Lotteries in Student Assignment: The Equivalence of Queueing and a Market-Based Approach*

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Abstract

Lotteries are often used in allocation processes because of considerations of fairness. This paper formally examines two competing methods of conducting a lottery in an assignment problem involving public resources. One example of this problem is from the centralized process by which students are assigned a high school seat in New York City. The main result is that a mechanism which places students in a queue based on a single lottery, random serial dictatorship, produces a distribution of matchings which is equivalent to a market-based mechanism based on multiple lotteries, the core mechanism from random property rights.

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1 Introduction

Concerns about fairness often influence the choice of allocation mechanisms. The perception of fairness is particularly important in the allocation of public resources, such as visas for immigration, public housing units, licenses or permits, or for the distribution of land, where individuals have the same priority for the object being assigned and monetary transfers are explicitly forbidden (see Elster (1992) for more examples). In these situations, rationing schemes employing lotteries are sometimes viewed as part of a simple and fair solution.[1] This paper examines a problem motivated by the perception of fairness involving lotteries in the allocation of public school seats in school admissions.

A large number of school districts around the world employ centralized mechanisms to assign students to school.[2] The public school systems in two large cities in the United States, Boston and New York City (NYC), have recently adopted new student assignment mechanisms inspired by the mechanism design literature on the allocation of indivisible goods (e.g. Roth and Sotomayor (1990) and Abdulkadiroğlu and Sönmez (2003)), and over half a million students have been assigned to school places through these new mechanisms.[3] The mechanism in NYC assigns students to public high schools and consists of two rounds: the Main and Supplementary round. The Supplementary round involves over 8,000 students, a little more than 10% of applicants, who are unassigned after the Main round. Starting in 2003, these students are asked to submit a rank order list over high schools with vacant capacity. All students have the same priority to attend any school in this round.

When many students are equivalent from the point of view of a school with limited space, an assignment mechanism must have a way to determine which applicants obtain a placement. Random tie-breaking by assigning each student a unique lottery number to be compared when

[1]See, for instance, Weitzman (1977) and Che and Gale (2007) for comparisons of non-market rationing schemes versus market assignment schemes.

[2]See Abdulkadiroğlu and Sönmez (2003) for descriptions of several systems in the United States, Balinski and Sönmez (1999) for a description of Turkish college admissions, Chiu and Weng (2007) for a description of the Chinese college admissions system, and Lavy (2007) for details on the public school choice plan in Tel-Aviv, Israel. In England, lotteries are brought up explicitly as the 'fairest solution', see "Debate Over School Lottery System," BBC News, 2007, http://news.bbc.co.uk/2/hi/uk_news/england/sussex/6745069.stm.

[3]A large theoretical literature on school choice problems has developed following Abdulkadiroğlu and Sönmez (2003). See Abdulkadiroğlu, Pathak, and Roth (2008), Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005), Abdulkadiroğlu, Pathak, Roth, and Sönmez (2006), Chen and Sönmez (2006), Erdil and Ergin (2008), Ergin and Sönmez (2006), Kesten (2005) and Pathak and Sönmez (2008).

students are otherwise tied (hereafter, *single lottery*) treats each student symmetrically and also preserves the dominant strategy incentive compatibility of the two mechanisms which Abdulkadiroğlu and Sönmez (2003) have suggested for school choice problems. However, alternative forms of tie-breaking can also be considered. In particular, each school may wish to conduct its own lottery when faced with students with equal claims to the school (hereafter, *multiple lotteries*).

During the course of the design of the new assignment mechanism, policymakers from the NYC Department of Education (DOE) believed that a mechanism based on a single lottery is less equitable than lotteries at each school. After community forums, one DOE official summarized the discussion:[4]

> Although students might not get their first choices, they were considered separately for each program. There was a rank order established and each student had an equal chance to be selected. [...] If we want to give each child a shot at each program, the only way to accomplish this is to run a new random. [...] I cannot see how the children at the end of the line are not disenfranchised totally if only one run takes place. I believe that one line will not be acceptable to parents. When I answered questions about this at training sessions, (it did come up!) people reacted that the only fair approach was to do multiple runs.

This paper formally investigates this perceived equity concern by examining two competing mechanisms for assigning school seats. The first mechanism, random serial dictatorship, places students in a queue based on a single lottery. It selects an ordering from a given distribution and assigns the first student her top choice, the second student his top choice among available schools, and so on.[5] The second mechanism, the core mechanism from random property rights, is a market-based mechanism based on multiple lotteries. It selects an ordering of students from a given distribution and sets that order as the priority for the first school, selects another ordering from the same distribution and sets it as the priority for the second school, and so on. Then the mechanism finds a core allocation in the market with these property rights by

[4]See Pathak (2007) for additional statements by DOE policymakers about the perceived fairness of multiple lotteries relative to single lotteries.

[5]This mechanism is also known as random priority.

executing trades among students using the top trading cycles algorithm.[6] The main result of this paper is that a random serial dictatorship is equivalent to the core mechanism from random property rights. For any preference profile, the distribution of assignments is exactly the same whether there is a single lottery or a lottery at each school, followed by the core mechanism.

The rest of this paper is organized as follows. The next subsection briefly mentions related literature. Section 2 presents the formal model and describes the environment in NYC as well as other potential applications. Section 3 begins with a simple example and states the main result. Section 4 outlines the proof of the theorem, and describes the argument for two other examples. Finally, the last section concludes with discussion of the implications of the result together with potential extensions. Proofs are contained in the appendix, along with a richer example.

Related literature

This paper is most closely related to the classic paper of Abdulkadiroğlu and Sönmez (1998), who establish the equivalence between a random serial dictatorship and the core from random endowments in problems involving the allocation of indivisible goods, known as house allocation problems. A random serial dictatorship and the core from random endowments are both mechanisms based on a single lottery. The technique to prove their result is based on the construction of a one-to-one mapping between each of the $n!$ orderings of n agents and each of the $n!$ possible arrangements of the endowment of n houses.

In contrast, this paper is interested in the comparison between mechanisms with single and multiple lotteries. With multiple lotteries, a student can be ordered in different ways by each school, and may even receive the highest priority for more than one school. If there are n

[6]The top trading cycles algorithm was first defined for the housing market model (attributed to David Gale by Shapley and Scarf (1974)), where each agent is endowed with an indivisible good, such as a house. The next section precisely defines a version of top trading cycles algorithm for the problem. Mechanisms based on Gale's top trading cycles also played an important role in the policy discussion in Boston about student assignment. First, a student task force in charge of making recommendations to the Boston Public Schools committee, in their September 2004 report, recommended that Boston change their assignment mechanism to one based on the top trading cycles algorithm immediately. For the report from the student task force, see http://boston.k12.ma.us/assignment/TFreport.pdf (page 15). Second, during school committee deliberation on new assignment mechanisms, a mechanism based on top trading cycles was one of two mechanisms discussed in public hearings. The school committee eventually adopted a mechanism based on the student-proposing deferred acceptance algorithm. Details are presented in Abdulkadiroğlu, Pathak, Roth, and Sönmez (2006).

students and schools, there are $(n!)^n$ possible ways that students may be ordered at schools. The proof here cannot use the fact that the number of possible endowments is equal to the number of possible serial dictatorships. Rather, the number of possible ways students may be ordered with multiple lotteries is $(n!)^{n-1}$ times the number of ways they may be ordered with a single lottery. This feature of the problem introduces additional challenges and prevents us from directly using the argument of that paper.

Another important paper is Sönmez and Ünver (2005) who study the house allocation model with existing tenants introduced by Abdulkadiroğlu and Sönmez (1999). They show the equivalence of a mechanism based on finding the core of a sister economy and a particular form of the top trading cycles procedure. Their model and result is a generalization of Abdulkadiroğlu and Sönmez (1998). The same challenge remains, however, as their comparison is between two mechanisms based on a single lottery.

2 Model

2.1 Random assignment

This section first defines the basic model in terms of an assignment problem, and then discusses its relationship to student assignment in New York City and other applications. In the model, there is a group of n students who can be assigned to only one school, and each student has strict preferences over the set of schools. Formally, a **random assignment problem** is a triple (I, S, P) where I is a finite set of students, S is a finite set of schools each with one seat where $|I| = |S| = n$, and $P = (P_i)_{i \in I}$ is a profile of student preferences where P_i is the strict preference ordering of student i over the set of objects S. Let R_i be the weak counterpart to P_i and let $P_{-i} = (P_{i'})_{i' \in I \setminus i}$. The main assumption in this formulation is that each school has one seat, so a school with multiple seats can be thought of as distinct school-seats. Section 2.3 discusses the importance of this modeling choice.

A **matching** μ is a bijection from I to S. Let μ_i denote the assignment of agent i under μ. A matching is **Pareto efficient** if there is no other matching which assigns each student a weakly more preferred school and at least one student a strictly more preferred school.

A **mechanism** is a systematic procedure for producing a matching for each problem (I, S, P). Let ϕ denote a mechanism which maps preference profile P to matching, and $\phi_i(P)$

denote the assignment of i in the matching $\phi(P)$. A mechanism is Pareto efficient if for every problem, it selects a Pareto efficient matching. A mechanism is **strategy-proof** (dominant-strategy incentive compatible) if

$$\forall i, \forall \hat{P}_i, \text{ and } \forall Q_{-i}, \quad \phi_i(P_i, Q_{-i}) R_i \phi_i(\hat{P}_i, Q_{-i}).$$

A stochastic mechanism is a mechanism which produces a probability distribution over matchings. A stochastic mechanism is strategy-proof if it is dominant-strategy incentive compatible when students submit their ordinal ranking over schools. A stochastic mechanism is Pareto efficient if it places positive probability only on matchings that are Pareto efficient.

2.2 Gale's top trading cycles

Gale's **top trading cycles** algorithm was initially described to find the core allocation in a problem where there are agents who are initially endowed with indivisible goods or "houses" (Shapley and Scarf (1974)). In our assignment problem, students are not initially assigned to schools. To employ this procedure, property rights to trade schools must be determined by the mechanism. One approach studied by Abdulkadiroğlu and Sönmez (1998) is to randomly endow each student with a school and compute the core matching for this induced economy using the top trading cycles algorithm.[7]

Our motivation comes from a different procedure where students are ordered by each school via priorities, which define the property rights for a school. Let **priorities at school** s, denoted π_s, be a bijection from the set of students to $\{1, ..., n\}$. Each function π_s induces a strict ordering of students at school s. If $\pi_s(i) < \pi_s(j)$ for some $i, j \in I$, then student i receives higher priority than student j at school s. The notation:

$$\pi_s : i_1, i_2, i_3$$

is shorthand for $\pi_s(i_1) = 1, \pi_s(i_2) = 2$, and $\pi_s(i_3) = 3$. Let $\pi = (\pi_s)_{s \in S}$ denote a **priority structure**. A priority structure generated from a single lottery has the property that $\pi_s(i) =$

[7] Pápai (2000) and Pycia and Ünver (2008) study generalizations of the top trading cycles procedure for different classes of problems.

$\pi_{s'}(i)$ for any two schools $s, s' \in S$. Multiple lotteries do not impose this restriction.

Start with a problem (I, S, P) together with priority structure π. The version of Gale's top trading cycles this paper works with is described as follows:

Step 1: Each student points to her favorite school and each school points to the student who has the highest priority. Since the number of students and school seats is finite, there is at least one cycle (a **cycle** is an ordered list of students and schools $\{i_1 \to s_1 \to i_2 \to s_2 ... \to i_m \to s_m \to i_1\}$ where student i_1 points to s_1, school s_1 points to student i_2, ..., and s_m points to i_1). In each cycle, every student is assigned to the school she points to, and the student and school are removed from the problem. We will sometimes say the student and school have left the market. If there are remaining students, go to the next step.

In general, at

Step t: Each remaining student points to her favorite school among the remaining schools and each remaining school points to the student with highest priority among the remaining students. There is at least one cycle. Every student in a cycle is assigned the school that she points to, and the student and the school are removed. If there are remaining students, go to the next step.

The algorithm terminates when all students are assigned a school.

This mechanism is Pareto efficient and strategy-proof as a direct mechanism (Roth and Postlewaite (1977)). It will be denoted the **core mechanism**.

2.3 School choice problem

When each school has more than one seat, there are many ways to adapt the top trading cycles algorithm. Abdulkadiroğlu and Sönmez (2003) introduce a mechanism based on the top trading cycles algorithm where the number of seats at each school is measured with a counter. At each step of the algorithm, if a school has remaining capacity, it points to the student who has the highest priority. If it is involved in a cycle, then a student is assigned to one of its school seats and the count of its seats is reduced by one. If the count of a school's seats becomes zero, the school no longer participates in any of the subsequent steps of the algorithm.[8]

[8]Kesten (2005) proposes another adaptation of top trading cycles algorithm for the school choice problem.

The approach in this paper to extend the top trading cycles algorithm to schools with multiple seats parallels the approach of Roth (1985) who defines a many-to-one matching model (college admissions problem) in terms of a one-to-one matching model (marriage problem) for two-sided matching models. Consider a problem (P, I, S) where q_s denotes the capacity of school s. Suppose there is a priority π_s at each school. Consider a version of the top trading cycles algorithm applied to the following economy:

Step 1: Initialization

For each school s with capacity q_s, index seats such that there is a first seat, second seat, and so on. At school s, let s^j be the j^{th} seat at school s, where $j = 1, \ldots, q_s$. For each school seat at s, set π_s as the priority. Let $\hat{S}$ denote the set of school seats and let $\hat{\pi}$ be the set of priorities for the school seats. For each student, construct preference ordering $\hat{P}_i$ from P_i by replacing every occurrence of school s is by the ordered list $\{s^1, ..., s^{q_s}\}$.

Step 2: Algorithm

Run Gale's top trading cycles algorithm on the problem $(I, \hat{S}, \hat{P})$ with priority structure $\hat{\pi}$.

It is straightforward to show that this mechanism yields the same outcome as the version defined in Abdulkadiroğlu and Sönmez (2003). That is, for any economy (I, S, P) where a school may have more than one seat and priority structure π, both versions produce the same matching.

