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LOCAL PROJECTIONS AND VARS ESTIMATE THE SAME IMPULSE RESPONSES

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We prove that local projections (LPs) and Vector Autoregressions (VARs) estimate the same impulse responses. This nonparametric result only requires unrestricted lag structures. We discuss several implications: (i) LP and VAR estimators are not conceptually separate procedures; instead, they are simply two dimension reduction techniques with common estimand but different finite-sample properties. (ii) VAR-based structural identification—including short-run, long-run, or sign restrictions—can equivalently be performed using LPs, and vice versa. (iii) Structural estimation with an instrument (proxy) can be carried out by ordering the instrument first in a recursive VAR, even under noninvertibility. (iv) Linear VARs are as robust to nonlinearities as linear LPs.

KEYWORDS: External instrument, impulse response function, local projection, proxy variable, structural vector autoregression.

1. INTRODUCTION

MODERN DYNAMIC MACROECONOMICS STUDIES the propagation of structural shocks (Frisch (1933), Ramey (2016)). Central to this impulse-propagation paradigm are impulse response functions—the dynamic response of a macro aggregate to a structural shock. Following Sims (1980), Bernanke (1986), and Blanchard and Watson (1986), Structural Vector Autoregression (SVAR) analysis remains the most popular empirical approach to impulse response estimation. Over the past decade, however, starting with Jordà (2005), local projections (LPs) have become an increasingly widespread alternative econometric approach.

How should we choose between SVAR and LP estimators of impulse responses? Unfortunately, so far there exists little theoretical guidance as to which method is preferable in practice. Conventional wisdom holds that SVARs are more efficient, while LPs are more robust to model misspecification. Examples of the former statement can be found in the textbook treatment of Kilian and Lütkepohl (2017, Chapter 12.8) and the survey of Ramey (2016, p. 84), while the latter statement is expressed by Jordà (2005, p. 162), Ramey (2016, p. 83) and Nakamura and Steinsson (2018, pp. 80–81), among others.[1]

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[1]In the online postscript to her handbook chapter, Ramey corrects the claims and restates the relationship between LP and VAR estimands following the findings of this paper.

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Kilian and Lütkepohl (2017) and Stock and Watson (2018, p. 944), however, caution that these remarks are not based on formal analysis and call for further research. It is also widely believed that LPs invariably require a measure of a "shock" (perhaps obtained from an auxiliary SVAR model), so that SVAR estimation is required to implement nonrecursive structural identification schemes such as long-run or sign restrictions.[2] Finally, when applied to the same empirical question, LP- and VAR-based approaches sometimes give substantively different results (Ramey (2016)). Existing simulation studies provide useful guidance on particular approaches to local projections or VARs, but differences in implementation details cause these studies to reach disparate conclusions (Meier (2005), Kilian and Kim (2011), Brugnolini (2018), Nakamura and Steinsson (2018), Choi and Chudik (2019)).

The central result of this paper is that linear local projections and VARs in fact estimate the exact same impulse responses in population. Specifically, any LP impulse response function can be obtained through an appropriately ordered recursive VAR, and any (possibly nonrecursive) VAR impulse response function can be obtained through a LP with appropriate control variables. This result applies to all common implementations of local projections used in the literature (Jordà (2005, 2009), Ramey (2016)). While the result concerns linear estimators, we essentially only require the nonparametric assumption that the data are weakly stationary and that the lag structures in the two specifications are unrestricted. In particular, we do not impose restrictions on the linearity or dimensionality of the underlying data generating process (DGP). Intuitively, a VAR model with sufficiently large lag length captures all covariance properties of the data. Hence, iterated VAR(∞) forecasts coincide with direct LP forecasts. Since impulse responses are just forecasts conditional on specific innovations, LP and VAR impulse response estimands coincide in population. Furthermore, we prove that if only a fixed number p of lags are included in the LP and VAR, then the two impulse response estimands still approximately agree out to horizon p (but not further), again without imposing any parametric assumptions on the data generating process.

The equivalence of VAR and LP estimands has several implications for structural estimation in applied macroeconometrics.

First, LPs and VARs are not conceptually different methods; instead, they are simply two particular linear projection techniques that share the same estimand but differ in their finite-sample properties. At short impulse response horizons, the two methods are likely to approximately agree if the same lag length is used for both methods. However, with finite lag lengths, the two methods may give substantially different results at long horizons.

Second, structural estimation with VARs can equally well be carried out using LPs, and vice versa. Structural identification—which is a population concept—is logically distinct from the choice of finite-sample estimation approach. In particular, we show concretely how various popular "SVAR" identification schemes—including recursive, long-run, and sign identification—can just as easily be implemented using local projection techniques. Ultimately, our results show that LP-based structural estimation can succeed if and only if SVAR estimation can succeed.

Third, valid structural estimation with an instrument (IV, also known as a proxy variable) can be carried out by ordering the IV first in a recursive VAR à la Kilian (2006) and Ramey (2011). This is because the LP-IV estimand of Stock and Watson (2018) can equivalently be obtained from a recursive (i.e., Cholesky) VAR that contains the IV.

[2]See the reviews by Ramey (2016) and Kilian and Lütkepohl (2017, Chapter 12.8).

Importantly, the "internal instrument" strategy of ordering the IV first in a VAR yields valid impulse response estimates even if the shock of interest is noninvertible, unlike the well-known "external instrument" SVAR-IV approach (Stock (2008), Stock and Watson (2012), Mertens and Ravn (2013)).[3] In particular, this result goes through even if the IV is contaminated with measurement error that is unrelated to the shock of interest.

Fourth, in population, linear local projections are exactly as "robust to nonlinearities" in the DGP as linear VARs. We show that their common estimand may be formally interpreted as a best linear approximation to the underlying, perhaps nonlinear, impulse responses.

In summary, in addition to clarifying misconceptions in the literature about the relationship between the LP and VAR estimands, our results allow applied researchers to separate the choice of identification scheme from the choice of estimation approach. Researchers who prefer the intuitive regression interpretation of the LP impulse response estimator can apply our methods for imposing "SVAR" identifying restrictions such as short-run, long-run, and sign restrictions. Researchers who instead prefer the explicit multivariate model of the VAR estimator can apply our results on how to use instruments/proxies without requiring invertible shocks, as in LP-IV.

Literature. While the existing literature has pointed out connections between LPs and VARs, our contributions are to establish a formal equivalence result that does not require extraneous functional form assumptions and to derive implications for structural identification of impulse responses. Jordà (2005, Section I.B) and Kilian and Lütkepohl (2017, Chapter 12.8) show that, under the assumption of a finite-order VAR model, VAR impulse responses can be estimated consistently through LPs. In contrast, our equivalence result between these two linear estimation methods does not restrict the data generating process itself to be linear or finite-dimensional. While Dufour and Renault (1998, equation (3.17)) discuss a similar result in the context of testing for Granger causality, we go further by demonstrating how causal structural VAR orderings map into particular choices of LP control variables, and vice versa.[4] Moreover, to our knowledge, our results on long-run/sign identification, LP-IV, and best linear approximations have no obvious parallels in the preceding literature.[5]

In this paper, we focus exclusively on identification and point estimation of impulse responses. Plagborg-Møller and Wolf (2019) provided identification results for variance/historical decompositions using IVs/proxies. We do not consider questions related to inference, and instead refer to Jordà (2005), Kilian and Lütkepohl (2017), and Stock and Watson (2018).

Outline. Section 2 presents our core result on the population equivalence of local projections and VARs in a reduced-form setting. Section 3 traces out the implications for structural estimation. We illustrate our equivalence results with a practical application to IV-based identification of monetary policy shocks in Section 4. Section 5 concludes

[3] In contemporaneous work, Noh (2018) also included the IV as an internal instrument in a VAR; our result offers additional insights by drawing connections to LP-IV and to the general LP/VAR equivalence.

[4] Jordà, Schularick, and Taylor (2020) informally discussed the connection between control variables and recursive SVARs.

[5] Kilian and Lütkepohl (2017, Chapter 12.8) presented alternative arguments for why it is a mistake to assert that finite-order LPs are generally more "robust to model misspecification" than finite-order VAR estimators. They do not appeal to the general equivalence of the LP and VAR estimands, however.

by summarizing the takeaways for empirical practice. Some proofs are relegated to Appendix A, and details are presented in the Online Supplementary Material (Plagborg-Møller and Wolf (2021)) in the Appendix B.[6]

2. EQUIVALENCE BETWEEN LOCAL PROJECTIONS AND VARS

This section presents our core result: Local projections and VARs estimate the same impulse response functions in population. First, we establish that local projections are equivalent with recursively identified VARs when the lag structure is unrestricted. Then we extend the argument to (i) nonrecursive identification and (ii) finite lag orders, and we illustrate the results graphically. Finally, we discuss an in-sample asymptotic equivalence result that complements the population analysis.

Our analysis in this section is "reduced form" in that it does not assume any specific underlying structural/causal model; we merely manipulate linear projections of stationary time series. We will discuss implications for causal identification in Section 3.

2.1. *Main Result*

Suppose the researcher observes data $w_t = (r_t', x_t, y_t, q_t')'$, where r_t and q_t are, respectively, $n_r \times 1$ and $n_q \times 1$ vectors of time series, while x_t and y_t are scalar time series. We are interested in the dynamic response of y_t after an impulse in x_t. The vector time series r_t and q_t (which may each be empty) will serve as control variables. Readers who wish to have a structural interpretation in mind may think of x_t as predetermined with respect to y_t and r_t as predetermined with respect to $\{x_t, y_t\}$. However, our reduced-form equivalence result below does not require any such predeterminedness assumptions. The precise roles of the controls r_t and q_t will become clear in equations (1) and (3) below.

