Two's Company, Three's an Equilibrium: Strategic Voting and Multicandidate Elections

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Abstract

In this paper, we characterize equilibria in games of electoral competition between three or more office-seeking candidates. Recognizing that electoral equilibrium involves both candidates' and voters' strategies, we first prove existence of pure strategy electoral equilibria when candidates seek to maximize their vote share. Accordingly, the main difficulty with electoral equilibria is multiplicity. We prove that, even after restricting attention to subgame perfect Nash equilibria in weakly undominated strategies, the set of electoral equilibria is very large. We provide several characterizations of candidates' equilibrium platforms, including a set of conditions under which equilibrium platforms are located in the minmax set. We also examine welfare implications of the results, connections between the noncooperative equilibria, the core, and the uncovered set. Finally, we show that the paper's results do not depend upon whether any or all of the candidates are maximizing their vote-share, plurality, or their probability of victory.

Keywords: *Multicandidate elections, undominated equilibrium, spatial competition, minmax set.*

1 Introduction

> "The formal theory of elections, originated by Hotelling [1929], Downs [1957], and Black [1958], has developed despite a well-known and unresolved foundational issue arising in the basic model: the existence of equilibrium in multidimensional policy spaces." (Duggan and Jackson [2005], p.1)

In this paper, we prove the existence of pure strategy, subgame perfect Nash equilibria in electoral competition with three or more candidates. In addition to resolving the foundational issue described by Duggan and Jackson, the paper's results have the virtue of simplicity. By including the voters' voting strategies in the definition of electoral equilibria, it is quite to simple to construct electoral equilibria: the well-known generic nonexistence of pure strategy equilibria in complete information models of multidimensional electoral competition results directly from the standard presumption that voters vote sincerely. Thus, analogous to the results of Farquharson [1969], Shepsle and Weingast [1984], Banks [1985], Miller [1995], and others, incorporating strategic voters within our theory of electoral competition results in a well-defined set of possible equilibrium outcomes.

Importantly, the results highlight a theoretical and empirical difficulty for the study of elections with more than two candidates or parties. Simply put, the results indicate that there are "too many" electoral equilibria in such systems. As opposed to the parsimony of Black's median voter theorem (Black [1948]), which offers a precise prediction for unidimensional electoral competiton between two candidates and the general failure of (pure strategy) electoral equilibria to exist when voters are assumed to vote sincerely, the allowance for strategic voting creates a new issue: so long as there are three or more candidates, *almost any* observed profile of party platforms is consistent with a pure strategy, subgame perfect Nash equilibrium in which no voters is using a weakly dominated voting strategy. Accordingly, from a theoretical side, the results of this paper demonstrate that existence of a pure strategy Nash equilibrium in electoral competition in which voters do not use weakly dominated strategies is a problem only when there are *exactly* two parties.

From an empirical viewpoint, the results of this paper imply that the hypothesis that an observed profile of election platforms was the result of equilibrium behavior by vote- or office-seeking candidates is *unfalsifiable* without auxiliary data about voters' voting strategies. For example, the main result of this paper is that, without data demonstrating that some candidate *would have received more votes by a unilateral change in his or her platform*, then any profile of platforms in which at least three candidate's platforms differ from each other is consistent with a pure strategy subgame perfect Nash equilibrium in which voters are using weakly undominated voting strategies. In addition to attempting to inform the testing of different theories of electoral competition, the multiplicity of equilibrium policy profiles can be viewed as a call for more work that explicitly includes other aspects of electoral competition (*e.g.*, activists, campaigns, more detailed models of electoral institutions, and interest groups) in our theoretical study of elections.[1]

The seminal results of Plott [1967], McKelvey [1976], Schofield [1977, 1978], Rubinstein [1979], McKelvey and Schofield [1987], and others have highlighted the sensitivity of pure strategy equilibrium existence in two-candidate electoral competition to the distribution of voters' preferences. In particular, under the assumption of sincere voting behavior, pure strategy electoral equilibria in two-candidate competition generically do not exist unless the policy space is unidimensional.[2] Furthermore, since sincere voting is clearly the most defensible behavioral assumption from a strategic perspective in two-alternative voting games (for example, sincere voting is weakly undominated in this case), this existence problem is not ameliorated by a more explicit incorporation of the voters within the model. Maintaining the assumption of sincere voting, this existence problem extends to multicandidate competition – even in the unidimensional case.[3] In contrast with the two candidate case, however, sincere voting is not uniquely defensible as a behavioral starting point when there are more than two candidates. Indeed, whereas sincere voting

[1] Of course, such aspects have been considered by scholars. A (clearly non-exhaustive) list of examples is Aldrich [1983], Ware [1992], Skaperdas and Grofman [1995], David F. Damore [1997], Wright and Riker [1997], Carsey and Layman [1999], McGann et al. [2002], Miller and Schofield [2003], and Schofield and Sened [2005].

[2] The question of mixed-strategy Nash equilibria has also recently attracted attention (*e.g.*, Laffond et al. [1993], Dutta and Laslier [1999], Banks et al. [2002], and Duggan and Jackson [2005]).

[3] Two important works in this tradition are Cox [1987] and Shepsle [1991].

in the two-candidate case will always produce a Nash equilibrium in the voting game following any pair of policy announcements by the candidates, this is not necessarily the case when there are three or more candidates. Because this fact is well known and has been famously applied in several theories of voting (*e.g.*, McKelvey and Ordeshook [1972], Austen-Smith and Banks [1996]), it is somewhat surprising that relaxing the assumption of sincere voting is a relatively new innovation in the literature on multicandidate spatial competition. Furthermore, both empirical (*e.g.*, Cox [1997]) and experimental (*e.g.*, McKelvey and Ordeshook [1985], Forsythe et al. [1993], and Forsythe et al. [1996]) research has indicated that voters do account for other voters' preferences and likely behaviors when determining the instrumental impact of their own vote choice.

In this paper, we demonstrate formally that, not only does the incorporation of strategic voting within the theory of electoral competition between $K \geq 2$ candidates or parties yield existence of pure strategy equilibria, but indeed produces "too many" equilibria. We then investigate several refinements of electoral equilibria, focusing on equilibria in which voters use weakly undominated voting strategies.

The main formal results of the paper are:

1. Convergent equilibria (*i.e.*, equilibria in which all candidates announce the same platform) are exceptional: they require a very high degree of consensus among the voters' policy preferences and are also equilibria under the assumption of sincere voting (Proposition 1).

2. *Any* policy profile with at least 3 distinct platforms is part of some equilibrium (Theorem 2).

3. *Any* policy that is part of a majority rule cycle can be observed as the winning policy in equilibrium (Theorem 3).

4. Under some reasonable assumptions, there exist pure-strategy subgame perfect equilibria in weakly undominated strategies in which platforms are located in the *minmax set* (Proposition 4). The minmax set is centrally located and relatively "small."

5. Though convergence in equilibrium is exceptional, the minmax set represents a centrist re-

finement, particularly in two- and three-candidate elections (Proposition 7).

1.1 Related Literature

Duggan and Jackson [2005] also examine multidimensional and multicandidate electoral competition. In line with this paper, Duggan and Jackson examine equilibria in which the voters' behaviors are endogenous and focus attention on weakly undominated voting strategies.[4] They provide numerous powerful results, examining a wide array of candidate and voter motivations, as well as demonstrating a link between the uncovered set and equilibria of two-candidate electoral competition. Due to the breadth of applicability of their existence results, Duggan and Jackson establish existence of *mixed strategy* equilibria, whereas the results provided here establish existence of *pure strategy* equilibria.[5]

McKelvey and Patty [2006] examine a model of probabilistic voting in which voters are allowed to vote strategically in multicandidate, multidimensional elections. They prove an asymptotic equilibrium existence result, showing that, for a large enough electorate, there exists a convergent equilibrium in which all candidates announce the policy that maximizes the sum of the electorates' expected policy payoffs, which is the "usual suspect" for such an equilibrium in these models.[6] They prove that their equilibrium is unique for two-candidate elections, but they do not investigate the possibility of asymmetric equilibria when there are more than two candidates. The main contribution of their work is the inclusion of strategic voting within a probabilistic framework. However, while McKelvey and Patty demonstrate that this does not eliminate (at least not asymptotically) the "usual suspect" equilibrium profile, they also do not inquire as to whether the allowance for strategic probabilistic voting behaviors produces more equilibrium policy profiles.

Schofield [2004] argues in favor of local Nash equilibria as the proper solution concept for elec-

[4] Technically, Duggan and Jackson use a slightly weaker notion of undominated voting strategies, which they dub "dominated*." The difference is minor, and coincides in two-candidate competition.

[5] Similarly, Duggan and Fey [2005] examine the possibility of policy motivations by candidates, but in two candidate elections.

[6] For example, see Hinich [1977], Coughlin and Nitzan [1981a,b], Coughlin [1992], Lin et al. [1999], Banks and Duggan [2004].

toral games.[7] Proposition 1 of this paper demonstrates that, under the assumption that voters use weakly undominated strategies, policy coincidence in equilibrium is exceptional: full policy coincidence in equilibrium requires not only that a core exists – in fact, the use of weakly undominated voting strategies implies that policy coincidence occurs in equilibrium only if a super-majority of voters have identical ideal points. Furthermore, the required super-majority is an increasing function of the number of candidates.