Rather than working with this particular adaptation of the top trading cycles algorithm, this paper focuses on the canonical version of the algorithm as initially considered by Shapley and Scarf (1974), and the associated direct mechanism. This modeling choice has an implication for what is meant by lotteries at each school. In particular, one may consider the Abdulkadiroğlu and Sönmez (2003) version of top trading cycles with counters, and define a school-specific lottery to be a single lottery draw which applies to all of the seats at a school. That is, if there are n schools with k seats each and $m = nk$ total students, then there are $(m!)^n$ possible orderings with school-specific lotteries. When each school is treated as a collection of school seats, as in the formulation here, there are $(m!)^m$ possible orderings. We defer the argument for the alternate adaptation of top trading cycles with counters for future work, and focus on the case where each school has one seat in this paper.[9]

[9] Extensive simulation evidence suggests that the equivalence in this paper also extends to this alternative

2.4 School admissions in New York City

In New York City, student placement consists of two rounds.[10] In the Main round, the majority of students are assigned schools. Each year, about 8,000 students, however, are unassigned. The district decided that the best way to place these remaining students was through a second round, known as the Supplementary round. In this problem, each school is indifferent between all students. The motivating quotes in the introduction correspond to discussion regarding the entire allocation procedure.

It is worth noting that as in NYC, a common feature of school choice problems is that school priorities are not strict. For instance, in Boston, the highest priority at a school is given to students who have a sibling at the school and live in the walk zone surrounding the school. The next priority is given to students with only a sibling, followed by students in the walk zone, and finally the remaining students. There are often many students in each category. Virtually every school choice plan in the United States uses lotteries to construct strict orderings over students.[11] In cities where there are specialized programs for academically talented students (as in Chicago IL), the situation exactly corresponds to the environment of the Supplementary round in New York City where each student is given the same priority at each program.

2.5 Other applications

The issue of single versus multiple lotteries in this paper was motivated by student assignment. However, this issue also has potential applications in other problems. One application is in queueing problems with a fixed number of agents and number of objects (e.g. Maniquet (2003)). If individuals have preferences for tickets to various events, they could either form a single queue for tickets, or place their names on multiple queues for different events and then trade amongst

version of the top trading cycles algorithm and method of randomization.

[10]Full details of public high school admissions in NYC are contained in Abdulkadiroğlu, Pathak, and Roth (2008).

[11]A partial list of US school districts with choice plans where lotteries are used to break ties includes Albany NY, Anchorage AK, Berkeley CA, Boston MA, Brockton MA, Cambridge MA, Champaign IL, Charlotte-Mecklensburg NC, Clarke County GA, Columbus OH, Denver CO, Durham NC, Escambia County FL, Eugene OR, Framingham MA, Irvine CA, Jackson County FL, Lee County FL, Los Banos CA, Malden MA, Miami-Dade FL, New Haven CT, Palo Alto CA, Palm Beach FL, Portland OR, Rochester NY, San Diego CA, San Francisco CA, Seattle WA, St. Lucie FL, Tacoma WA, Tampa-St. Petersburg FL (Hillsborough and Pinellas Counties), Upper Marlboro MD, White Plains NY, Wilmington DE, and Wyandotte WA.

themselves if their name is called for a ticket.

Another set of applications include time-sharing and scheduling problems. For instance, consider sharing time on large scale computer servers with multiple processers. Each processor can handle different jobs and users have strict preferences over the single job they wish to pursue. A mechanism based on multiple lotteries would treat each processer as determining its own property rights for claims on the processor's time. Moulin (2003) describes other problems involving managing common property resources where the issue of single versus multiple lotteries is potentially relevant.

3 Main Result

3.1 Simple Example

Let us first illustrate the model with an informal description of the two mechanisms. Section 3.2 contains the formal definitions.

Example 1. Consider a random assignment problem with 3 students, i_1, i_2, and i_3, and 3 schools, s_1, s_2, and s_3. Suppose the student preferences are:

$$P_{i_1} : s_3, s_1, s_2$$
$$P_{i_2} : s_2, s_3, s_1$$
$$P_{i_3} : s_3, s_2, s_1.$$

Consider the mechanism based on a single lottery. Suppose the lottery generates the following ordering:

$$i_1, i_3, i_2.$$

A serial dictatorship (or queue) with this ordering produces the following matching:

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 \\ s_3 & s_1 & s_2 \end{pmatrix},$$

where student i_1 obtains her top choice, student i_2 obtains her last choice, and student i_3 obtains her second choice.

The mechanism based on multiple lotteries requires lottery draws for each of the three schools. Suppose the first lottery yields the ordering above, and the second and third lottery yield:

$$\begin{aligned} \text{second lottery } &: i_3, i_1, i_2 \\ \text{third lottery } &: i_3, i_2, i_1. \end{aligned}$$

If these orderings are used for the priority structure, then π is defined by:

$$\begin{aligned} \pi_{s_1} &: i_1, i_3, i_2 \\ \pi_{s_2} &: i_3, i_1, i_2 \\ \pi_{s_3} &: i_3, i_2, i_1. \end{aligned}$$

The core mechanism yields the following matching:

$$\nu = \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix},$$

where students i_2 and i_3 obtain their top choice, and student i_1 obtains her second choice.

In this example, for the particular realization of the single lottery, student i_2 who is last in the lottery receives her last choice in μ. With multiple lotteries, student i_2 was "given a shot at each program" and even though she does not win the lottery at the second and third school, she obtains her top choice in ν. Of course, these observations apply to the particular orderings realized by the lotteries in this example. Both single and multiple lotteries lead to a distribution over matchings. The main result of this paper is that the distribution over the matchings from either single or multiple lotteries is identical. After introducing the notation to state the result in general, this example will be used to illustrate some ideas of the proof.

3.2 Two Competing Mechanisms

The choice $Ch_i(S')$ of a student $i \in I$ from a set of schools $S' \subseteq S$ is the most preferred school among those in S'. That is,

$$Ch_i(S') = s' \quad \Leftrightarrow \quad s' \in S' \text{ and } s' P_i s \text{ for all } s \in S' \backslash s'.$$

Let **ordering** $f : \{1, 2, ..., n\} \rightarrow I$ be a bijection and $\mathcal{F}$ be the class of all such bijections, where $|\mathcal{F}| = n!$. For any $f \in \mathcal{F}$, student $f(1)$ is first, student $f(2)$ is second, and so on. For some number k and student i, we have that $f(k) = i$ if and only if $f^{-1}(i) = k$. We sometimes say student i is ordered ahead of student j if $f(i) < f(j)$.

Given any ordering $f \in \mathcal{F}$ of students, define the **serial dictatorship** induced by f, and denoted by ψ^f as:

$$\begin{aligned}
\psi^f_{f(1)} &= Ch_{f(1)}(S), \\
\psi^f_{f(2)} &= Ch_{f(2)}(S \backslash \{\psi^f_{f(1)}\}), \\
&\vdots \\
\psi^f_{f(i)} &= Ch_{f(i)}(S \backslash \cup_{j=1}^{i-1} \{\psi^f_{f(j)}\}), \\
&\vdots \\
\psi^f_{f(n)} &= Ch_{f(n)}(S \backslash \cup_{j=1}^{n-1} \{\psi^f_{f(j)}\}).
\end{aligned}$$

This notation simply means that the student who is first in the ordering $f(1)$ obtains her first choice among the set of schools S, the student who is next $f(2)$ obtains her top choice among the remaining schools, and so on. Denote the matching corresponding to the outcome of the serial dictatorship for ordering f of students as m^{ψ^f}.

A **random serial dictatorship** is a stochastic mechanism ψ^{rsd} defined as:

$$\psi^{\text{rsd}} = \sum_{f \in \mathcal{F}} \frac{1}{n!} m^{\psi^f}.$$

Each serial dictatorship is selected with equal probability, or equivalently, an ordering is randomly chosen with uniform distribution and the induced serial dictatorship is used.

Recall that priority structure $\pi = (\pi_s)_{s \in S}$ is a collection of functions $\pi_s : I \to \{1, 2, ..., n\}$ such that for school k and students $i, j \in I$, the statement $\pi_s(i) < \pi_s(j)$ means that student i is given higher priority than student j at school s. Let Π be the set of all priority structures. Note that $|\Pi| = (n!)^n$. Denote the core mechanism with priority structure π as φ^π and the matching corresponding to the outcome of the core mechanism in a market with priority structure π as matching m^{φ^π}. It is easy to see that if for all schools s, priority π_s implies the same ordering as f, then the core mechanism yields the same matching as a serial dictatorship with ordering f.

Given any priority structure, the core mechanism produces a Pareto efficient allocation. Consider the following stochastic mechanism, **core from random property rights**, φ^{core}, defined as:

$$\varphi^{\text{core}} = \sum_{\pi \in \Pi} \frac{1}{(n!)^n} m^{\varphi^\pi}.$$

This mechanism selects each possible priority structure with equal probability and then determines the outcome of the top trading cycles algorithm for the induced market.

The main result of the paper is:

Theorem 1. *For any profile of student preferences,* $\varphi^{core} = \psi^{rsd}$.

4 Overview of the proof

To illustrate the structure of the proof, let us return to the example in Section 3.1. With three students, there are $3! = 6$ possible orderings that can result from a single lottery. The expected matching produced by a random serial dictatorship is:

$$\frac{1}{3}\begin{pmatrix} i_1 & i_2 & i_3 \\ s_3 & s_2 & s_1 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} i_1 & i_2 & i_3 \\ s_3 & s_1 & s_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix}.$$

When there are multiple lotteries, there are $(3!)^3 = 216$ possible priority structures for the schools. The method of the proof is to map each of the 6 orderings produced by a single lottery to a set of priority structures with cardinality $(3!)^2 = 36$ or $\frac{1}{6}$ of the total of 216 which could be generated by multiple lotteries. The mapping has the property that the matching produced by the serial dictatorship with the ordering from a particular single lottery is the same as the matching produced by the core mechanism with any priority structure in the set of priority structures which correspond to the particular single ordering. In the construction, each single ordering corresponds to a set of priority structures, and any priority structure in this set corresponds only to this particular single ordering. This feature of the construction, which is the most difficult, guarantees that each single ordering only corresponds to this set of priority structures.

Formally, for each ordering of students f, the proof constructs a set of priorities structures $\Pi(f)$ with three properties:

Property 1: For any $\pi \in \Pi(f)$, $m^{\psi^f} = m^{\varphi^\pi}$.

Property 2: For any f, $|\Pi(f)| = (n!)^{n-1}$.

Property 3: For any $f_1, f_2 \in \mathcal{F}$, if $f_1 \neq f_2$, then $\Pi(f_1) \cap \Pi(f_2) = \emptyset$.

The first property states that the matching, m^{ψ^f}, which corresponds to the serial dictatorship for the ordering f is the same matching as m^{φ^π}, the matching produced by the core mechanism, for any priority structure $\pi \in \Pi(f)$. The second property is a statement about the frequencies of matchings produced by the two mechanisms. Since $|\mathcal{F}| = n!$ and $|\Pi| = (n!)^n$, and the random serial dictatorship induces a probability $\frac{1}{n!}$ on matching m^{ψ^f} for each f, the first and second condition together imply that each f corresponds to a set of priority structures $\Pi(f)$ with $(n!)^{n-1}$ elements, which induce a lottery with probability $\frac{(n!)^{n-1}}{(n!)^n} = \frac{1}{n!}$ on matching m^{φ^π}. The third property states that each f defines a set of priority structures $\Pi(f)$, where there is no "double counting," or overlap between the two sets of priority structures for any two different orderings. These three properties together demonstrate that both mechanisms induce the same probability distribution over matchings.

For instance, in Example 1, consider the single lottery outcome i_1, i_2, i_3 and corresponding ordering f_1, which yields matching

$$m^{\psi^{f_1}} = \begin{pmatrix} i_1 & i_2 & i_3 \\ s_3 & s_2 & s_1 \end{pmatrix}.$$

Consider the following procedure to build $\Pi(f_1)$. At school s_3, student i_1 is given highest priority and the remaining students are placed in arbitrary order. This generates two possibilities for priorities at school s_3:

$$\pi_{s_3} : i_1, i_2, i_3 \quad \text{or} \quad \pi_{s_3} : i_1, i_3, i_2.$$

Next, suppose that i_2 is ordered ahead of i_3 at school s_2. This generates three possibilities for priorities at school s_2:

$$\pi_{s_2} : i_2, i_1, i_3 \quad \text{or} \quad \pi_{s_2} : i_2, i_3, i_1 \quad \text{or} \quad \pi_{s_2} : i_1, i_2, i_3.$$

Finally, let the possible priorities at the remaining school, s_1, be any arbitrary ordering of students. With three students, this corresponds to 6 possible priorities at s_1.

This procedure for constructing $\Pi(f_1)$ generates a set of priority structures with 36 elements. It is easy to verify that the core mechanism with any $\pi \in \Pi(f_1)$ produces $m^{\psi^{f_1}}$. The construction of $\Pi(f_1)$ is also easy to describe in this example: student $f_1(1)$ is given the highest priority at the school she obtains in matching $m^{\psi^{f_1}}$, student $f_1(2)$ is given higher priority than any student who follows her in the ordering f_1 at the school she obtains in $m^{\psi^{f_1}}$, and since there is no student who follows student $f_1(3)$ in the ordering, the set of priorities at the school she obtains can be any arbitrary ordering of students.

Unfortunately, this procedure to construct $\Pi(f_1)$ does not satisfy all three properties. To see why, consider the single lottery outcome i_2, i_1, i_3 and corresponding ordering f_2. The approach just described for f_1 yields two possible priorities at s_2 corresponding to ordering f_2:

$$\pi_{s_2} : i_2, i_1, i_3 \quad \text{or} \quad \pi_{s_2} : i_2, i_3, i_1$$

while at school s_3 there are three possible priorities such that i_1 is ahead of i_3:

$$\pi_{s_3} : i_1, i_2, i_3 \quad \text{or} \quad \pi_{s_3} : i_1, i_3, i_2 \quad \text{or} \quad \pi_{s_3} : i_2, i_1, i_3$$

and at school s_1 the priorities are any arbitrary ordering of students.

In this case, $|\Pi(f_2)| = 36$ and the core mechanism with any $\pi \in \Pi(f_2)$ produces $m^{\psi^{f_2}}$.

