 0 by the assumption of the lemma. So, under every arbitrary optimal policy, after every history where $\sigma$ is played a strategy different from $\sigma$ is played with probability 1, so $\sigma$ is not a limiting strategy. □

**Lemma A.4.** If $\sigma$ is not a selective equilibrium there are $s' \in S$ and $\epsilon, C \in \mathbb{R}_{++}$ such that

$$\frac{\nu(KL^\Theta_{s^m}(\sigma, \epsilon))}{1 - \nu(KL^\Theta_{s^m}(\sigma, \epsilon))} > C \implies \sigma(s') \notin BR(s', \nu).$$

**Proof.** Since $\sigma$ is not a selective memory equilibrium, there is an $s' \in S$ such that $\sigma(s') \notin BR(s', \Delta(KL^\Theta_{s^m}(\sigma)))$. By the upper-hemicontinuity of the best reply map $BR(s, \cdot)$ and the compactness of $KL^\Theta_{s^m}(\sigma, \epsilon)$ established in Lemma A.2, this implies that there are $\epsilon, C \in \mathbb{R}_{++}$ such that if $\frac{\nu(KL^\Theta_{s^m}(\sigma, \epsilon))}{1 - \nu(KL^\Theta_{s^m}(\sigma, \epsilon))} > C$ then $\sigma(s') \notin BR(s', \nu)$. □

For every $\sigma \in A^S$, and $s' \in S$ let

$$M_\sigma(s') = \min_{y \in Y} \left( -\sum_{s \in S} \zeta(s) \sum_{y \in Y} m_{s'}(s, \sigma(s), y)p^*_{\sigma(s), s}(y) \log p_{\sigma(s), s}(y) \right)$$

denote the divergence of the weighted memory minimizers under strategy $\sigma$.

**Proof of Theorem 1.** Suppose towards a contradiction that $\sigma$ is a limit strategy under the optimal policy $\pi$, but not a selective memory equilibrium. Then by Lemma A.4 there are $s' \in S$ and $\epsilon, C \in \mathbb{R}_{++}$ such that if $\frac{\nu(KL^\Theta_{s^m}(\sigma, \epsilon))}{1 - \nu(KL^\Theta_{s^m}(\sigma, \epsilon))} > C$ then $\sigma(s') \notin BR(s', \nu)$. Let $h_t$ be a
history with positive probability. We show that if the agent plays $\sigma$ at every period after $h_t$ almost surely the belief $\mu_t$ reaches a region where no optimal policy prescribes $\sigma$ after signal $s'$. By Lemma A.3 this is enough to obtain the desired conclusion.

By the strong law of large numbers, the frequency $f(h_t)(s, a, y)$ converges a.s. on $h_t$ to

$$\lim_{\tau \to \infty} f(h_\tau)(s, a, y) = \begin{cases} \zeta(s)p_{a,s}^*(y) & \text{if } a = \sigma(s) \\ 0 & \text{otherwise} \end{cases}.$$ 

Define $\tilde{p}(\sigma, s') \in \Delta(S \times A \times Y)$ to be the induced distribution over remembered experiences

$$\tilde{p}(\sigma, s')(s, a, y) = \begin{cases} \frac{\zeta(s)m_{\sigma}(s, \sigma(s), y)p_{\sigma(s), y}(y)}{\sum_{\sigma', y} \zeta(s)m_{\sigma}(s, \sigma(s), y)p_{\sigma(s), y}(y)} & \text{if } a = \sigma(s) \\ 0 & \text{otherwise} \end{cases}.$$ 

Since for every two periods $\tau' > \tau$ and $B \subseteq \{1, \ldots, \tau'\}$ the probability of recalling $B$ at time $\tau'$ conditional on the actual history $h_{\tau'}$ is independent of the history $r_\tau$ recalled at period $\tau$, i.e. $\mathbb{P}[r_{\tau'} = B|h_{\tau'}] = \mathbb{P}[r_{\tau'} = B|h_{\tau}, r_\tau]$, by the second Borel-Cantelli lemma, for every $\varepsilon \in \mathbb{R}_{++}$, $s' \in S$ and $k \in \mathbb{N}_{++}$ almost surely there is a $\tau > t$ such that $s_\tau = s'$, $|r_\tau| > k$ and $||\hat{f}(h_\tau, r_\tau) - \tilde{p}(\sigma, s')||_{\infty} < \varepsilon$.

Let $\varepsilon', \kappa \in \mathbb{R}_{++}$ be such that

$$\kappa > \inf_{\{p' \notin KL_{\hat{f}_t}^{\Theta, m}(\sigma, \varepsilon)\}} \left( -\sum_{s \in S} \zeta(s) \sum_{y \in Y} p_{\sigma(s), s}^*(y) m_{\sigma}(s, \sigma(s), y) \log p_\sigma(s)(y) \right) - M_\sigma(s')$$

and

$$\kappa < \sup_{\{p' \in KL_{\hat{f}_t}^{\Theta, m}(\sigma, \varepsilon')\}} \left( -\sum_{s \in S} \zeta(s) \sum_{y \in Y} p_{\sigma(s), s}^*(y) m_{\sigma}(s, \sigma(s), y) \log p_\sigma(s)(y) \right) - M_\sigma(s').$$

So, by Lemma A.1

$$\frac{\mu(KL_{\hat{f}_t}^{\Theta, m}(\sigma, \varepsilon)|h_\tau(r_\tau))}{1 - \mu(KL_{\hat{f}_t}^{\Theta, m}(\sigma, \varepsilon)|h_\tau(r_\tau))} \geq \frac{\mu(KL_{\hat{f}_t}^{\Theta, m}(\sigma, \varepsilon))}{1 - \mu(KL_{\hat{f}_t}^{\Theta, m}(\sigma, \varepsilon'))} \exp \left( |r_\tau| \left( -\inf_{\{p \notin KL_{\hat{f}_t}^{\Theta, m}(\sigma, \varepsilon)\}} D(\hat{f}(h_\tau, r_\tau) \| p) - \sup_{\{p \in KL_{\hat{f}_t}^{\Theta, m}(\sigma, \varepsilon')\}} D(\hat{f}(h_\tau, r_\tau) \| p) \right) \right).$$

The last expression goes to $+\infty$ as $|r_\tau| \to \infty$, and by the definitions of $\kappa$ and $\varepsilon'$
\[ - \lim_{t \to \infty} \sup_{\{p \in KL^\Theta, m'(\sigma, \varepsilon)\}} - \sum_{(s, a, y)} \hat{f}(h_\tau, r_\tau) (s, a, y) \log(p_{a,s}(y)) \\
+ \lim_{t \to \infty} \inf_{\{p \in KL^\Theta, m'(\sigma, \varepsilon)\}} - \sum_{(s, a, y)} \hat{f}(h_\tau, r_\tau) (s, a, y) \log(p_{a,s}(y)) \\
= - \sup_{\{p \in KL^\Theta, m'(\sigma, \varepsilon)\}} \sum_{s \in S} \sum_{y \in Y} m'_s(s, \sigma, y) p^*_\sigma(s, y) \log p'_\sigma(s, y) \\
+ \inf_{\{p \notin KL^\Theta, m'(\sigma, \varepsilon)\}} \sum_{s \in S} \sum_{y \in Y} m'_s(s, \sigma, y) p^*_\sigma(s, y) \log p_{\sigma(s), s}(y) > \frac{K}{2} > 0. \square \\

**Proof of Proposition 1.** We show that only data generating processes \( p \) for which \( p_{\sigma(s), s} = p^*_{\sigma(s), s} \) are memory-weighted KL-minimizers after signal \( s \).