For now, we only make the following standard nonparametric regularity assumption.[7]

ASSUMPTION 1: *The data $\{w_t\}$ are covariance stationary and purely nondeterministic, with an everywhere nonsingular spectral density matrix and absolutely summable Wold decomposition coefficients.*

In particular, we assume nothing about the underlying causal structure of the economy, as this section is concerned solely with properties of linear projections.[8]

As an expositional device, we impose an additional assumption of joint Gaussianity.

ASSUMPTION 2: *$\{w_t\}$ is a jointly Gaussian vector time series.*

The Gaussianity assumption is made purely for notational simplicity, as this allows us to write conditional expectations instead of linear projections. If we drop the Gaussianity assumption, all calculations below hold with projections in place of conditional expectations.

We will show that, in population, the following two approaches estimate the *same* impulse response function of y_t with respect to an innovation in x_t.[9]

[6] Online Appendix and replication files in Plagborg-Møller and Wolf (2021).

[7] The restriction to nonsingular spectral density matrices rules out overdifferenced data. We conjecture that this restriction could be relaxed using the techniques in Almuzara and Marcet (2017).

[8] Assumption 1 allows the time series to be discrete or censored, though structural interpretation of the linear impulse response estimand in such cases requires care; cf. the discussion below Proposition 1.

[9] We write linear projections on the span of infinitely many variables as an infinite sum. This is justified under Assumption 1, since we can invert the Wold representation to obtain a VAR(∞) representation.

1. LOCAL PROJECTION. Consider for each $h = 0, 1, 2, \dots$ the linear projection

$$y_{t+h} = \mu_h + \beta_h x_t + \gamma_h' r_t + \sum_{\ell=1}^{\infty} \delta_{h,\ell}' w_{t-\ell} + \xi_{h,t}, \tag{1}$$

where $\xi_{h,t}$ is the projection residual, and $\mu_h, \beta_h, \gamma_h, \delta_{h,1}, \delta_{h,2}, \dots$ the projection coefficients.

DEFINITION 1: The *LP impulse response function* of y_t with respect to x_t is given by $\{\beta_h\}_{h\geq 0}$ in equation (1).

Effectively, this defines the LP impulse response estimand at horizon h as

$$\beta_h = E\big(y_{t+h} \mid x_t = 1, r_t, \{w_\tau\}_{\tau<t}\big) - E\big(y_{t+h} \mid x_t = 0, r_t, \{w_\tau\}_{\tau<t}\big). \tag{2}$$

Notice that the projection (1) controls for the contemporaneous value of r_t but not of q_t. Notice also that we do not require x_t to be a predetermined "shock" variable in this section, although such additional assumptions may be important for interpreting β_h structurally, as discussed in Section 3 below. Importantly, the formulation (1) is general enough to cover all common empirical implementations of local projections.[10]

2. VAR. Consider the multivariate linear "VAR(∞)" projection

$$w_t = c + \sum_{\ell=1}^{\infty} A_\ell w_{t-\ell} + u_t, \tag{3}$$

where $u_t \equiv w_t - E(w_t \mid \{w_\tau\}_{-\infty<\tau<t})$ is the projection residual, and $c, A_1, A_2, \dots$ the projection coefficients. Let $\Sigma_u \equiv E(u_t u_t')$, and define the Cholesky decomposition $\Sigma_u = BB'$, where B is lower triangular with positive diagonal entries. Consider the corresponding recursive SVAR representation

$$A(L)w_t = c + B\eta_t,$$

where $A(L) \equiv I - \sum_{\ell=1}^{\infty} A_\ell L^\ell$ and $\eta_t \equiv B^{-1}u_t$. Notice that r_t is ordered first in the VAR, while q_t is ordered last.[11] Define the lag polynomial $\sum_{\ell=0}^{\infty} C_\ell L^\ell = C(L) \equiv A(L)^{-1}$. Noting that x_t and y_t are the $(n_r + 1)$-th and $(n_r + 2)$-th elements in w_t, we now introduce the following familiar definition.

[10]This includes: (i) estimating reduced-form impulse responses via LP and then rotating them using estimates of the impact impulse response matrix from an auxiliary SVAR as in Jordà (2005, 2009), (ii) projections on an exogenous shock $x_t = \varepsilon_{j,t}$ (e.g., Ramey (2016), Nakamura and Steinsson (2018)) (in this case control variables are often omitted, though they may increase efficiency), and (iii) projections on an endogenous covariate x_t while controlling for confounding variables r_t (e.g., Jordà, Schularick, and Taylor (2013)). The fact that options (ii) and (iii) are covered by (1) is immediate, while Chang and Sakata (2007) show that option (i) is equivalent in population to directly projecting on the shock x_t identified by the auxiliary SVAR as in option (ii).

[11]The relative ordering of y_t and q_t in the SVAR representation does not matter for our results, since it can be verified that the $(n_r + 1)$-th column of B is equivariant with respect to this ordering. Similarly, if x_t is ordered after y_t in the SVAR representation, then the equivalence result below still obtains as long as we additionally control for y_t on the right-hand side of (1) (so in particular $\beta_0 = 0$).

DEFINITION 2: The *VAR impulse response function* of y_t with respect to an innovation in x_t is given by $\{\theta_h\}_{h\geq 0}$, where

$$\theta_h \equiv C_{n_r+2,\bullet,h} B_{\bullet,n_r+1},$$

and $\{C_\ell\}$ and B are defined above.

Here, $C_{i,\bullet,h}$, say, refers to the ith row of C_h, while $B_{\bullet,j}$ is the jth column of B.

Note that our definitions of the LP and VAR estimands include infinitely many lags of w_t in the relevant projections; we consider the case of finitely many lags in Section 2.3. Note also that we take the use of the control variables r_t and q_t as given in this section, as controls are common in applied work. We will discuss structural justifications for the use of such controls in Section 3.

Although LP and VAR approaches are often viewed as conceptually distinct in the literature, they in fact estimate the same population impulse response function.

PROPOSITION 1: *Under Assumptions 1 and 2, the LP and VAR impulse response functions are equal, up to a constant of proportionality*: $\theta_h = \sqrt{E(\tilde{x}_t^2)} \times \beta_h$ *for all* $h = 0, 1, 2, \dots,$ *where* $\tilde{x}_t \equiv x_t - E(x_t \mid r_t, \{w_\tau\}_{-\infty<\tau<t})$.

That is, any LP impulse response function can equivalently be obtained as an appropriately ordered recursive VAR impulse response function. Conversely, any recursive VAR impulse response function can be obtained through a LP with appropriate control variables. We comment on nonrecursive identification schemes below. The constant of proportionality in the proposition depends on neither the response horizon h nor on the response variable y_t. The reason for the presence of this constant of proportionality is that the implicit LP innovation $\tilde{x}_t$, after controlling for the other right-hand side variables, does not have variance 1. If we scale the innovation $\tilde{x}_t$ to have variance 1, or if we consider relative impulse responses θ_h/θ_0 (as further discussed below), the LP and VAR impulse response functions coincide.

The intuition behind the result is that a VAR(p) model with $p \to \infty$ is sufficiently flexible to perfectly capture all covariance properties of the data (Lewis and Reinsel (1985), Inoue and Kilian (2002)). Thus, iterated forecasts based on the VAR coincide perfectly with direct forecasts $E(w_{t+h} \mid w_t, w_{t-1}, \dots)$. Since both recursive VAR and LP impulse responses are just linear functions of these direct reduced-form forecasts, they coincide. Although the intuition for this equivalence result is simple, its implications do not appear to have been generally appreciated in the literature on impulse response estimation, as discussed earlier in Section 1.

PROOF: The proof of the proposition relies only on least-squares projection algebra. First, consider the LP estimand. By the Frisch–Waugh theorem, we have that

$$\beta_h = \frac{\operatorname{Cov}(y_{t+h}, \tilde{x}_t)}{E(\tilde{x}_t^2)}. \tag{4}$$

For the VAR estimand, note that $C(L) = A(L)^{-1}$ collects the coefficient matrices in the Wold decomposition

$$w_t = \chi + C(L)u_t = \chi + \sum_{\ell=0}^{\infty} C_\ell B \eta_t, \quad \chi \equiv C(1)c.$$

As a result, the VAR impulse responses equal

$$\theta_h = C_{n_r+2,\bullet,h} B_{\bullet,n_r+1} = \mathrm{Cov}(y_{t+h}, \eta_{x,t}), \tag{5}$$

where we partition $\eta_t = (\eta'_{r,t}, \eta_{x,t}, \eta_{y,t}, \eta'_{q,t})'$ the same way as $w_t = (r'_t, x_t, y_t, q'_t)'$. By $u_t = B\eta_t$ and the properties of the Cholesky decomposition, we have[12]

$$\eta_{x,t} = \frac{1}{\sqrt{E(\tilde{u}^2_{x,t})}} \times \tilde{u}_{x,t}, \tag{6}$$

where we partition $u_t = (u'_{r,t}, u_{x,t}, u_{y,t}, u'_{q,t})'$ and define[13]

$$\tilde{u}_{x,t} \equiv u_{x,t} - E(u_{x,t} \mid u_{r,t}) = \tilde{x}_t. \tag{7}$$

From (5), (6), and (7) we conclude that

$$\theta_h = \frac{\mathrm{Cov}(y_{t+h}, \tilde{x}_t)}{\sqrt{E(\tilde{x}^2_t)}},$$

and the proposition now follows by comparing with (4). *Q.E.D.*

In the special case where x_t represents a "shock," in the sense that $E(x_t \mid r_t, \{w_\tau\}_{\tau<t}) = 0$, the LP estimand β_h coincides also with the impulse response estimand φ_h from a distributed lag regression $y_t = a + \sum_{\ell=0}^{\infty} \varphi_\ell x_{t-\ell} + \omega_t$ (understood as a linear projection); see Baek and Lee (2020). Note that in this special case, the LP estimand is unchanged if we drop all control variables in equation (1). However, the projection coefficient φ_h differs from the LP (and VAR) estimand if x_t correlates with r_t or with lags of the data (Alloza, Gonzalo, and Sanz (2019)).