However, $\Pi(f_1) \cap \Pi(f_2) \neq \emptyset$, since

$$\pi_{s_1}: i_2, i_1, i_3$$
$$\pi_{s_2}: i_2, i_3, i_1$$
$$\pi_{s_3}: i_1, i_3, i_2,$$

is an element in both $\Pi(f_1)$ and $\Pi(f_2)$, which shows that the third property is not satisfied.

Rather, the key challenge is to construct $\Pi(f)$ so that it satisfies the third property. Our approach is to show how for any $\pi \in \Pi(f)$, it is possible to identify f. The procedure for recovering f is by observing the sets of cycles which form in each step of the top trading cycles algorithm for the priority structure π.

The details of the construction for a general problem are in the Appendix. Some of the main ideas can be illustrated via simple examples. Before specifying the construction for the first example, let us consider two polar cases, one where students have identical preferences and one where each student has a different top choice, so there is no conflict of interest. It is useful to analyze these two cases before returning to our first example, which builds on both cases.

Example 2. *Identical Student Preferences*
Suppose there are 3 students, i_1, i_2, and i_3, and 3 schools, s_1, s_2, and s_3, where the student preferences are identical:

$$P_{i_1}: s_1, s_2, s_3$$
$$P_{i_2}: s_1, s_2, s_3$$
$$P_{i_3}: s_1, s_2, s_3.$$

In this case, each possible matching is produced by a random serial dictatorship with equal probability. Therefore, the expected matching is:

$$\frac{1}{6}\begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_3 & s_2 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} i_1 & i_2 & i_3 \\ s_2 & s_1 & s_3 \end{pmatrix}$$

$$+\frac{1}{6}\begin{pmatrix} i_1 & i_2 & i_3 \\ s_2 & s_3 & s_1 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} i_1 & i_2 & i_3 \\ s_3 & s_1 & s_2 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} i_1 & i_2 & i_3 \\ s_3 & s_2 & s_1 \end{pmatrix}.$$

The serial dictatorship for each ordering f corresponds to a distinct matching. Moreover, for any priority structure, at each step of the top trading cycles algorithm, each student who is not yet assigned points to the same school, so the only cycles which form involve one student and one school. For any ordering f, in the construction, the priorities at school s_1 are:

$$\pi_{s_1} : f(1), f(2), f(3) \quad \text{or} \quad \pi_{s_1} : f(1), f(3), f(2).$$

This corresponds to two possible priorities. Since the first student in the ordering, student $f(1)$, receives her top choice, where she is ordered at any other school does not affect which assignment she receives. To ensure that the matching produced by the top trading cycles algorithm is the same as the matching produced by the serial dictatorship with ordering f, at school s_2, student $f(2)$ must receive higher priority than student $f(3)$. This condition implies three possible priorities at school s_2:

$$\pi_{s_2} : f(1), f(2), f(3) \quad \text{or} \quad \pi_{s_2} : f(2), f(1), f(3) \quad \text{or} \quad \pi_{s_2} : f(2), f(3), f(1).$$

Finally, in the top trading cycles algorithm, school s_3 points to the student with the highest priority. Since the priorities at s_1 are such that student $f(1)$ is ahead of $f(3)$ and at s_2 student $f(2)$ is ahead of $f(3)$, the choice of priority at school s_3 does not change the assignment of student $f(3)$, so this can be any of 6 possible priorities for school s_3.

In this example, with this procedure for any f, it is easy to check that for any $\pi \in \Pi(f)$, the matchings are identical, $m^{\psi^f} = m^{\varphi^\pi}$. Moreover, since there are two possible π_{s_1}, three possible π_{s_2}, and 6 possible π_{s_3}, the size of the set of priority structures $|\Pi(f)| = 36$. To establish the third property of the set $\Pi(f)$, since each f corresponds to a distinct matching, f can be recovered by examining the steps of the top trading cycles algorithm for any $\pi \in \Pi(f)$. In the first step the only student who is part of a cycle is the first student in the ordering f which corresponds to π. In the second step, the only student who is part of a cycle is the second student in the ordering f. Finally, in the last step, the only student who is part of a cycle is the last student in the ordering f, which corresponds to π.

Two features of this example merit additional discussion. First, with identical student preferences, each student after the first prefers a school assigned immediately before her in the serial dictatorship to her own assignment. This allows us to partition the students into three sets. For example, place student $f(1)$ in the first partition, student $f(2)$ in the second partition, and student $f(3)$ in the third partition. In the general construction, we define priorities within the set of students and schools in a partition. The main constraint is that no student in the second or third partition obtains a school under the top trading cycles algorithm before a student in the first partition and no student in the third partition obtains a school before a student in the second partition. This feature allows us to separately identify which students are part of which partition by observing the order in which cycles form during the execution of the top trading cycles algorithm.

This example also illustrates a feature of the construction which is helpful for showing the second property: $|\Pi(f)| = (n!)^{n-1}$. Consider the case where there are n students, and the first three students have the same preferences as in our example. For any ordering f where these three students are ordered before the rest of the students, at school s_1, there are $(n-1)!$ ways to order the students after student $f(1)$ is given the highest priority. At school s_2, the construction allows any priority such that student $f(2)$ is given higher priority than $f(3)$. In this case, either student $f(2)$ is ordered first, and there are $(n-1)!$ priorities of the remaining students, or student $f(1)$ is ordered first followed by $f(2)$, and there are $(n-2)!$ priorities of the remaining students. This corresponds to $(n-1)! + (n-2)! = \frac{n!}{n-1}$ possible priorities. Finally, at school s_3, the construction requires that student $f(3)$ is given higher priority than any student who follows her in the ordering. Either i) student $f(3)$ obtains the highest priority and the remaining $n-1$ students are ordered arbitrarily afterwards generating $(n-1)!$ orderings, ii) student $f(3)$ obtains the second highest priority after either $f(1)$ or $f(2)$, and the remaining students are ordered arbitrarily which corresponds to $2(n-2)!$ orderings, or iii) student $f(3)$ is ordered third after student $f(1)$ and $f(2)$ (in any order) which corresponds to $2(n-3)!$ priorities. The total number of priorities at school s_3 is: $(n-1)! + 2(n-2)! + 2(n-3)! = \frac{n!}{n-2}$.

More generally, if there are $k-1$ students who can be ordered in arbitrary permutations before the k^{th} student, and $n-k$ who follow in arbitrary permutation, then the number of ways to choose $k-1$ out of n students multiplied by the number of ways to order the remaining

$n-k$ students is simply:

$$\frac{n!}{(n-(k-1))!} \cdot (n-k)! = \frac{n!}{(n-k+1)}.$$

This observation simplifies the calculation in the case where all students have identical preferences. Of course without identical student preferences, this logic does not directly apply, but it remains the key idea behind establishing second property of the construction.

The next example is one where there is no conflict of interest among students. This example is important because the construction builds on subproblems where there is no conflict of interest.

Example 3. *No Conflict of Interest*
Suppose there are 3 students, i_1, i_2, and i_3, and 3 schools, s_1, s_2, and s_3. The student preferences are:

$$P_{i_1} : s_1, s_2, s_3$$
$$P_{i_2} : s_2, s_3, s_1$$
$$P_{i_3} : s_3, s_1, s_2.$$

In this preference profile, no two students have the same top choice, so there is no conflict of interest. As a result, there is only one efficient matching, where each student receives her top choice:

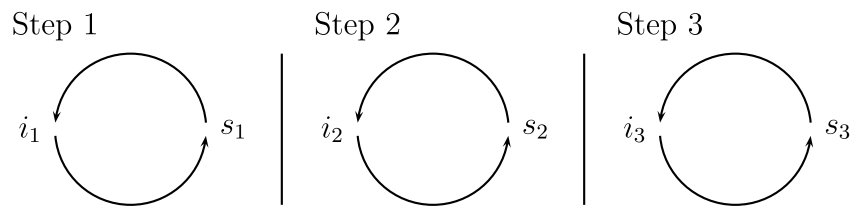
$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix}.$$

Since the core mechanism is efficient, with any priority structure π it produces μ. This example illustrates part of the difficulty in constructing $\Pi(f)$ for given f so that property 3 is satisfied.

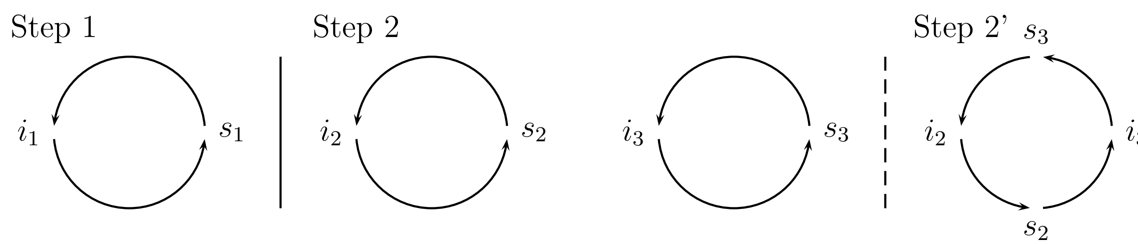
It is easiest to illustrate $\Pi(f)$ for this problem by thinking about how to recover f for any $\pi \in \Pi(f)$. Consider the ordering f_1, where $f_1(1) = i_1$, $f_1(2) = i_2$, and $f_1(3) = i_3$. To recover this ordering from the execution of the top trading cycles algorithm for some $\pi \in \Pi(f_1)$, consider priority structures with the following characteristic: students i_2 and i_3 are not assigned in any cycle before student i_1 and student i_3 is not assigned in a cycle before student i_2.

This allows for four cases of sets of cycles to form in the execution of the top trading cycles algorithm:

Case 1: i_1 is assigned to s_1 in the first cycle; after that, i_2 is assigned to s_2 in the second cycle; after that i_3 is assigned to s_3 in the third cycle

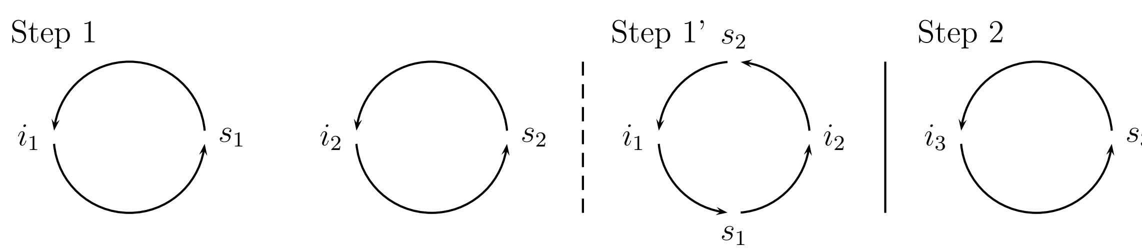


Case 2: i_1 is assigned to s_1 in the first cycle; after that, i_2 and i_3 are assigned to s_2 and s_3 respectively in the second set of cycles



In this case, the set of cycles in the second step are either as illustrated in Step 2, where there are two separate cycles each with only one student and one school, or in Step 2' where there is one cycle involving both of the remaining students.

Case 3: i_1 and i_2 are assigned to s_1 and s_2, respectively, in the first set of cycles; after that, i_3 is assigned to s_3 in the second cycle



In this case, the first set of cycles which may form are either as illustrated in Step 1, where there are two separate cycles each with only one student and school, or in Step 1' where both students i_1 and i_2 belong to the same cycle.

Case 4: i_1, i_2, and i_3 are assigned to s_1, s_2, and s_3, respectively, in the first set of cycles

In this case, there are six possible sets of cycles that may form where all three students are assigned in the first step of the top trading cycles algorithm. These are:

i) $i_1 \rightarrow s_1 \rightarrow i_1$, $i_2 \rightarrow s_2 \rightarrow i_2$, and $i_3 \rightarrow s_3 \rightarrow i_3$,

ii) $i_1 \rightarrow s_1 \rightarrow i_1$ and $i_2 \rightarrow s_2 \rightarrow i_3 \rightarrow s_3 \rightarrow i_2$,

iii) $i_1 \rightarrow s_1 \rightarrow i_2 \rightarrow s_2 \rightarrow i_1$ and $i_3 \rightarrow s_3 \rightarrow i_3$,

iv) $i_1 \rightarrow s_1 \rightarrow i_3 \rightarrow s_3 \rightarrow i_1$ and $i_2 \rightarrow s_2 \rightarrow i_2$,

v) $i_1 \rightarrow s_1 \rightarrow i_2 \rightarrow s_2 \rightarrow i_3 \rightarrow s_3 \rightarrow i_1$,

vi) $i_1 \rightarrow s_1 \rightarrow i_3 \rightarrow s_3 \rightarrow i_2 \rightarrow s_2 \rightarrow i_1$.

Let us examine each of these four cases. For the first case, to ensure that student i_1 is assigned in the first cycle and student i_2 is assigned in a cycle which forms only after i_1 is assigned, it must be the case that at s_1 and s_2:[12]

$$\pi_{s_1} : i_1, \ldots \quad \text{and} \quad \pi_{s_2} : i_1, i_2, \ldots$$

At school s_3, there are two possibilities:

$$\pi_{s_3} : i_1, i_2, \ldots \quad \text{or} \quad \pi_{s_3} : i_2, \ldots$$

In both possibilities, student i_2 is ordered ahead of i_3 at school s_3 so i_3 will not be assigned in a cycle until after i_2 is assigned to s_2. There are a total of 6 possible priority structures for this first case.

For the second case, at school s_1:

$$\pi_{s_1} : i_1, \ldots \tag{1}$$

[12] The notation "..." means that the rest of the priority is an arbitrary ordering of the remaining students.

In the construction, the remaining priorities are

$$\pi_{s_2} : i_1, i_2, \ldots \quad \text{and} \quad \pi_{s_3} : i_1, i_3, \ldots$$

as in Step 2, or

$$\tilde{\pi}_{s_2} : i_3, \ldots \quad \text{and} \quad \tilde{\pi}_{s_3} : i_1, i_2, \ldots$$

as in Step 2'. Note that either with π_{s_2} and π_{s_3} or $\tilde{\pi}_{s_2}$ and $\tilde{\pi}_{s_3}$, students i_2 and i_3 will not be assigned in cycles until after i_1 is assigned. There are a total of 6 possible priority structures for this second case.