Suppose that \( p \) is such that for some \( \hat{s} \in S \), \( p_{\sigma(\hat{s}), \hat{s}} \neq p^*_{\sigma(\hat{s}), \hat{s}} \). By the Gibbs inequality, we have

\[ \sum_{y \in Y} p^*_{\sigma(s), s}(y) \log p_{\sigma(s), s}(y) \geq \sum_{y \in Y} p^*_{\sigma(s), s}(y) \log p_{\sigma(s), s}(y) \]

for all \( s \in S \), with a strict inequality for \( s = \hat{s} \). This, together with \( d(\hat{s}, \hat{s}) = 0 \) and \( f(0) > 0 \), implies that

\[ \sum_{s \in S} \sum_{y \in Y} m'_s(s, \sigma, y) p^*_{\sigma(s), s}(y) \log p_{\sigma(s), s}(y) = \sum_{s \in S} \sum_{y \in Y} m'_s(s, \sigma, y) p^*_{\sigma(s), s}(y) \log p_{\sigma(s), s}(y) \\
> \sum_{s \in S} \sum_{y \in Y} m'_s(s, \sigma, y) p^*_{\sigma(s), s}(y) \log p^*_{\sigma(s), s}(y) \\
= \sum_{s \in S} \sum_{y \in Y} m'_s(s, \sigma, y) p^*_{\sigma(s), s}(y) \log p^*_{\sigma(s), s}(y) \\
\]

proving that \( p \notin KL^\Theta, m'(\sigma) \). \( \square \)

**Proof of Proposition 2.** We first derive the long-run belief for \( \tilde{m} \in \{m, m'\} \). Because the memory function \( \tilde{m} \) and the probability distribution over outcomes \( p^* \) are independent of the agent’s action we suppress the dependence of \( p \) and \( \tilde{m} \) on \( a \), so that for every \( \sigma \),

\[ KL^\Theta, \tilde{m}(\sigma) = \arg\min_{p \in \Delta(Y)} \sum_{y \in Y} \log(p(y)) \tilde{m}(y) p^*(y). \]

Taking first-order conditions of the associated Lagrangian yields that there is a unique \( p \in KL^\Theta, \tilde{m}(\sigma) \) given by

\[ p^\tilde{m}(y) = \frac{\tilde{m}(y) p^*(y)}{\sum_{z \in Y} \tilde{m}(z) p^*(z)}. \]
Thus \( p^m(y) = w(y)p^m(y) \) where \( w(y) = \frac{f(y)}{\sum_{z \in Y} f(z)p^*(z)} \) is non-decreasing, so the function
\[
z \mapsto \sum_{x \leq z} (p^m(x) - p^m(x)) = \sum_{x \leq z} p^m(x)(w(x) - 1)
\]
is quasi-convex. As it equals 0 for \( z < \min Y \) and for \( z \geq \max Y \), it is negative for \( s \in [\min Y, \max Y] \), so \( p^m \) dominates \( p^m \) in first-order stochastic dominance. Every long-run action must be optimal given \( p^m \) by Theorem 1, so the action taken by the agent must be higher than the objectively optimal action. The second part of the statement is just the special case in which \( m(y) = 1 \) for all \( y \in Y \).

**Proof of Proposition 3.** Because \( (y_1, y_2) \) are subjectively independent conditional on the value of \( p_1 \), the learning problem decouples across the two dimensions. By Proposition 2 the long-run belief about \( p_1 \) is higher than the true distribution \( p_1^* \). The probability with which an outcome is remembered is independent of the second component, so the agent learns \( p_2^* \). They infer \( q \) to be
\[
q(y_2) = \frac{p_2^*(y_2) - \alpha p_1(y_2)}{1 - \alpha}.
\]
Thus \( q - q^* = \frac{\alpha}{1 - \alpha} (p_1^* - p_1) \), and as \( p \) is greater than \( p^* \) in first-order stochastic dominance, it follows that \( q \) is lower than \( q^* \) in first-order stochastic dominance. \( \square \)

**Proof of Proposition 4.** If \( a = 1 \) is not a selective memory equilibrium with perfect memory, then \( \sum_{y \in Y} u(y)p_1^*(y) < u(y_0) \). If \( a = 1 \) is a selective memory equilibrium with selective memory \( m(y) = h(|y - E_{p_1^*}(y)|) \), then
\[
u(y_0) \leq \frac{\sum_{y \in Y} p_1^*(y)h(|y - E_{p_1^*}(y)|)u(y)}{\sum_{y \in Y} p_1^*(y)h(|y - E_{p_1^*}(y)|)}.
\]
We prove that this cannot be the case by showing that the distribution
\[
\hat{p}_1(y) := \frac{p_1^*(y)h(|y - E_{p_1^*}(y)|)u(y)}{\sum_{z \in Y} p_1^*(z)h(|z - E_{p_1^*}(y)|)}
\]
is second-order stochastically dominated by \( p^* \). To see this, observe that as \( p_1^* \) is symmetric around \( E_{p_1^*}(y) \) and \( y \mapsto h(|y - E_{p_1^*}(y)|) \) is symmetric around \( r \) it follows that \( \hat{p}_1 \) is symmetric around \( r \). As \( h \) is increasing it follows that \( \hat{p}_1 - p_1^* \) changes its sign from positive to negative and back to positive so \( \sum_{y < z} p_1^*(y) \) and \( \sum_{y < z} \hat{p}_1(y) \) cross only once, at \( r \). This implies that \( \hat{p}_1 \) is a mean-preserving spread of \( p_1^* \), so the risky action is perceived as more risky. As \( u \) is
concave, this implies that
\[
\sum_{y \in Y} u(y)p^*(y) \geq \frac{\sum_{y \in Y} p^*_1(y)h(|y - r|)}{\sum_{y \in Y} p^*_1(y)h(|y - r|)}.
\]
and the risky action cannot be a selective memory equilibrium.