Caplin and Nalebuff [1988, 1991a] consider electoral stability in the presence of two alternatives, one of which is possibly a privileged status quo. The provide relatively weak sufficient conditions for the distribution of preferences and the super-majority proportion required to overturn the status quo to ensure that some policy is indeed stable (*i.e.*, unbeatable) as the status quo. As discussed in Section 5 of this paper, Caplin and Nalebuff's use of the minmax set provides a useful starting point for defining a multicandidate solution set in spatial election games.[8]

The work of Osborne and Slivinsky [1996] and Besley and Coate [1997] establish equilibrium existence in electoral competition through the presumption that candidates can not credibly commit to implementing anything other than their own most-preferred policy. In this setting, the strategic choice of any candidate is simply whether to to stand for office or remain on the sidelines. An additional connection between Besley and Coate [1997] and the setting examined here is the refinement of equilibria by requiring that voters use weakly undominated voting strategies.[9]

This paper provides a number of important results. First, in contrast with the two-candidate, multidimensional case (where pure strategy electoral equilibria often fail to exist), the problem with equilibrium existence in this setting is that they are *too* prevalent. A set of our main results illustrates that, even after restricting attention to subgame perfect Nash equilibria in weakly undominated strategies, most profiles of policy announcements can occur in equilibrium.

The second set of results do rule out some policy profiles as equilibria when voters use weakly

[7] The use of local Nash equilibrium is similarly adopted in Patty [2006, 2005] and Duggan [2000].

[8] Recently, Schofield [1996, 1999] has forwarded the notion of the *heart* as a multicandidate solution set. The *heart* explicitly incorporates bargaining and policy motivations, neither of which are dealt with in this paper.

[9] De Sinopoli and Turrini [2002] extend the application of this refinement within the Besley and Coate [1997] framework to the candidate's strategies.

undominated voting strategies (for example, as mentioned above, Proposition 1 rules out equilibria exhibiting full policy coincidence except in cases where the voters have extremely homogeneous policy preferences). Some of these results depend upon a further intuitive refinement – symmetry, which requires that all candidates receive the same vote share at the equilibrium policy profile. Taken together, the results highlight the following fact: any significant restriction of the set of equilibrium policy profiles will be the result of a refinement similar to symmetry (*i.e.*, one based on the vector of candidates' payoffs at the equilibrium policy profile) rather than refinements of the voters' behaviors (*e.g.*, subgame perfection and weakly undominated strategies) or, surprisingly, refinements directly based upon the policies chosen by the candidates. This final point is most clearly made by Theorem 2, which states that any policy profile in which at least 3 payoff-distinct policy positions are announced can be supported as a subgame perfect Nash equilibrium in which voters use weakly undominated voting strategies. Thus, sensible and *a priori* interesting refinements such as requiring that at least one candidate choose a policy in the minmax set or any other set (*e.g.*, the Pareto set or the uncovered set) will *not* meaningfully refine the set of equilibrium policy profiles.

2 The Model

We consider a model of electoral competition with a policy space, denoted by $X \subset \mathbf{R}^M$, with $M \geq 1$. There are two collections of players: N is a set of $n \geq 3$ voters, and $\mathcal{K}$ is a set of K candidates. We make the reasonable assumption that there are fewer candidates than voters, $K < n$, and that there are more available policies than there are candidates, $K < |X|$.[10] Each voter $i \in N$ is characterized by a *type*, t_i, the set of which is denoted by $T = X$. We denote the *type profile* by $\tau = \{t_i\}_{i \in N}$. Throughout, we assume that τ is common knowledge between the voters and the candidates.

Each voter's payoff depends on the winning candidate's policy and the voter's type, t_i, accord-

[10]The number of policies may be infinite, for example.

ing to function $u : X \times T \to \mathbf{R}$. We assume that each voter's utility function is given by

$$u(x,t) = -||x-t||, \tag{1}$$

where $||\cdot||$ denotes the l^2-norm (or "Euclidean norm").[11] Thus, for any voter $i \in N$, t_i denotes voter i's most preferred-policy (*i.e.*, $\{t_i\} = \text{argmax}_{x \in \mathbf{R}^M} u(x,t)$) and is referred to as i's *ideal point*. For simplicity, this point is assumed to exist and be unique for each type $t \in T$.[12] For any type profile τ and any two distinct positions $x, y \in X$, let $n_\tau(x,y) = |\{i \in N : u(x,t_i) < u(y,t_i)\}|$ denote the number of voters who *strictly* prefer y to x.

Finally, the following notion of payoff-distinctness is central to the presentation of the paper's main results. In words, we refer to a set of policies as *payoff-distinct* if no voter is indifferent between any pair of policies in the set. Note that, for any finite integer $K > 1$ and for any ideal point $t \in \mathbf{R}^M$, almost all finite collections of K alternatives are payoff-distinct given any ideal point $t \in \mathbf{R}^M$.[13] Accordingly, for any finite set of n voters' ideal points, $\tau \in T^n$, this condition is similarly satisfied for almost all finite collections of K alternatives.

Definition 1 *For any type profile τ and any set of m policies $Y = \{x_1, \ldots, x_m\}$, with $m \in \{2,3,\ldots\}$ and $Y \subseteq X$, Y is referred to as* payoff-distinct *if, for each pair $j,k \in \{1,2,\ldots,m\}$ with $j \neq k$, the following holds:*

$$\forall i \in N, \; u_i(x_j,t_i) \neq u_i(x_k,t_i).$$

[11] The arguments in this paper do not rely on the functional form of u. The main requirement that one would want to impose on u is that it have a well-defined and unique maximizing policy in X for each type $t \in T$. Relaxing this assumption does not lead to any real increase in generality. However, it would greatly complicate the statement of the appropriately extended version of Proposition 1.

[12] Since allowing for multiple most-preferred policies for any given type would make the requirement that voting strategies be weakly undominated *less* demanding, relaxing this assumption would not change the results in any interesting way: more policy profiles will be supportable as pure strategy equilibria in weakly undominated voting strategies. Relaxing the assumption that a most-preferred policy exists would not change the results in any interesting way either: it would simply add to the notational complexity of the paper, since the set of candidates is assumed to be finite.

[13] The satisfaction is satisfied by "almost all" finite collections of K alternatives in the sense that that the set of K policies that do not satisfy it are contained in a set with Lebesgue measure zero in $\left(\mathbf{R}^M\right)^M$.

For any $k \in \{2, \ldots, K\}$, a finite collection of policies, $x \in X^K$, is said to "contain k payoff-distinct policies" if there exists a subset of x containing k policies such that x is payoff-distinct.

The Electoral Game. As is traditional in spatial models of elections, the game begins with each candidate $k \in \mathcal{K}$ announcing a policy platform, $x_k \in X$. These choices are announced simultaneously by all candidates, after which the voters observe the profile of these announcements, denoted by $x = (x_1, x_2, \ldots, x_K) \in X^K$. In addition, for any profile x and any candidate $d \in \mathcal{K}$, we denote the set of profiles that are identical to x with respect to all elements except possibly the d^{th} by $D(x, d)$.[14] The set of all profiles that differ from x by no more than one element is denoted by $\mathbf{D}(x) = \cup_{d \in \mathcal{K}} D(x, d)$. Finally, for any two profiles $x, y \in X$, let $\delta(x, y) \subseteq \mathcal{K}$ be a set-valued function indicating the indices of the elements that differ between x and y (*e.g.*, $\delta((6, 2, 9), (5, 2, 7)) = \{1, 3\}$).

After x is made common knowledge, the voters each simultaneously pick exactly one candidate to vote for, with voter i's choice denoted by $v_i \in \mathcal{K}$. The profile of vote choices for all voters is denoted by $a = \{a_i\}_{i \in N}$ and the profile of vote choices for all voters other than $i \in N$ is denoted by a_{-i}. The *voting strategy* for player i is denoted by $v_i : X^K \to \mathcal{K}$, with the vote choice of voter i after announcement of $\tilde{x} \in X^K$ being denoted by $v_i(\tilde{x})$. A profile of voting strategies for all voters $i \in N$, is denoted by $v = \{v_i\}_{i \in N}$.[15]

Given a profile of vote choices a, the set of candidates who receive the most votes is denoted by $W(a) \subset \mathcal{K}$. The winning candidate is denoted by $w \in W(a)$ and is chosen randomly and fairly from $W(a)$. The winner's policy announcement, x_w, is implemented as the final policy outcome.

Each candidate k's payoff is equal to the number of votes received by k. Using $\mathbf{1}[Z]$ to denote the indicator function that takes on value 1 whenever Z is true and 0 otherwise, this payoff function

[14] The notation is intended to denote "the set of possible **D**eviations from x by candidate d."