Proposition 1 implies that linear LPs are exactly as "robust to nonlinearities" as linear VAR methods, in population. This is because, while the equivalence result concerns linear estimation methods, our argument was nonparametric in that it did not rely on functional form assumptions on the true data generating process, such as linearity or finite dimensionality. In the Online Appendix, we prove that the common LP/VAR estimand can be interpreted as a "best linear approximation" to the true, possibly nonlinear, structural/causal impulse responses. Of course, this best linear approximation may bear little resemblance to the impulse responses in the underlying nonlinear model, which will generally depend on the history and magnitudes of current and past shocks, unlike the linear impulse responses.[14]

In conclusion, LPs and VARs should not be thought of as conceptually different methods—they are simply two particular linear projection techniques with a shared estimand. LPs and VARs offer two equivalent ways of arriving at the same population parameter (4), or equivalently (2), up to a scale factor that does not depend on the horizon h.

[12] B is lower triangular, so the (n_r+1)-th equation in the system $B\eta_t = u_t$ is $B_{n_r+1,1:n_r}\eta_{r,t} + B_{n_r+1,n_r+1}\eta_{x,t} = u_{x,t}$, with obvious notation. Since $\eta_{x,t}$ and $\eta_{r,t}$ are uncorrelated, we find $B_{n_r+1,n_r+1}\eta_{x,t} = u_{x,t} - E(u_{x,t} \mid \eta_{r,t}) = u_{x,t} - E(u_{x,t} \mid u_{r,t}) = \tilde{u}_{x,t}$. Expression (6) then follows from $E(\eta^2_{x,t}) = 1$.

[13] Observe that $u_{x,t} - \tilde{x}_t = E(x_t \mid r_t, \{w_\tau\}_{-\infty<\tau<t}) - E(x_t \mid \{w_\tau\}_{-\infty<\tau<t}) = E(u_{x,t} \mid r_t, \{w_\tau\}_{-\infty<\tau<t}) = E(u_{x,t} \mid u_{r,t}, \{w_\tau\}_{-\infty<\tau<t}) = E(u_{x,t} \mid u_{r,t})$.

[14] See Kilian and Vigfusson (2011) as an example of a model in which the common linear estimand of local projections and VARs is not the structural object of interest.

2.2. *Extension: Nonrecursive Specifications*

Our equivalence result extends straightforwardly to the case of nonrecursively identified VARs. Above we restricted attention to recursive identification schemes, as the VAR directly contains a measure of the impulse x_t. In a generic structural VAR identification scheme, the impulse is some—not necessarily recursive—rotation of reduced-form forecasting residuals. Thus, let us continue to consider the VAR (3), but now we shall study the propagation of *some* rotation of the reduced-form forecasting residuals,

$$\bar{\eta}_t \equiv b' u_t, \tag{8}$$

where b is a vector of the same dimension as w_t. Under Assumptions 1 and 2, we can follow the same steps as in Section 2.1 to establish that the VAR-implied impulse response at horizon h of y_t with respect to the innovation $\bar{\eta}_t$ equals—up to scale—the coefficient $\bar{\beta}_h$ of the linear projection

$$y_{t+h} = \bar{\mu}_h + \bar{\beta}_h (b' w_t) + \sum_{\ell=1}^{\infty} \bar{\delta}'_{h,\ell} w_{t-\ell} + \bar{\xi}_{h,t}, \tag{9}$$

where the coefficients are least-squares projection coefficients and the last term is the projection residual. Thus, any recursive or nonrecursive SVAR(∞) identification procedure is equivalent with a local projection (9) on a particular linear combination $b' w_t$ of the variables in the VAR (and their lags). For recursive orderings, this reduces to Proposition 1. We give concrete examples of the mapping from nonrecursive VAR to the rotation vector b in Section 3.2 as well as in the Online Appendix.

2.3. *Extension: Finite Lag Length*

Whereas our main equivalence result in Section 2.1 relied on infinite lag polynomials, we now prove an equivalence result that holds when only finitely many lags are used. Specifically, when p lags of the data are included in the VAR and as controls in the LP, the impulse response estimands for the two methods agree approximately out to horizon p, but generally not at higher horizons. Importantly, this result is still entirely nonparametric, in the sense that we do not impose that the true DGP is a linear or finite-order VAR.

First, we define the finite-order LP and VAR estimands. We continue to impose Assumptions 1 and 2. Consider any lag length p and impulse response horizon h.

1. LOCAL PROJECTION. The local projection impulse response estimand $\beta_h(p)$ is defined as the coefficient on x_t in a projection as in (1), except that the infinite sum is truncated at lag p. Again, we interpret all coefficients and residuals as resulting from a least-squares linear projection.
2. VAR. Consider a linear projection of the data vector w_t onto p of its lags (and a constant), that is, the projection (3) except with the infinite sum truncated at lag p. Let $A_\ell(p)$, $\ell = 1, 2, \dots, p$, and $\Sigma_u(p)$ denote the corresponding projection coefficients and residual variance. Define $A(L; p) \equiv I - \sum_{\ell=1}^{p} A_\ell(p)$ and the Cholesky decomposition $\Sigma_u(p) = B(p)B(p)'$. Define also the inverse lag polynomial $\sum_{\ell=0}^{\infty} C_\ell(p) L^\ell = C(L; p) \equiv A(L; p)^{-1}$ consisting of the reduced-form impulse responses implied by $A(L; p)$. Then the VAR impulse response estimand at horizon h is defined as

$$\theta_h(p) \equiv C_{n_r+2,\bullet,h}(p) B_{\bullet,n_r+1}(p);$$

cf. the definition in Section 2.1 with $p = \infty$.

Note that the VAR(p) model used to define the VAR estimand above is "misspecified," in the sense that the reduced-form residuals from the projection of w_t on its first p lags are not white noise in general.

We now state the equivalence result for finite p. The statement of the result is a simple generalization of Proposition 1, which can be thought of as the case $p = \infty$. Define the projection residual $\tilde{x}_t(p) \equiv x_t - E(x_t \mid r_t, \{w_\tau\}_{t-p\leq\tau<t}) = x_t - \sum_{\ell=0}^{p} \varrho_\ell(p)' w_{t-\ell}$ (where the last $n_q + 2$ elements of $\varrho_0(p)$ are zero). Let also the operator $\text{Cov}^p(\cdot, \cdot)$ denote the covariance between any variables in the VAR that *would hypothetically* obtain if the data in fact followed a VAR(p) model with the parameters $(A(L; p), \Sigma_u(p))$ defined above.

PROPOSITION 2: *Impose Assumptions 1 and 2. Let the nonnegative integers h, p satisfy $h \leq p$. Then $\theta_h(p) = \sqrt{E(\tilde{x}_t(p)^2)} \times \beta_h(p) + \phi_h(p)$, where the remainder is given by $\phi_h(p) = \{E(\tilde{x}_t(p)^2)\}^{-1/2} \sum_{\ell=p-h+1}^{p} \{\text{Cov}(y_{t+h}, w_{t-\ell}) - \text{Cov}^p(y_{t+h}, w_{t-\ell})\} \varrho_\ell(p)$.*

PROOF: Please see Section A.1. *Q.E.D.*

Thus, if long lags of the data do not help to predict the impulse variable x_t (that is, when $\varrho_\ell(p) = 0$ for all $\ell \geq p - h + 1$), then the population LP and VAR impulse response estimands agree at all horizons $h \leq p$, although generally not at horizons $h > p$. This finding would not be surprising if the true DGP were assumed to be a finite-order VAR (as in Jordà (2005, Section I.B), and Kilian and Lütkepohl (2017, Chapter 12.8)), but we allow for general covariance stationary DGPs. The reason why the result still goes through is that a VAR(p) obtained through least-squares projections perfectly captures the autocovariances of the data out to lag p (but not further), and these are precisely what determine the LP estimand.[15] For example, if $p = 2$, then $\text{Cov}^p(y_t, x_{t-2}) = \text{Cov}(y_t, x_{t-2})$, but generally $\text{Cov}^p(y_t, x_{t-3}) \neq \text{Cov}(y_t, x_{t-3})$.

Proposition 2 implies that LP and VAR impulse response estimands will agree approximately at short horizons for a wide range of empirically relevant DGPs. If, as in many applications, x_t is a direct measure of a "shock," and thus uncorrelated with r_t and all past data, then necessarily $\phi_h(p) = 0$ and so the LP/VAR equivalence holds exactly out to horizon h. More generally, the LP estimand projects y_{t+h} onto $\tilde{x}_t(p)$; thus, the projection depends on the first $p + h$ autocovariances of the data. The estimated VAR(p) generally does not precisely capture the autocovariances of the data at lags $p + 1, \dots, p + h$, and so the LP and VAR may not agree exactly. However, as we illustrate in Section 2.4, empirically relevant DGPs often have $\varrho_\ell(p) \approx 0$ for long lags ℓ, since it is typically only the first few lags of the data w_t that contribute substantially to forecasting x_t. But then $\phi_h(p) \approx 0$ for $h \ll p$, so the LP/VAR equivalence holds approximately at short horizons (in particular, $\phi_0(p) = 0$).

In conclusion, even if we use "too short" a lag length p, the LP and VAR impulse response estimands only disagree materially at horizons longer than p. This is a comforting fact in applications where the questions of interest revolve around short-horizon impulse responses.

2.4. *Graphical Illustration*

In this section, we graphically illustrate our previous theoretical results in the context of a particular data generating process: the structural macro model of Smets and Wouters

[15] Baek and Lee (2020) proved a similar result for the related but distinct setting of single-equation autoregressive distributed lag models with a white noise exogenous regressor.