**Proof of Proposition 5.** Suppose towards a contradiction that \( \sigma \) is a limit strategy under the optimal policy \( \pi \), but not a selective memory equilibrium. Then by Lemma A.4 there are \( s' \in S \) and \( c, C \in \mathbb{R}_+ \) such that if
\[
\frac{\nu(KL_{s'}^{\theta,m}(\sigma,c))}{1 - \nu(KL_{s'}^{\theta,m}(\sigma,c))} > C
\]
then \( \sigma(s') \notin BR(s', \nu) \). Let \( h_t \) be a history with positive probability. We show that if the agent plays the strategy \( \tilde{\pi} \) that coincides with \( \pi \) until \( h_t \) and prescribes \( \sigma \) at every period after \( h_t \), then almost surely \( \mu_t \) reaches a region where no optimal policy prescribes \( \sigma \) after signal \( s' \). By Lemma A.3 this is enough to obtain the desired conclusion.

Under strategy \( \tilde{\pi} \), by the Strong Law of Large Numbers, we have
\[
\lim_{\tau \to \infty} f(h_\tau)(s, a, y) = \begin{cases} 
\zeta(s)p^*_{a,s}(y) & \text{if } a = \sigma(s) \\
0 & \text{otherwise}
\end{cases}.
\] (10)
amost surely on the cylinder \( h_t \). Now we express the posterior as a function of the observed frequencies, and show that it concentrates on the memory-weighted KL minimizers, so that the result follows from the upperhemicontinuity of \( BR \).

Let
\[
\kappa > \inf_{\{\rho \neq KL_{s'}^{\theta,m}(\sigma,c)\}} \left( -\sum_{s \in S} \zeta(s) \sum_{y \in Y} p^*_{\sigma(s),s}(y)m_{s'}(s, \sigma(s), y) \log p'_{\sigma(s),s}(y) \right) - M_{\sigma}
\]
and choose \( c' < c \) such that
\[
\kappa/2 < \sup_{\{\rho' \in KL_{s'}^{\theta,m}(\sigma,c')\}} \left( -\sum_{s \in S} \zeta(s) \sum_{y \in Y} p^*_{\sigma(s),s}(y)m_{s'}(s, \sigma(s), y) \log p'_{\sigma(s),s}(y) \right) - M_{\sigma}.
\]
By equation (10) and the definition of \( \kappa \) and \( c' \) almost surely on the cylinder \( h_t \) we have

\[
C = \lim_{t \to \infty} \sup_{\{p' \in KL_{s'}^{\Theta,m}(\sigma,c')\}} - \sum_{(s,a,y)} f(h_t(s,a,y))m(s,a,y) \log(p_{a,s}(y)) \\
- \lim_{t \to \infty} \inf_{\{p' \notin KL_{s'}^{\Theta,m}(\sigma,c)\}} - \sum_{(s,a,y)} f(h_t(s,a,y))m(s,a,y) \log(p_{a,s}(y)) \\
= \sup_{\{p' \in KL_{s'}^{\Theta,m}(\sigma,c')\}} \sum_{s \in S} \sum_{y \in Y} m_{s'}(s,\sigma(s),y)p_{\sigma(s),s}(y) \log p'_{\sigma(s),s}(y) \\
- \inf_{\{p' \notin KL_{s'}^{\Theta,m}(\sigma,c)\}} \sum_{s \in S} \sum_{y \in Y} m_{s'}(s,\sigma(s),y)p_{\sigma(s),s}(y) \log p'_{\sigma(s),s}(y) > \kappa/2 > 0.
\]

By Lemma A.1 we have

\[
\frac{\mu(KL_{s'}^{\Theta,m}(\sigma,c)|(h_t))}{1 - \mu(KL_{s'}^{\Theta,m}(\sigma,c)|(h_t))} \\
\geq \frac{\mu\left(KL_{s'}^{\Theta,m}(\sigma,c')\right) \exp\left(\sup_{\{p' \in KL_{s'}^{\Theta,m}(\sigma,c')\}} - \sum_{(s,a,y)} t f(h_t(s,a,y))m(s,a,y) \log(p_{a,s}(y))\right)}{\exp\left(\inf_{\{p' \notin KL_{s'}^{\Theta,m}(\sigma,c)\}} - \sum_{(s,a,y)} t f(h_t(s,a,y))m(s,a,y) \log(p_{a,s}(y))\right)} \\
= \frac{\mu\left(KL_{s'}^{\Theta,m}(\sigma,c)\right)}{\exp(tC)},
\]

which goes to \( \infty \) as \( C > 0 \).

\[\Box\]

**Proof of Proposition 6.** To prove part (1), let \( \sigma \in A^S \) be a uniformly strict Berk-Nash equilibrium, and let \( p' \) be an arbitrarily element of \( KL_{s'}^{\Theta,1} \). Since \( \sigma \) is a uniformly strict Berk-Nash equilibrium, for all \( s \in S \) we have \( \{\sigma(s)\} = BR(s,\delta_{p'}) \). Moreover, by assumption (1), \( p^*_{\sigma(s),s}(y) = 0 \) implies \( p'_{\sigma(s),s}(y) = 0 \), so\(^{33}\)

\[M := \max_{(s,y) \in S \times Y} \frac{p'_{\sigma(s),s}(y)}{p^*_{\sigma(s),s}(y)} < \infty.\]

Define \( \tilde{m} \) by \( \tilde{m}_{s'}(s,a,y) = \frac{p'_{a,s}(y)}{M p^*_{a,s}(y)} \). Then for an agent with full support prior and memory function \( \tilde{m} \) the memory-weighted KL minimizers for strategy \( \sigma \) after signal \( s' \) are the elements

\[^{33}\text{We use the convention that } 0/0 = 0.\]
argmin \sum_{s \in S} \zeta(s) \sum_{y \in Y} \tilde{m}_s(s, \sigma(s), y)p^*_\sigma(s),s(y) \log p\sigma(s),s(y) \\
= \argmin \sum_{s \in S} \zeta(s) \sum_{y \in Y} \frac{p'_\sigma(s),s(y)}{M} \log p\sigma(s),s(y) \\
= \argmin \sum_{s \in S} \zeta(s) \sum_{y \in Y} \frac{p'_\sigma(s),s(y)}{M} \log p\sigma(s),s(y) \\

Thus \( p' \) minimizes the KL divergence for all \( s' \in S \), so \( \sigma \) is a selective memory equilibrium.

Part (2), the converse direction, is trivial: simply take \( \Theta' \) to be a singleton \( p \) with \( p_{a,s}(y) = \hat{p}_{a,s}(y) \) for some \( \hat{p}' \in KL_{\Theta}^{\sigma}(\sigma) \).

Proof of Remark. To prove the statements we give three examples with a null signal space \( S \).