[15] We restrict attention to pure voting strategies. The proofs make abundantly clear that this restriction is without consequence.

for candidate $k \in \mathcal{K}$, given a profile of voting strategies, v, is formally defined as

$$\pi_k(x; v) = \sum_{i \in N} \mathbf{1}[v_i(x) = k].$$

When the context is clear, the notation $\pi_k(a)$ will be used to denote the number of votes received by candidate k in a profile of vote choices a.

Electoral Competition as Two Games In One. An equilibrium model of electoral competition involves (at least) two components: the choice of platforms by the candidates, followed by the choice of whom to vote for by the voters.[16] Given a profile of announcements, $x \in X^K$, we refer to the voting game after the announcement of x as the *x-voting game*. As will be made clear below, any subgame perfect equilibrium of the electoral competition game must involve Nash equilibrium behavior by the voters in the x-voting game for all $x \in X$. Similarly, given a profile of voting strategies v, we refer to the candidates' platform choice game as the *v-positioning game*. It is important to note that an electoral competition equilibrium, (x^*, v^*), describes behavior in both of these components, *including the hypothetical behaviors of the voters following announcements other than x^**. We now use these two components to construct a formal definition of equilibrium in the model of electoral competition.

Definition 2 *For any profile of announcements $x \in X^K$, a profile of vote choices, $a^* \in \mathcal{K}^N$, is a Nash equilibrium of the x-voting game if, for each voter $i \in N$ and all $\hat{a} \in \mathcal{K}^N$ with $\hat{a}_{-i} = a^*_{-i}$,*

$$\frac{1}{|W(\hat{a})|} \sum_{w \in W(\hat{a})} u(x_w, t_i) \leq \frac{1}{|W(a^*)|} \sum_{w \in W(a^*)} u(x_w, t_i).$$

Definition 3 *For any profile of voting strategies $v : X^K \to \mathcal{K}^N$, a profile of announcements,*

[16] Of course, a model of electoral competition could include other components as well, including the participation decisions of potential candidates and voters.

$x^* \in X^K$, is a Nash equilibrium of the v-positioning game if, for each candidate $k \in \mathcal{K}$,

$$\pi_k(\hat{x}_k, x^*_{-k}; v) \leq \pi_k(\hat{x}^*_k, x^*_{-k}; v).$$

Definition 4 *A pair of strategy profiles (x^*, v^*) is a* Nash equilibrium of the electoral game *if*

1. *x^* is a Nash equilibrium of the v^*-positioning game and*

2. *$v^*(x^*)$ is a Nash equilibrium of the x^*-voting game.*

Definition 5 *A pair of strategy profiles (x^*, v^*) is a* subgame perfect Nash equilibrium of the electoral game *if*

1. *x^* is a Nash equilibrium of the v^*-positioning game and*

2. *$v^*(x)$ is a Nash equilibrium of the x-voting game for all $x \in X^K$.*

The set of subgame perfect Nash equilibria for an electoral game with type profile τ is denoted by $SPN(\tau)$.

2.1 Weakly Undominated Voting

An appealing refinement of Nash equilibrium is to require that no player i choose an action a that is dominated in the sense that there exists an action b such that b never yields a lower payoff for i than does a and that, in at least one situation (in terms of the other players' actions) yields a strictly higher payoff for i than is yielded by action a in that situation. A Nash equilibrium in which no player uses a strategy that calls for the use of a dominated action is referred to as a Nash equilibrium in weakly undominated strategies.[17]

Given a profile of platforms $x \in X^K$ and any voter $i \in N$, a vote choice $a_i \in \mathcal{K}$ is *weakly undominated* if either

[17] This refinement is also referred to as "undominated Nash equilibrium," particularly in the implementation literature (*e.g.*, Palfrey and Srivastava [1991] and Jackson [1992]). Unfortunately, while "undominated Nash equilibrium" is succinct, it is also ambiguous, sometimes being used to describe a Nash equilibrium that is not Pareto dominated by any other Nash equilibrium (*e.g.*, Le Breton and Weber [1997], Codognatoy and Ghosalz [2003]).

1. there exists $\hat{a}_i \in \mathcal{K}$ such that $u(x_{\hat{a}_i}, t_i) < u(x_{a_i}, t_i)$ or

2. for all $\hat{a}_i \in \mathcal{K}$, $u(x_{\hat{a}_i}, t_i) = u(x_{a_i}, t_i)$.

In words, a vote choice is weakly undominated if the voter is either not voting for (one of) his or her least-preferred candidate(s) or if the voter is indifferent between all candidates. For any announcement profile x and type $t \in T$, let $V^W(x,t) \subset \mathcal{K}$ denote the set of weakly undominated vote choices. Extending the notion of weak undomination to strategies, a voting strategy v_i is weakly undominated if it chooses a weakly undominated vote choice for every profile of announcements: $v_i(x) \in V^W(x, t_i)$ for all $x \in X^K$. The set of weakly undominated voting strategies for a voter with type t_i is denoted by $\bar{V}^W(t_i)$ and the set of all profiles of weakly undominated voting strategies for a type profile τ is denoted by $\bar{V}^W(\tau)$.

Definition 6 *A pair of strategy profiles* (x^*, v^*) *is a* subgame perfect Nash equilibrium in weakly undominated strategies *(UE) if*

1. *x^* is a Nash equilibrium of the v^*-positioning game,*

2. *$v^*(x)$ is a Nash equilibrium of the x-voting game for all $x \in X^K$, and*

3. *v^* is a profile of weakly undominated voting strategies.*

Denote the set of UE, for type profile τ, by

$$UE(\tau) = \{(x, v) \in SPN(\tau) : v \in \bar{V}^W(\tau)\}.$$

The following fact is key to our analysis.

Fact 1 *For any type profile τ and announcement profile x^*, there exists a Nash equilibrium in which every voter i uses a weakly undominated action, $v_i \in V^W(x^*, t_i)$.*

3 Electoral Equilibrium Existence

It is well-known that the existence of positioning equilibria under the presumption that voters vote sincerely is difficult to guarantee. Allowing for strategic voting ameliorates the problem greatly, though, as the choice of voters' behaviors offers a significant "degree of freedom" in supporting a profile of candidate announcements, x, as a Nash equilibrium.

3.1 Subgame Perfect Nash Equilibria

The first result is immediate and provided as justification of restrictions that we will examine afterward. Theorem 1 states that *any* profile of announcements can be supported as a subgame perfect Nash equilibrium.

Theorem 1 *For any type profile τ and any announcement profile $x \in X^K$, there exists a profile of voting strategies v^x such that $(x, v^x) \in SPN(\tau)$.*

Proof: Fix τ and $x \in X^K$. Let $\hat{v}$ be a profile of constant functions, with $\hat{v}_i(y) = 1$ for all $i \in N$ and $y \in X^K$ (*i.e.*, $\hat{v}$ is the profile of voting strategies in which each voter always votes for candidate 1, regardless of the candidates' policy announcements). Clearly, $x \in X^K$ is a Nash equilibrium of the $\hat{v}$-positioning game (all candidates are indifferent between all announcement profiles in X^K). Similarly, it is clear that $\hat{v}$ is a Nash equilibrium of the x-voting game, since the presumption that $n \geq 3$ implies that no voter can be pivotal, given that the the other $n - 1$ voters are voting for the same candidate. Accordingly, letting $v^x = \hat{v}$, it follows that $(x, v^x) \in SPN(\tau)$. ∎

The proof of Theorem 1 highlights a well-known "game-theoretic quirk" of voting games: anything can be supported as a Nash equilibrium in any non-unanimity, anonymous voting game with any number of alternatives. This quirk can be eliminated in (almost all) two-alternative elections by requiring that voters use weakly undominated strategies: in a two alternative voting game, this refinement requires that any voter with a strict preference between the two alternatives vote for his or her most-preferred alternative.[18] The next section explores the effect on equilibrium announcement profiles when voters are required to use weakly undominated voting strategies.

3.2 Equilibria in Weakly Undominated Voting Strategies

The next proposition links equilibrium policy coincidence with weakly undominated voting strategies with equilibrium with sincere voting. Let $v^{\mathbf{sin}}$ denote the profile of sincere voting strategies in which each voter randomizes with equal probability between all candidates offering him or her the highest utility.[19]

Proposition 1 *For any type profile τ and any convergent position profile $\hat{x} = (x, x, \ldots, x)$, there exists a profile of voting strategies v^* such that $(\hat{x}, v^*) \in UE(\tau)$ if and only if $(\hat{x}, v^{\mathbf{sin}})$ is a Nash equilibrium.*

In words, the proposition states that, so long as there is no equilibrium with sincere voting under a type profile τ, then *no* UE involves full policy coincidence: an equilibrium position profile must involve at least two payoff-distinct positions.

The next result, Proposition 2, provides necessary and sufficient conditions for UE announcement profiles to contain exactly two payoff-distinct elements. The most interesting sufficient condition requires four or more candidates: any policy profile with two payoff-distinct policy announcements can be supported in a weakly undominated Nash equilibrium so long as each policy announcement is announced by at least two candidates.