Of course, there are other possible priority structures which generate the cycles which correspond to the second case. The construction chooses these priorities to ensure there is a way to distinguish f_1 from the ordering $f_2 : i_1, i_3, i_2$. There are two types of priorities to consider. For π_{s_2} and π_{s_3}, the cycles form the first time school s_2 points to student i_2 and the first time school s_3 points to student i_3. The ordering of the two students can be determined by using the index of the schools which point to the students. For $\tilde{\pi}_{s_2}$ and $\tilde{\pi}_{s_3}$, school s_2 points to i_3 in the algorithm in a step before school s_3 points to i_2: $i_2 \to s_2 \to i_3 \to s_3 \to i_2$. The procedure implies that ordering is i_2 and then i_3.

In contrast, there is a different set of priorities structures which correspond to $f_2 : i_1, i_3, i_2$ in the second case. The priorities at s_1 are the same as in (1). However, for school s_2 and s_3, they are as follows:

$$\pi^{f_2}_{s_2} : i_1, i_3, \ldots \quad \text{and} \quad \pi^{f_2}_{s_3} : i_1, i_2, \ldots \tag{2}$$

or

$$\tilde{\pi}^{f_2}_{s_2} : i_1, i_3 \ldots \quad \text{and} \quad \tilde{\pi}^{f_2}_{s_3} : i_2, \ldots. \tag{3}$$

Note that when priorities are as in (2), the following cycle forms: $i_3 \to s_3 \to i_2 \to s_2 \to i_3$. Since s_2 and s_3 are assigned in this cycle the first time that s_3 points to i_2 and the first time s_2 points to i_3, the index of the schools is used to order the two students such that i_3 is ahead of i_2.

Likewise, when priorities are as in (3), the following cycle forms: $i_3 \to s_3 \to i_2 \to s_2 \to i_3$. Here, school s_3 points to student i_2 in a step before this cycle forms, so i_2 is ordered after i_3.

Returning to the third case for f_1, the priorities at s_1 and s_2 are:

$$\pi_{s_1} : i_1, \ldots \quad \text{and} \quad \pi_{s_2} : i_2, \ldots.$$

In the construction, the reason i_1 is given the top priority at s_1 and i_2 at s_2, rather than i_1 at s_2 and i_2 at s_1 is that under either scenario in the first step of top trading cycles algorithm with these priorities, i_1 and i_2 are part of cycles and are assigned. Since i_1 and i_2 are both given the top priority, the index of the school which points to the student is used to determine the corresponding ordering. When s_1 points to i_1 and s_2 points to i_2, student i_1 is be ordered before i_2. In contrast, for ordering $f_3 : i_2, i_1, i_3$ in this third case, i_2 is given top priority at s_1 and i_1 is given top priority at s_2.

At s_3, the priority is

$$\pi_{s_3} : i_1, \ldots \quad \text{or} \quad \pi_{s_3} : i_2, \ldots$$

The third case defines a total of 16 priorities structures.

Finally, in the fourth case, since all students are assigned in the first set of cycles, any permutation such that a distinct student is given the highest priority at each school ensures that all students are assigned in the first set of cycles. This case is related to the setting of Abdulkadiroğlu and Sönmez (1998). As in the second and third case, the index of the school which points to the student in the cycle in which the student is assigned is used to recover the ordering. Therefore, the priority is

$$\pi_{s_1} : i_1, \ldots \quad \text{and} \quad \pi_{s_2} : i_2, \ldots \quad \text{and} \quad \pi_{s_3} : i_3, \ldots$$

which defines 8 priority structures. Since the lowest indexed school s_1 points to i_1, the second lowest indexed school s_2 points to i_2 and the third lowest indexed school s_3 points to i_3, the corresponding ordering is i_1, i_2, i_3. The total number of priorities structures defined for these four cases is $6 + 6 + 16 + 8 = 36$, as desired.

This example illustrates a number of lessons. When there is a set of students in order such that there is no conflict of interest, there are many possibilities for the priorities at the schools. The main principle to construct the priorities when there is no conflict of interest is to ensure that it is possible to recover the ordering from the execution of the top trading cycles algorithm. If a set of students is assigned in a set of cycles before another set of students is assigned in a set of cycles, then the students in the first set are ordered before the students in the second set. Within a set of students who are part of the same set of cycles, the construction mirrors the description of Step 2 or Step 2' in the second case.

The construction partitions the set of students according to the ordering into cases where there is no conflict of interest among the students and applies the ideas of this example. For the first partition of students, the application of ideas in Example 3 is straightforward. For subsequent partitions, there is one additional complication. The priority structure must be such that cycles involving students in the second partition can be identified separately from cycles involving students in the first partition. Since the first student in each partition of students after the first partition of students prefers the assignment of a student in an earlier cycle over her assignment, this student is given priority at a school in a way that she prevents any subsequent cycles from forming until she is assigned. In the execution of the top trading cycles algorithm, this student points to the school she prefers over her assignment, and the priority she receives prevents any students who follow her in the ordering from forming cycles and obtaining their assignment before the school she prefers is assigned.[13] This makes it possible to identify partitions of students for which there is no conflict of interest.

With these two examples described, let us return to Example 1.

Example 1. (cont.)

There are 6 possible orderings of the 3 students.

- f_1: i_1, i_2, i_3

  Partition the students in order of f_1 into sets of students where there is no conflict of interest. Under ψ^{f_1}, the first two students in the ordering both receive their top choice. When it is student i_3's turn, her top and second choice are both unavailable, so there is a new partition. Let I_i be the set of students who are in partition i. $I_1 = \{i_1, i_2\}$, and $I_2 = \{i_3\}$. A new subproblem is defined following the ordering whenever there is a student who prefers a school obtained by some student in the previous subproblem over

[13] The example in the Appendix illustrates how this is handled in the construction.

the school she obtains from the serial dictatorship. Let S_i denote the schools that are assigned in subproblem i. $S_1 = \{s_2, s_3\}$ and $S_2 = \{s_1\}$.

For the students in I_1, there are two possible sets of cycles which may form. First, both students may obtain their assignment in the same step of the top trading cycles algorithm. In this case, they are placed at the same priority level, and as in Example 3, the first student in the ordering obtains the highest priority at the lowest indexed school in S_1, and the second student in the ordering obtains the highest priority at the next indexed school. This defines the following priorities for the two schools:

$$\pi_{s_2} : i_1, \ldots \quad \text{and} \quad \pi_{s_3} : i_2, \ldots$$

The second possibility for the students in I_1 is to define priorities such that i_1 is assigned in a cycle which forms in the first step of top trading cycles algorithm. After that, i_2 is assigned in a cycle which forms in the next step. This defines the following priorities for the two schools:

$$\pi_{s_2} : i_1, i_2, \ldots \quad \text{and} \quad \pi_{s_3} : i_1, \ldots$$

Next consider the students in I_2. Since i_3 is the only student to place at school s_1, and she prefers s_2 and s_3 over s_1, under the top trading cycles algorithm, she is not assigned until s_2 and s_3 are assigned because she points to one of these schools before pointing to s_1. Therefore, for any arbitrary priority at s_1, student i_3 is assigned only after i_1 and i_2, so let π_{s_1} be any arbitrary ordering of students. With 6 possible priorities at s_1 and $4 + 2 = 6$ priorities at s_2 and s_3, there are 36 possible priority structures, such that f_1 can be recovered from the execution of the top trading cycles algorithm.

- f_2: i_2, i_1, i_3

  Since i_2 and i_1 both receive their top choice under the serial dictatorship, i_3 is part of the second partition. Hence, $I_1 = \{i_2, i_1\}$ and $I_2 = \{i_3\}$, and the corresponding schools are $S_1 = \{s_2, s_3\}$ and $S_2 = \{s_1\}$.

  For students in I_1, there are two possible sets of cycles which may form. When i_2 and i_1 form a set of cycles in the first step, since i_2 is ahead of i_1 in the ordering f_2, she obtains the highest priority at the lowest indexed school in S_1, and i_1 obtains highest priority at the other school. This defines the following priorities for the two schools:

$$\pi_{s_2}: i_2, \ldots \quad \text{and} \quad \pi_{s_3}: i_1, \ldots$$

Another alternative is to define priorities such that i_2 is assigned in a cycle which forms in the first step of the top trading cycles algorithm, and after that i_1 is assigned in a cycle which forms in the next step. This defines the following priorities:

$$\pi_{s_2}: i_2, \ldots \quad \text{and} \quad \pi_{s_3}: i_2, i_1, \ldots$$

Just as for f_1, at school s_1, the priority is arbitrary because the remaining student i_3 does not exit the market in a cycle before schools s_2 and s_3 are assigned. There are 36 possible priority structures, such that f_2 can be recovered from the execution of the top trading cycles algorithm.

- f_3: i_1, i_3, i_2

  Since i_3 prefers s_3 over her assignment from the serial dictatorship, she begins the second partition. Likewise, since i_2 prefers s_2 over her assignment, she begins the third partition. Hence, $I_1 = \{i_1\}$, $I_2 = \{i_3\}$, and $I_3 = \{i_2\}$, and $S_1 = \{s_3\}$, $S_2 = \{s_2\}$, and $S_3 = \{s_1\}$. Since there is only one student in each subproblem, the only cycles are those involving the student and the school she is assigned to be ψ^{f_3}. This implies that at s_3, student i_1 must be given priority ahead of the other two students, at s_2, student i_3 must be given priority ahead of i_2, and the priority at s_1 is arbitrary. There are 36 priority structures defined in this way, and for each priority structure, the top trading cycles algorithm requires three steps to complete where the ordering of students is equal to the step in which the student is assigned.

- f_4: i_2, i_3, i_1

  The students are partitioned as $I_1 = \{i_2, i_3\}$ and $I_2 = \{i_1\}$, and $S_1 = \{s_2, s_3\}$ and $S_2 = \{s_1\}$. For the students in I_1, there are two possible sets of cycles which may form. First, both students obtain their assignment in the same step of the top trading cycles algorithm. This defines the following priorities for the two schools:

$$\pi_{s_2} : i_2, \ldots \quad \text{and} \quad \pi_{s_3} : i_3, \ldots$$

The second possibility for the students in I_1 is to define priorities such that i_2 is assigned in a cycle which forms in the first step of the top trading cycles algorithm, and after that i_3 is assigned in a cycle which forms in the next step. This defines the following priorities for the two schools:

$$\pi_{s_2} : i_2, \ldots \quad \text{and} \quad \pi_{s_3} : i_2, i_3, \ldots$$

As before, the priority at school s_1 is arbitrary. There are 36 possible priority structures, such that f_4 can be recovered from the execution of the top trading cycles algorithm.

- f_5: i_3, i_1, i_2

  The partition of students is $I_1 = \{i_3\}$ and $I_2 = \{i_1, i_2\}$, and $S_1 = \{s_3\}$ and $S_2 = \{s_1, s_2\}$. One new feature of this case is that even though i_2 does not prefer i_3's assigned school, she is part of I_2 because she follows student i_1 who does prefer i_3's assigned school in the ordering f_5. For the students in I_1, the only possible priority is

$$\pi_{s_3} : i_3, \ldots$$

  For the second subproblem, there are two possibilities:

$$\pi_{s_1} : i_1, \ldots \quad \text{and} \quad \pi_{s_2} : i_2, \ldots$$

  or

$$\pi_{s_1} : i_1, \ldots \quad \text{and} \quad \pi_{s_2} : i_1, i_2, \ldots$$

  where student i_3 can be placed anywhere in the priorities at these two schools. In the first case, i_1 and i_2 forms a set of cycles only after i_3 is assigned. In the second case, i_1 forms a cycle only after i_3 is assigned, and i_2 forms a cycle only after i_1 is assigned. There are 36 possible priority structures, such that f_5 can be recovered from the execution of the top trading cycles algorithm.

- f_6: i_3, i_2, i_1.

  The partition of students is $I_1 = \{i_3, i_2\}$ and $I_2 = \{i_1\}$, and $S_1 = \{s_2, s_3\}$ and $S_2 = \{s_1\}$. For the students in I_1, there are two possible sets of cycles which may form. First, both students may obtain their assignment under the same step of the top trading cycles algorithm. This defines the following priorities for the two schools:

  $$\pi_{s_2} : i_3, \ldots \quad \text{and} \quad \pi_{s_3} : i_2, \ldots$$

  The second possibility is to define priorities such that i_3 is assigned in a cycle which forms in the first step of the top trading cycles algorithm, and after that i_2 is assigned in a cycle which forms in the next step. This defines the following priorities for the two schools:

  $$\pi_{s_2} : i_3, i_2, \ldots \quad \text{and} \quad \pi_{s_3} : i_3, \ldots$$

  The priority at school s_1 is arbitrary. There are 36 possible priority structures, such that f_6 can be recovered from the execution of the top trading cycles algorithm.

This description completely specifies the construction in this case. Each set of priority structures has 36 elements. For each priority structure, it is possible to recover the ordering from the execution of the top trading cycles algorithm. Since each $\pi \in \Pi(f)$ corresponds to only one f, it is easy to directly verify that the third property is satisfied. Thus, the priorities satisfy the three desired properties.

5 Discussion and Conclusion

This paper has shown the equivalence of two competing mechanisms in a random assignment problem. If policymakers are worried about perceptions of fairness between single and multiple lotteries as in NYC, then this result allows them to inform the public that either method produces the same exact distribution of matchings.

It is surprising that seemingly different allocation mechanisms produce the same exact distribution over matchings. In contrast, the equivalence of single and multiple lotteries does not extend to other matching mechanisms such as those based the student-proposing deferred acceptance algorithm, where the tie-breaking procedure may have significant welfare consequences (see Abdulkadiroğlu, Pathak, and Roth (2008) and Erdil and Ergin (2008)).

5.1 Extensions

The main theorem here, together with Abdulkadiroğlu and Sönmez (1998), implies that the following three mechanisms are equivalent: 1) the core mechanism from random property rights, 2) random serial dictatorship, and 3) the core mechanism from random endowments. The connection between these three seemingly different mechanisms suggests a deeper relationship between efficiency and strategy-proofness in the random assignment model which is worth exploring.