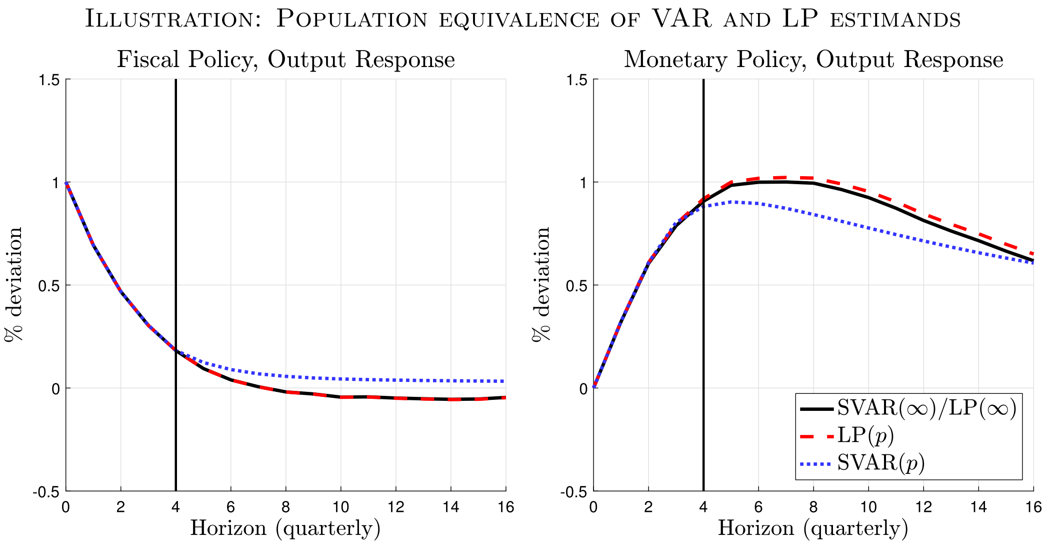

FIGURE 1.—LP and VAR impulse response estimands in the structural model of Smets and Wouters (2007). Left panel: response of output to a government spending innovation. Right panel: response of output to a negative interest rate innovation. The horizontal line marks the horizon p after which the finite-lag-length LP(p) and VAR(p) estimands diverge.

(2007). We abstract from sampling uncertainty and throughout assume that the econometrician actually observes an infinite amount of data.[16] Since this section is merely intended to illustrate the properties of different projections, we do not comment on the relation of the projection estimands to true *structural* model-implied impulse responses. We formally discuss structural identification in Section 3.

The left panel of Figure 1 shows LP and VAR impulse response estimands of the response of output to an innovation in government spending. We assume the model's government spending innovation is directly observed by the econometrician, who additionally controls for lags of output and the fiscal spending innovation itself. This experiment is therefore similar in spirit to that of Ramey (2011). As ensured by Proposition 1, the LP(∞) and VAR(∞) estimands—that is, with infinitely many lags as controls—agree at all horizons. Since by assumption the "impulse" variable x_t is a direct measure of the government spending innovation, we have $\tilde{x}_t(p) = x_t$. Thus, any LP(p) estimand for finite p also agrees with the LP(∞) limit at all horizons. Finally, we observe that the impulse responses implied by a VAR(4) *exactly* agree with the true population projections up until horizon $h = 4$, as predicted by Proposition 2.

The right panel of Figure 1 shows LP and VAR impulse response estimands for the response of output to an innovation in the nominal interest rate. Here, the model's innovation is not directly observed by the econometrician, only the interest rate. The LP specifications control for the contemporaneous value of output and inflation as well as lags of output, inflation, and the nominal interest rate; as discussed, this set of control variables is equivalent to ordering the interest rate last in the VAR. Thus, the experiment

[16]We use the Dynare replication of Smets and Wouters (2007) kindly provided by Johannes Pfeifer. The code is available at: https://sites.google.com/site/pfeiferecon/dynare. We truncate the model-implied vector moving average representation at a large horizon ($H = 350$), and then invert to obtain a VAR(∞).

emulates the familiar monetary policy shock identification analysis of Christiano, Eichenbaum, and Evans (2005), although we, at least for the purposes of this section, interpret the projections purely in a reduced-form way. Again, the LP(∞) and VAR(∞) estimands agree at all horizons. Now, however, the "impulse" $\tilde{x}_t(p)$ upon which the LP(p) and VAR(p) methods project depends on the lag length p, so these estimands differ from the LP(∞)/VAR(∞) estimands. Furthermore, the LP(p) and VAR(p) estimands also differ from each other, as the remainder term $\phi_h(p)$ in Proposition 2 is not exactly zero. Nevertheless, because distant lags of w_t do not contribute substantially to forecasting x_t in this DGP, all impulse response estimands are nearly identical until the truncation horizon $p = 4$, consistent with the discussion in Section 2.3.

2.5. *Sample Equivalence*

In addition to being identical *conceptually* and in *population*, we show in the Online Appendix that the difference between the local projection and VAR impulse response estimators converges to zero asymptotically *in sample* when large lag lengths are used in the regression specifications. Formally, let $\hat{\beta}_h(p)$ and $\hat{\theta}_h(p)$ denote the least-squares estimators of the LP and VAR specifications (1)–(3) if we include p lags of the data in the VAR and on the right-hand side of the local projection. Under standard nonparametric regularity conditions (Lewis and Reinsel (1985)), the sample analogue of the population equivalence result in Section 2.1 holds: There exists a constant of proportionality $\hat{\kappa}$ such that, at any fixed horizon h, the distance $|\hat{\theta}_h(p) - \hat{\kappa}\hat{\beta}_h(p)|$ tends to zero in probability asymptotically, provided that the lag length p tends to infinity with the sample size at an appropriate rate. We relegate the details of this result to the Online Appendix.

Combining Proposition 1 and the logic in Newey (1994), we conjecture that the least-squares local projection and VAR impulse response estimators $\hat{\beta}_h(p)$ and $\hat{\theta}_h(p)$ are equally asymptotically *efficient* (at a fixed horizon h) under weak regularity conditions, provided that the lag length p tends to infinity at an appropriate rate with the sample size. In finite samples, and with finite lag lengths, the two estimation methods are likely to agree approximately at short horizons due to Proposition 2, but the choice between the procedures at long horizons requires navigating a bias-variance trade-off.[17] Much more research is warranted on the optimal way to resolve the bias-variance trade-off in practice.[18]

3. STRUCTURAL IDENTIFICATION OF IMPULSE RESPONSES

We now show that our result on the population equivalence of LP and VAR impulse response functions has important implications for structural identification. The problem of structural identification is a population concept and thus logically distinct from the choice of finite-sample dimension reduction technique. We apply our equivalence result to show that popular "SVAR" identification schemes—including short-run restrictions,

[17]A VAR with p lags extrapolates its long-run responses from the first p sample autocovariances, which are estimated more precisely than the long-lag sample autocovariances used by the local projection estimator. Hence, the VAR impulse response estimates at long horizons typically have lower variance than local projections, but potentially higher asymptotic bias if the lag length is misspecified (though this depends on the magnitude of the misspecification, see Schorfheide (2005), and Kilian and Kim (2011)).

[18]The forecasting literature offers extensive guidance on the choice between direct forecasts (analogous to LP) and iterated forecasts (analogous to VAR) (Schorfheide (2005), Chevillon (2007), Marcellino, Stock, and Watson (2006), Pesaran, Pick, and Timmermann (2011), McElroy (2015)).

long-run restrictions, and sign restrictions—can equivalently be carried out using LPs. Conversely, invertibility-robust structural estimation with an external instrument (proxy) is also possible using VARs, not just LPs.

3.1. *Structural Model*

To discuss structural identification, we impose a linear but otherwise general semiparametric Structural Vector Moving Average (SVMA) model. This model does not restrict the *linear* transmission mechanism of shocks to observed variables (see the Online Appendix for a discussion of nonlinear models). SVMA models have been analyzed by Stock and Watson (2018), Plagborg-Møller and Wolf (2019), and many others. The class of SVMA models encompasses all discrete-time, linearized DSGE models as well as all stationary SVAR models.

ASSUMPTION 3: *The data $\{w_t\}$ are driven by an n_ε-dimensional vector $\varepsilon_t = (\varepsilon_{1,t}, \dots, \varepsilon_{n_\varepsilon,t})'$ of exogenous structural shocks*,

$$w_t = \mu + \Theta(L)\varepsilon_t, \quad \Theta(L) \equiv \sum_{\ell=0}^{\infty} \Theta_\ell L^\ell, \tag{10}$$

where $\mu \in \mathbb{R}^{n_w \times 1}$, $\Theta_\ell \in \mathbb{R}^{n_w \times n_\varepsilon}$, and L is the lag operator. $\{\Theta_\ell\}_\ell$ is assumed to be absolutely summable, and $\Theta(x)$ has full row rank for all complex scalars x on the unit circle. For notational simplicity, we further assume normality of the shocks:

$$\varepsilon_t \overset{i.i.d.}{\sim} N(0, I_{n_\varepsilon}). \tag{11}$$

Under these assumptions, w_t is a nonsingular, strictly stationary jointly Gaussian time series, consistent with Assumptions 1 and 2 in Section 2. The (i, j) element $\Theta_{i,j,\ell}$ of the $n_w \times n_\varepsilon$ moving average coefficient matrix Θ_ℓ is the impulse response of variable i to shock j at horizon ℓ.

The researcher is interested in the propagation of the structural shock $\varepsilon_{1,t}$ to the observed macro aggregate y_t. Since y_t is the $(n_r + 2)$-th element in w_t, the parameters of interest are $\Theta_{n_r+2,1,h}$, $h = 0, 1, 2, \dots$. In line with applied work, we also consider *relative* impulse responses $\Theta_{n_r+2,1,h}/\Theta_{n_r+1,1,0}$. This may be interpreted as the response in y_{t+h} caused by a shock $\varepsilon_{1,t}$ of a magnitude that raises x_t by one unit on impact.

3.2. *Implementing "SVAR" Identification Using LPs*

In this subsection, we show that LP methods are as applicable as VAR methods when implementing common identification schemes. Our main result in Section 2.1 implies that LP-based causal estimation can succeed if and only if SVAR-based estimation can succeed. We will exhibit several concrete and easily implementable examples of this equivalence.

