

When Optimists Need Credit: Asymmetric Disciplining of Optimism for Asset Prices

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Abstract

Heterogeneity of beliefs has been suggested as a major contributing factor to the recent financial crisis. This paper theoretically evaluates this hypothesis. Similar to Geanakoplos (2009), I assume that optimists have limited wealth and take on leverage in order to take positions in line with their beliefs. To have a significant effect on asset prices, they need to borrow from traders with moderate beliefs using loans collateralized by the asset itself. Since moderate lenders do not value the collateral as much as optimists do, they are reluctant to lend, providing an endogenous constraint on optimists' ability to leverage and to influence asset prices. I demonstrate that optimism concerning the likelihood of bad events has no or little effect on asset prices because these types of optimism are disciplined by this constraint. Instead, optimism concerning the relative likelihood of good events could have significant effects on asset prices. This *asymmetric disciplining of optimism* is robust to allowing for state contingent loans and short selling of the asset. These results emphasize that what investors disagree about matters for asset prices, to a greater extent than the level of disagreement.

I then use a dynamic extension to show how the asymmetric disciplining result interacts with the speculative component of asset prices identified in Harrison and Kreps (1978). When optimists have limited wealth, belief heterogeneity can lead to speculative asset price "bubbles" but only if it concerns the relative likelihood of good events. The asymmetric disciplining result shows that the size of the bubble depends on the type of belief heterogeneity, and that bubbles can come to an end because of a shift in belief heterogeneity towards the likelihood of bad events.

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1 Introduction

Belief heterogeneity and optimism have been suggested as contributing factors to the recent financial crisis. Shiller (2005), Reinhart and Rogoff (2008) and Gorton (2008), along with many other commentators, have identified the optimism of a fraction of investors as a potential cause for the increase in prices in the housing and the complex security markets in the run-up to the crisis. As noted by Geanakoplos (2009), for the optimism of a fraction of investors to have a significant effect on asset prices, they need to leverage their investments by borrowing from less optimistic investors- from moderate lenders. Most borrowing in financial markets is collateralized, and optimists often use the asset itself as collateral (e.g., mortgages, REPOs, or asset purchases on margin). This represents a puzzle because moderate lenders do not value the collateral (the asset) as much as optimists do, which might make them reluctant to lend. Put differently, belief heterogeneity implies an endogenous constraint on optimists' ability to leverage and to influence asset prices.

The purpose of this paper is to understand the implications of this constraint for asset prices. I construct an equilibrium model in the asset and the loan market, and I show that certain types of optimism, specifically those concerning the likelihood of bad events, have no or little effect on asset prices because they are disciplined by the endogenous financial constraints. Instead, optimism concerning the relative likelihood of good events could have significant effects on asset prices, because these types of optimism are unchecked by these constraints.

To illustrate the effect of different types of optimism, consider a simple example in which a single risky asset is traded. There are three future states, good, normal and bad, in which the asset price will respectively be high, average and low. Moderate lenders assign an equal probability, $1/3$, to each state, while optimists have a greater expected valuation of the asset. Optimists borrow from moderate lenders using loans collateralized by the asset. More specifically, the asset and collateralized loans are traded in a competitive market (as in Geanakoplos, 2009), and loans are no-recourse in the sense that payment is only enforced by the collateral pledged for the loan. Loans of different sizes (per unit collateral pledged) are available for trade, and the loan to value ratio is endogenously determined in equilibrium. For the baseline setting, suppose that loans are non-contingent, that is, they promise the same payment in all future states, and that the asset cannot be short sold. These assumptions arguably provide a good starting point, because collateralized loans (e.g., mortgages, REPOs) typically do not have many contingencies in their payoffs; and short selling of many assets other than stocks (and some stocks) is difficult and costly.¹

In this setting, there are two different ways in which optimists can be optimistic about the asset. For the first case, suppose optimists assign a probability less than $1/3$ to the bad state, and equal probabilities to the normal and the good states. That is, optimists are optimistic

¹For a description of the short market, see, for example, Jones and Lamont (2001), D'Avolio (2002), Duffie, Garleanu, and Pedersen (2002), Ofek and Richardson (2002), Lamont and Stein (2004), and Asquith, Pathak, and Ritter (2005).

because they think bad events are unlikely. For the second case, suppose optimists agree about the probability, $1/3$, of the bad state, but they think the good state is more likely than the normal state. Moreover, construct the two cases such that optimists have the same valuation of the asset, so that the *level* of the optimism is the same but the *type* of the optimism is different.²

My main results, Theorem 1 and Theorem 3, show that the asset price in the first case of this example is always lower than in the second case (and strictly so for the appropriate range of parameters). In other words, optimism is *asymmetrically disciplined* by endogenous financial constraints: optimism concerning the probability of bad states is disciplined more than optimism concerning the relative likelihood of good states.