Proposition 2 *For any type profile τ and any position profile x^* containing exactly 2 payoff-distinct positions, there exists a profile of voting strategies v^* such that $(x^*, v^*) \in UE(\tau)$ if and only if either of the following holds for x^*:*

[18] Individual indifference is the reason that this refinement does not eliminate the quirk for all two-alternative voting games. If, in a two-alternative voting game with alternatives x and y, the number of voters who are indifferent between x and y is no less than the absolute value of the difference between the number of individuals who strictly prefer x to y and the number who strictly prefer y to x, then either x or y can be supported as a Nash equilibrium in weakly undominated strategies.

[19] As discussed by Duggan and Jackson [2005], this canonical assumption about individual "tie-breaking" can lead to nonexistence of (even mixed strategy) equilibrium. That fact makes Proposition 1 even more discriminating.

1. *each element of x^* appears more than once (*i.e.*, no policy is announced by only one candidate) or*

2. *denoting the policy announced by only one candidate by y and the policy announced by the remaining $K-1$ candidates by z,*

$$y \in \underset{x \in X}{\operatorname{argmax}}\ n_\tau(z, x).$$

Proof: The proof consists of three parts: (*i*) sufficiency of condition 1, (*ii*) sufficiency of condition 2, and (*iii*) necessity of at least one of the two conditions. For all steps, let τ be a fixed type profile.

(*i*): *Sufficiency of condition 1.* Suppose that x^* contains exactly 2 payoff-distinct positions, y and z, each of which appears more than once. Satisfaction of this implies $K \geq 4$. Without loss of generality, suppose $x_1^* = x_2^* = y$ and $x_3^* = x_4^* = z$. The presumption that y and z are payoff-distinct implies that, for any $i \in N$,

$$\left[\{1,2\} \subseteq V^W(x^*, t_i)\right] \Leftrightarrow \left[\{3,4\} \not\subseteq V^W(x^*, t_i)\right].$$

Let v^* be defined as follows. For each $i \in N$ and all $\hat{x} \in \mathbf{D}(x^*)$, let[20]

$$v_i^*(\hat{x}) = \begin{cases} 1 & \text{if } 1 \in V^W(x^*, t_i) \text{ and } 1 \notin \delta(\hat{x}, x^*) \\ 2 & \text{if } 1 \in V^W(x^*, t_i) \text{ and } 1 \in \delta(\hat{x}, x^*) \\ 3 & \text{if } 3 \in V^W(x^*, t_i) \text{ and } 3 \notin \delta(\hat{x}, x^*) \\ 4 & \text{if } 3 \in V^W(x^*, t_i) \text{ and } 3 \in \delta(\hat{x}, x^*) \end{cases}.$$

Notice that

$$\left[j \in V^W(x^*, t_i)\ \&\ \hat{x} \in \mathbf{D}(x^*)\right] \Rightarrow \left[j \in V^W(\hat{x}, t_i)\right],$$

so that v^* produces a weakly undominated vote choice for all $i \in N$ and all $\hat{x} \in \mathbf{D}(x^*)$. For any

[20]Note that this definition will work for any $K \geq 4$.

$x \notin \mathbf{D}(x^*)$, let $v^*(x)$ be any weakly undominated Nash equilibrium for the x-voting game. The profile x^* is a Nash equilibrium of the v^*-positioning game because all candidates' payoffs are unchanged by any unilateral deviation from x^*.

(*ii*): *Sufficiency of condition 2.* Suppose that candidate 1 announces y, and the remaining candidates (2, 3, ..., K) all announce $z \neq y$, so that $x^* = (y, z, \ldots, z)$. By the assumption that y and z are payoff-distinct implies that, for any $i \in N$,

$$\left[1 \in V^W(x^*, t_i)\right] \Leftrightarrow \left[\{2, 3\} \not\subseteq V^W(x^*, t_i)\right].$$

Let $v^*(\hat{x})$ be defined as follows. For each $i \in N$ and all $\hat{x} \in \mathbf{D}(x^*)$, let

$$v_i^*(\hat{x}) = \begin{cases} 1 & \text{if } 1 \in V^W(\hat{x}, t_i) \\ 2 & \text{if } 1 \notin V^W(\hat{x}, t_i) \text{ and } 2 \notin \delta(\hat{x}, x^*) \\ 3 & \text{if } 1 \notin V^W(\hat{x}, t_i) \text{ and } 2 \in \delta(\hat{x}, x^*) \end{cases}.$$

Finally, for any $x \notin \mathbf{D}(x^*)$, let $v^*(x)$ be any weakly undominated Nash equilibrium for the x-voting game. The profile x^* is a Nash equilibrium of the v^*-positioning game because all candidates' payoffs are unchanged by any unilateral deviation from x^*. It is straightforward to verify that no candidate strictly benefits from deviating from x^* (and candidates 1 and 2 may lose votes if they deviate).

(*iii*): *Necessity of at least one of the conditions.* Consider a profile of announcements in which candidate 1 announces y and the remaining candidates (2, 3, ..., K) all announce $z \neq y$, so that $x^* = (y, z, \ldots, z)$. Now suppose that v^* is a profile of weakly undominated voting strategies such that $(x^*, v^*) \in UE(\tau)$. Accordingly, there must exist no $y' \neq y$ such that y^* is strictly preferred to z by strictly more voters than strictly prefer y to z. Formally, for all $y' \in X$,

$$n_\tau(z, y') \leq n_\tau(z, y),$$

satisfaction of is equivalent to $y \in \operatorname{argmax}_{x \in X} n_\tau(z, x)$.

Finally, any profile x^* having two payoff-distinct elements that does not satisfy (up to relabeling the candidates) the supposition x^* that candidate 1 announces some position y and the other $K-1$ candidates all choose an identical position $z \neq y$ must satisfy condition 1: each of the two payoff-distinct elements of x^* must appear more than once in x^*. ∎

3.3 Divergent Equilibria: The Main Result

Having considered convergent equilibria and equilibria in which candidates announce two payoff-distinct platforms, we now consider equilibria in which candidates announce at least three payoff-distinct platforms. The main result of this section (and the paper) is that *any* announcement profile containing at least 3 payoff-distinct platforms can be supported as a subgame perfect Nash equilibrium in weakly undominated strategies. Before proceeding to the discussion and proof this results, we state and prove two lemmas that simultaneously simplify and clarify the proof of the main theorem. The first lemma states that, given any announcement profile with at least 3 payoff-distinct elements, the set of weakly undominated vote choices for any voter i maintains at least one common element following a unilateral deviation by any candidate k. Furthermore, there exists at least one common element for voter i *other than* k. This lemma is key to the proof of the main result: for any announcement profile with at least 3 payoff-distinct elements, no candidate can unilaterally deviate in such a way as to make himself or herself the unique undominated vote choice for *any* voter i. This conclusion plays a key role in ensuring that, any given announcement profile x^* with at least 3 payoff-distinct positions, one can construct a profile of weakly undominated strategies that does not reward any candidate for unilaterally deviating from x^*.

Lemma 1 *For $K \geq 3$, let $x^* \in X^K$ contain at least three payoff-distinct elements. Then, for any candidate $d \in \mathcal{K}$, any announcement profile $\hat{x} \in D(x^*, d)$, and any voter $i \in N$,*

$$\left(V^W(x^*, t_i) \cap V^W(\hat{x}, t_i)\right) \setminus \{d\} \neq \emptyset.$$

Proof: For any $i \in N$, there exist $j^i, k^i \in \mathcal{K} \setminus \{d\}$, such that $u(x^*_{k^i}, t_i) > u(x^*_{j^i}, t_i)$. Similarly, since $j^i \neq d$ and $k^i \neq d$, $u(\hat{x}_{k^i}, t_i) = u(x^*_{k^i}, t_i) > u(x^*_{j^i}, t_i) = u(\hat{x}_{j^i}, t_i)$. Accordingly, $k^i \in \big(V^W(x^*, t_i) \cap V^W(\hat{x}, t_i)\big)$, as was to be shown. ∎

The next lemma states that, for any announcement profile x^* in which each voter has at least 2 weakly undominated vote choices and any candidate k, one can construct a weakly undominated Nash equilibrium to the x^*-voting game such that candidate k receives *no* votes. This result complements Lemma 1 in the proof of the main result. In particular, Lemma 2 ensures that we can construct an *equilibrium* profile of weakly undominated voting strategies that does not reward any candidate for unilaterally deviating from any announcement profile x^* with at least 3 payoff-distinct elements.