Identification Under Invertibility. Standard SVAR analysis assumes (partial) invertibility, that is, the ability to recover the structural shock of interest, $\varepsilon_{1,t}$, as a function of only current and past macro aggregates:

$$\varepsilon_{1,t} \in \operatorname{span}\big(\{w_\tau\}_{-\infty<\tau\le t}\big). \tag{12}$$

A given SVAR identification scheme then identifies as the candidate structural shock a particular linear combination of the Wold forecast errors:

$$\tilde{\varepsilon}_{1,t} \equiv b' u_t, \tag{13}$$

where the chosen identification scheme gives the vector b as a function of the reduced-form VAR parameters $(A(L), \Sigma_u)$, or equivalently the Wold decomposition parameters $(C(L), \Sigma_u)$. Under invertibility, there must exist a vector b such that $\tilde{\varepsilon}_{1,t} = \varepsilon_{1,t}$, so SVAR identification can in principle succeed (Fernández-Villaverde, Rubio-Ramírez, Sargent, and Watson (2007), Wolf (2020)).

We now illustrate through three examples that common SVAR identification schemes are equally as simple to implement using LP methods. We first consider a standard recursive scheme covered by our benchmark analysis in Section 2.1. The second and third examples involve long-run and sign restrictions and require the general equivalence result of Section 2.2.

EXAMPLE 1—Recursive Identification: Christiano, Eichenbaum, and Evans (2005) identified monetary policy shocks through a recursive ordering. They assume that their observed data $\{w_t\}$ follow an invertible SVMA model, that is, the condition (12) holds for all shocks in the system (10). They then additionally impose a temporal ordering on the set of variables w_t: Output, consumption, investment, wages, productivity, and the price deflator do not respond within the period to changes in the policy rate (Federal Funds Rate), which itself in turn does not react within the period to changes in profits and money growth. In the notation of Section 2.1, the assumed ordering corresponds to the Federal Funds Rate as the impulse variable x_t, real and price variables as the controls r_t, and financial variables collected in the vector q_t. Thus, for the purposes of structural interpretation, it is now explicitly assumed that x_t and r_t are predetermined with respect to y_t. Christiano, Eichenbaum, and Evans implemented their structural analysis through the recursive VAR (3). By our main result, they could have equivalently estimated the LP regression (1) and collected the regression coefficients $\{\beta_h\}_{h\geq 0}$. The population estimand would have been the same.

EXAMPLE 2—Long-run identification: Blanchard and Quah (1989) identified the effects of demand and supply shocks using long-run restrictions in a bivariate system. Let gdp_t and unr_t denote log real GDP (in levels) and the unemployment rate, respectively. Then $\Delta\text{gdp}_t \equiv \text{gdp}_t - \text{gdp}_{t-1}$ is log GDP growth. Blanchard and Quah impose that $w_t \equiv (\Delta\text{gdp}_t, \text{unr}_t)'$ follows the SVMA model in Assumption 3 with $n_\varepsilon = 2$ shocks, where the first shock is a supply shock, the second shock a demand shock, and both shocks are invertible; cf. (12). They then additionally impose the identifying restriction that the long-run effect of the demand shock on the *level* of output is zero, that is, $\sum_{\ell=0}^{\infty} \Theta_{1,2,\ell} = 0$.

While Blanchard and Quah impose their long-run restriction on an SVAR model to estimate impulse responses, the extended equivalence result in Section 2.2 implies that the same restriction can be equivalently implemented using an LP approach. To see how, consider, for a large horizon H, the "long difference" projection

$$\text{gdp}_{t+H} - \text{gdp}_{t-1} = \tilde{\mu}_H + \tilde{\beta}'_H w_t + \sum_{\ell=1}^{\infty} \tilde{\delta}'_{H,\ell} w_{t-\ell} + \tilde{\xi}_{H,t}. \tag{14}$$

Intuitively, this projection uncovers the linear combination of the data that best explains long-run movements in GDP. By assumption, such explanatory power can only come from the supply shock. Thus, to estimate impulse responses with respect to the supply shock, we can run the local projection (9) with $b = \tilde{\beta}_H$ and with y_t given by the response variable of interest (either Δgdp_t or unr_t). Indeed, we show formally in Section A.2 that, as $H \to \infty$, this procedure correctly identifies the impulse responses $\Theta_{i,1,h}$ with respect to the supply shock, up to a constant scale factor. In this way, relative impulse responses $\Theta_{i,1,h}/\Theta_{1,1,0}$ are correctly identified.[19] To estimate relative impulse responses $\Theta_{i,2,h}/\Theta_{1,2,0}$ to the demand shock, the researcher can choose any vector $\tilde{b}$ such that $\tilde{b}'b = 0$, and then implement the local projection (9) with $\tilde{b}$ in lieu of b.

In finite samples, the mean squared error performance of the proposed procedure relative to the conventional SVAR(p) approach of Blanchard and Quah (1989) will depend on the tuning parameters H and p, and on whether the low-frequency properties of the data are well approximated by a low-order VAR model.[20] For researchers who prioritize bias over variance, the LP approach to long-run restrictions has the advantage that it does not extrapolate long-run impulse responses from short-run autocorrelations, as a finite-order VAR does.

EXAMPLE 3—Sign Identification: Uhlig (2005) and Rubio-Ramírez, Waggoner, and Zha (2010) considered set-identification via sign restrictions on impulse responses. For concreteness, suppose we are interested in the impulse response of real GDP growth y_t to a monetary shock $\varepsilon_{1,t}$ at horizon h. As before, assume that the full set of observed data $\{w_t\}$ follows an SVMA system (10) where all shocks are invertible. As a very simple example of sign restrictions, we may impose the identifying restriction that the scalar variable r_t (say, the nominal interest rate) responds positively to a monetary shock at all horizons $s = 0, 1, \dots, \bar{H}$.

The traditional SVAR approach to sign identification proceeds as follows. By invertibility, the monetary shock $\varepsilon_{1,t}$ is related to the Wold forecast errors u_t through $\varepsilon_{1,t} = b'u_t$, where $b \in \mathbb{R}^{n_w}$ is an unknown vector. If we knew b, the structural impulse responses of any variable $w_{i,t}$ to $\varepsilon_{1,t}$ could be obtained as the linear combination b of the reduced-form impulse responses of $w_{i,t}$ from a VAR in w_t. To impose the sign restrictions, we search over all possible vectors b such that (i) the r_t impulse responses are positive at all horizons $s = 0, 1, \dots, \bar{H}$ and (ii) the impact impulse response of r_t is normalized to 1 (other normalizations are also possible). Once we have determined the set of possible b's, we can then use the VAR to compute the corresponding set of possible impulse responses of y_t with respect to $b'u_t$.

By the logic in Section 2.2, we can alternatively impose sign restrictions using an LP approach. We simply estimate the reduced-form impulse responses using LPs instead of a VAR. Consider the coefficient vector $\check{\beta}_h$ obtained from the projection

$$y_{t+h} = \check{\mu}_h + \check{\beta}_h' w_t + \sum_{\ell=1}^{\infty} \check{\delta}_{h,\ell}' w_{t-\ell} + \check{\xi}_{h,t}.$$

The above LP yields the reduced-form impulse responses $\check{\beta}_h$ of y_t to the Wold forecast errors u_t. Exactly as in the VAR approach, we now seek the linear combination $b'\check{\beta}_h$ that

[19]Absolute impulse responses can be identified by rescaling the identified shock so it has variance 1.

[20]Christiano, Eichenbaum, and Vigfusson (2006) and Mertens (2012) made the related point that SVAR-based long-run identification could employ nonparametric estimators of the long-run variance matrix instead of the VAR estimator.

equals the structural impulse response to the monetary shock $\varepsilon_{1,t} = b'u_t$. To find the set of b's consistent with the sign restrictions, the natural analogue of the VAR approach is as follows. For each horizon $s = 0, 1, \dots, \bar{H}$, store the coefficient vector $\ddot{\beta}_s$ from the projection

$$r_{t+s} = \ddot{\mu}_s + \ddot{\beta}_s' w_t + \sum_{\ell=1}^{\infty} \ddot{\delta}_{s,\ell}' w_{t-\ell} + \ddot{\xi}_{s,t}.$$

The coefficients $\ddot{\beta}_s$ measure the reduced-form impulse responses of r_t to u_t, so sign restrictions on the structural impulse responses of r_t amount to linear inequality restrictions on these coefficients. Consequently, the *largest possible* response of y_{t+h} to a monetary shock that raises r_t by one unit on impact can be obtained as the solution to the linear program[21]

$$\sup_{b \in \mathbb{R}^{n_w}} b'\check{\beta}_h \quad \text{subject to} \quad b'\ddot{\beta}_0 = 1,$$
$$b'\ddot{\beta}_s \geq 0, s = 1, \dots, \bar{H}.$$

To compute the smallest possible impulse response, replace the supremum with an infimum.[22] In population, this LP-based procedure recovers exactly the same identified set as analogous sign restrictions in an SVAR.

In the Online Appendix, we show how to perform local projection identification with a completely general system of sign, zero, and magnitude restrictions on multiple variables, shocks, and horizons, as in the SVAR frameworks of Rubio-Ramírez, Waggoner, and Zha (2010) and Kilian and Murphy (2012). There we also argue that narrative restrictions as in Antolín-Díaz and Rubio-Ramírez (2018) and Ludvigson, Ma, and Ng (2020b) can be exploited in a local projection framework, by recasting those restrictions in the instrumental variable framework analyzed in Section 3.3 below.

Our examples demonstrate that invertibility-based identification need not be thought of as "SVAR identification," contrary to standard practice in textbooks and parts of the literature. As a matter of identification (i.e., in population), the two methods succeed or fail together. Ideally, researchers ought to decide on the identification scheme separately from how they decide on the finite-sample dimension reduction technique. The former choice should be based on economic theory. The latter choice should be based on the researcher's preferences over bias and variance as well as on features of the DGP.