1. Suppose that \( Y = \{-1, 1\} \), the probability of 1 is 0.5 regardless of \( a \), and that the agent does not have selective memory, but is misspecified, with \([0, .2] \cup [.8, 1]\) as the support of the prior beliefs over the probability of 1 under both actions. Then both .2 and .8 are KL minimizers, which cannot arise from selective memory with a full support prior. This is immediate if \( m = 0 \) for some experience, and follows from full support and the strict convexity of the memory-weighted KL divergence if \( m \gg 0 \).

2. Suppose that \( Y = \{-1, 0, 1\} \), the probability over outcomes is uniform regardless of \( a \), with \( \Theta = \{(1/3, 1/3, 1/3), (1/3, 1/6, 1/2)\} \) and \( m(a, y) = 1_{y=-1} \). Then both \((1/3, 1/3, 1/3)\) and \((1/3, 1/6, 1/2)\) are memory-weighted KL minimizers, but they can never be both KL minimizers with perfect memory.

3. Suppose \( Y = \{-1, 1\} = A \) and \( u(a, y) = ya \). Then if \( m(-1, a) = 0 < m(1, a) \) for all \( a \in A \), and the agent has a full-support prior over the action-independent outcome distributions, the only selective memory equilibrium is \( a = 1 \) even if the true probability of 1 under both actions is less than 1/2 so that the objectively optimal action is \(-1\).

Proof of Proposition 8. We first derive the long-run belief for a given subjective memory function \( \hat{m} \). Because the memory function \( m \) and the probability distribution over outcomes \( p^* \) are independent of the agent’s action we suppress the dependence of \( p \) and \( m \) on \( a \), so that for every \( \sigma \),

\[
KL^{\Theta,m}(\sigma) = \argmin_{p \in \Delta(Y)} \sum_{y \in Y} \log(p(y)\hat{m}(y))m(y)p^*(y).
\]

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Taking first-order conditions of the associated Lagrangian shows there is a unique element \( p \) of \( KL^{\Theta,m,\hat{m}}(\sigma) \), given by
\[
p(y) = \frac{\frac{m(y)}{\hat{m}(y)}p^*(y)}{\sum_{z \in Y} \frac{m(z)}{\hat{m}(z)}p^*(z)}.
\]
Thus the long-run beliefs under the subjective memory function \( m \) and subjective memory function \( \hat{m} \) are the same as those of a fully naïve agent with memory function \( \tilde{m}(y) = \frac{m(y)}{\hat{m}(y)} \) who is not aware of their selective memory. Note that for \( \hat{m}_\alpha(y) = \alpha + (1 - \alpha)m(y) \) and \( \alpha > \alpha' \),
\[
\frac{\frac{m(y)}{\hat{m}_\alpha(y)}}{\frac{m(y)}{\hat{m}_{\alpha'}(y)}} = \frac{\hat{m}_{\alpha'}(y)}{\hat{m}_\alpha(y)} = \frac{\alpha + (1 - \alpha)m(y)}{\alpha' + (1 - \alpha')m(y)}
\]
is increasing in \( m(y) \) and hence in \( y \). This lets us apply Proposition 2 to conclude that the long-run belief under the subjective memory function \( \hat{m}_\alpha \) will be higher in FOSD than that under the subjective memory function \( \hat{m}_{\alpha'} \).

The proof of Theorem 2 builds on the techniques of Esponda, Pouzo, and Yamamoto [2021]. The key complication is that even when the action process converges beliefs remain stochastic, so we lack a counterpart to their Theorem 1. Instead, we prove a lemma on the convergence of the belief distribution that, paired with our assumption of expected utility maximization, lets us mimic the other steps of their proof. Let \((d_t)_{t \in \mathbb{N}}\) be a sequence of empirical joint distributions over actions and outcomes.

**Lemma A.5.** If for some \( \alpha \in \Delta(A) \), \( d_t(a,y) \) converges to \( \alpha(p^*_a(y)) \) for all \((a,y) \in A \times Y\), then the distribution \( \mu_t \) of time \( t \) beliefs weakly converges to \( F_\alpha \in \Delta(\Delta(\Theta)) \), and \( F \) is continuous.

**Proof.** Given a length \( l \) history \( (a_i,y_i)_{i=1}^l \), let \( n_{a,y}(l) = \sum_{i=1}^l \mathbb{1}_{a,y}(a_i,y_i) \) be the number of times the action-outcome pair \((a,y)\) realized. Then the recalled history at time \( t \) is distributed as a product of multinomial distributions:
\[
P[\hat{h}_t = (a_i,y_i)_{i=1}^n] = \prod_{(a,y) \in A \times Y} \binom{d_t(a,y)t}{n_{a,y} \hat{h}_t} (m_t(a,y))^{n_{a,y} \hat{h}_t} (1 - m_t(a,y))^{d_t(a,y)t - n_{a,y} \hat{h}_t}.
\]

By the Poisson limit theorem, the probability of recalling \((a,y)\) \( c \) times converges to
\[
\lim_{t \to \infty} \left( \frac{d_t(a,y)t}{c} \right)^c (1 - m_t(a,y))^{d_t(a,y)t - c} = e^{-\lambda_{a,y} \frac{c}{c!}}
\]

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where \( \lambda_{a,y} = \lim_{t \to \infty} d_t(a, y) m_t(a, y) = \alpha(a) p^*_a(y) k m(a, y) \). Thus the number of times this experience is recalled converges to a random variable \( N_{a,y} \) that is Poisson distributed with parameter \( \lambda_{a,y} \), and these random variables are independent across \((a, y)\) pairs. Furthermore, for every \( c \in \mathbb{N} \) the probability that \( N_{a,y} = c \) depends continuously on \( \alpha \).

Moreover, let \( f : \Delta(\Theta) \to \mathbb{R} \) be a continuous and bounded function, and \((\alpha_n)_{n \in \mathbb{N}} \in \Delta(A)\) be a sequence converging to \( \alpha^* \). Let \( \epsilon \in \mathbb{R}_{++} \). Since all the \( N_{a,y} \) have Poisson distributions, there exists a \( K \in \mathbb{N} \) such that \( \mathbb{P}_{\alpha^*}[\max_{a,y \in A \times Y} N_{a,y} > K] < \epsilon \). Let \( M \in \mathbb{N} \) be such that \( \mathbb{P}_{\alpha_n}[\max_{a,y \in A \times Y} N_{a,y} > M] < 2\epsilon \) and \( |\mathbb{P}_{\alpha_n}[N_{a,y} = c] - \mathbb{P}_{\alpha_n}[N_{a,y} = c]| < \epsilon \) for all \( a, y \in A \times Y \), for all \( c \leq K \) and \( n > M \). Then for all \( n > M \), we have

\[
\left| \int_{\Delta(\Theta)} f(\nu) dF_{\alpha_n} - \int_{\Delta(\Theta)} f(\nu) dF_{\alpha} \right| < \max_{\nu \in \Delta(\Theta)} f(\nu)(K|A \times Y| + 1)\epsilon \tag{11}
\]

so \( F_{\alpha_n} \) weakly converges to \( F_{\alpha} \).