Lemma 2 *For any candidate $k \in \mathcal{K}$ and any announcement profile x^* such that, for all $i \in N$, $V^W(x^*, t_i)$ contains at least 2 elements, there exists a profile of vote choices $a^{x^*,k}$ such that $a^{x^*,k}$ is a weakly undominated Nash equilibrium of the x^*-voting game and $\pi_k(a^{x^*,k}) = 0$.*

Proof: Fix some $\varepsilon > 0$ and consider the following alteration of the x^*-voting game: for each voter $i \in N$, redefine voter i's payoff from candidate k's position as follows:

$$u(x^*_k, t_i) = \begin{cases} \min_{j \in \mathcal{K}}[u(x^*_j, t_i)] & \text{if } \min_{j \in \mathcal{K}}[u(x^*_j, t_i)] < \max_{j \in \mathcal{K}}[u(x^*_j, t_i)] \\ \min_{j \in \mathcal{K}}[u(x^*_j, t_i)] - \varepsilon & \text{otherwise} \end{cases}.$$

In this altered game, for each voter i, the set of weakly undominated vote choices is simply $V^W(x^*, t_i) \setminus \{k\}$. By the supposition that $V^W(x^*, t_i)$ contains at least 2 elements for each voter i, the set of weakly undominated vote choices in the altered game is nonempty and a subset of his or her weakly undominated vote choices in the original game for each voter i. By Fact 1, the set of weakly undominated Nash equilibria for this altered game is nonempty. All that remains to be shown is that a weakly undominated Nash equilibrium of the altered game is also a weakly undominated Nash equilibrium of the original game. Denote a weakly undominated Nash equilibrium of the altered game by a'.

Verification that a' is also an equilibrium of the original game requires verifying that no voter i can strictly increase his or her payoff by switching his or her vote, a'_i, to the candidate of interest, k. The reason that no voter has such an incentive is that no voter can be pivotal for candidate k under a'. This follows because a' is defined such such that $a'_i \neq k$ for any $i \in N$: thus, the assumption that $n \geq K$ implies that $\max_{j \in \mathcal{K}} \pi_j(a') \geq 2$. Accordingly, since $\pi_k(a') = 0$, no voter has an incentive to vote for candidate k under any such vector a'. Accordingly, any Nash equilibrium in weakly undominated choices in the altered game, a', is also a Nash equilibrium in weakly undominated choices in the original game in which $\pi_k(a') = 0$. Thus, letting $a^{x^*,k} = a'$, the lemma follows. ∎

With these preliminary lemmas in hand, we are in a position to state and prove Theorem 2, which states that any announcement profile containing at least three payoff-distinct positions can be supported as a UE. Before stating the result, we first discuss the intuition behind it. The presumption that an announcement profile x^* has three or more payoff-distinct elements implies that, for any voter $i \in N$, the set of weakly undominated vote choices in the x^* voting game, $V^W(x^*, t_i)$, contains at least two elements. Lemma 1 implies that, for any voter i with type $t_i \in T$, it the case that, following any *unilateral* deviation by any candidate k from x^*, x', there exists at least one element of $V^W(x^*, t_i)$ other than k that is also an element of $V^W(x', t_i)$. Lemma 2 then implies that it is possible to construct a Nash equilibrium in which voter i votes for such a candidate at x'. Accordingly, candidate k can not strictly benefit from deviating to x'. Finally, the construction of the profile of voting strategies for policies that differ from x^* by more than one element is straightforward – for the voting game associated with such a policy, simply choose any weakly undominated Nash equilibrium.[21]

Theorem 2 *For any type profile τ and any position profile x^* containing at least 3 payoff-distinct positions, there exists a voting strategy v^* such that $(x^*, v^*) \in UE(\tau)$.*

[21]This point highlights one potential avenue for further research. One could examine the construction of weakly undominated voting equilibria that support announcement profiles under some equilibrium notion that is stronger than Nash (*e.g.*, coalition-proof Nash equilibrium; Bernheim et al. [1987]).

Proof: The proof is constructive (*i.e.*, it defines a v^* satisfying the conclusion of the theorem). To this end, fix a type profile τ and an announcement profile $x^* \in X^K$ such that x^* has at least 3 payoff-distinct elements.

Let $v^*(x^*)$ be any weakly undominated Nash equilibrium profile of vote choices for the x^*-voting game. Now consider any $\hat{x}$ such that $\hat{x} \in D(x^*, d)$ for some $d \in \mathcal{K}$, with $x^* \neq \hat{x}$. By Lemma 1, for each voter $i \in N$, there exists some candidate $k_d^i \in \mathcal{K} \setminus \{d\}$ such that $k_d^i \in V^W(x^*, t_i) \cap V^W(\hat{x}, t_i)$. Accordingly, set $v_i^*(\hat{x}) = k_d^i$. If such a v^* is not a Nash equilibrium of the $\hat{x}$-voting game, then Lemma 2 ensures that we can alter $v^*(\hat{x})$ so as to create an equilibrium profile of weakly undominated vote choices at $\hat{x}$, $\hat{v}^*(\hat{x})$ such that $\pi_d(\hat{x}; \hat{v}^*) = 0$. After altering v^* to such a $\hat{v}^*$ if necessary and denoting the result simply by v^*, it follows that $\pi_d(\hat{x}; v^*) \leq \pi_d(x^*; v^*)$. Since the deviating candidate was chosen arbitrarily, it follows that x^* is a Nash equilibrium of the v^*-positioning game.

Finally, to complete the construction of v^*, for any $\hat{x}'$ such that $\hat{x}'$ and x^* differ by more than one component, let v^* be any Nash equilibrium in weakly undominated strategies for the $\hat{x}'$-voting game. Since v^* is a Nash equilibrium in weakly undominated strategies for every x-voting game, it follows that $(x^*, v^*) \in UE(\tau)$, as was to be shown. ∎

The main implication of Theorem 2 is far-reaching: regardless of the distribution of voters' types, there exists essentially no restriction on the set of nonconvergent announcements that can be observed in equilibrium, even if we refine the set of voting equilibria to require the use of weakly undominated strategies following any given announcement. This fact has welfare implications, as we discuss in Section 4.1.

4 Social-Choice & Game-Theoretic Refinements

There are several other approaches one might consider when attempting to construct a smaller (and perhaps more reasonable or appealing) equilibrium set of policy outcomes. In this section, we examine restrictions based on properties of policy outcomes, in terms of social welfare as well as candidates' incentives.

4.1 Condorcet Cycles, Welfare, and the Uncovered Set

For any type profile τ, let $\mathcal{P}(\tau) \subseteq X$ denote the *Pareto set* with respect to τ. Formally, a policy x belongs to $\mathcal{P}(\tau)$ if and only if there exists no other policy y such that, given τ, no voter receives a strictly lower payoff from y and some voter receives a strictly higher payoff from y than from x. If $|X \setminus \mathcal{P}(\tau)| \geq 3$ (*i.e.*, there are at least three points outside the Pareto set), Theorem 2 implies that there exist equilibria in which no candidate announces a policy in the Pareto set. However, suppose that we assume that at least one candidate announces a policy in $\mathcal{P}(\tau)$. Theorem 2 does not necessarily imply that a policy outside of the Pareto set can win in a weakly undominated Nash equilibrium, given this restriction.

Unfortunately, the next result, Theorem 3, implies that *any* policy comprising part of a majority rule cycle within a profile of announcements can be supported as the winning outcome in a UE, regardless of whether the policy is Pareto dominated by some other announced policy. Prior to stating the result, define the following binary relations on X:

- The *strict majority preference relation* on X with respect to τ, $\succ_\tau$, defined as follows:

$$y \succ_\tau x \Leftrightarrow n_\tau(x, y) > n_\tau(y, x).$$

- The *covering relation* on X with respect to τ, C_τ, defined as follows:

$$yC_\tau x \Leftrightarrow [z \succ_\tau y \Rightarrow z \succ_\tau x] .$$

Then, using $\succ_\tau$ and C_τ, recall the following solution concepts within X for any type profile τ:

- The *core* with respect to τ, denoted by $\mathcal{C}o(\tau)$, is defined as follows:

$$\mathcal{C}o(\tau) = \left\{ x \in X : \max_{y \in X} n_\tau(x, y) \leq \frac{n}{2} \right\}.$$

- The *uncovered set* (Miller [1977]) with respect to τ, denoted by $UC(\tau)$, is defined as follows:

$$\mathcal{UC}(\tau) = \{x \in X : \nexists y \in X \text{ such that } yC_\tau x\}.$$

Theorem 3 *For any type profile τ and any position profile x^* containing a set of 3 payoff-distinct positions $\Delta = \{y_1, y_2, y_3\} \subset x^*$ with $y_1 \succ_\tau y_2 \succ_\tau y_3 \succ_\tau y_1$ and any $z \in \Delta$, there exists a voting strategy v^* such that $(x^*, v^*) \in UE(\tau)$, $W(v(x^*)) = w$ is a singleton, and $x_w = z$.*

For any τ in which the core is empty, Theorem 3 implies that any policy $x \in X$ for which there exists no UE in which x is the winning outcome must not belong to the uncovered set (*i.e.*, x must be covered with respect to τ by some other policy $y \in X$). This is stated in the following corollary.

Corollary 1 *Consider any type profile τ such that $\mathcal{C}o(\tau) = \emptyset$. For any uncovered policy $y \in \mathcal{UC}(\tau)$, there exists a pair of strategy profiles $(x^*, v^*) \in UE(\tau)$ with $y \in x^*$ and $j \in W(v^*(x^*))$ such that $x_j^* = y$.*

4.2 Candidate Objectives

An alternative approach to refining the set of equilibria involves assuming that candidates seek to maximize their probability of winning office, rather than maximizing the share of votes received. In general, these two candidate objectives do not yield the same equilibria (see Patty [2006, 2005] for more on this topic). It turns out that when candidates pursue office, rather than votes *per se*, we can refine the set of policy profiles that can be supported in a UE.