Beyond Invertibility. If the invertibility assumption (12) is violated, then identification strategies that erroneously assume invertibility—independent of whether they are implemented using VARs or LPs—will not measure the true impulse responses.[23] Instead, these

[21] To consider impulse responses to a *one-standard-deviation* monetary shock, replace the equality constraint in the linear program by the constraint $b' \operatorname{Var}(u_t)^{-1} b = 1$. The resulting linear-quadratic program with inequality constraints is similar to those in Gafarov, Meier, and Montiel Olea (2018) and Giacomini and Kitagawa (2020).

[22] We focus on computing the *bounds* of the identified set. An alternative approach is to sample from the identified set, as is commonly done in the Bayesian SVAR literature (Rubio-Ramírez, Waggoner, and Zha (2010)).

[23] Several recent papers have demonstrated how to perform valid semistructural identification without assuming invertibility; cf. the references in Plagborg-Møller and Wolf (2019). Often such methods rely on LP or VAR techniques to compute relevant linear projections, without interpreting the VAR disturbances (i.e., Wold innovations) as linear combinations of the contemporaneous true shocks.

methods will measure the impulse responses to a white noise disturbance that is a linear combination of current and lagged true structural shocks:

$$\tilde{\varepsilon}_{1,t} = \vartheta(L)\varepsilon_t. \tag{15}$$

The properties of the lag polynomial $\vartheta(L)$ are characterized in detail in Fernández-Villaverde et al. (2007) and Wolf (2020). Combining (10) and (15), we see that, in general, both LP and VAR impulse response estimands are linear combinations of contemporaneous and lagged true impulse responses. Thus, projection on a given identified impulse $\tilde{\varepsilon}_{1,t}$ correctly identifies impulse response functions (up to scale) if and only if $\tilde{\varepsilon}_{1,t}$ affects the response variable y_t *only* through the contemporaneous true structural shock $\varepsilon_{1,t}$. Trivially, this is the case if $\tilde{\varepsilon}_{1,t}$ is a function only of $\varepsilon_{1,t}$ (the invertible case); less obviously, the same is also true if $\tilde{\varepsilon}_{1,t}$ is only contaminated by shocks that do not directly affect the response variable y_t.[24] Instrumental variable identification, discussed next, is the leading example of this second case.

3.3. *Identification and Estimation With Instruments*

Instruments (also known as proxy variables) are popular in semistructural analysis. We here use our main result in Section 2 to show that the influential local projection instrumental variable estimation procedure is equivalent to estimating a VAR with the instrument ordered first. This is true irrespective of the underlying structural model.

An instrumental variable (IV) is defined as an observed variable z_t that is contemporaneously correlated only with the shock of interest $\varepsilon_{1,t}$, but not with other shocks that affect the macro aggregate y_t of interest (Stock (2008), Stock and Watson (2012), Mertens and Ravn (2013)).[25] More precisely, given Assumption 3, the IV exclusion restrictions are that

$$\operatorname{Cov}\big(z_t, \varepsilon_{j,s} \mid \{z_\tau, w_\tau\}_{-\infty<\tau<t}\big) \neq 0 \quad \text{if and only if} \quad \text{both } j = 1 \text{ and } t = s. \tag{16}$$

Stock and Watson (2018, p. 926) refer to this assumption as "LP-IV$^\perp$," and it is routinely made in theoretical and applied work, as reviewed by Ramey (2016) and Stock and Watson (2018). The assumption requires that, once we control for all lagged data, the instrument is not contaminated by other structural shocks or by lags of the shock of interest.

Without loss of generality, we can use projection notation to phrase the IV exclusion restrictions (16) as follows.

ASSUMPTION 4:

$$z_t = c_z + \sum_{\ell=1}^{\infty} (\Psi_\ell z_{t-\ell} + \Lambda_\ell w_{t-\ell}) + \alpha \varepsilon_{1,t} + v_t, \tag{17}$$

where $\alpha \neq 0$, $c_z, \Psi_\ell \in \mathbb{R}$, $\Lambda_\ell \in \mathbb{R}^{1\times n_w}$, $v_t \overset{i.i.d.}{\sim} N(0, \sigma_v^2)$, and v_t is independent of ε_t at all leads and lags. The lag polynomial $1 - \sum_{\ell=1}^{\infty} \Psi_\ell L^\ell$ is assumed to have all roots outside the unit circle, and $\{\Lambda_\ell\}_\ell$ is absolutely summable.

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[24] In particular, this means that neither invertibility nor recoverability (as defined in Plagborg-Møller and Wolf (2019)) are *necessary* for successful semistructural inference on impulse response functions.

[25] We focus on the case of a single IV. If multiple IVs for the same shock are available, Plagborg-Møller and Wolf (2019) show that (i) the model is testable, and (ii) all the identifying power of the IVs is preserved by collapsing them to a certain (single) linear combination.

Crucially, the assumption allows the IV to be contaminated by the independent measurement error v_t. In some applications, we may know by construction of the IV that the lag coefficients Ψ_ℓ and Λ_ℓ are all zero (so z_t satisfies assumption "LP-IV" of Stock and Watson (2018, p. 924), without controls); obviously, such additional information will not present any difficulties for any of the arguments that follow.

LP-IV. The Local Projection Instrumental Variable (LP-IV) approach estimates the impulse responses to the first shock using a two-stage least squares version of LP. Loosely, Mertens (2015), Jordà, Schularick, and Taylor (2015, 2020), Leduc and Wilson (2017), Ramey and Zubairy (2018), and Stock and Watson (2018) proposed to estimate the LP equation (1) using z_t as an IV for x_t. To describe the two-stage least-squares estimand in detail, define $W_t \equiv (z_t, w_t')'$ and consider the "reduced-form" IV projection

$$y_{t+h} = \mu_{\mathrm{RF},h} + \beta_{\mathrm{RF},h} z_t + \sum_{\ell=1}^{\infty} \delta'_{\mathrm{RF},h,\ell} W_{t-\ell} + \xi_{\mathrm{RF},h,t} \tag{18}$$

for any $h \geq 0$. Consider also the "first-stage" IV projection[26]

$$x_t = \mu_{\mathrm{FS}} + \beta_{\mathrm{FS}} z_t + \sum_{\ell=1}^{\infty} \delta'_{\mathrm{FS},\ell} W_{t-\ell} + \xi_{\mathrm{FS},t}. \tag{19}$$

Notice that the first stage does not depend on the horizon h. As in standard cross-sectional two-stage least-squares estimation, the LP-IV estimand is then given by the ratio $\beta_{\mathrm{LPIV},h} \equiv \beta_{\mathrm{RF},h}/\beta_{\mathrm{FS}}$ of reduced-form to first-stage coefficients (e.g., Angrist and Pischke (2009, p. 122)).[27]

Stock and Watson (2018) showed that, under Assumptions 3 and 4, the LP-IV estimand $\beta_{\mathrm{LPIV},h}$ correctly identifies the *relative* impulse response $\Theta_{n_r+2,1,h}/\Theta_{n_r+1,1,0}$. Importantly, this holds whether or not the shock of interest $\varepsilon_{1,t}$ is invertible in the sense of equation (12).

We now use our main result from Section 2.1 to show that the LP-IV impulse responses can equivalently be estimated from a recursive VAR that orders the IV first. As in Section 2, this result imposes no functional form assumptions on the underlying structural model and does not yet impose any exclusion restrictions on the IV z_t (although such assumptions are of course required to interpret the estimand causally, as discussed below).

COROLLARY 1: *Let Assumptions 1 and 2 hold for the expanded data vector $W_t \equiv (z_t, w_t')'$ in place of w_t. Assume also that $\beta_{\mathrm{FS}} \neq 0$, cf. (19). Consider a recursively ordered* SVAR(∞) *in the variables $(z_t, w_t')'$, where the instrument is ordered first (the ordering of the other variables does not matter). Let $\tilde{\theta}_{y,h}$ be the SVAR-implied impulse response at horizon h of y_t with respect to the first shock. Let $\tilde{\theta}_{x,0}$ be the SVAR-implied impact impulse response of x_t with respect to the first shock.*

Then $\tilde{\theta}_{y,h}/\tilde{\theta}_{x,0} = \beta_{\mathrm{LPIV},h}$.

PROOF: Let $\tilde{z}_t \equiv \alpha\varepsilon_{1,t} + v_t$ and $a \equiv \sqrt{E(\tilde{z}_t^2)} = \sqrt{\alpha^2 + \sigma_v^2}$. Proposition 1 states that $\tilde{\theta}_{y,h} = a \times \beta_{\mathrm{RF},h}$ for all h, and $\tilde{\theta}_{x,0} = a \times \beta_{\mathrm{FS}}$. The claim follows. *Q.E.D.*

[26] As always, the coefficients and residuals in (18)–(19) should be interpreted as linear projections.

[27] In the overidentified case with multiple IVs, the IV estimand can no longer be written as this simple ratio; we focus on a single IV as in most of the applied literature.

This essentially reduced-form result implies that, if we additionally impose the *structural* Assumptions 3 and 4, valid identification of relative structural impulse responses can equivalently be achieved through LP-IV or through an "internal instrument" recursive SVAR with the IV ordered first.[28] Importantly, under Assumptions 3 and 4, these equivalent estimation strategies are valid even when the shock of interest $\varepsilon_{1,t}$ is not invertible (Stock and Watson (2018)). Intuitively, although adding the IV z_t to the VAR does not render the shock $\varepsilon_{1,t}$ invertible, the only reason that the shock may be noninvertible with respect to the *expanded* information set $\{z_\tau, w_\tau\}_{-\infty<\tau\leq t}$ is the presence of the measurement error v_t in the IV equation (17).[29] But this independent measurement error merely leads to attenuation bias in the estimated impulse responses, and the bias (in percentage terms) is the same at all response horizons and for all response variables. Thus, it does not contaminate estimation of *relative* impulse responses.