More generally, Theorem 1 considers the above setting with a continuum of states (rather than three), and shows that the asset is priced according to a mixture of moderate and optimistic beliefs: the moderate beliefs are used to assess the likelihood of default states, while the optimistic beliefs are used to assess the conditional likelihood of non-default states. More precisely, the asset price can be written as:

$$p = \frac{1}{1+r} (\Pr_{\text{moderate}} [v < \bar{v}] E_{\text{moderate}} [v \mid v < \bar{v}] + \Pr_{\text{moderate}} [v \geq \bar{v}] E_{\text{optimistic}} [v \mid v \geq \bar{v}]), \quad (1)$$

where r is the interest rate on a benchmark asset, the random variable v captures the future value of the asset, and \bar{v} is the endogenously determined default threshold value, that is, collateralized loans in this economy default when the asset value v falls below \bar{v} . The notation $\Pr_{\text{moderate}} [v < \bar{v}]$ captures the probability of the event $\{v < \bar{v}\}$ with respect to the moderate beliefs, and $E_{\text{optimistic}} [v \mid v \geq \bar{v}]$ captures the expected value of the asset conditional on the event $\{v \geq \bar{v}\}$ with respect to the optimistic beliefs.

The expression in (1) further illustrates that optimism is asymmetrically disciplined. This asymmetric disciplining result is robust to allowing for more general collateralized loans and short selling. In particular, Sections 5 and 6 of this paper show that the asset price in these more general settings can also be represented with an expression similar to (1). While the details of the expressions depend on the type of the contracts available for trade, it remains true that optimism about bad states is disciplined more than optimism concerning the relative likelihood of good states.

The intuition for the asymmetric disciplining result is related to the asymmetry in the shape of the debt contract payoffs. These contracts make the same full payment in non-default states, but they make losses in default states. Consequently, any disagreement about the probability of default states translates into a disagreement about how to value the debt contracts, which

²For an example of case one type of optimism, consider the last quarter of 2008, when a main dimension of disagreement was whether the upcoming recession would be a depression or a garden variety recession. For an example of case two type of optimism, consider the Internet technology and the tech stocks in 1990s, when a main dimension of disagreement was how profitable the Internet technology would be.

in turn tightens optimists' financial constraints. In contrast, disagreements about the relative likelihood of non-default states do not tighten the financial constraints.

More specifically, in the above example (for the relevant range of parameters) collateralized loans that are traded in equilibrium default in the bad state but not in the normal or the good states. This implies that these loans always trade at an interest rate with a *spread* over the benchmark rate, which compensates the lenders for expected losses in case of default. Moreover, in a competitive loan market, the spread on a loan is just enough to compensate the lenders for their expected losses according to their moderate beliefs. Nonetheless, in the first case of the example, this spread appears too high to optimists. This is because optimists assign a lower probability to the bad state, and thus they find it more likely that they will pay the spread. Therefore, optimists believe they will pay a higher expected interest rate than the benchmark rate, which discourages them from borrowing and leveraging their investments. This lowers optimists' demand for the asset and leads to an equilibrium price closer to the moderate valuation. In contrast, in the second case of the example, the spread appears fair to optimists because they agree about the probability of the bad state. This encourages optimists to borrow and leverage their investments, increases their demand for the asset, and leads to an equilibrium price closer to their valuation.

The asymmetric disciplining characterization of asset prices lends itself to a number of comparative statics results regarding the effect of a change in the *level* and the *type* of belief heterogeneity. Earlier work by Miller (1977) has suggested a link between the level of belief heterogeneity and asset prices. According to this mechanism, belief heterogeneity and limited short selling leads to an overvaluation of the asset (relative to the average valuation of the population) because the asset is held by the most optimistic investors. This mechanism has been recently emphasized and empirically tested by a growing literature in finance, e.g., Chen, Hong and Stein (2002), Diether, Malloy and Scherbina (2002) and Ofek and Richardson (2003). In contrast to this literature, the level of belief heterogeneity in this model has ambiguous effects on the asset price. This is because, while an increase in optimists' optimism tends to increase the price, an increase in moderate lenders' pessimism tends to decrease the price through the tightening of financial constraints. This observation suggests that the Miller mechanism may not apply in markets in which optimists finance their asset purchases by borrowing from less optimistic investors.