The next result, Proposition 3, states a simple necessary and sufficient condition for the existence of a UE exhibiting full policy coincidence when candidates seek to maximize their probability of winning the election. Any such UE must involve all candidates announcing some position in the core with respect to the type profile τ, $\mathcal{C}o(\tau)$. As is well known, the core is frequently empty in multidimensional spatial settings. In such a case, then, the following result implies that there exists *no* UE exhibiting full policy convergence when candidates are office-seekers.

Proposition 3 *Suppose that candidates maximize their probability of victory. Then, for any type profile* τ *and any convergent position profile* $\hat{x} = (x, x, \ldots, x)$*, there exists a profile of voting strategies* v^* *such that* $(\hat{x}, v^*) \in UE(\tau)$ *if and only if* $x \in \mathcal{C}o(\tau)$.

Proof: The proof consists of two parts: sufficiency and necessity.

Sufficiency: ($x \in \mathcal{C}o(\tau) \Rightarrow \exists v^*$ such that $(\hat{x}, v^*) \in UE(\tau)$). Fix τ, and since the hypothesis is null otherwise, suppose that $\mathcal{C}o(\tau) \neq \emptyset$. Choose any $x \in \mathcal{C}o(\tau)$ and let $\hat{x} = (x, x, \ldots, x)$. Then consider the following profile of voting strategies. Since all candidates' positions are identical, any vote choice by any voter i following the announcement of $\hat{x}$ is weakly undominated. Thus, let $v_i^*(\hat{x}) = 1$ for all voters $i \in N$.

Consider any candidate $d \in \mathcal{K}$ and any $x' \in D(\hat{x}, d)$. Then, for any $i \in N$, let

$$v_i^*(x') = \begin{cases} 1 & \text{if } d \neq 1 \text{ or } 1 \in V^W(x', t_i) \\ 2 & \text{if } d = 1 \text{ and } 2 \in V^W(x', t_i) \\ d & \text{otherwise.} \end{cases}$$

For all such policy profiles x', each voter is clearly casting a weakly undominated ballot. Furthermore, by the presumption that $x \in \mathcal{C}o(\tau)$ and the fact that the specification of v_i^* implies that a voter i will vote for the deviating candidate d if and only if $u_i(x'_d, t_i) > u_i(x, t_i)$, the number of votes received by d must be represent no more than half of membership of N. Accordingly, this is less than the votes received by candidate 1 (candidate 2 if $d = 1$), implying that d does not win following the announcement of x'. (Note that it is irrelevant whether d has a positive probability of winning at $\hat{x}$.)

Finally, the specification of v for x'' that differ from $\hat{x}$ on more than one component is irrelevant: for each such policy profile, simply choose a profile of voting strategies such that no voter is casting a weakly dominated ballot.

Necessity: ($\exists v^*$ such that $(\hat{x}, v^*) \in UE(\tau) \Rightarrow x \in \mathcal{C}o(\tau)$). Fix τ and x, with profile $\hat{x} = (x, x, \ldots, x)$. The presumption that $(\hat{x}, v^*)$ is a UE implies that, for any candidate $d \in \mathcal{K}$ and any

$x' \in D(\hat{x}, d)$, it must be the case that

$$|\{i \in N : v_i^*(x') = d\}| < \max_{j \neq d} |\{i \in N : v_i^*(x') = j\}|,$$

implying that

$$|\{i \in N : v_i^*(x') = d\}| < \frac{n}{2},$$

while the presumption that v^* involves weakly undominated strategies for all voters $i \in N$ implies that, for each voter $i \in N$ such that $v_i^*(x') \neq d$, it must be the case that $u(x, t_i) \geq u(x'_d, t_i)$. Since this holds for all x'_d, it follows that $x \in \mathcal{C}o(\tau)$. ∎

4.3 Dominance Solvability

An additional refinement that one might impose is the use of *iteratively weakly undominated* voting strategies. Such an examination would be more compatible with the arguments of Farquharson [1969], for example. However, the fact that many plurality voting games are not dominance solvable, along with the results of De Sinopoli [2000] and Dhillon and Lockwood [2004], indicate that this refinement is weaker than one might hope for. Theorem 4 states that the substantive conclusions of Theorem 3 survive this stronger refinement of voting equilibria.

Theorem 4 *For any type profile τ and any position profile x^* containing a set of 3 payoff-distinct positions $\Delta = \{y_1, y_2, y_3\} \subset x^*$ such that*

$$\lambda_\tau(y_i, \Delta \setminus \{y_i\}) < \frac{1}{2} \text{ for all } i \in \{1, 2, 3\} \tag{2}$$

implies that for each $y \in \Delta$, there exists a subgame perfect Nash equilibrium in iteratively weakly undominated voting strategies, (x^*, v^y) in which $W(v^y(x^*)) = \{y\}$.

Proof: Theorem 3 implies that the vector of voting strategies v^y can be constructed. Theorem 2 of Dhillon and Lockwood [2004] implies that satisfaction of (2) is a sufficient condition for the

x^*-voting game to not be dominance solvable. ∎

5 The Minmax Set

It is not too strong to summarize the previous sections as providing uniformly negative results about the predictive power of game theoretic models electoral politics. Of course, this tenor does not resonate with the relative (inductive) predictability of real-world electoral outcomes (*e.g.*, Gelman and King [1993], Adams et al. [2004], Schofield and Sened [2005]). Accordingly, we now presume that there are a continuum of voters and examine a promising candidate for further refinement of the set of equilibria that we have examined thus far.[22] This refinement, the *minmax set*, is motivated by the fact that, for a broad class of distributions, the minmax set presents a (preference-profile-specific) bound on equilibrium platforms. Furthermore, the minmax set is both centrally located in X and relatively "small."[23] Before presenting our results for this refinement, we note a set of technical modifications to the framework examined in the previous sections.

1. The notion of a type profile, τ, is replaced by a *type distribution*, which is represented by a probability density function $f : X \to \mathbf{R}_+$. We maintain the assumption that this distribution is common knowledge to all voters and candidates, and require that f be *log-concave*.[24]

2. A profile of voting strategies is now a function $v : X \times X^K \to \mathcal{K}$.[25] The realization of the voting strategy v for a voter with ideal point $t \in X$, given a platform profile $x \in X^K$, is denoted by $v(t, x)$.

[22]The presumption of a continuum of voters is more than an analytical simplification. This paper's focus on equilibria in weakly undominated voting strategies and the results presented above clarify that the fact that no voter can be pivotal when he or she is merely one of an uncountably infinite set of voters will not "enlarge" the set of equilibria in any meaningful way. Put another way, the results presented above did not depend on there being a finite number of voters. The assumption that n was finite actually made the proofs "harder," if only marginally so.

[23]It should be noted, however, that the leverage is not obtained without cost – in particular, the results of the earlier sections imply the existence of equilibria that do not fall within the minmax set (*e.g.*, the discussion on p. 7).

[24]A probability distribution function f is log-concave if $\log[f]$ is a concave function. Many probability distributions are log-concave, including the uniform, normal, exponential, and many elements of the Beta and Gamma families.

[25]Note that this is a slight restriction, in the sense that it requires that voters with the same ideal point use the same voting strategy. As with the restriction of attention to pure voting strategies (and for the same reason), this restriction is without consequence.

3. The payoff function for any candidate $k \in \mathcal{K}$, given the type distribution, f, and the voting strategy is now redefined as

$$\pi_k(x; v) = \int_{t \in X} \mathbf{1}[v(t, x) = k] f(t) dt.$$

4. Finally, for any pair of policies $(x, y) \in X^2$, we denote the proportion of voters who strictly prefer policy y to policy x by $n(x, y)$.[26]

5.1 The Minmax Set and Equilibrium Platforms

We focus on equilibria satisfying two criteria. The first requirement is *symmetry*, which requires that all parties or candidates receive the same proportion of votes.

Definition 7 *For any type profile τ, a pair of strategy profiles $(x; v)$ is* symmetric *if, for all $(j, k) \in \mathcal{K}^2$,*

$$\pi_j(x; v) = \pi_k(x; v) = \frac{1}{K}.$$

The second requirement is that some candidates choose positions inside the minmax set (Kramer [1977]). This set is defined as follows. Let $\tilde{n}(x) = \sup_y n(x, y)$ be the "vulnerability level" of x. And, let $m^* = \inf_x \tilde{n}(x)$. This is the *minmax number* – the minimum vulnerability level. Finally, let $\mathcal{M} = \{x | \tilde{n}(x) = m^*\}$. This is the *minmax set* – the set of policies with vulnerability level equal to the minmax number.

Caplin and Nalebuff [1991a] show that, given a convex set of alternatives X and a log-concave distribution of voter ideal points, the minmax number is bounded from above by $1 - 1/e \approx 0.632$.[27] In the simplest case, where X is unidimensional, $\mathcal{M}$ is simply the median voter and $m^* = 1/2$.