To prove our results on the convergence of the time average of actions we start by writing the empirical frequency of actions in a recursive way. Indeed, we have \( \alpha_{t+1}(h_{t+1})(a) = \alpha_t(h_t)(a) + \frac{1}{t+1}(\delta_{a_{t+1}}(a) - \alpha_t(h_t)(a)) \). By adding and subtracting the expectation of \( \delta_{a_{t+1}}(a) \), we get

\[
\alpha_{t+1}(h_{t+1})(a) = \alpha_t(h_t)(a) + \frac{1}{t+1} \left( \mathbb{E}[\delta_{a_{t+1}}(a)|\mu_{t+1}] - \alpha_t(h_t)(a) \right) \\
+ \frac{1}{t+1} \left( \delta_{a_{t+1}}(a) - \mathbb{E}[\delta_{a_{t+1}}(a)|\mu_{t+1}] \right) \\
= \alpha_t(h_t)(a) + \frac{1}{t+1} \left( \mathbb{E}[\delta_{a_{t+1}}(a)|\mu_{t+1}] - \alpha_t(h_t)(a) \right) \tag{12}
\]

where the equality follows from the fact that the agent uses the stationary pure policy \( \pi \). To deal with the discontinuity of \( \mathbb{E}[\delta_{a_{t+1}}(a)|\mu_{t+1}] \) as a function of \( \mu_{t+1} \), we see equation (12) as a particular case of the differential inclusion

\[
\alpha_{t+1}(h_{t+1})(a) \in \left\{ \alpha_t(h_t)(a) + \frac{1}{t+1} (\delta_{a'}(a) - \alpha_t(h_t)(a)) : a' \in BR(\mu_{t+1}) \right\}.
\]

Set \( \tau_0 = 0 \) and \( \tau_t = \sum_{i=1}^{t} \frac{1}{i} \) for all \( t \in \mathbb{N} \). The continuous-time interpolation of \( \alpha_t \) is the function \( w : \mathbb{R}_+ \to \Delta(A) \)

\[
w(\tau_t + c) = \alpha_t + c \frac{\alpha_{t+1} - \alpha_t}{\tau_{t+1} - \tau_t}, \quad c \in \left[0, \frac{1}{t+1}\right]. \tag{13}
\]
Proof of Theorem 2. Let $\Pi_o$ be the set of selections from the correspondence $BR$, and let $\chi_\alpha$ be the distribution of actions induced by the distribution of beliefs $F_\alpha$ and some best response selection $\rho \in \Pi_o$: 

$$\chi_\alpha = \{ \alpha' \in \Delta(A) : \exists \rho \in \Pi_o \text{ such that for all } a \in A, \alpha'(a) = F_\alpha(\{\mu : a = \rho(\mu)\}) \}.$$

We will use the theory of stochastic approximation for differential inclusions (Benaim, Hofbauer, and Sorin [2005]) to show that (13) can be approximated by a solution to

$$\hat{\alpha}_t \in \chi_{\alpha_t} - \alpha_t. \quad (14)$$

A solution to the differential inclusion (14) with initial point $x^* \in \Delta(A)$ is a mapping $x : \mathbb{R}_+ \to \Delta(A)$ that is absolutely continuous over compact intervals such that $x(0) = x^*$ and (14) is satisfied for almost every $t$. Since $F$ by Lemma A.5 is upper hemicontinuous, a solution exists. Let $X_{T^*}^T$ be the set of solutions to (14) over $[0,T]$ with initial conditions $x^*$.

We next establish that the continuous-time interpolation of $\alpha$ defined in (13) can in the long run be approximated arbitrarily well by a solution to (14). By definition, $w$ is Lipschitz continuous of order 1 (to see this observe that $||\alpha_{t+1} - \alpha_{t+1}||_x < 1/(t+1)$), and therefore absolutely continuous, and $\alpha_t$ is uniformly bounded because it takes values in $\Delta(A)$. Thus by Proposition 1.3 and Theorem 4.2 in Benaim, Hofbauer, and Sorin [2005] with their $U_t$ identically equal to 0 and $\delta_t = d_\rho(F_\alpha, F_{\alpha_t})$ that is guaranteed to converge to 0 by the assumed convergence of $\alpha_t$ and Lemma A.5,

$$\lim_{t \to \infty} \inf_{\hat{\alpha} \in X_{T_t}^T} \sup_{0 \leq s \leq T} ||w(t + s) - \hat{\alpha}(s)|| = 0 \ \mathbb{P}_x\text{-almost surely for all } T \in \mathbb{N}. \quad (15)$$

If $\alpha \in \Delta(A)$ is not a stochastic memory equilibrium, there is $a \in A$ with $\alpha(a) > \chi_\alpha(a)$. Let $K = \alpha(a) - \chi_\alpha(a)$. By Lemma A.5, there exists $\varepsilon \in \mathbb{R}_{++}$ such that for all $\alpha' \in B_\varepsilon(\alpha)$, $\chi_{\alpha'}(a) < D_\alpha(a) + K/4$ and $\alpha'(a) > D_\alpha(a) + K/4$. Therefore, for every initial condition $\alpha^* \in B_\varepsilon(\alpha)$ and every solution in $X_{T_t}^T$, $\alpha(a)$ decreases at rate at least $K/2$ until it leaves $B_\varepsilon(\alpha)$. So there exists $T \in N$ such that for every initial condition $\alpha^* \in B_\varepsilon(\alpha)$ and every solution in $X_{T_t}^T$, the differential equation leaves $B_\varepsilon(\alpha)$ by time $T$.

We conclude the proof by combining an argument similar to Proposition 1 of Esponda, Pouzo, and Yamamoto [2021] with equation (15) to rule out convergence to a non-equilibrium point. Let $h$ be a sample path on which the convergence of equation (15) happens. We will prove that $\alpha_t$ does not converge to $\alpha$ on this sample path. Since this set has probability 1,
α_t can only converge to stochastic memory equilibria.

Let ˆT ∈ N be such that on that history

\[ \inf_{\alpha \in \mathcal{X}_{w(\hat{T})}} \sup_{0 \leq s \leq \hat{T}} ||w(\hat{T} + s) - \tilde{\alpha}|| \leq \varepsilon/2. \]  

(16)

If there is no t > ˆT such that w_t ∈ B_{\varepsilon/2}(\alpha), α_t does not converge to α. But if w_t ∈ B_{\varepsilon/2}(\alpha) for some t > ˆT, then the differential equation leaves B_{\varepsilon}(\alpha) by time T + t, and by (16), w leaves B_{\varepsilon}(\alpha), so α_t does not stay in B_{\varepsilon}(\alpha) forever and it cannot converge to α. This proves Theorem 2.