Another direction to consider is the role of randomization in assignment problems with more general priority structures. Just as Sönmez and Ünver (2005) generalize the result of Abdulkadiroğlu and Sönmez (1998) in a house allocation model with existing tenants (Abdulkadiroğlu and Sönmez (1999)), a similar generalization of the main result here may be possible.

The paper has also focused on a specific version of the top trading cycles algorithm in the definition of the core mechanism. Simulation evidence also suggests that the equivalence result holds in the version of top trading cycles with counters described in Abdulkadiroğlu and Sönmez (2003). This is left for future work.

5.2 Conclusions

From a design perspective, it is possible to consider the merits of other mechanisms for the Supplementary round in NYC. While a random serial dictatorship is strategy-proof and ex-post efficient, other mechanisms (see, for instance, Hylland and Zeckhauser (1979) or Bogomolnaia and Moulin (2001)) may achieve outcomes which have better efficiency properties.[14] An active frontier is understanding the efficiency and incentive properties of these mechanisms and how

[14]To compare the (potentially unfeasible) ordinally efficient allocation, Pathak (2007) compares the distribution of matchings from a random serial dictatorship to the distribution produced by probabilistic serial assuming that participants report the same preferences in both mechanisms. The magnitude of the difference is small: about 4,999 students (out of 8,255) receive their top choice from a random serial dictatorship, while 5,016 students receive their top choice from probabilistic serial, a difference of 0.3%.

they compare to random serial dictatorships (see Che and Kojima (2008), Kojima and Manea (2008) and Abdulkadiroğlu, Che, and Yasuda (2008)).

Finally, this paper illustrates how game theory is becoming increasingly useful for the design of markets and other types of allocation mechanisms. The paper shows how perceptions of fairness might influence the choice of allocation mechanisms, and how experience designing mechanisms in real environments can spur new questions about the properties of allocation mechanisms. As economists gain experience with practical allocation mechanisms, hopefully this will continue to enrich our understanding of the theoretical properties of these mechanisms.

Appendix

It is convenient to describe the construction along with the following example:

Example

There are 8 students and 8 schools each with one seat. The strict student preferences are:

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}	P_{i_7}	P_{i_8}
s_3	s_1	s_2	s_1	s_4	s_3	s_1	s_3
.	.	.	s_5	.	s_6	s_5	s_8
.	.	.	.	.	.	s_7	.
.	.	.	.	.	.	.	.

This means that the top choice of student i_1 is s_3, and the rest of the rank ordering is some arbitrary list of the remaining students. Let f_1 be the ordering where students are ordered according to their index: $f_1(1) = i_1$, $f_1(2) = i_2$, and so on.

Step 1: Define Subproblems

Given ordering of students f and serial dictatorship ψ^f, partition the students into sets $I_1, I_2, \ldots$ as follows:

Step 1.) Starting with student $f(1)$, process each student in order of f until it is the turn of a student i for whom her assignment ψ_i^f is less preferred than some other school which has been assigned to a student ordered before i. The first partition I_1 includes all students before student i, and i belongs to the second partition. If such a student does not exist, then I_1 consists of all students. Otherwise, proceed to step 2.

In general,

Step t.) Starting with the next student, add students to I_t in order of f until it is the turn of a student i for whom her assignment ψ_i^f is less preferred than a school assigned to another student in I_t. I_t consists of all students processed in this step before student i, and i belongs to I_{t+1}. If such a student does not exist, there are a total of t sets. Otherwise, proceed to step $t+1$.

Since there are a finite number of students, this procedure produces a finite number of partitions: $I_1, I_2, \ldots, I_T$. Let S_t be the set of schools assigned under ψ^f to students in I_t, let n_t be the number of students in I_t, and let N_t denote the total number of students in sets $I_1, I_2, ..., I_t$. That is, $S_t = \cup_{i \in I_t} \psi_i^f$, $n_t = |I_t|$, and $N_t = \sum_{i=1}^t n_i$. Note that n_t is also the number of schools assigned in step t and N_t is the total number of schools assigned in steps from 1 to t. Let the problem involving the set of students and schools in step t be subproblem t.

This procedure partitions the students into sets with two properties:

i) There is no "conflict of interest" among students within a partition: no student in I_t prefers the school assigned to another student in I_t.

ii) The student ordered first by f among those in I_t for $t > 1$ prefers some school in S_{t-1} over her assignment from the serial dictatorship ψ^f.

The first property allows us to simplify the construction by considering subproblems involving students with no conflict of interest and the schools they are assigned in the serial dictatorship. The second property implies that there is at least one student in I_t who points to some school S_{t-1} in the top trading cycles algorithm before pointing to the school she is assigned to by ψ^f. These students play a key role in constructing $\Pi(f)$ because they are given priority at certain schools in a way that prevents any student who follows them in f from forming a cycle under the top trading cycles algorithm and obtaining an assignment before any student who precedes them in f. It is useful to collect all such students in I_t in another set I_t^*. Specifically, for $t > 1$, let I_t^* be the set of **unsatiated** students in I_t, those who prefer some $s \in S_{t-1}$ over their assignment from ψ^f:

$$I_t^* = \{i \in I_t : s P_i \psi_i^f \text{ for some } s \in S_{t-1}\}.$$

For each $t > 1$, the set I_t^* is non-empty by construction. The remaining students in I_t are **satiated** in step $t-1$. Each satiated student prefers the school she receives from ψ^f over any school in S_{t-1}.

Example (cont.)
The first partition of students is $I_1 = \{i_1, i_2, i_3\}$. Since student i_4 prefers s_1 over $\psi_{i_4}^{f_1} = s_5$, the second partition is $I_2 = \{i_4, i_5, i_6\}$. Since student i_7 prefers s_5 to $\psi_{i_7}^{f_1} = s_8$, the third partition is

$I_3 = \{i_7, i_8\}$. The set of schools assigned to the students in each partition are: $S_1 = \{s_1, s_2, s_3\}$, $S_2 = \{s_4, s_5, s_6\}$, and $S_3 = \{s_7, s_8\}$. The sets of unsatiated students are: $I_2^* = \{i_4, i_6\}$ and $I_3^* = \{i_7, i_8\}$. Finally, $n_1 = 3, n_2 = 3$, and $n_3 = 2$, and $N_1 = 3, N_2 = 6$, and $N_3 = 8$.

The next two steps apply to students within a subproblem, so we focus on subproblem t.

Step 2: Define Potential Orderings (Priority Skeleton) for Students

The construction identifies a particular feature of the orderings of students in I_t which form the basic structure for the priorities for each school in S_t, which does not depend on the students or schools in the other partitions. Before proceeding, we need notation for the permutations of the elements of a set. It is convenient to express such permutations in terms of column vectors. Let $\mathbb{P}(i_1, ..., i_{n_t})$ be the set of permutations of length 0 to n_t of the set $\{i_1, ..., i_{n_t}\}$. With the column vector convention,

$$\mathbb{P}(i_1, ..., i_{n_t}) = \left\{ \emptyset, (i_1), \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}, \begin{pmatrix} i_2 \\ i_1 \end{pmatrix}, \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix}, \begin{pmatrix} i_1 \\ i_3 \\ i_2 \end{pmatrix}, \begin{pmatrix} i_2 \\ i_1 \\ i_3 \end{pmatrix}, \begin{pmatrix} i_2 \\ i_3 \\ i_1 \end{pmatrix}, \begin{pmatrix} i_3 \\ i_1 \\ i_2 \end{pmatrix}, \begin{pmatrix} i_3 \\ i_2 \\ i_1 \end{pmatrix}, \dots \right\}.$$

Given this definition, a **priority skeleton** for subproblem t, denoted M_t, is a matrix consisting of columns of a particular form. Let $\mathcal{M}_t$ be a set of matrices with typical element $M_t \in \mathcal{M}_t$ which has dimension $n_t \times n_t$ and is of the form:

$$M_t = \begin{bmatrix} i_{N_{t-1}+1} & i_{N_{t-1}+2} & \dots & i_{N_{t-1}+n_t} \\ \emptyset & \mathbb{P}(i_{N_{t-1}+1}) & \vdots & \mathbb{P}(i_{N_{t-1}+1}, \dots, i_{N_{t-1}+n_t-1}) \\ \vdots & \vdots & & \vdots \\ \emptyset & \emptyset & \dots & \emptyset \end{bmatrix}.$$

It is convenient to write $M_t = \{(m_j)_{j \in \{1,...,n_t\}}\}$ where m_j is the j^{th} column of the matrix M_t, and m_{ij} is the element in the i^{th} row and j^{th} column of M_t.

Example (cont.)

The set of priority skeletons for the three subproblems, $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$, are matrices of the form:

$$M_1 = \begin{bmatrix} i_1 & i_2 & i_3 \\ \emptyset & \mathbb{P}(i_1) & \mathbb{P}(i_1, i_2) \\ \emptyset & \emptyset & \emptyset \end{bmatrix}, \quad M_2 = \begin{bmatrix} i_4 & i_5 & i_6 \\ \emptyset & \mathbb{P}(i_4) & \mathbb{P}(i_4, i_5) \\ \emptyset & \emptyset & \emptyset \end{bmatrix}, \quad \text{and} \quad M_3 = \begin{bmatrix} i_7 & i_8 \\ \emptyset & \mathbb{P}(i_7) \end{bmatrix}.$$

Step 3: From Priority Skeleton to Orderings for Schools (Priority Assignment)

The priority skeletons encode information only about students. The next step of the construction is to relate the columns of a priority skeleton to particular schools. As in the previous step, we focus on subproblem t. For each t, fix some priority skeleton M_t.

To specify which ordering of students applies to which school, define the **priority assignment** function α:

$$\alpha : S \rightarrow \{1, ..., n\}.$$

For subproblem t we will assign values to this function for $s \in S_t$, which is how a particular ordering of students, formed from a column of the priority skeleton, becomes the priority for the school.

To specify values of α, we define a partition of the students $I_t = I_t^1, ..., I_t^r, ...$ with the following properties:

1) Students in I_t^r each immediately following one another in the ordering f.

2) All students in I_t^r are ordered by f before students in $I_t^{r'}$ for $r < r'$.

Case 1: First Subset of Students, I_t^1

For each column of M_t, find the student who is in the largest non-empty row. Since the first student I_t, student $i_{N_{t-1}+1}$, is in the first column of M_t and followed by no other students, she is among this set of students. From this set of students, place the subset who form the longest sequence of students who immediately follow $i_{N_{t-1}+1}$ in ordering f into I_t^1, and any of the remaining students in O_t^2 (i.e. these students are "out of order").

Let S_t^1 be the set of schools assigned to I_t^1 by ψ^f, and partition I_t^1 into the set who are satiated, $I_t^{1,+} = I_t^1 \cap (I_t \backslash I_t^*)$, and unsatiated, $I_t^1 \backslash I_t^{1,+}$.

a) First Partition, I_1^1

Since there are no unsatiated students in the first partition, the priority assignments in subproblem 1 are slightly different than for step $t > 1$. For step 1, order the schools in $S_1^1 = \{s^1, s^2, ..., s^{n_1^1}\}$ from smallest to largest index. Set $\alpha(s^1) = 1, \alpha(s^2) = 2$, and so on.

b) Subsequent Partitions, I_t^1 for $t > 1$

For $t > 1$, define priority assignments as follows:

i) Unsatiated students ($I_t^1 \backslash I_t^{1,+}$)

For each $i \in I_t^1 \cap I_t^{1,+}$, if i is not the last student in I_t^1, find the school assigned to the next student in I_t^1 according to f. If i is the last student in I_t^1, find the school assigned to the first student in I_t^1. (Note: if there is only one student in I_t^1, then this student is both the first and last student in I_t^1.) Place this school into set $S_t^{1,*}$. Iterate for each member of $I_t^1 \cap I_t^{1,+}$ to construct $S_t^{1,*}$.

For the first student i in $I_t^1 \cap I_t^*$, find the lowest indexed school s in $S_t^{1,*}$, and set $\alpha(s) = f^{-1}(i)$. Continue in this way for the second student in $I_t^1 \cap I_t^*$ and so on.

ii) Satiated students ($I_t^{1,+}$)

Find the first satiated student $i \in I_t^{1,+}$. If the student is not the last student in I_t^1, find the school s that is assigned to the next student in I_t^1 and set $\alpha(s) = f^{-1}(i)$. If i is the last student in I_t^1, find the school s assigned to the first student in I_t^1 and set $\alpha(s) = f^{-1}(i)$. Iterate for each member of $I_t^{1,+}$.

When this process is complete for both unsatiated and satiated students in I_t^1, the priority assignment is defined for each school in S_t^1.

The next step is to trim the priority skeleton M_t to eliminate the students assigned to the schools which just received a priority assignment. First, remove columns where the first entry is some student in I_t^1. Second, in each column, delete all entries involving a student in I_t^1 and advance the row entries up so all of the empty entries are at the bottom of the column. Let M_t^2 be the new priority skeleton. If M_t^2 consists only of empty elements, then stop.

Case 2: Subsequent Subsets of Students, I_t^r for $r > 1$

For each column of M_t^r, find the student who is in the largest non-empty row. Collect this set of students and consider the union of this set with O_t^r. Compute the largest sequence of distinct students who immediately follow in order after the last student in I_t^{r-1}. Let I_t^r be this set of students and O_t^{r+1} be the remaining set of students.

Let S_t^r be the set of schools assigned to I_t^r by ψ^f, and $I_t^{r,+} = I_t^r \cap O_t^r$. The students in $I_t^{r,+}$ are the subset of students in I_t^r that we came across in an earlier substep.

i) Students in $I_t^r \backslash I_t^{r,+}$

For each $i \in I_t^r \backslash I_t^{r,+}$, if i is not the last student in I_t^r, find the school s that is assigned to the next student in I_t^r according to f. If i is the last student in I_t^r, let s be the school that is assigned to the first student in I_t^r. Place s into the set $S_t^{r,*}$. Iterate for each member of $I_t^r \backslash I_t^{r,+}$ to construct $S_t^{r,*}$

For the first student i in $I_t^r \backslash I_t^{r,+}$, find the lowest indexed school s in $S_t^{r,*}$, and set $\alpha(s) = f^{-1}(i)$. Continue in this way for the second student in $I_t^r \backslash I_t^{r,+}$ and so on.

ii) Students in $I_t^{r,+}$

Find the first student $i \in I_t^{r,+}$. If the student is not the last student in I_t^r, find the school s that is assigned to the next student in I_t^r and set $\alpha(s) = f^{-1}(i)$. If i is the last student in I_t^r, find the school s assigned to the first student in I_t^r and set $\alpha(s) = f^{-1}(i)$. Iterate for each member of $I_t^{r,+}$.