In contrast, this model suggests that the type of the belief heterogeneity may be a more robust determinant of asset prices in these markets. To capture the effect of different types of belief heterogeneity, I formally define a notion of right-skewed (resp. left-skewed) optimism as a single-crossing condition on the hazard rates of optimists' belief distributions. In the above described example with three states, the optimism in the second case is more right-skewed than the optimism in the first case. This is because, in the second case, the optimistic hazard rate at the bad state is higher (since optimists are not optimistic about the probability of the bad state), but the hazard rate at the normal state is lower (since optimists are optimistic about

the relative likelihood of the good and the normal states). Theorem 3 shows that an increase in this type of right-skewness of belief heterogeneity unambiguously increases the asset price, because a given level of optimism is disciplined less by financial constraints when it is more right-skewed. In addition, Theorem 4 shows that the level of belief heterogeneity also has an unambiguous effect on the asset price if the type of the heterogeneity is also accounted for. In particular, in response to an increase in belief heterogeneity, the asset price increases if the additional heterogeneity concerns the relative likelihood of non-default states, while it decreases if the additional heterogeneity concerns the probability of default states. These results suggest that *what investors disagree about* matters for asset prices, to a greater extent than the level of their disagreement.

While the baseline model with non-contingent contracts and no short selling is a good starting point, it is important to verify the robustness of the asymmetric disciplining characterization to more general settings, especially because allowing for a richer set of contracts introduces new economic forces. Theorem 5 shows that a version of the asymmetric disciplining result continues to apply in the setting in which debt contracts can be fully contingent. The optimal contingent contract (collateralized by one unit of the asset) is such that optimistic borrowers give up the asset completely if the state realization is below a threshold level, while paying nothing if the state is above the threshold. While this threshold contract is different than a non-contingent contract, it has the same feature of making a fixed payment (namely, zero) for all relatively good states. Consequently, optimism about the relative likelihood of good states does not lead to heterogeneity in the valuation of the optimal contingent contract. It follows that these types of optimism do not tighten optimists' financial constraints, and thus they lead to a higher asset price. In contrast, optimism about the relative likelihood of states below the threshold level tightens optimists' financial constraints and leads to a lower asset price.

The setting with contingent contracts reveals a surprising result: the equilibrium asset price can exceed the valuation of even the most optimistic investor.³ Intuitively, the ability to fine-tune their borrowing enables optimists to take loans which they perceive to be even more favorable than borrowing at the benchmark interest rate. Optimists concentrate all of their payments at the bad states (which they find the least likely), and thus they expect to make a relatively small payment. Consequently, optimists continue to demand the asset when the price exceeds their valuation (which is calculated according to the benchmark rate), because they finance some of the purchase with contingent contracts which they perceive to be very attractive. This result creates a presumption that finer levels of financial engineering of loans can potentially have a large impact on asset prices.

Another natural question is whether the asymmetric disciplining result generalizes to the

³As discussed below, it is well known that the asset price can exceed the optimistic valuation in a dynamic setting with belief heterogeneity (through a different mechanism). However, when contingent contracts are available, the asset price can exceed the optimistic valuation even in the static setting.

class of assets that can be short sold (e.g., the majority of stocks). While short selling reduces the general overvaluation of the asset, Theorem 6 shows that a version of asymmetric disciplining applies also in this case. The critical observation is that moderate investors that wish to short sell the asset face an endogenous borrowing constraint similar to the one faced by optimists. In particular, to short sell the asset, moderates need to borrow the asset from optimists. Moreover, since borrowing in this economy is collateralized, they need to use the riskless bond that they hold as collateral in their short contracts. But optimists value the bond (the collateral) relatively less, which might make them reluctant to lend the asset. Hence, belief heterogeneity represents an endogenous constraint on moderates' ability to short sell.