Proof of Lemma 1. Call H' the effective set of histories that can be reached with positive probability starting from the empty history. To show that η_α has a unique invariant distribution we will show it is irreducible on H' and that all states are positive recurrent.

To see that the chain is irreducible, take an arbitrary recalled history h = (a_i, y_i)_{i=1}^n ∈ H'. At every recalled history h, the probability of a transition to the empty history is bounded from below by:

\[ Q := \prod_{(a,y) \in A \times Y} \exp(-\alpha(a)p^*_a(y)k[m(a,y) + r]) > 0, \]

and we immediately see that the Markov chain is irreducible on the set of histories that can be reached with positive probability starting from the empty history. Moreover, for any h' ∈ H' there is τ ∈ N such that the probability of a simple path of length τ from the empty history to h' is some M > 0. Therefore the expected time of return to h' is bounded from above by \( \tau + \sum_{i=0}^{\infty} (1 - QM)^i \leq \infty \), so h is positive recurrent.

To prove Theorem 3 we extend Doeblin’s Theorem\(^{34}\) to non-homogeneous Markov chains whose entries in one column are uniformly bounded away from 0.\(^{35}\) Recall that an element of \( \mathbb{R}^{N \times N} \) is (row) stochastic if and only if each entry is non-negative and the entries of each row sum up to 1, that is, \( r_{ij} \geq 0 \) and \( \sum_{j \in N} r_{ij} = 1 \) for all \( j \in N \). For every \( R \in \mathbb{R}^{N \times N} \) let

\[ \|R\|_{TV} = \sup_{i \in N} \sum_{j \in N} r_{ij}. \]

Given \( \varepsilon > 0 \), let \( T_\varepsilon \) be the set of stochastic matrices \( R \) for which there exists \( l \in N \) such

\[^{34}\text{Doeblin’s theorem says that if all entries of a stochastic matrix } R \text{ are at least } \varepsilon, \text{ then the chain converges to its invariant distribution } \pi \text{ at rate at least } 1 - \varepsilon: \|R^n \mu - \pi\|_{TV} \leq (2(1 - \varepsilon)^n) \|\mu\|_{TV}.\]

\[^{35}\text{Cerreia-Vioglio, Corrao, and Lanzani [2021] proves a related extension for finite-state chains.}\]
that $r_{ij} \geq \varepsilon$ for all $j \in \mathbb{N}$. Define $V_0 = \{x \in \mathbb{R}^\mathbb{N} : \sum_i x_i = 0\}$. The next lemma follows immediately from the opening arguments in the proof of Theorem 2.2.1 in Stroock [2013] by setting $n = 1$.\footnotemark

**Lemma A.6.** If $R \in \mathcal{T}_\varepsilon$ and $x \in V_0$, then $xR \in V_0$ and $\|xR\|_{TV} \leq (1 - \varepsilon) \|x\|_{TV}$.

**Lemma A.7.** If $\{R_t\}_{t=1}^m \subseteq \mathcal{T}_\varepsilon$, then for each $x \in V_0$

$$x \left( \prod_{t=1}^m R_t \right) \in V_0 \quad \text{and} \quad \|x \left( \prod_{t=1}^m R_t \right)\|_{TV} \leq (1 - \varepsilon)^m \|x\|_{TV}. \quad (17)$$

We proceed by induction.

**Inductive Step.** The statement follows from Lemma A.6.

**Initial Step.** We assume the statement holds for $m \in \mathbb{N}$. We show it holds for $m + 1$. Define $y = x \left( \prod_{t=1}^m R_t \right)$. By Lemma A.6 and since $R_{m+1} \in \mathcal{T}_\varepsilon$, this implies that $y \in V_0$, $yR_{m+1} \in V_0$ and $x \left( \prod_{t=1}^{m+1} R_t \right) = x \left( \prod_{t=1}^m R_t \right) R_{m+1} = yR_{m+1} \in V_0$. By inductive hypothesis and Lemma A.6, this implies that

$$\left\| x \left( \prod_{t=1}^{m+1} R_t \right) \right\|_{TV} = \|yR_{m+1}\|_{TV} \leq (1 - \varepsilon) \|y\|_{TV} = (1 - \varepsilon) \left\| x \left( \prod_{t=1}^m R_t \right) \right\|_{TV} \leq (1 - \varepsilon) (1 - \varepsilon)^m \|x\|_{TV} = (1 - \varepsilon)^{m+1} \|x\|_{TV},$$

proving the inductive step.

**Proposition 9.** If $\{P_t\}_{t \in \mathbb{N}} \subseteq \mathcal{T}_\varepsilon$, and $P_t \to P^*$, then there is $\bar{\xi} \in \Delta(\mathbb{N})$ and $K \in \mathbb{N}$ such that for each $\xi \in \Delta(\mathbb{N})$, $\xi_n \to \bar{\xi}$, where $\xi_n = \left( \prod_{t=1}^n P_t \right) \xi$ and

$$\|\xi_n - \bar{\xi}\|_{TV} \leq 3 (1 - \varepsilon)^n \quad \forall n > K. \quad (18)$$

Moreover,

$$\bar{\xi} P^* = \bar{\xi}. \quad (19)$$

Given Lemma A.7 the result is a straightforward extension of Theorem 4.14 of Seneta [2006] to the case of a countable state space.

\footnotetext{Given a sequence $\{P_t\}_{t \in \mathbb{N}}$ of stochastic matrices, $\prod_{t=1}^{n+1} P_t = \prod_{t=1}^n P_t P_{n+1}$ for all $n \in \mathbb{N}$.}
For a given \( \alpha \), let \( \chi_\alpha \) denote the distributions over actions induced by an optimal policy \( \pi \) and \( F_\alpha : \chi_\alpha = \{ \alpha' \in \Delta(A) : \exists \rho \in \Pi_\rho \text{ such that for all } a \in A, \alpha'(a) = F_\alpha(\{ \mu : a \in \rho(\mu) \}) \} \).

**Proof of Theorem 3.** We will use stochastic approximation for differential inclusions (Benaïm, Hofbauer, and Sorin [2005]) to show that (13) can be approximated by

\[
\hat{\alpha}_t \in \mathbb{E}_{H_\alpha} [\alpha^h_t] - \alpha_t. \tag{20}
\]

**Claim 1.** Suppose there is \( \alpha \in \Delta(A) \) such that \( d_t(a, y) \) converges to \( \alpha(a)p^*_a(y) \) for all \( (a, y) \in A \times Y \). The distribution of time \( t \) belief \( \mu_t \) conditional on a recalled history \( h' \) weakly converges to \( F_{\alpha, h'} \in \Delta(\Delta(\Theta)) \), and \( F_{\alpha, h'} \) is continuous.