Since at least one school is assigned a value for their priority assignment at each substep and at least one column is removed from the priority skeleton in each substep, the procedure is terminates in a finite number of steps. For step t, there will be a priority assignment for each $s \in S_t$ to some number in $\{N_{t-1}+1, N_{t-1}+2, ..., N_{t-1}+n_t\}$.

Example (cont.)

For subproblem 1, suppose the priority skeleton is:

$$M_1 = \begin{bmatrix} i_1 & i_2 & i_3 \\ \emptyset & i_1 & i_2 \\ \emptyset & \emptyset & i_1 \end{bmatrix}.$$

In this case, $I_1^1 = \{i_1\}$, so $S_1^1 = \{s_3\}$. Hence, $\alpha(s_3) = 1$. The trimmed priority skeleton for substep 2 is

$$M_1^2 = \begin{bmatrix} i_2 & i_3 \\ \emptyset & i_2 \end{bmatrix}.$$

$I_1^2 = \{i_2\}$, $S_1^2 = \{s_1\}$ and $O_1^2 = \emptyset$, so $\alpha(s_1) = 2$. The trimmed priority skeleton for substep 3 is

$$M_1^3 = \begin{bmatrix} i_3 \end{bmatrix}.$$

$I_1^3 = \{i_3\}$, so $\alpha(s_2) = 3$.

For subproblem 2, suppose the priority skeleton is:

$$M_2 = \begin{bmatrix} i_4 & i_5 & i_6 \\ \emptyset & i_4 & \emptyset \\ \emptyset & \emptyset & \emptyset \end{bmatrix}.$$

Here, i_4 and i_6 are the students who are at the largest non-empty row of each column. Since i_6 does not immediately follow i_4 in the ordering, i_6 is out of order. Therefore, $I_2^1 = \{i_4\}$ and $O_2^2 = \{i_6\}$. As a result, $S_2^1 = \{s_5\}$ and $\alpha(s_5) = 4$. The trimmed priority skeleton for substep 2 is

$$M_2^2 = \begin{bmatrix} i_5 & i_6 \\ \emptyset & \emptyset \end{bmatrix}$$

Here $I_2^2 = \{i_5, i_6\}$. Since $O_2^2 = \{i_6\}$, we have that $I_2^2 \backslash I_2^{2,+} = \{i_5\}$. Find the school that is assigned to the next student according to f in I_2^2. The next student is i_6, and she is assigned s_6, so $\alpha(s_6) = 5$. For each student in $I_2^{2,+} = \{i_6\}$, i_6 is the last student in I_2^2, so we find the school assigned assigned to the first student in I_2^2. This student is i_5 and the school she receives is s_4, so $\alpha(s_4) = 6$.

For subproblem 3, suppose the priority skeleton is:

$$M_3 = \begin{bmatrix} i_7 & i_8 \\ \emptyset & \emptyset \end{bmatrix}.$$

Notice that $I_3^1 = \{i_7, i_8\}$ and $I_3^{1,+} = \{i_8\}$. First process students in $I_3^1 \backslash I_3^{1,+} = \{i_7\}$. For this student, find the school that is assigned to the next student in I_3^1. The next student is i_8 and the school she receives in s_8, so set $\alpha(s_8) = 7$. Next, process students in $I_3^{1,+}$. Since i_8 is the last student in I_3^1, find the first student in I_3^1. This student is i_7 and the school she receives is s_7, so $\alpha(s_7) = 8$.

To summarize, for the priority skeleton we considered for each subproblem, the priority assignment for each school is:

$$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \\ 2 & 3 & 1 & 6 & 4 & 5 & 8 & 7 \end{pmatrix}.$$

Step 4: Define Priorities for Schools Using Skeleton and Assignment

For step t and priority skeleton M_t, the priority assignment α is defined for each $s \in S_t$. Next, define a set of priorities for each school as follows:

1) The first student in I_t

Let i be the first student in I_t. She is the $(N_{t-1}+1)^{\text{th}}$ student in the ordering f. Find the school s which corresponds to the $(N_{t-1}+1)^{\text{th}}$ priority assignment: $\alpha(s) = N_{t-1} + 1$.

The set of priorities for school s is:

$$\Pi_s = \Big[\pi_s \quad | \quad \pi_s(i) < \pi_s(k), \quad \forall k \in \cup_{\tau=t}^{T} I_\tau \backslash i\Big].$$

Any ordering at school s which gives i higher priority that any student who follows her according to f is part of this set. Furthermore, the students who precede student i (that is, any student in $\cup_{\tau=1}^{t-1} I_\tau$) can be placed anywhere in the ordering.

j) The j^{th} student in I_t

Let j be the $(N_{t-1}+j)^{\text{th}}$ student in the ordering f, which is the j^{th} student in the set I_t. Let s be such that $\alpha(s) = N_{t-1} + j$. Let $|m_j|$ be the number of non-empty rows of column j of the priority skeleton M_t.

The set of priority structures for school s is:

$$\Pi_s = \Big[\pi_s \quad | \quad \pi_s(m_{ij}) < \pi_s(m_{(i+1)j}), \text{ for } i = 1,..,|m_j| - 1$$
$$\text{and } \pi_s(j) < \pi_s(k) \quad \forall k \in \cup_{\tau=t}^{T} I_\tau \setminus \cup_{i=1}^{|m_j|} m_{ij}\Big]$$

Any priority for school s in this set can give a student in any previous step (those students in $\cup_{\tau=1}^{t-1} I_\tau$) priority anywhere at the school. Students in subsequent steps (i.e. $\cup_{\tau=t+1}^{T} I_\tau$) together with students in I_t who are not in the priority skeleton $I_t \setminus \cup_{i=1}^{|m_j|} m_{ij}$ must be ordered after student j.

At this stage, given priority skeleton M_t, students I_t, and schools S_t, we have defined the set of priorities for each school in S_t which corresponds to M_t. Next, construct another set of priorities by considering a different priority skeleton $M_t \in \mathcal{M}_t$. Finally, to define the entire set $\Pi(f)$, repeat the procedure for each subproblem t.

Example (cont.)

The set of priority structures that correspond to the particular priority skeleton defined earlier are

s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
i_1	i_1	i_1	$[I_1]$	$[I_1]$	$[I_1]$	$[I_1 \cup I_2]$	$[I_1 \cup I_2]$
i_2	i_2	$\cdot$	i_6	i_4	i_4	i_8	i_7
$\cdot$	i_3	$\cdot$	$\cdot$	$\cdot$	i_5	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

where $[I_1]$ means any student in I_1 can be anywhere in the priority at the school. For instance, at school s_6, any student in I_1 can either be given higher priority than i_4, placed in between i_4 and i_5 or placed after i_5. Likewise, at school s_7, any student in $I_1 \cup I_2$ can be given higher or lower priority than i_8. In the set described, note the relationship between the priority skeleton and the priorities. The example illustrates how the set of priorities structures is defined by re-arranging the columns and inverting the rows of the priority skeleton and then assigning them to particular schools.

The number of priority structures which correspond to this priority skeleton is:

$$\underbrace{(6)!(5)!(7)!}_{\text{subproblem 1}} \cdot \underbrace{\frac{8!}{(5)}\frac{8!}{(5)}\frac{8!}{(5)(4)}}_{\text{subproblem 2}} \cdot \underbrace{\frac{8!}{(2)}\frac{8!}{(2)}}_{\text{subproblem 3}}$$

To construct the entire set $\Pi(f)$, repeat the procedure for each possible priority skeleton.

Step 5: Properties of $\Pi(f)$

The first property we establish is that the matching produced by the core mechanism for any $\pi \in \Pi(f)$ is the same as the matching produced by the serial dictatorship ψ^f.

Lemma 1. For any $\pi \in \Pi(f)$, we have that $m^{\varphi^\pi} = m^{\psi^f}$.

Proof. Each $i \in I_1$ prefers her assignment, ψ_i^f, to all other schools matched to students in I_1 by ψ^f. Since ψ^f is a serial dictatorship, each $i \in I_1$ must receive her top choice. Consider the priority structure π at the schools in S_1. Begin with the students who obtain the top priority for any school in S_1. By construction, a subset of these students who are immediately in order following the first student in f each receive the top priority at a school that some student in the subset desires as her top choice. In the first step of the top trading cycles algorithm, each student in this subset points to the school they desire and since there is no conflict of interest among these students, a set of cycles forms involving these students and their top choice school. Once this cycle forms and the students and schools involved are removed, the highest priority for the remaining schools in S_1 will be given to students in I_1 who have not yet been assigned. A subset of these students, following the last student who left the market in a cycle in the previous substep, will each point to the school for which some student among those remaining in I_1 receives the top priority. Under the next step of φ^π, a set of cycles will form involving these students as each points to her top choice. Once the schools assigned in these cycles are removed, iterate these arguments for the remaining schools in S_1. By construction, the highest priorities are given to students remaining in I_1 who have not yet been assigned. When each points to their top choice, a subset of these form cycles and leave the market until every student in I_1 has been assigned. Thus, for $i \in I_1$, we have that $\psi_i^f = \varphi_i^\pi$.

Since each student only receives one school, once the assignments for students in I_1 have been finalized, the position of these students in the priorities for any subsequent schools does not matter for the allocation received by students who are not in I_1. Once all students in I_1 have been processed, we iterate the arguments from step 1 for students in I_2. Every student in I_2 must prefer her assignment over any school remaining in the market. For the priority structure π at the schools in S_2, when a cycle forms, the highest priority for the remaining schools in the step are given to students in the step who have not yet been assigned. This fact combined with the fact that there is no conflict of interest among students in I_2 ensures that each student $i \in I_2$ receives the same school under ψ^f as φ^π.

Once φ^π is fixed for all students in the second subproblem, the argument can be iterated for students in I_3 and so on. This establishes the claim. □

The next step is to count the number of priority structures which corresponds to each f. In the priority assignment step, we simply re-arrange columns of the priority skeleton M_t. This is the key to counting the number of elements in $\Pi(f)$ in a simple way.

Lemma 2. $|\Pi(f)| = (n!)^{n-1}$ for all $f \in \mathcal{F}$.

Proof. Given f, consider the set $\Pi(f)$ and its restriction to the schools assigned in the first subproblem, which we denote $\Pi_1(f)$. Given the priority skeleton M_1, the values of the priority assignment α for the schools in the first step are $\{1, ..., n_1\}$. The exact mapping between schools and $\{1, ..., n_1\}$ may be different for another member of $\mathcal{M}_1$. To note this dependence on M_1 and α, subscript each school's priority by π_α, leaving the dependence of α on M_1 implicit. Since α for the first subproblem is a bijection between S_1 and $\{1, ..., n_1\}$, we can ignore which school is mapping to the particular column of the priority skeleton M_1 for the purposes of counting.

The entire set of priorities for the schools takes a form which mimics the priority skeleton:

$$\Pi_1(f) = \Big[\Pi_1 \text{ s.t. } \pi_\alpha(f(1)) < \pi_\alpha(i) \quad \text{for any } i \in [\cup_{t=1}^T I_t] \backslash [f(1)], \tag{4}$$

$$\pi_\alpha(f(2)) < \pi_\alpha(i) \quad \text{for any } i \in [\cup_{t=1}^T I_t] \backslash [\cup_{j=1}^2 f(j)], \tag{5}$$

$$\vdots$$

$$\pi_\alpha(f(n_1)) < \pi_\alpha(i) \quad \text{for any } i \in [\cup_{t=1}^T I_t] \backslash [\cup_{j=1}^{n_1} f(j)]\Big]. \tag{6}$$

In subproblem 1, consider the possible priorities of the form corresponding to (4). Restriction (4) admits $(n-1)!$ orderings of students, as it specifies only that $f(1)$ receives the top priority for the school. For the next student, restriction (5) admits $(n-1)!+(n-2)!$ orderings, corresponding to whether $f(2)$ receives the top priority and the remaining $(n-1)$ students are ordered arbitrarily or $f(1)$ receives the top priority, $f(2)$ receives the second priority, and the remaining $(n-2)$ students are ordered arbitrarily, respectively. Consider the k^{th} student in step 1, where $\pi_\alpha(f(k)) < \pi_\alpha(i)$ for any $i \in [\cup_{t=1}^{T} I_t] \backslash [\cup_{j=1}^{k-1} f(j)]$. This restriction admits $(n-1)!+(k-1)(n-2)!+(k-1)(k-2)(n-3)!+...+(k-1)!(n-k)! = \sum_{l=1}^{k-1} \frac{(k-1)!}{(k-l)!}(n-l)! = \frac{n!}{n-(k-1)}$ priorities. Thus,

$$|\Pi_1(f)| = \left[\frac{n!}{n} \cdot \frac{n!}{n-1} \cdot \dots \cdot \frac{n!}{n-(n_1-1)}\right] = \prod_{i_1=0}^{n_1-1} \frac{n!}{n-i_1}.$$

Follow the same reasoning for subproblems 2,...,T. For subproblem t,

$$|\Pi_t(f)| = \left[\frac{n!}{n-\sum_{j=1}^{t-1} n_j} \cdot \frac{n!}{n-\sum_{j=1}^{t-1} n_j - 1} \cdot \dots \cdot \frac{n!}{n-\sum_{j=1}^{t-1} n_j - n_t - 1)}\right] = \prod_{i_t=0}^{n_t-1} \frac{n!}{n-\sum_{j=1}^{t-1} n_j - i_t}.$$

Finally, putting each step together:

$$|\Pi(f)| = \prod_{i_1=0}^{n_1-1} \frac{n!}{n-i_1} \cdot \prod_{i_2=0}^{n_2-1} \frac{n!}{n-n_1-i_2} \cdot \dots \cdot \prod_{i_t=0}^{n_t-1} \frac{n!}{n-\sum_{j=1}^{t-1} n_j - i_t} = (n!)^{n-1},$$

which completes the proof of the claim. □

Step 6: Define the Inverse Mapping

Next, we construct the inverse mapping to recover f. For some f and $\pi \in \Pi(f)$, let us begin by carefully executing the top trading cycles algorithm:

Step 1) Simultaneously remove all cycles of students which form. When a student is assigned as part of a cycle, we say that the student has left the market. Let C_1^1 be the set of these cycles and let G_1^1 be the set of schools assigned in the cycles in set C_1^1. This is the first substep of step 1.