[26] The log-concavity of f implies for any pair of policies x and y with $x \neq y$, the share of voters who are indifferent between x and y is zero. Thus, the fraction of voters who strictly prefer x to y is $n(y, x) = 1 - n(x, y)$.

[27] Their Theorem 1 states that with a ρ-concave distribution of ideal points, the *mean* point of the density f has a vulnerability level of $1 - \left[\frac{M+1/\rho}{M+1+1/\rho}\right]^{M+1/\rho}$. This expression is monotonically increasing in n, less than .64, and has a limit as $\rho \to 0$ (*i.e.*, log-concavity) of $1 - 1/e$. However, the mean point is not necessarily in the minmax set.

Their result allows us to derive the following propositions.[28]

The first result is that if the minmax set contains at least K points, then there exists a symmetric equilibrium in which all platforms are located inside.

Proposition 4 *Suppose $|\mathcal{M}| \geq K$. Let x be any vector of candidate positions such that $x_k \in \mathcal{M}$ for all k, and $x_k \neq x_j$ for all k and j. Then there exists an equilibrium in which $\pi_k = 1/K$ for all k.*

Proof: Let $\beta_k(x)$ be the number of voters who rank candidate k last given x. Evidently, for each candidate k, $\beta_k(x) \leq \tilde{n}(x_k) = m^* \leq .64$. This means, in particular, that $\tilde{n}(x_k) = m^* \leq 2/3$. Thus, the fraction of voters that do *not* rank candidate k last is $1 - \beta_k(x) \geq 1/3$. Thus, we can always construct a voting equilibrium with $\pi_k = 1/K$ for all k.

Consider voting strategies satisfying the following: (1) no voter votes for their least favorite candidate; (2) if the candidates adopt the positions specified by x, then voters divide their votes so $\pi_k = 1/K \leq 1/3$ for all k; (3) if any candidate k deviates, then all voters who would have voted for it under x vote for one of the other candidates that they do not rank last, so $\pi_k = 0$.

To show the existence of an allocation of votes satisfying (1) and (2), we construct a voting partition of X iteratively. Pick any partition $\{X_k\}_{j=1}^{K}$ of X such that $v_i = k$ if $t_i \in X_k$ and $\pi_k = 1/K$ for each k. Consider each X_j ($j = 1, \ldots, K$) in order. For each j, let μ_{-j} be the measure of voters in X_j who rank candidate j last. If $\mu_{-j} = 0$, then iteration j is finished. Otherwise, the minmax set and log-concavity imply the existence of a measure of at least $\overline{\mu}_{-j} = (K-1)/K - .64 + \mu_{-j}$ voters not voting for candidate j who do not rank candidate j last. Since $K \geq 3$, $\overline{\mu}_{-j} > \mu_{-j}$. Now for the set of voters in X_j ranking candidate j last, we can find a set of measure μ_{-j} of voters not voting for candidate j who do not rank candidate j last. Finally, exchange the votes of these two sets. Thus no voters in X_j choose their lowest-ranked candidate and no voters who switch votes to candidate j rank it last, while the voting outcome remains $\pi_k = 1/K$ for each k.

[28] For an overview of other applications of these developments, see Caplin and Nalebuff [1991b].

Thus each voter is choosing a weakly undominated strategy, and no candidate wants to deviate, so we have a subgame perfect equilibrium in weakly undominated strategies. ∎

The intuition of Proposition 4 is that since $m^* < 2/3$, then at least $1/3 > 1/K$ voters do *not* rank each candidate last. This makes it possible for $1/K$ voters not to choose a weakly dominated strategy in voting for each candidate.[29]

There are subgame perfect equilibria in which some candidates adopt the same positions in $\mathcal{M}$, but not all of these candidates can win in such equilibria. Thus, equilibria featuring policy convergence are not necessarily symmetric. As an example, suppose that $K = 3$ and $x_1^* = x_2^*$. Then all voters who prefer x_3^* to x_2^* must vote for candidate 3. This is true for at least 36% of voters, so candidate 3 wins with certainty if candidates 1 and 2 split the remaining 64% evenly. Note, however, that this argument does not hold when there are at least three distinct platforms.

There can also be subgame perfect equilibria where some candidates locate "far away" from the minmax set, but such candidates cannot win in a plurality-rule contest. In particular, if candidate k is ranked last by at least $(K-1)/K$ of the voters, then it cannot come in first.

One issue with Proposition 4 is that the number of candidates may exceed the size of the minmax set. This will be true when $M = 1$ (in which case the minmax set is simply the median voter), and in some cases for higher dimensions as well.[30] Caplin and Nalebuff [1988] show that if f is uniform and X is a triangle, then the minmax set is the triangle's center of gravity. For such cases, as well as others in which the minmax set is not open, it is still possible to show the existence of a symmetric equilibrium in which some platforms are in the minmax set.

Proposition 5 *There exists an equilibrium in which $x_k \in \mathcal{M}$ for some k, $x_k \neq x_j$ for $i \neq j$, and $\pi_k = 1/K$ for all k.*

Proof: Without loss of generality, suppose that $x_1^* \in \mathcal{M}$. Let $\beta_k(x)$ be the number of voters who rank candidate k last given x. We construct a vector of K platforms x such that $\beta_k(x) < 1 - 1/K$

[29] The result can be also be stated more generally in terms of the minmax number, rather than the minmax set.

[30] Kramer [1977] argues that the size of the minmax set is increasing in the level of social agreement, and so more uniform distributions of voter ideal points will tend to have smaller minmax sets.

for each k. Since $m^* < .64$, this holds trivially for x_1. Observe that for any platform z, there exists another platform y such that $n(z, y) > 0.36$. Let $z = x_1$ and choose any such y as x_2. Repeat this operation for $k > 2$ by substituting x_{k-1} for z and setting $x_k = y$, where $x_k \neq x_j$ for all $j < k$. To ensure that distinct platforms can be chosen, note that because f is continuous and preferences are Euclidean, for any y' and z and $\alpha \in (0, 1)$, $n(z, \alpha z + (1 - \alpha)y')$ is strictly increasing in α. Thus there are an infinite number of points ($\{y'\}$) satisfying the desired criteria for y.

As constructed, at least $.36$ of citizens prefer each x_k to some x_j ($j \neq k$). Thus, $\beta_k(x) < 1 - 1/K$ for $K \geq 3$. We may therefore apply the argument from Proposition 4, establishing the result. ∎

This result uses the continuity of f, which ensures that for any platform z there exists a nearby platform y that many voters will prefer to z. Since $m^* \geq 1/2$ for all n, it is always possible to find y such that $n(z, y)$ is near $1/2$. As a result, y will not be the least-preferred choice of enough voters, which makes the construction of a symmetric equilibrium possible.

5.2 A Partial Analytical Characterization of the Minmax Set

For many distributions, the minmax set is difficult to characterize analytically. However, we may exploit a relationship between the minmax set and Pareto sets to derive an analytical bound of the minmax set. To see this, denote by $B_z(p)$ the ball centered around z containing proportion p of voter ideal points. The next result shows that the minmax set must be contained within the intersection of all $B_z(m^*)$.

Proposition 6 $\mathcal{M} \subseteq \cap_{z \in X} B_z(m^*)$.

Proof: Let $P(m^*)$ be any Pareto set that contains at least proportion m^* of voter ideal points. Since preferences are Euclidean and f is continuous and log-concave, for any $x \notin P(m^*)$, there exists some $x' \in P(m^*)$ such that $n(x, x') > m^* + \epsilon$ for some $\epsilon > 0$. Thus, $\mathcal{M} \subseteq P(m^*)$. Since any $B_z(m^*)$ is a Pareto set, $\mathcal{M}$ must belong to all such balls. ∎

With a log-concave distribution of ideal points, $\mathcal{M}$ is then a subset of all balls $B(2/3)$, since $\mathcal{M}$ lies in the intersection of all Pareto sets with more than m^* voters in them, and $m^* < 2/3$. Thus, for equilibria of the type identified in Propositions 4 and 5, one can derive an "outer bound" on platform locations simply by constructing the intersection of balls with probability mass $2/3$.

5.3 Examples of the Minmax Set

The following numerical example illustrates the preceding construction. The program, written in R, first draws a random sample of N points from a multivariate normal distribution. The distribution is centered at $\mathbf{0}$ and has uniform variance and no covariance.[31] It then estimates the boundary of the intersection of the two-thirds balls, and categorizes points as inside or outside this boundary.