**Proof.** Given a length \( l \) history \( (a_i, y_i)_{i=1}^l \), let \( n_{a,y}(a_i, y_i) \) be the number of times the action-outcome pair \( (a, y) \) appears in \( (a_i, y_i)_{i=1}^l \). Given an history \( h' = (\hat{a}_i, \hat{y}_i)_{i=1}^l \) recalled in period \( t - 1 \), the recalled history at time \( t \) is distributed as a product of multinomial distributions:

\[
\mathbb{P}_t \left( (a_i, y_i)_{i=1}^l \right) = \prod_{(a,y) \in A \times Y} \left( d_t(a, y) t \right) \left( n_{a,y}(a_i, y_i)_{i=1}^l \right) \\
\times \left( m_t(a, y) + r \left( \mathbb{I}_{(a,y) \in (\hat{a}_i, \hat{y}_i)_{i=1}^l} + \mathbb{I}_{(a,y) \notin (\hat{a}_i, \hat{y}_i)_{i=1}^l} \mathbb{I}_{(a=\pi(h'))p^*_a(y)} \right) \right)^n_{a,y}(a_i, y_i)_{i=1}^l \\
\times \left( 1 - m_t(a, y) - r \left( \mathbb{I}_{(a,y) \in (\hat{a}_i, \hat{y}_i)_{i=1}^l} + \mathbb{I}_{(a,y) \notin (\hat{a}_i, \hat{y}_i)_{i=1}^l} \mathbb{I}_{(a=\pi(h'))p^*_a(y)} \right) \right)^d_{t}(a, y) t - n_{a,y}(a_i, y_i)_{i=1}^l).
\]

By the Poisson limit theorem, the probability that \( (a, y) \) is recalled \( n_{a,y} \) times when the previous recalled history was \( h' = (\hat{a}_i, \hat{y}_i)_{i=1}^l \) converges to

\[
e^{-\lambda(h')_{a,y}} \frac{\lambda(h')_{a,y}^{n_{a,y}}}{n_{a,y}!},
\]

where \( \lambda(h')_{a,y} = \lim_{t \to \infty} d_t(a, y) t (m_t(a, y) + r \left( \mathbb{I}_{(a,y) \in (\hat{a}_i, \hat{y}_i)_{i=1}^l} + \mathbb{I}_{(a,y) \notin (\hat{a}_i, \hat{y}_i)_{i=1}^l} \mathbb{I}_{(a=\pi(h'))p^*_a(y)} \right) \right)^n_{a,y}(a_i, y_i)_{i=1}^l \\
= \alpha(a)p^*_a(y) k (m(a, y) + r \left( \mathbb{I}_{(a,y) \in (\hat{a}_i, \hat{y}_i)_{i=1}^l} + \mathbb{I}_{(a,y) \notin (\hat{a}_i, \hat{y}_i)_{i=1}^l} \mathbb{I}_{(a=\pi(h'))p^*_a(y)} \right) \right) .
\]

Thus the random number of times \( (a, y) \) is recalled when the previous history is \( h' \) converges to a random variable \( N_{a,y}(h') \) that is Poisson distributed with parameter \( \lambda(h')_{a,y} \). Moreover, let \( f : \Delta(\Theta) \to \mathbb{R} \) be a continuous and bounded function, and \( (\alpha_n)_{n \in \mathbb{N}} \in \Delta(A) \) be a sequence.
converging to \( \alpha^* \). Let \( \varepsilon \in \mathbb{R}_{++} \). Since all the \( N_{a,y}(h') \) have Poisson distributions, there exists a \( K \in \mathbb{N} \) such that \( \mathbb{P}_{\alpha^*}[\max_{a,y} N_{a,y}(h') > K] < \varepsilon \). Let \( M \in \mathbb{N} \) be such that 
\[
\mathbb{P}_{\alpha}[\max_{a,y} N_{a,y}(h') > M] < 2\varepsilon \quad \text{and} \quad \mathbb{P}_{\alpha}[N_{a,y}(h') = c] - \mathbb{P}_{\alpha}[N_{a,y}(h') = c] < \varepsilon \quad \text{for all} \\
a, y \in A \times Y, \quad \text{for all} \quad c \leq K \quad \text{and} \quad n > M.
\]
Then for all \( n > M \), we have
\[
\left| \int_{\Delta(\Theta)} f(\nu)dF_{\alpha_n,h'} - \int_{\Delta(\Theta)} f(\nu)dF_{\alpha,h'} \right| < \max_{\nu \in \Delta(\Theta)} f(\nu)(K|A \times Y| + 1)\varepsilon \tag{22}
\]
so \( F_{\alpha_n,h'} \) weakly converges to \( F_{\alpha,h'} \). \( \square \)

We now apply Proposition 9 to prove that the distribution of recalled histories converges to \( \mathcal{H}_\alpha \). Lemma 1 shows that the transition matrices over histories converge. Moreover, from the definition of \( m_t \), the transition from histories that contain the same experiences (but possibly a different number of each) is the same, and since the set of experiences is finite, this convergence is uniform, as required by Proposition 9. Transition to the null history always has positive probability, so the transition matrices are in \( T_{\varepsilon} \) and thus the Proposition guarantees convergence to the stationary distribution \( \mathcal{H}_\alpha \). \( \square \)

**Definition 9.** A strategy \( \sigma \) is **stable** if for every \( \varepsilon \in (0, 1) \), and every prior \( \nu \) with support \( \Theta \) there is an \( n \) such that if \( \sigma \) is used in the first \( n \) periods the probability that the best reply at period \( n + 1 \) is \( \sigma \) is larger than \( 1 - \varepsilon \).

**Proposition 10.** Every uniformly strict selective memory equilibrium is stable.

**Proof of Proposition 10.** If \( \sigma \) is a uniformly strict selective memory equilibrium, the upperhemicontinuity of the best reply map \( BR(s, \cdot) \) and the compactness of \( KL_{s}^{\Theta,m}(\sigma, \varepsilon) \) established in Lemma A.2 imply that there are \( C \in \mathbb{R}_{++} \) and \( \varepsilon \) such that for all \( s \in S \) if 
\[
\nu(KL_{s}^{\Theta,m}(\sigma, \varepsilon)) > C(1 - \nu(KL_{s}^{\Theta,m}(\sigma, \varepsilon))) \quad \text{then} \quad \{\sigma(s)\} = BR(s, \nu).
\]