Next, simultaneously remove all cycles that do not involve a student who desires a school assigned to a student in cycle C_1^1. Place these cycles into C_1^2 and let G_1^2 be the corresponding set of schools. Continue simultaneously removing all cycles that do not involve a student who desires a school assigned to a student in the cycles that have taken place in the step.

Suppose there are r_1 substeps of step 1. Define the set of cycles which form in this step as: $C_1 = \{C_1^1, C_1^2, ..., C_1^{r_1}\}$, where cycles in the set C_1^1 form before cycles in the set C_1^2 form, and so on. Let $G_1 = \{G_1^1, G_1^2, ..., G_1^{r_1}\}$ be the corresponding set of schools assigned in these cycles. When each potential cycle that forms in the market involves a student who prefers a school assigned to a student in G_1, proceed to the next step.

In general,

Step t) Simultaneously remove all cycles and place them into C_t^1. Let G_t^1 be the set of schools assigned in the cycles in C_t^1. Next, simultaneously remove all cycles that do not involve a student who prefers a school assigned to a student in C_t^1 over what she receives in the cycle. Place these cycles into the set C_t^2 and define the corresponding G_t^2. Stop when all cycles involve at least one student who was pointing to a school that was assigned in step t.

Suppose there are r_t substeps of step t. Order the sets of cycles by the substep in which they form: $C_t = \{C_t^1, C_t^2, ..., C_t^{r_t}\}$ and define the corresponding set of schools $G_t = \{G_t^1, G_t^2, ..., G_t^{r_t}\}$. When each cycle that forms in the market involves a student who prefers a school in G_t to what she receives in the cycle, proceed to the next step.

This procedure stops when no more students remain. For each step t, let I_{C_t} be the set of students involved in the collection of cycles C_t and let $I_{C_t^r}$ be the set of students involved in the set of cycles in set C_t^r.

For each step $t > 1$ and substep $r = 1$, define $I^+_{C_t^1}$ as the subset of students in $I_{C_t^1}$ who are satiated: they prefer their school assignment to any school assigned in G_{t-1}.

$$I^+_{C_t^1} = \{i \in I_{C_t^1} \quad \text{s.t.} \quad \varphi_i^\pi P_i s \quad \text{for all } s \in G_{t-1}\}.$$

Let $G_t^{1,*}$ be the set of schools who point to students who are unsatiated. These students are in $I_{C_t^1} \backslash I^+_{C_t^1}$, and $G_t^{1,*}$ are schools which point to these students in a cycle in C_t^1.

For each step $t = 1, ..., T$ and substep $r > 1$, let s^i be the school that points to student i in the cycle where the student leaves the market. Define the set of students $I^+_{C^r_t}$ as those who are on a higher position in the priority ordering for the school that points to them in the cycle where they leave the market than the students in $I_{C^{r-1}_t}$. More precisely,

$$I^+_{C^r_t} = \{i \in I_{C^r_t} \quad \text{s.t.} \quad \nexists i' \in I_{C^{r-1}_t} \text{ where } \pi_{s^i}(i') < \pi_{s^i}(i)\}.$$

Let $G^{r,*}_t$ be the set of schools who point to students in $I_{C^r_t} \backslash I^+_{C^r_t}$ in cycles C^r_t. The role of unsatiated students $I^1_t \backslash I^{1,+}_t$ in the first substep for steps $t > 1$ is analogous to the role of students in $I^r_t \backslash I^{r,+}_t$.

Example (cont.)
From the set of priority structures, consider a particular priority structure π_1:

π_{s_1}	π_{s_2}	π_{s_3}	π_{s_4}	π_{s_5}	π_{s_6}	π_{s_7}	π_{s_8}
$\boxed{i_1}$	$\boxed{i_1}$	$\boxed{i_1}$	i_2	$\boxed{i_4}$	i_3	i_6	i_2
$\boxed{i_2}$	$\boxed{i_2}$	i_2	i_3	i_8	$\boxed{i_4}$	i_3	i_5
i_4	$\boxed{i_3}$	i_5	$\boxed{i_6}$	i_1	i_1	i_2	i_1
i_5	i_6	i_6	i_7	i_3	i_2	$\boxed{i_8}$	i_4
i_8	i_5	i_3	i_4	i_2	$\boxed{i_5}$	i_7	i_3
i_7	i_7	i_4	i_8	i_6	i_8	i_5	i_6
i_3	i_8	i_7	i_7	i_1	i_7	i_4	$\boxed{i_7}$
i_6	i_4	i_8	i_5	i_5	i_6	i_1	i_8

where boxed entries are the non-empty entries of the priority skeleton.

In the first substep of step 1 of the top trading cycles algorithm, the sets are:

$$C^1_1 = \{i_1 \rightarrow s_3 \rightarrow i_1\}, \quad G^1_1 = \{s_3\}, \quad I_{C^1_1} = \{i_1\}.$$

The second substep of step 1, the sets are:

$$C^2_1 = \{i_2 \rightarrow s_1 \rightarrow i_2\}, \quad G^2_1 = \{s_1\}, \quad I_{C^2_1} = \{i_2\}.$$

Moreoever,

$$I^+_{C^2_1} = \emptyset, \quad G^{2,*}_1 = \{s_1\}.$$

In the next substep, i_4 points to s_5 because her top choice school s_1 has left the market, and school s_5 points to her. However, since student i_4 prefers the assignment of student i_2 to her own assignment, any cycle involving her is not removed until after all cycles involving students who do not prefer the assignment of another student in a cycle in the step are removed. In the third substep, the sets are:

$$C_1^3 = \{i_3 \to s_2 \to i_3\}, \quad G_1^3 = \{s_s\}, \quad I_{C_1^3} = \{i_3\}, \quad I_{C_1^3}^+ = \emptyset,$$

where

$$I_{C_1^3}^+ = \emptyset, \quad G_1^{2,*} = \{s_2\}.$$

At this stage, each possible cycle involves one student who prefers the assignment of a student over the school the student receives. This begins the second step:

$$C_2^1 = \{i_4 \to s_5 \to i_4\}, \quad G_2^1 = \{s_5\}, \quad I_{C_2^1} = \{i_4\}.$$

Moreoever,

$$I_{C_2^1}^+ = \emptyset, \quad G_2^{1,*} = \{s_5\}.$$

For the next substep, the sets are:

$$C_2^2 = \{i_5 \to s_4 \to i_6 \to s_6 \to i_5\}, \quad G_2^2 = \{s_4, s_6\}, \quad I_{C_2^2} = \{i_5, i_6\}.$$

Since i_6 is ordered ahead of i_4 at school s_4, the sets are:

$$I_{C_2^2}^+ = \{i_6\}, \quad I_{C_2^2} \backslash I_{C_2^2}^+ = \{i_5\}, \quad G_2^{2,*} = \{s_6\}.$$

Student i_7 is part of the next step because she prefers a s_5, the school assigned to student i_4 over her assigned school. There is only one substep in the third step:

$$C_3^1 = \{i_7 \to s_7 \to i_8 \to s_8 \to i_7\}, \quad G_3^1 = \{s_7, s_8\}, \quad I_{C_3^1} = \{i_7, i_8\}.$$

Since i_8 is satiated, the sets are:

$$I^+_{C^1_3} = \{i_8\}, \quad I_{C^1_3} \backslash I^+_{C^1_3} = \{i_7\}, \quad G^{1,*}_3 = \{s_8\}.$$

This completes the definition of the sets from the execution of top trading cycles for π_1.

With these sets in hand, it is possible to define the inverse mapping $g : \Pi \to \mathcal{F}$. For any π,

1.) Construct the sets $\{C^1_t, C^2_t, ..., C^{r_t}_t\}$ and $\{G^1_t, G^2_t, ..., G^{r_t}_t\}$ and the corresponding sets $\{C_t\}$ and $\{G_t\}$. For $t > 1$, construct $I^+_{C^1_t}$ and $G^{1,*}_t$. For each $t = 1, ..., T$ and $r > 1$, construct $I^+_{C^r_t}$ and $G^{r,*}_t$.

2.) For any t, order the students in I_{C_t} before the students in $I_{C_{t+1}}$.

3.) For any t and $r > 1$, order the students in $I_{C^r_t}$ before the students in $I_{C^{r+1}_t}$.

4.) Order the students in $I_{C^1_t}$. There is a different procedure for $t = 1$ and $t > 1$.

   a.) Order the students in $I_{C^1_1}$ based on the index of the school which points to them in the cycle in C^1_1. The student pointed to by the lowest indexed school is first, followed by the student pointed to by the second lowest indexed school, and so on.

   b.) For $I_{C^1_t}$, look first at the unsatiated students in $I_{C^1_t} \backslash I^+_{C^1_t}$. Order them based on the index of the school which points to them in a cycle in C^1_t. Without loss of generality, write $I_{C^1_t} \backslash I^+_{C^1_t} = \{\tilde{i}_1, \tilde{i}_2, ..., \tilde{i}_\ell\}$ as the ordering of students.
   To complete the ordering of students in $I^+_{C^1_t}$, begin with $\tilde{i}_2$. Find the school that $\tilde{i}_2$ points to, and the student i' whom this school points to in a cycle in C^1_t. If $i' \in I_{C^1_t} \backslash I^+_{C^1_t}$, then this student has already been processed, and move to $\tilde{i}_3$. Otherwise, place i' immediately before $\tilde{i}_2$ and find which school student i' points to. Let i'' be the student who this school points to. If $i'' \in I_{C^1_t} \backslash I^+_{C^1_t}$, then this student has already been processed, and move to $\tilde{i}_3$. Otherwise, place i'' immediately before i' in the part of the order which orders $I_{C^1_t}$. Proceed in a similar way until encountering a student in $I_{C^1_t} \backslash I^+_{C^1_t}$, at which point proceed to $\tilde{i}_3$. Repeat this procedure for each of the students in $\tilde{i}_3, ..., \tilde{i}_\ell$.

Finally, consider student $\tilde{i}_1$ and find the cycle she belongs to. Find the school that $\tilde{i}_1$ points to and the student i' that this school points to. If $i' \in I_{C_t^1} \backslash I^+_{C_t^1}$, then this student is already processed and the procedure stops. If $i' \in I^+_{C_t^1}$, then order this student at the very end of the part of the order which orders students in $I_{C_t^1}$, and find the student i'' who is pointed to by the school that i' points to in the cycle. If $i'' \in I_{C_t^1} \backslash I^+_{C_t^1}$, then this student is already processed and terminate the procedure. If $i'' \in I^+_{C_t^1}$, order i'' before i' and proceed in a similar way until encountering a student in $I_{C_t^1} \backslash I^+_{C_t^1}$. This student will already be handled, so terminate the procedure.

At the conclusion of this process, all students in $I_{C_t^1}$ will be ordered.

5.) For $r > 1$, first order the students in $I_{C_t^r} \backslash I^+_{C_t^r}$ based on the index of the school which points to them in a cycle in C_t^r. Without loss of generality, write $I_{C_t^r} \backslash I^+_{C_t^r} = \{\tilde{i}_1, \tilde{i}_2, ..., \tilde{i}_\ell\}$ as the ordering of students.

To complete ordering of the remaining students in $I^+_{C_t^r}$ begin with $\tilde{i}_2$. Find the school that $\tilde{i}_2$ points to, and the student i' whom this school points to in a cycle in C_t^r. If $i' \in I_{C_t^r} \backslash I^+_{C_t^r}$, then this student has already processed, and move to $\tilde{i}_3$. Otherwise, place i' immediately before $\tilde{i}_2$ and find which school student i' points to. Let i'' be the student who this school points to. If $i'' \in I_{C_t^r} \backslash I^+_{C_t^r}$, then this student has already been processed, and move to $\tilde{i}_3$. Otherwise, place i'' immediately before i' in the part of the ordering (that which orders students in $I_{C_t^r}$). Proceed in a similar way until encountering a student in $I_{C_t^r} \backslash I^+_{C_t^r}$, at which point proceed to $\tilde{i}_3$. Repeat this procedure for each of the students in $\tilde{i}_3, ..., \tilde{i}_\ell$.

Finally, consider student $\tilde{i}_1$ and find the cycle she belongs to. Find the school that $\tilde{i}_1$ points to and the student i' that this school points to. If $i' \in I_{C_t^r} \backslash I^+_{C_t^r}$, then this student is already processed and the process ends. If $i' \in I^+_{C_t^r}$, then order this student at the very end of the part of the order (that which orders students in $I_{C_t^r}$), and find the student i'' who is pointed to by the school that i' points to in the cycle. If $i'' \in I_{C_t^r} \backslash I^+_{C_t^r}$, then this student is already processed and we terminate the procedure. If $i'' \in I^+_{C_t^r}$, order i'' immediately before i' and proceed in a similar way until encountering a student in $I_{C_t^r} \backslash I^+_{C_t^r}$. This student will already be handled, so terminate the procedure.

This process orders the students in $I_{C_t^r}$ for all $r > 1$.

At the conclusion of this procedure, there will be an ordering f for a set of priorities π.