The severity of this constraint depends on the type of belief heterogeneity. If the future state is above a threshold level, then the value of the asset is greater than the value of the posted collateral, and moderates default on the short contract. Hence, a short contract effectively promises the same payment in all states above a threshold state. Consequently, if the belief heterogeneity is about the relative likelihood of states above the threshold, then moderates cannot bet on their pessimism by selling the short contract. Put differently, to bet on these types of pessimism, moderates need to choose a short contract with a higher level of collateral (that has a higher default threshold). Hence, these types of short sales are more difficult to leverage, which leads to an asset price closer to the optimistic valuation. In contrast, if the belief heterogeneity is about the probability of bad states, then moderates are able to make leveraged bets on their pessimism, which leads to an asset price closer to the moderate valuation.

While the results described so far concern a static setting, the asymmetric disciplining mechanism naturally interacts with the speculative component of asset prices identified in Harrison and Kreps (1978). I consider a dynamic extension of the baseline model to analyze this interaction. In a dynamic economy in which the identity of optimists changes over time, a speculative phenomenon obtains as the current optimists purchase the asset not only because they believe it will yield greater dividend returns, but also because they expect to make capital gains by selling the asset to future optimists. The asset price exceeds the present discounted valuation of the asset with respect to the beliefs of any trader because of the resale option value introduced by the speculative trading motive. As Scheinkman and Xiong (2003) note, this resale option value may be reasonably called a "speculative bubble." This setup is the starting point of the dynamic extension, which introduces the additional element of optimists' financial constraints. The dynamic model reveals that, when optimists need to purchase the asset by borrowing from moderate lenders, belief heterogeneity can lead to speculative asset price bubbles, but only if it concerns the relative likelihood of non-default states. When this is the case, however, the resale option value can increase the size of the speculative component of the asset price considerably because large positions can be financed by credit collateralized by the speculative asset. This is because moderate lenders' valuation, as well as optimists' valuation, features a speculative component. Put differently, in a speculative episode, moderate

lenders agree to finance optimists' purchase of the asset by extending large loans because they think, should the optimist default on the loan, they can sell the collateral (the asset) to another optimist in the next period. The asymmetric disciplining characterization shows that the size of the bubble depends on the skewness of belief heterogeneity. This result also shows that bubbles can come to an end because of a shift in belief heterogeneity towards the likelihood of default states.

The closest work to my paper is by Geanakoplos (2009), who considers the determination of leverage and asset prices in a model with two continuation states and traders with a continuum of belief types. In contrast, I consider a model with a continuum of continuation states and traders with two belief types (optimists and moderates). My assumptions are relevant for understanding a range of situations, including the effect of different types of belief disagreements on asset prices, leverage, and the default frequency of equilibrium loans. In particular, while Geanakoplos (2009) illustrates that an increase in belief heterogeneity can decrease asset prices considerably, my paper shows that an increase in the level of belief heterogeneity generally has ambiguous effects on asset prices, and identifies the skewness of belief heterogeneity as an important determinant of asset prices. In the model considered by Geanakoplos (2009), the increase in the level of heterogeneity decreases asset prices because the additional heterogeneity is concentrated on default states. An increase in the level of heterogeneity in that model would rather increase asset prices if the additional heterogeneity were concentrated on good states. Moreover, in the two state model analyzed in Geanakoplos (2009), loans that are traded in equilibrium are always fully secured with respect to the worst case scenario, i.e., there is no default. This feature makes it impossible to analyze the effect of belief heterogeneity on the default frequency and riskiness of equilibrium loans, which is one of the topics that I consider. In addition, my paper extends the model in Geanakoplos (2009) by allowing for short selling, and characterizes the effect of belief heterogeneity in this more general setting.⁴

The relationship of my paper to the literatures initiated by Miller (1977) and Harrison and Kreps (1978) have already been discussed.⁵ A related literature concerns the plausibility of the heterogeneous priors assumption in financial markets. The market selection hypothesis, which goes back to Alchian (1950) and Friedman (1953), posits that investors with incorrect

⁴Other related papers that concern the effect of collateral constraints on leverage or asset prices include Hart and Moore (1994), Geanakoplos (1997, 2003), Kiyotaki and Moore (1997), Caballero and Krishnamurty (2001), Gromb and Vayanos (2002), Fostel and Geanakoplos (2008), Brunnermeier and Pedersen (2009), Brunnermeier and Sannikov (2009), Ashcraft, Garleanu, and Pedersen (2010), and He and Xiong (2010). My paper is also related to a large literature which concerns the determination of leverage, asset prices and investment in economies with a different type of financial friction based on asymmetric information. An incomplete list includes Townsend (1979), Diamond (1984), Myers and Majluf (1984), Greenwald, Stiglitz and Weiss (1984), Bernanke and Gertler (1989), Shleifer and Vishny (1992, 1997), Greenwald and Stiglitz (1993), Bernanke, Gertler, and Gilchrist (1996, 1998), Holmstrom and Tirole (1997). On the empirical side, a number of recent studies document the variation in leverage and its effect on asset prices (see, for example, Adrian and Shin, 2009).