The categorization algorithm works as follows. Along each dimension in a given sample, points are sorted from lowest to highest. The distances between $\mathbf{0}$ and the points at which two-thirds of the points have lower and higher values provide two estimates of the bounds of $\cap_{z \in X} B_z(m^*)$. This is based on the fact that a ball centered sufficiently far from $\mathbf{0}$ along this dimension has an arc approximated by an orthogonal hyperplane intersecting either of the aforementioned points. Each point with a rank order higher than one third of the points along this dimension is therefore within the ball $B_z(2/3)$, where the element of z in this dimension is ∞; likewise, each point with a rank lower than two thirds of the points along this dimension is within $B_z(2/3)$, where the element of z in this dimension is $-\infty$. Note that these two-thirds boundaries provide the smallest estimate of the theoretical radius of $\cap_{z \in X} B_z(2/3)$ along this dimension, as they are the limiting case for all circles. Due to radial symmetry, finding the value in one dimension provides an estimate of the radius in all directions, but estimating in more than one direction provides greater accuracy. The estimated radius is calculated by averaging across both estimates from all dimensions. Distances from $\mathbf{0}$ can be easily calculated for each point, and all points further than the radius are classified as outside $\cap_{z \in X} B_z(2/3)$.

Table 1 summarizes the results from 150 trials. The program was run 10 times for each combi-

[31] Code available upon request.

nation of dimensions $(1, \ldots, 5)$ and variance $(1, 2, 3)$. Each trial drew $N = 10,000$ points, except for the five-dimension trials, which drew $N = 100,000$ for greater accuracy. As intuition suggests, the proportion of points within all $B_z(2/3)$ is about one third for the unidimensional case. Note that for each dimension, the proportion within the ball is invariant with respect to variance, and that for each variance level, the proportion within the ball is invariant with respect to dimension. The table clearly shows that the proportion of points in the intersection falls dramatically as the number of dimensions increases. Even with three dimensions, only about 2% of the points are located within our estimated upper bound of the minmax set.[32]

[Table 1 here]

In some cases, the bounds on the minmax set suggested by Proposition 6 can be derived analytically. Suppose that $X \equiv \left\{ z \mid -1 \leq z_1 \leq 1, 0 \leq z_2 \leq \sqrt{1 - x_1^2} \right\}$ is a unit half-circle, and $f(z) = 2/\pi$ is uniform over z. Using the same argument as in the proof of Proposition 6, it is straightforward to derive some Pareto sets which bound the minmax set.

Consider the following four Pareto sets; $P_1 \equiv \{z \in X \mid z_1 \leq \alpha_1\}$, $P_2 \equiv \{z \in X \mid z_1 \geq \alpha_2\}$, $P_3 \equiv \{z \in X \mid z_2 \leq \alpha_3\}$, and $P_4 \equiv \{z \in X \mid z_2 \geq \alpha_4\}$. We are therefore interested in finding α_k such that each P_k has probability mass $2/3$. The integral of the boundary of X is $[\sqrt{1 - z^2} + \sin^{-1} z]/2 + c$. From this, it is easily seen that $\alpha_1 \approx 0.265$ and $\alpha_3 \approx 0.553$. By symmetry, $\alpha_1 = -\alpha_2 = \alpha_4$. Taking the intersection of these sets yields a rectangular superset of the minmax set with an area of 0.153, which accounts for about 9.7% of the probability mass of the ideal points under f.

5.4 Extension: The Centrality of Platforms

Our results suggest that the minmax set deserves some attention as a "solution set" for multi-dimensional electoral competition. The main intuition behind our results has been the importance

[32]One possible objection to this example is that the symmetry of the normal distribution implies the satisfaction of the Plott [1967] conditions, which are sufficient for the existence of a majority rule core. We have modified the simulation slightly to accommodate multivariate Type I extreme value distributions. The results, available upon request from the authors, are qualitatively similar to those presented in Table 1, and accordingly omitted.

of not being the lowest-ranked candidate by at least $1/K$ of the voters. Here we push that logic a step further, and derive a more general result on the centrality of equilibrium platform locations.

As before, let $B_z(p)$ be a ball centered at z containing proportion p of the ideal points, and let $r(p)$ be its radius. Let $B_z^3(p)$ be the ball centered at z with radius $3r(p)$. The next result uses weak dominance (but not the minmax set) to establish that in a symmetric equilibrium, if all platforms but one are centrally located in the sense of being located within some $B_z(p)$, then none can be outside of $B_z^3(p)$.

Proposition 7 *Let $p > (K-1)/K$. For any symmetric equilibrium and $B_z(p)$, if $|\{x_k \mid x_k \in B_z(p)\}| = K-1$, then $x_k \in B_z^3(p)$ for all k.*

Proof: Let the equilibrium platform profile be x, and let $\beta_k(x)$ be the number of voters who rank candidate k last given x. Suppose to the contrary (and without loss of generality) that $x_1, \ldots, x_{K-1} \in B_z(p)$ and $x_K \notin B_z^3(p)$. Clearly, all voters in $B_z(p)$ rank candidate K last. It follows that $\beta_K(x) > (K-1)/K$, or equivalently that less than $1/K$ do not rank candidate K last. But for candidate K to win under weakly undominated strategies, at least $1/K$ of the voters must not rank it last: contradiction. ∎

To see the intuition for Proposition 7, let $K = 3$ and suppose that candidates 1 and 2 both locate inside $B_z(2/3)$ and candidate 3 locates outside $B_z^3(2/3)$. Then all voters in $B_z(2/3)$ must rank candidate 3 lowest, which implies that less than one third of voters do not rank candidate 3 lowest. Thus, candidate 3 cannot win without some voters using weakly dominated strategies, and no symmetric equilibrium exists.

This result applies to all values of p and any ball center z, and therefore restricts the extent to which platforms may be dispersed across X. Taking $p = 2/3$ allows us to connect the result with those of the minmax set. If $K-1$ platforms are located in the minmax set, then by Proposition 6 they are also located within $\cap_{z\in X} B_z(2/3)$. In a symmetric equilibrium, the remaining platform is therefore located within $\cap_{z\in X} B_z^3(2/3)$.

Note finally that since Proposition 7 bounds the locations of possible election winners, it also has implications for games of electoral competition with endogenous and costly entry. Consider a citizen-candidate model with strategic voters. Given the presence of at least two "centrist" citizen-candidates, no sufficiently extreme citizen can enter and expect to win with positive probability. Thus the result provides some modest limits on the set of candidates that can be expected in citizen-candidate competition.

6 Conclusion and Future Work

From a broader perspective, the results of this paper, combined with the general failure of equilibria to exist when voters vote sincerely, indicate that making useful predictions about electoral competition between vote-maximizing candidates or parties requires understanding (and specifying) how voters make their vote choices. While this point might appear obvious, the results here bring to the forefront the degree to which specificity is required. Even requiring that no voter use a weakly dominated strategy does not rule out any policy being observed in equilibrium. In this regard, approaches, such as probabilistic voting, that involve a complete specification of voters' behaviors for all profiles of announcement profiles and derive (more) precise predictions about equilibrium policy outcomes represent a "step forward" relative to a more agnostic approach in which the requirement for a profile to be a prediction is that it be consistent with (even a refined) Nash equilibrium profile of candidates' and voters' strategies.

Future Work. One of the goals of future work should be restricting the set of policies that can, in equilibrium, be the predicted *winner*. At first, one might suppose that this can be accomplished by making more restrictions on "reasonable" voting equilibria. For example, in a finite electorate, requiring that voting strategies be measurable with respect to voters' types (implying a relatively high, but plausible, level of coordination within the electorate) can rule out some of the profiles of voting strategies used here. Another direction would involve focusing on the considerations about voting equilibria expressed in Myerson and Weber [1993]. This seems appealing, but it must be remembered that the approach taken in Myerson and Weber [1993] is probabilistic – they assume that all two-way-tie pivot probabilities are strictly positive. As they note (Myerson and Weber [1993], p.104), relaxing this assumption does not undermine the internal consistency of their analysis, but does admit a larger set of voting equilibria.[33]

Finally, the results here suggest that we consider including more of the structural details of electoral competition in pursuit of refining our theoretical predictions regarding the platforms that will be offered by office-seeking candidates. Immediate possibilities include candidate entry, pre-election polling, and other representations of voters' coordinating and signaling devices prior to the election. The results presented in this paper highlight both the difficulties behind, and need for, such extensions of the formal theory of first-past-the-post electoral competition.

[33]Indeed, a complete relaxation of this assumption would generate a larger set of voting equilibria than is the ultimate focus of this paper. In particular, their *ordering condition* (p. 105) will be satisfied by *any* Nash equilibrium of the voting game, including equilibria in which some or all voters are using weakly dominated strategies.

Table 1

Approximate Bounds of Minmax Set

Normally distributed ideal points

Dimensions	N	Variance	Mean Radius of $\cap_{z\in X} B_z(2/3)$	Mean Proportion in $\cap_{z\in X} B_z(2/3)$
1	10,000	1	0.4367	0.3334
		2	0.6078	0.3334
		3	0.7478	0.3329
2	10,000	1	0.4310	0.0886
		2	0.6077	0.0887
		3	0.7463	0.0888
3	10,000	1	0.4300	0.0202
		2	0.6087	0.0208
		3	0.7451	0.0206
4	10,000	1	0.4288	0.0038
		2	0.6123	0.0044
		3	0.7477	0.0040
5	100,000	1	0.4308	0.0007
		2	0.6094	0.0008
		3	0.7461	0.0007

* averaged over ten trials per dimension/variance set.

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