IV identification is therefore an example of a setting where SVAR analysis works even though invertibility fails (including the partial invertibility notion of Forni, Gambetti, and Sala (2019), and Miranda-Agrippino and Ricco (2019)). The "internal instrument" recursive SVAR(∞) procedure estimates the right *relative* impulse responses despite the fact that no invertible structural VAR model generally exists under our assumptions. Our result implies that it is valid to include an externally identified shock in an SVAR even if the shock is measured with (independent) error, as long as the noisily measured shock is ordered first.[30]

SVAR-IV. Unlike the noninvertibility-robust procedure of ordering the IV first in a VAR, the popular SVAR-IV (also known as proxy-SVAR) procedure (Stock (2008), Stock and Watson (2012), Mertens and Ravn (2013)) is only valid under invertibility. This procedure uses an SVAR to identify the shock of interest as

$$\tilde{\varepsilon}_{1,t} \equiv \frac{1}{\sqrt{\operatorname{Var}(\tilde{z}_t^\dagger)}} \times \tilde{z}_t^\dagger,$$

where $\tilde{z}_t^\dagger$ is computed as a linear combination of the reduced-form residuals u_t from a VAR in w_t alone (i.e., excluding the IV from the VAR):

$$\tilde{z}_t^\dagger \equiv E(\tilde{z}_t \mid u_t) = E(\tilde{z}_t \mid \{w_\tau\}_{-\infty<\tau\leq t}).$$

If Assumptions 3 and 4 and the invertibility condition (12) hold, then SVAR-IV is valid. In fact, in this case SVAR-IV removes any attenuation bias, thus correctly identifying *absolute* (not just relative) impulse responses.[31] However, in the general noninvertible

[28]Plagborg-Møller and Wolf (2019) showed that point identification of *absolute* impulse responses—and thus variance decompositions—can be achieved under a further *recoverability* assumption that is mathematically and substantively weaker than assuming invertibility.

[29]Note that, even though Assumption 4 allows z_t to be correlated with lags of w_t, noninvertibility of $\varepsilon_{1,t}$ is entirely consistent with Theorem 1 of Stock and Watson (2018). That theorem states that if the shock is noninvertible, then it is *possible to construct an example* of an IV $\check{z}_t$ satisfying $E(\check{z}_t\varepsilon_{j,t}) = 0$ for all $j \neq 1$ and $E(\check{z}_t\varepsilon_{j,t-\ell} \mid \{w_\tau\}_{\tau<t}) \neq 0$ for some j and $\ell \geq 1$ (so $\check{z}_t$ does not satisfy Assumption 4).

[30]Romer and Romer (2004) and Barakchian and Crowe (2013) included an externally identified monetary shock in an SVAR, but they order it last, which assumes additional exclusion restrictions. Kilian (2006), Ramey (2011), Miranda-Agrippino (2017), and Jarociński and Karadi (2020), among others, mentioned the strategy of ordering an IV first in an SVAR, but these papers do not consider the noninvertible case.

[31]Consistent with our analytical results, Carriero, Mumtaz, Theodoridis, and Theophilopoulou (2015) observed in a calibrated simulation study that, under invertibility, SVAR-IV correctly identifies *absolute* impulse response functions, while direct projections on the IV suffer from attenuation bias.

case, SVAR-IV misidentifies the shock as $\tilde{\varepsilon}_{1,t} \neq \varepsilon_{1,t}$.[32] Plagborg-Møller and Wolf (2019, Appendix B.4) characterized the bias of SVAR-IV under noninvertibility and showed that the invertibility assumption can be tested using the IV.

Summary. The relative impulse responses obtained from the LP-IV procedure of Stock and Watson (2018) are nonparametrically identical to the relative impulse responses from a recursive SVAR with the IV ordered first (an "internal instrument" approach). Assuming an SVMA model and the IV exclusion restrictions, these procedures correctly identify relative structural impulse responses, irrespective of the invertibility of the shock of interest. This allows researchers to exploit VAR estimation techniques while relying on the same invertibility-robust identifying restrictions as the popular two-stage least squares implementation of LP-IV. In contrast, the SVAR-IV procedure of Stock and Watson (2012) and Mertens and Ravn (2013) (an "external instrument" approach) requires invertibility.[33]

4. EMPIRICAL APPLICATION

We finally illustrate our theoretical equivalence results by empirically estimating the dynamic response of corporate bond spreads to a monetary policy shock. We adopt the specification of Gertler and Karadi (2015), who, using high-frequency financial data, obtain an external instrument for monetary policy shocks.[34] Because of possible noninvertibility (Ramey (2016), Plagborg-Møller and Wolf (2019)), we do not consider the external SVAR-IV estimator, but instead implement direct projections on the IV through (i) local projections and (ii) an "internal instrument" recursive VAR, following the logic of Corollary 1. In both cases, our vector of macro control variables exactly follows Gertler and Karadi (2015); it includes output growth (log growth rate of industrial production), inflation (log growth rate of CPI inflation), the 1-year government bond rate, and the Excess Bond Premium of Gilchrist and Zakrajšek (2012) as a measure of the part of the corporate bond spread that is unrelated to default. The data is monthly and spans January 1990 to June 2012.[35]

Figure 2 shows that LP-IV and "internal instrument" VAR impulse response estimates agree at short horizons, but diverge at longer horizons, consistent with Proposition 2. The figure shows point estimates of the response of the Excess Bond Premium to the monetary policy shock, for different projection techniques and different lag lengths. For all specifications, the Excess Bond Premium initially increases after a contractionary monetary policy shock, consistent with the results in Gertler and Karadi (2015). The left panel shows results for LP(4) and VAR(4) estimates. Up until horizon $h = 4$, the estimated impulse responses are closely aligned. At longer horizons, the iterated VAR structure enforces a smooth return to 0, while direct local projections give more erratic impulse responses. The right panel shows an analogous picture for LP(12) and VAR(12) estimates:

[32] The VARX approach of Paul (2019) is equivalent with SVAR-IV under Assumptions 1 and 2.

[33] SVAR-IV does have one advantage over LP-IV (and thus also over the "internal instruments" VAR approach): Provided the shock is invertible, SVAR-IV does not require z_t to only be correlated with lagged shocks through *observed* lagged variables as in Assumption 4, cf. Stock and Watson (2018, Section 2.1).

[34] The external IV z_t is constructed from changes in 3-month-ahead futures prices written on the Federal Funds Rate, where the changes are measured over short time windows around Federal Open Market Committee monetary policy announcement times. See Gertler and Karadi (2015) for details on the construction of the IV and a discussion of the exclusion restriction.

[35] The data were retrieved from: https://www.aeaweb.org/articles?id=10.1257/mac.20130329.

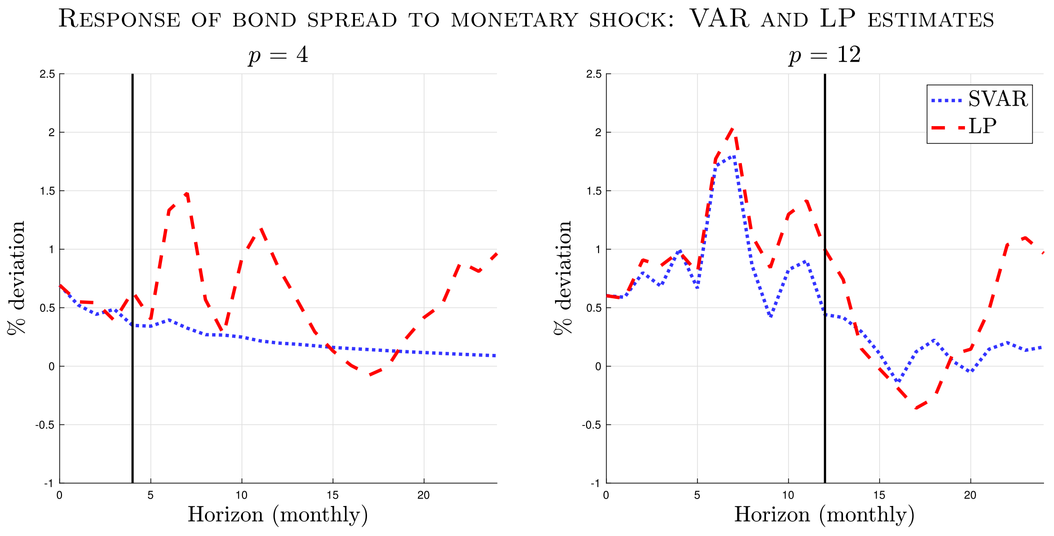

FIGURE 2.—Estimated impulse response function of the Excess Bond Premium to a monetary policy shock, normalized to increase the 1-year bond rate by 100 basis points on impact. Left panel: lag length $p = 4$. Right panel: $p = 12$. The horizontal line marks the horizon p after which the VAR(p) and LP(p) estimates may diverge substantially.

The estimated impulse responses agree closely until horizon $h = 12$, but they diverge at longer horizons.

These results provide an empirical illustration of our earlier claim that LP and VAR estimates are closely tied together at short horizons.[36] The results further illustrate that the larger the lag length used for estimation (in both the LP and VAR specifications), the more impulse response horizons will exhibit agreement between LP and VAR estimates, consistent with Section 2.3. As this exercise is merely meant to illustrate our theoretical results, we refrain from conducting statistical tests of the validity of the different regression models.