⁵There is a large literature that analyzes the effect of heterogeneous beliefs on asset prices or bubbles. In addition to the above mentioned papers, an incomplete list includes Varian (1985,1989), Harris and Raviv (1993), Allen, Morris, and Postlewaite (1993), Detemple and Murthy (1994), Wang (1994), Morris (1996), Kandel and Pearson (1995), Zapatero (1998), Basak (2000), Hong and Stein (2003), Gollier (2007), Banerjee (2007).

beliefs should be driven out of the market as they would consistently lose money. Thus, this hypothesis suggests that investors that remain in the long run should have accurate (and common) beliefs. Recent research has emphasized that the market selection hypothesis does not apply for incomplete markets, that is, traders with inaccurate beliefs may have a permanent presence when asset markets are incomplete.⁶ Of particular interest for my paper is the work by Cao (2009), who considers a similar economy in which markets are endogenously incomplete because of collateral constraints. Cao (2009) shows that belief heterogeneity in this economy remains in the long run, thus providing theoretical support for my central assumptions. Another strand of literature concerns whether investors' Bayesian learning dynamics would eventually lead to accurate, and thus common, beliefs. Recent work (e.g., by Acemoglu, Chernozhukov and Yildiz, 2009) has emphasized the limitations of Bayesian learning in generating long run agreement.⁷

The organization of the rest of this paper is as follows. Section 2 introduces the baseline version of the model and defines the collateral equilibrium. Section 3 characterizes the collateral equilibrium and presents the asymmetric disciplining result. Section 4 establishes the comparative statics of the collateral equilibrium with respect to the type and the level of belief heterogeneity. Sections 5 and 6 generalize the asymmetric disciplining result to settings with respectively contingent debt contracts and short selling of the asset. Section 7 introduces the dynamic extension and presents the results for speculative bubbles. Section 8 concludes. The paper ends with several appendices that present the proofs omitted from the main text.

2 Environment and Equilibrium

Consider a two period economy with a single consumption good in which a continuum of risk neutral traders have endowments in period 0 but they need to consume in period 1. The resources can be transferred between periods by investing either in a risk-free bond, denoted by B , or a risky asset, denoted by A . Bond B is supplied elastically at a normalized price 1 in period 0. Each unit of the bond yields $1 + r$ units of the consumption good in period 1. Asset A is in fixed supply, which is normalized to 1. The asset pays dividend only once (in units of the consumption good), and it pays it in period 1. The dividend payment of each unit of the asset is denoted by $v(s)$. Taking the set of all possible states as $\mathcal{S} = [s^{\min}, s^{\max}] \subset \mathbb{R}$, I assume that the function $v : \mathcal{S} \rightarrow \mathbb{R}_{++}$ is strictly increasing and continuously differentiable.⁸ I

⁶See, for example, DeLong, Shleifer, Summers and Waldman (1990,1991), Blume and Easley (1992, 2006), Sandroni (2000), Kogan, Ross, Wang, and Westerfield (2006), Beker and Espino (2008), and Cao (2009).

⁷For further discussion on the merits of the common prior assumption in economic theory, see Bernheim (1986), Aumann (1986,1998), Varian (1989), Morris (1995), and Gul (1998).

⁸Note that the state space could be equivalently defined as $v(\mathcal{S}) = [v(s^{\min}), v(s^{\max})]$ over asset payoffs, so the value function $v(\cdot)$ is redundant in this section. Put differently, without loss of generality, the value function can be taken to be the identity function $v(s) = s$. I introduce the value function $v(\cdot)$ because this will considerably simplify the analysis of the dynamic model in Section 7, in which the value function will be endogenously determined.

denote the price of the asset by p .