By the Law of Large Numbers and the finiteness of \( S \times A \times Y \), for every \( C \) there exists \( n_1 \in \mathbb{N} \) such that if \( \sigma \) is used for the first \( n \) periods, \( n > n_1 \)
\[
\mathbb{P}(\|f(h_{n_1})(s,a,y) - f^*(s,a,y)\| < C) > 1 - \frac{\varepsilon}{2} \tag{23}
\]
where
\[
f^*(s,a,y) = \begin{cases} 
\zeta(s)p^*_a(s,y) & \text{if } a = \sigma(s) \\
0 & \text{otherwise}
\end{cases}
\]
Define
\[
\tilde{p}(\sigma, s')(s,a,y) = \frac{\zeta(s)m_{\sigma}(s,\sigma(s),y)p^*_{\sigma(s),a}(y)}{\sum_{\hat{s} \in Y, \hat{a} \in S} \zeta(\hat{s})m_{\sigma}(\hat{s},\sigma(\hat{s}),\hat{y})p^*_{\sigma(\hat{s}),\hat{a}}(\hat{y})}
\]

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if \( a = \sigma(s) \) and \( \tilde{p}(\sigma, s')(s, a, y) = 0 \) otherwise. Since for every \( \tau' \) and \( \tau \) with \( \tau' > \tau \) and every history \( B \in H \) \( \mathbb{P}[r_{\tau'} = B|h_{\tau'}] = \mathbb{P}[r_{\tau'} = B|h_{\tau'}, r_{\tau}] \), by the second Borel-Cantelli lemma and equation (23), for every \( \varepsilon \in \mathbb{R}_{++}, s' \in S \) and \( k \in \mathbb{N}_{++} \) there is a \( n_{k,C} > n_1 \) such that for all \( \tau \geq n_{k,C} \)

\[
\mathbb{P}\left[|r_{\tau}| > k \text{ and } \|\hat{f}(h_{\tau}, r_{\tau}) - \tilde{p}(\sigma, s')\|_{\infty} < C\right] > 1 - \varepsilon.
\]  

(24)

As in the proof of Theorem 1 choose \( k, C' \in \mathbb{R}_{++} \) and \( \varepsilon' < \varepsilon \) such that

\[
\kappa < \inf_{f: \exists s' \in S} \frac{\sup_{\|f - \tilde{p}(\sigma, s')\|\leq C'} \left( -\sum_{s \in S} \zeta(s) \sum_{y \in Y} p_{\sigma(s), s}(y) \log p'_{\sigma(s), s}(y) \right)}{1 - \nu(KL_{\sigma', s'}^\Theta, m(\sigma, \varepsilon')|h_t)} \exp \left( |r_{\tau}| \left( -\sum_{s \in S, a, y \in Y} f(s, a, y) \log p'_{a, s}(y) \right) \right). 
\]

On the set of remembered histories where \( \{|r_{\tau}| > k \text{ and } \|\hat{f}(h_{\tau}, r_{\tau}) - \tilde{p}(\sigma, s')\|_{\infty} < C\} \) identified by equation (24), Lemma A.1 implies that

\[
\frac{\nu(KL_{\sigma', s'}^\Theta, m(\sigma, \varepsilon)|h_t)}{1 - \nu(KL_{\sigma', s'}^\Theta, m(\sigma, \varepsilon)|h_t)} \geq \frac{\nu(KL_{\sigma', s'}^\Theta, m(\sigma, \varepsilon))}{1 - \nu(KL_{\sigma', s'}^\Theta, m(\sigma, \varepsilon'))} \exp \left( |r_{\tau}| \left( -\sup_{\|f' \leq \tilde{p}(\sigma, s')\|} D(\hat{f}(h_{\tau}, r_{\tau})||\log(p')) \right) \right)
\]

\[
\geq \frac{\nu(KL_{\sigma', s'}^\Theta, m(\sigma, \varepsilon))}{1 - \nu(KL_{\sigma', s'}^\Theta, m(\sigma, \varepsilon'))} \exp (k\kappa).
\]

The result follows by setting \( n = n_{k', C'} \) where \( k' = \left\lfloor \log \left( \frac{\nu(KL_{\sigma', s'}^\Theta, m(\sigma, \varepsilon))}{1 - \nu(KL_{\sigma', s'}^\Theta, m(\sigma, \varepsilon'))} / \kappa \right) \right\rfloor + 1. \)

\[\square\]

References


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Example 9. [Limit distribution may not be a stochastic memory equilibrium] Suppose that $k = 1$ and that there are three actions, \{l, r, c\}. The outcome has a product structure: $Y = \{0, 1\} \times \{l, r\}$, and $u(a, y) = y_1$. The agent considers two possible data generating processes, $p$ and $q$, and $\mu$ assigns probability $\frac{1}{2}$ to both. Action $c$ delivers an expected payoff of $\frac{1}{2}$ under both $p$ and $q$, and it is very informative about the identity of the true data generating process through the realization of the second component of the outcome:

$$
\begin{align*}
    p_c(y_1, l) &= 0.45 = q_c(y_1, r) & \forall y_1 \in \{0, 1\} \\
    p_c(y_1, r) &= 0.05 = q_c(y_1, l) & \forall y_1 \in \{0, 1\}.
\end{align*}
$$

Action $l$ performs very well under $p$ and badly under $q$:

$$
\begin{align*}
    p_l(1, l) &= 0.701 & p_l(0, l) = 0.299 \\
    q_l(1, l) &= 0.1 & q_l(0, l) = 0.9.
\end{align*}
$$

Action $r$ performs very well under $q$ and badly under $p$:

$$
\begin{align*}
    p_r(1, r) &= 0.1 & p_r(0, r) = 0.9 \\
    q_r(1, r) &= 0.7 & q_r(0, r) = 0.3.
\end{align*}
$$

Moreover, let $p_c^* = \text{unif} Y$, $p_l^* = \delta_{(1,l)}$ and $p_r^* = \delta_{(1,r)}$. In the first period the agent will choose the safe action $c$. If the second component of the first period outcome is $l$, the agent plays action $l$ from the second period onward, and if it is $r$, the agent plays $r$ from the second period onward. Therefore

$$
a_t \xrightarrow{\alpha} \hat{\alpha} = \frac{1}{2} \hat{\delta}_l + \frac{1}{2} \hat{\delta}_r.
$$

However, $F\hat{\alpha}(\mu_0) > 0$, so that the induced limit frequency of action $c$ is larger than 0 and $\alpha$ is not a stochastic memory equilibrium. ▲

Example 10. Let $S = \{s_{-1}, s_1\}$, $A = \{-1, 1\}$, $Y = \{-1, 1\}$, $u(a, y) = ay$, $p_{s_{-1}, a}(-1) = 3/4$, and $p_{s_1, a}(-1) = 1/5$, and for all $p \in \Theta$, $s, s' \in S$ and $a, a' \in A$, $p_{s, a} = p_{s', a'}$. Under perfect memory the unique equilibrium is to always play 1. If the agent has similarity weighted memory with $f(d(s_{-1}, s_1)) = 1/10$, $f(0) = 1$, then the unique selective memory equilibrium is the objectively optimal $\sigma(s_{-1}) = -1$, $\sigma(s_1) = 1$. ▲