Example (cont.)
Once the sets are defined, part 2) of the procedure which defines the inverse mapping implies the following:

$$\underbrace{\{i_1, i_2, i_3\}}_{I_{C_1}} \Bigg| \underbrace{\{i_4, i_5, i_6\}}_{I_{C_2}} \Bigg| \underbrace{\{i_7, i_8\}}_{I_{C_3}}.$$

Part 3) implies the following:

$$\underbrace{i_1}_{I_{C_1^1}} | \underbrace{i_2}_{I_{C_1^2}} | \underbrace{i_3}_{I_{C_1^3}} \Bigg| \underbrace{i_4}_{I_{C_2^1}} | \underbrace{\{i_5, i_6\}}_{I_{C_2^2}} \Bigg| \underbrace{\{i_7, i_8\}}_{I_{C_3^1}}.$$

All that remains is to order the students $I_{C_2^2}$ and $I_{C_3^1}$. Since $I^+_{C_2^2} = \{i_6\}$ and $I_{C_2^2} \backslash I^+_{C_2^2} = \{i_5\}$, to follow the notation we write that $i_5 = \tilde{i}_1$. Since there is no $\tilde{i}_2$, there is no need to order the only student in $I_{C_2^2} \backslash I^+_{C_2^2}$. Since $i_5 = \tilde{i}_1$ points to school s_4 which points to $i_6 = i'$, and $i_6 \notin I_{C_2^2} \backslash I^+_{C_2^2}$, student i_6 is placed at the very end of the ordering of students in $I_{C_2^2}$. Therefore, i_5 is ahead of i_6.

To order the students in $I_{C_3^1}$, since $I^+_{C_3^1} = \{i_8\}$ and $I_{C_3^1} \backslash I^+_{C_3^1} = \{i_7\}$, we write that $i_7 = \tilde{i}_1$. Since there is no $\tilde{i}_2$, there is no need to order the only student in $I_{C_3^1} \backslash I^+_{C_3^1}$. Since $i_7 = \tilde{i}_1$ points to school s_7 which points to $i_8 = i'$, and $i_8 \notin I_{C_3^1} \backslash I^+_{C_3^1}$, student i_8 is placed at the very end of the ordering of students in $I_{C_3^1}$. Therefore i_7 is ahead of i_8. Hence, the ordering is as follows:

$$i_1 \;|\; i_2 \;|\; i_3 \Bigg| i_4 \;|\; \underbrace{i_5 \;|\; i_6}_{\text{From Part 5)}} \Bigg| \underbrace{i_7 \;|\; i_8}_{\text{From Part 4b)}},$$

which shows that $g(\pi_1) = f_1$.

Step 7: Establish that $g(\pi) = f$

Next, we show that this procedure recovers the same f for each $\pi \in \Pi(f)$.

Lemma 3. For any f and $\pi \in \Pi(f)$, we have that $g(\pi) = f$.

Proof. The proof proceeds by relating the steps involved in constructing $\Pi(f)$ to the inverse mapping. From Lemma 1, $m^{\varphi^\pi} = m^{\psi^f}$.

Claim. For all t and r, we have that $I_{C_t^r} = I_t^r$.

Proof. With our execution of the top trading cycles algorithm for π, no student in $\cup_{\tau>t} I_{C_\tau}$ leaves the market before any student in I_{C_t}. Under π, the first student in I_t who is also the first student in I_t^1 receives the highest priority for a school assigned in a cycle in C_t^1 among all students in I_t^1. When $t > 1$, the first student I_t^1 must be unsatiated. If there are multiple students who receive the top priority among the schools assigned to students in I_t^1, then the construction ensures that each cycle that forms among these students must involve at least one unsatiated student. Therefore, these cycles can be identified when the unsatiated student points to a school that had been assigned to a student in the previous step. Moreover, every satiated student in C_t^1 will not be assigned until the first student in the step has been assigned and this student is unsatiated. Thus, the students in $I_{C_t^1}$ cannot be part of I_{t-1} and must be part of I_t^1. Next, since no student in $\cup_{\rho>r} I_{C_t^\rho}$ leaves the market before any student in $I_{C_t^r}$ and the students in $I_{C_t^r}$ for $\rho > 1$ each prefer the school they receive under C_t^r to the schools assigned in some cycle in $C_t^1, ..., C_t^{r-1}$, these students in $I_{C_t^r}$ must be part of I_t^r. ◇

This claim implies that $I_{C_t} = I_{C_t^1} \cup I_{C_t^2} \cup ... \cup I_{C_t^{r_t}} = \cup_{r=1}^{r_t} I_t^r = I_t$.

Claim. For all t and r, we have that $G_t^r = S_t^r$.

Proof. This follows from the fact that $m^{\varphi^\pi} = m^{\psi^f}$, $I_{C_t^r} = I_t^r$, and that there is no conflict of interest among students within each step. ◇

This claim implies that $G_t = G_t^1 \cup G_t^2 \cup ... \cup G_t^{r_t} = \cup_{r=1}^{r_t} S_t^r = S_t$. Given $I_{C_t^r} = I_t^r$ and $G_t^r = S_t^r$ it is straightforward to see that the satiated students in the first substep are the same for $t > 1$: $I_{C_t^1}^+ = I_t^{1,+}$. Moreover, the set of students who receive higher priority at a school than a student assigned in an cycle in an earlier substep is equal to the set of students who are at a higher level in the substep: $I_{C_t^r}^+ = I_t^{r,+}$. This also implies that $S_t^{1,*} = G_t^{1,*}$ and $S_t^{r,*} = G_t^{r,*}$.

This leaves us to establish the last claim:

Claim. $g(\pi) = f$

Proof. The proof is by induction. Begin by examining step 1 and all of its substeps. Suppose $i \in I_1 = I_{C_1}$ is among the students who leave the market as part of a cycle in C_1^1 and that $f(k) = i$ for some number k. By construction of π, the school which points to i must be the k^{th} smallest indexed school in S_1^1. Since the mapping g places students in $I_{C_1^1}$ ahead of students in $I_{C_1^2}$ and the order within $I_{C_1^1}$ is based on the index of the school which points to the student, student i should be ordered k^{th} by $g(\pi)$. Iterate this argument for each student in $I_{C_1^1}$ to establish that $g(\pi)(k) = i$ and $f(k) = i$ for all $i \in I_{C_1^1}$.

Consider the students in $I_{C_1^2}$. Start with the ordering of students who are not on a higher level. Pick $\tilde{i}_k \in I_1^2 \backslash I_1^{2,+}$ and suppose that f orders $\tilde{i}_k$ as the k^{th} among students in $I_1^2 \backslash I_1^{2,+}$. Since $S_1^{2,*} = G_1^{2,*}$, the school in $G_1^{2,*}$ who points to $\tilde{i}_k$ must be the k^{th} smallest indexed school in $S_1^{2,*}$. Since g orders students in $G_1^{2,*}$ based on the index of the school which points to them, if i is k^{th} among $I_1^2 \backslash I_1^{2,+}$ according to f, it must be k^{th} according to g. Proceed to order each student in $I_1^2 \backslash I_1^{2,+}$ among themselves in this way.

Consider the students in $I_1^{2,+}$. There are two cases to deal with: 1) $k > 1$ and 2) $k = 1$. In the first case, consider $\tilde{i}_{k-1} \in I_1^2 \backslash I_t^{2,+}$ where $\tilde{i}_{k-1}$ is ordered $(k-1)^{\text{th}}$ among students in $I_t^2 \backslash I_1^{2,+}$. Find the student i ordered between $\tilde{i}_{k-1}$ and $\tilde{i}_k$ immediately before $\tilde{i}_k$ in f. By construction of π, $\tilde{i}_k$ wants the school that points to i and $i \in I_1^{2,+} = I_{C_1^2}^+$, so g will order her right before $\tilde{i}_k$. Next continue with the student i' who is right before i in f. By construction of π, i wants the school that points to i' and $i' \in I_1^{2,+} = I_{C_1^2}^+$, so g will order her right before i. Continue for each such student between $\tilde{i}_{k-1}$ and $\tilde{i}_k$ to show that these students will be ordered the same wayunder f and $g(\pi)$.

In the second case, $\tilde{i}_k = \tilde{i}_1$ is the first student $\in I_1^2 \backslash I_1^{2,+}$. Suppose $\tilde{i}_\ell$ is the last student in $I_1^2 \backslash I_1^{2,+}$. We will demonstrate that the students after $\tilde{i}_\ell$ in I_1^2 according to f will be ordered the same way under g. Let i be the last student according to f in I_1^2. By construction of π, $\tilde{i}_1$ points to the school which points to i, and $i \in I_1^{2,+} = I_{C_1^2}^+$ implies that under g, this student is also the last student in I_1^2. Find the student i' immediately before i and repeat the same argument. This will demonstrate that the students in $I_{C_1^2} = I_1^2$ are ordered in the same way under f and $g(\pi)$. Proceed in the same way for each C_t^r for $r > 2$ to show that each $i \in I_1^r$ is ordered the same way under f and g. This will establish our base case: the students in $I_1 = I_{C_1}$ are ordered the same way under f and $g(\pi)$.

Next, suppose that for each step $\tau \in \{2, ..., t-1\}$, for any student $i \in I_\tau$, we have shown that f and $g(\pi)$ have same the order for i. We will show that each $i \in I_t$ will have the same ordering under f and $g(\pi)$. The main issue is the distinction between satiated and unsatiated students in cycles C_t^1.

The argument follows a similar approach as that for students in step 1. Consider the students in $I_{C_t^1}$. Start with the ordering of students who are unsatiated. Pick $\tilde{i}_k \in I_t^1 \backslash I_t^{1,+}$ and suppose that f orders $\tilde{i}_k$ as the k^{th} among students in $I_t^1 \backslash I_t^{1,+}$. Since $S_t^{1,*} = G_t^{1,*}$, the school in $G_t^{1,*}$ who points to $\tilde{i}_k$ must be the k^{th} smallest indexed school in $S_t^{1,*}$. Since g orders students in $G_t^{1,*}$ based on the index of the school which points to them, if i is k^{th} among $I_t^1 \backslash I_t^{1,+}$ according to f, it must be k^{th} according to g. Proceed to order each student in $I_t^1 \backslash I_t^{1,+}$ in this way.

Consider the students in $I_t^{1,+}$. There are two cases to deal with: 1) $k > 1$ and 2) $k = 1$. In the first case, consider $\tilde{i}_{k-1} \in I_t^1 \backslash I_t^{1,+}$ where $\tilde{i}_{\ell-1}$ is ordered $(\ell-1)^{\text{th}}$ among students in $I_t^1 \backslash I_t^{1,+}$. Find the student i ordered between $\tilde{i}_{k-1}$ and $\tilde{i}_k$ immediately before $\tilde{i}_k$. By construction of π, $\tilde{i}_k$ wants the school that points to i and $i \in I_t^1 \backslash I_t^{1,+}$, so g will order her right before $\tilde{i}_\ell$. Next continue with the student i' who is right before i in f. By construction of π, i wants the school that points to i' and $i' \in I_t^1 \backslash I_t^{1,+}$, so g will order her right before i. Continue for each such student between $\tilde{i}_{k-1}$ and $\tilde{i}_k$ to show that these students will be ordered the same under f and $g(\pi)$.

In the second case, $\tilde{i}_k = \tilde{i}_1$ is the first student in $I_t^1 \backslash I_t^{1,+}$. Suppose $\tilde{i}_\ell$ is the last student in $I_t^1 \backslash I_t^{1,+}$. We will demonstrate that the students after $\tilde{i}_\ell$ in I_t^1 according to f will be ordered the same way under g. Let i be the last student in f in I_t^1. By construction of π, $\tilde{i}_1$ points to the school which points to i, and $\in I_t^1 \backslash I_t^{1,+}$ implies that under g, this student is also the last student in I_t^1. Find the student i' immediately before i and repeat the same argument. This will demonstrate that the students in $I_{C_t^1} = I_t^1$ are ordered in the same way under f and $g(\pi)$. Proceed in the same way for each C_t^r for $r > 1$ to show that each $i \in I_t^r$ is ordered the same way under f and g. This will include all students in I_t and show that they are ordered the same way under f and $g(\pi)$. $\diamond$

This claim completes proof of Lemma 3.

□

Let f_1 and f_2 be any two orderings such that $f_1 \neq f_2$. For all $\pi \in \Pi(f_1)$, we have shown that $g(\pi) = f_1$ and for all $\pi \in \Pi(f_2)$, we have shown that $g(\pi) = f_2$. There is no $\pi \in \Pi(f_1) \cap \Pi(f_2)$ because this would imply that $f_1 = g(\pi) = f_2$, which is only true for $f_1 = f_2$. Since Lemma 1 showed that $m^{\varphi^\pi} = m^{\psi^f}$ for all $\pi \in \Pi(f)$, Lemma 2 showed that $|\Pi(f)| = (n!)^{n-1}$, and we have shown that $\forall f_1 \neq f_2$, $\Pi(f_1) \cap \Pi(f_2) = \emptyset$. Hence, the construction satisfies the three properties, which completes the proof.

Table of Notation

π	priority structure, a strict priority for all schools
π_s	priority for school s
ψ^f	serial dictatorship with ordering f
φ^π	core mechanism when the priority structure is π
m^{ψ^f}	matching produced by the serial dictatorship with ordering f
m^{φ^π}	matching produced by the core mechanism when the priority structure is π
I_t	set of students in subproblem t
I_t^r	set of students in subproblem t, substep r
S_t	set of schools assigned to students in I_t
S_t^r	set of schools assigned to students in I_t^r
n_t	number of students in S_t
n_t^r	number of students in S_t^r
N_t	total number of students in $I_1 \cup \ldots \cup I_t$
I_t^*	set of unsatiated students in I_t
$\mathcal{M}_t$	set of priority skeletons for subproblem t
M_t	priority skeleton for subproblem t
m_j	column j of priority skeleton M_t
m_{ij}	entry in row i and column j of priority skeleton M_t
$\Pi(f)$	set of priority structures
$\Pi_t(f)$	set of priorities for schools in subproblem t
$I_t^{1,+}$	set of satiated students in I_t^1
$I_t^{r,+}$	set of students in a higher position in the priority skeleton among I_t^r, for $r > 1$
O_t^r	set of students who do not immediately follow the last student in I_t^{r-1}
C_t	set of cycles in step t
C_t^r	set of cycles in step t, substep r
G_t	set of schools assigned in cycles in C_t
G_t^r	set of schools assigned in cycles in C_t^r
I_{C_t}	set of students assigned in cycles C_t
$I_{C_t^r}$	set of students assigned in cycles C_t^r
$I_{C_t^1}^+$	set of satiated students in $I_{C_t^1}$
$I_{C_t^r}^+$	set of students in a higher position in priority ordering among $I_{C_t^r}$, for $r > 1$

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