5. CONCLUSION

We demonstrated a general equivalence of the local projection and VAR impulse response function estimands. Although these estimation methods are linear, the equivalence between them obtains nonparametrically, in the sense that we do not require linearity or other functional form assumptions on the data generating process. This result, which applies to all common implementations of local projections in the applied literature, has several implications for empirical practice:

1. VAR and local projection estimators of impulse responses should not be regarded as conceptually distinct methods—in population, they estimate the same thing, as long as we control flexibly for lagged data.
2. Local projections with p lags as controls and VAR(p) estimators agree approximately at impulse response horizons $h \leq p$. Hence, VARs that control for a large

[36]In unreported results, we confirm this finding for the impulse response functions of output growth, inflation, and the 1-year government bond rate. Ludvigson, Ma, and Ng (2020a), referencing our paper, reached a similar conclusion in a different empirical study.

number of lags, as recommended by Kilian and Lütkepohl (2017), will tend to agree at short and medium-long horizons with local projections that also control for a rich set of lags. However, at long horizons the methods may disagree substantially.

3. Structural identification is logically distinct from the dimension reduction choices that must be made for *estimation* purposes. It may be counterproductive to follow standard practice in assuming an SVAR model whenever the discussion turns to structural identification, as this conflates the population identification analysis and the dimension reduction technique of using a VAR estimator.
4. Any structural estimation method that works for SVARs can be implemented with local projections, and vice versa. For example, if a paper already relies on local projections for parts of the analysis, then an additional sign restriction identification exercise, say, can also be implemented in a local projection fashion.
5. If an instrument/proxy for the shock of interest is available, structural impulse responses can be consistently estimated by ordering the instrument first in a recursive VAR (an "internal instrument" approach), even if the shock of interest is noninvertible. In contrast, the popular SVAR-IV estimator (an "external instrument" approach) is only consistent under invertibility.
6. Linear local projections and linear VARs will continue to estimate the same impulse responses even when the true data generating process is nonlinear.
7. Although this paper has focused solely on population properties of estimators, the population equivalence result implies that no single estimation method should be expected to dominate in terms of mean squared error across every possible data generating process. In finite samples, and with finite lag lengths, researchers must navigate a bias-variance trade-off at long horizons. Much more research is warranted on this key issue for applied research.

We conclude that each of the following common assertions is, in fact, mistaken: (i) VAR impulse response estimators are generally more efficient than local projection estimators; (ii) local projections are generally more robust to misspecification than VARs; (iii) SVAR analysis is required when implementing nonrecursive, non-IV identification schemes; (iv) simple SVAR methods cannot be used when the structural shock of interest is noninvertible.

Our work points to several promising areas for future research. First, it would be useful to adapt the results in the present paper to nonlinear estimators, such as regressions with interactions or polynomial terms. Second, future research could consider data with unit roots or cointegration. Third, we only discussed the population properties of IV estimators, and thus ignored weak IV issues. Fourth, it would be interesting to generalize our LP-IV equivalence result to settings with multiple instruments/proxies. Finally, we have deliberately avoided questions related to estimation and inference.

APPENDIX A

A.1. *Equivalence Result With Finite Lag Length*

We here prove Proposition 2 from Section 2.3. We proceed mostly as in the proof of Proposition 1. As a first step, the Frisch–Waugh theorem implies that

$$\beta_h(p) = \frac{\operatorname{Cov}\big(y_{t+h}, \tilde{x}_t(p)\big)}{E\big(\tilde{x}_t(p)^2\big)}. \tag{20}$$

We remind the reader of the notation $\mathrm{Cov}^p(\cdot,\cdot)$, which denotes covariances of the data $\{w_t\}$ as implied by the (counterfactual) stationary "fitted" SVAR(p) model

$$A(L;p)w_t = B(p)\bar{\eta}_t, \quad \bar{\eta}_t \sim \mathrm{WN}(0,I), \tag{21}$$

that is, where $\bar{\eta}_t$ is truly white noise (unlike the residuals from the VAR(p) projection on the actual data). For example $\mathrm{Cov}^p(y_t, x_{t-1})$ denotes the covariance of y_t and x_{t-1} that *would* obtain if $w_t = (r_t', x_t, y_t, q_t')'$ were generated by the model (21) with parameters $A(L;p)$ and $B(p)$ obtained from the projection on the *actual* data, as defined in Section 2.3. We similarly define any covariances that involve $\bar{\eta}_t$. Note that stationarity of the VAR model (21) follows from Brockwell and Davis (1991, Remark 2, pp. 424–425).

It follows from the argument in Brockwell and Davis (1991, p. 240) that $\mathrm{Cov}^p(w_t, w_{t-h}) = \mathrm{Cov}(w_t, w_{t-h})$ for all $h \le p$ (see also Brockwell and Davis (1991, Remark 2, pp. 424–425) for the multivariate generalization of the key step in the argument). In words, the autocovariances implied by the "fitted" SVAR(p) model (21) agree with the autocovariances of the actual data out to lag p, although generally not after lag p.

Under the counterfactual model (21), we have the moving average representation $w_t = C(L;p)B(p)\bar{\eta}_t$, and thus

$$\theta_h(p) = C_{n_r+2,\bullet,h}(p)B_{\bullet,n_r+1}(p) = \mathrm{Cov}^p(y_{t+h}, \bar{\eta}_{x,t}), \tag{22}$$

where $\bar{\eta}_{x,t}$ is the (n_r+1)-th element of $\bar{\eta}_t$. Since $B(p)$ is lower triangular by definition, it is straightforward to show from (21) that

$$B_{n_r+1,n_r+1}(p)\bar{\eta}_{x,t} = x_t - E^p\big(x_t \mid r_t, \{w_\tau\}_{t-p\le\tau<t}\big) = x_t - E\big(x_t \mid r_t, \{w_\tau\}_{t-p\le\tau<t}\big) = \tilde{x}_t(p), \tag{23}$$

where $E^p(\cdot\mid\cdot)$ denotes linear projection under the inner product $\mathrm{Cov}^p(\cdot,\cdot)$, the second equality follows from the above mentioned equivalence of $\mathrm{Cov}^p(\cdot,\cdot)$ and $\mathrm{Cov}(\cdot,\cdot)$ out to lag p, and the last equality follows by definition. Since $\mathrm{Cov}^p(\bar{\eta}_{x,t}, \bar{\eta}_{x,t}) = 1$, equation (23) implies

$$B_{n_r+1,n_r+1}(p)^2 = \mathrm{Cov}^p\big(\tilde{x}_t(p), \tilde{x}_t(p)\big) = E\big(\tilde{x}_t(p)^2\big),$$

where the last equality again uses the equivalence of $\mathrm{Cov}^p(\cdot,\cdot)$ and $\mathrm{Cov}(\cdot,\cdot)$ out to lag p. Putting together (22), (23), and the above equation, we have shown that

$$\theta_h(p) = \frac{1}{\sqrt{E\big(\tilde{x}_t(p)^2\big)}} \times \mathrm{Cov}^p\big(y_{t+h}, \tilde{x}_t(p)\big).$$

If we compare with the expression (20), the desired conclusion follows from

$$\mathrm{Cov}^p\big(y_{t+h}, \tilde{x}_t(p)\big) - \mathrm{Cov}\big(y_{t+h}, \tilde{x}_t(p)\big) = \sum_{\ell=0}^{p} \underbrace{\big\{\mathrm{Cov}(y_{t+h}, w_{t-\ell}) - \mathrm{Cov}^p(y_{t+h}, w_{t-\ell})\big\}}_{=0 \text{ when } (t+h)-(t-\ell)\le p} \varrho_\ell(p),$$

where we have used the notation for the projection coefficients $\varrho_\ell(p)$ defined immediately above Proposition 2, and we have again appealed to the equivalence of $\mathrm{Cov}^p(\cdot,\cdot)$ with the covariance function of the actual data.

A.2. *Long-Run Identification Using Local Projections*

Here, we show that the LP-based long-run identification approach in Example 2 is valid. Define the Wold innovations $u_t \equiv w_t - E(w_t \mid \{w_\tau\}_{-\infty<\tau<t})$ and Wold decomposition

$$w_t = \chi + C(L)u_t, \quad C(L) \equiv I_2 + \sum_{\ell=1}^{\infty} C_\ell L^\ell. \tag{24}$$

Since both structural shocks are assumed to be invertible, there exists a 2×2 matrix B such that $\varepsilon_t = Bu_t$. Comparing (10) and (24), we then have $\Theta(1)B = C(1)$. Let $e_1 \equiv (1, 0)'$. Note that the Blanchard and Quah assumption $e_1'\Theta(1) = (\Theta_{1,1}(1), 0)$ implies

$$e_1'C(1) = e_1'\Theta(1)B = \Theta_{1,1}(1)e_1'B,$$

and, therefore,

$$e_1'C(1)u_t = \Theta_{1,1}(1) \times e_1'Bu_t = \Theta_{1,1}(1) \times \varepsilon_{1,t}.$$

By the result in Section 2.2, the claim in Example 2 follows if we show that

$$\lim_{H\to\infty} \tilde{\beta}_H' = e_1'C(1). \tag{25}$$

Define $\Sigma_u \equiv \mathrm{Var}(u_t)$. Applying the Frisch–Waugh theorem to the projection (14), and using $w_{1,t} = \Delta\mathrm{gdp}_t$, we find

$$\begin{aligned}\tilde{\beta}_H' &= \mathrm{Cov}(\mathrm{gdp}_{t+H} - \mathrm{gdp}_{t-1}, u_t)\Sigma_u^{-1} \\ &= \mathrm{Cov}\left(\sum_{\ell=0}^{H} w_{1,t+\ell}, u_t\right)\Sigma_u^{-1} = \sum_{\ell=0}^{H} \mathrm{Cov}(w_{1,t+\ell}, u_t)\Sigma_u^{-1}.\end{aligned} \tag{26}$$

On the other hand, the Wold decomposition (24) implies (recall that u_t is white noise)

$$\sum_{\ell=0}^{\infty} \mathrm{Cov}(w_{t+\ell}, u_t)\Sigma_u^{-1} = \sum_{\ell=0}^{\infty} C_\ell = C(1). \tag{27}$$

Comparing (26) and (27), we get the desired result (25).

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