Traders have heterogeneous priors about the return of the asset. In particular, there are two types of traders, *optimists* and *moderates*, respectively denoted by subscript $i \in \{1, 0\}$, with corresponding prior belief about the next period state given by the probability distribution F_i over S . Traders know each others' priors, and thus optimists and moderates agree to disagree. I normalize the population measure of each type of traders to 1, and I let α_i (resp. w_i) denote type i traders' period 0 endowment of the asset (resp. the consumption good). The asset endowments satisfy $\alpha_0 > 0$ and $\alpha_0 + \alpha_1 = 1$. An economy is denoted by the tuple $\mathcal{E} = (\mathcal{S}; v(\cdot); \{F_i\}_i; \{w_i\}_i; \{\alpha_i\}_i)$.

I adopt the following notion of optimism.

Definition 1 (Optimism Order). Consider two probability distributions H, \tilde{H} over $\mathcal{S} = [s^{\min}, s^{\max}]$ with density functions h, \tilde{h} that are continuous and positive over \mathcal{S} . The distribution \tilde{H} is more optimistic than H , denoted by $\tilde{H} \succ_O H$, if $\frac{1-\tilde{H}(s)}{1-H(s)}$ is strictly increasing; equivalently, if the following hazard rate inequality is satisfied for all $s \in (s^{\min}, s^{\max})$:

$$\frac{\tilde{h}(s)}{1-\tilde{H}(s)} < \frac{h(s)}{1-H(s)}. \quad (2)$$

The distribution \tilde{H} is weakly more optimistic than H , denoted by $\tilde{H} \succeq_O H$, if (2) is satisfied as a weak inequality.

Assumption (O). The probability distributions F_1 and F_0 have density functions f_1, f_0 that are continuous and positive over \mathcal{S} , and they satisfy $F_1 \succ_O F_0$.

The optimism order, \succ_O , concerns optimists' relative probability assessment for the *upper-threshold events* $[s, s^{\max}] \subset \mathcal{S}$, and it posits that optimists are increasingly optimistic for these events as the threshold level s is increased. It captures the idea that, the "better" the event becomes, the greater the optimism is for the event. This order, also known as the hazard rate order, is related to some well known regularity conditions. It is stronger than the first order stochastic order, that is, $F_1 \succ_O F_0$ implies F_1 dominates F_0 in the first order stochastic sense. However, it is weaker than the monotone likelihood ratio property, that is, if $\frac{f_1(s)}{f_0(s)}$ is strictly increasing over \mathcal{S} , then $F_1 \succ_O F_0$ (cf. Appendix A.1).

Let $E_i[\cdot]$ denote the expectation operator corresponding to the belief of a type i trader. Assumption (O) also implies $E_0[v(s)] < E_1[v(s)]$, that is, moderates value the asset less than optimists do. This further implies that moderates would like to short sell the asset in this economy, which is ruled out by assumption.

Assumption (S). Asset A cannot be short sold.

This assumption will be maintained for most of the paper (until Section 6). In reality, many assets other than stocks, and also some stocks, are difficult and costly to short sell (see, e.g., Jones and Lamont, 2001).

Given assumption (S), if there were no financial frictions, i.e., if optimists could freely borrow and lend at the going interest rate $1 + r$, they would bid up the price of the asset to the optimistic valuation, $\frac{E_1[v(s)]}{1+r}$. However, financial frictions may prevent optimists from increasing the asset price to this level. With financial frictions, the asset (in the baseline setting) trades at a price in the interval

$$\left[\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r} \right],$$

the exact location being determined by optimists' wealth and the type of the financial frictions. The financial frictions are microfounded through a collateralized loan market.

2.1 Financial Frictions and Collateral Equilibrium

I make a number of institutional assumptions for the loan market. First, I assume that loans in this economy must be secured by collateral owned by the borrower, and the court system enforces the transfer of collateral to the lender in case the borrower does not pay.⁹ Second, I assume that loans are non-recourse, that is, the borrower does not get further punishment than potential loss of collateral. Third, I also assume that the loans are non-contingent, that is, they promise the same payment in all future states. These assumptions arguably provide a good starting point, because most REPO loans and some mortgages are non-recourse, and they do not have many contingencies.

Formally, a *unit debt contract*, denoted by $\varphi \in \mathbb{R}_+$, is a promise of φ units of the consumption good in period 1 by the borrower, collateralized by 1 unit of the asset A (which the borrower owns).¹⁰ In period 1, the borrower defaults on the unit debt contract φ if and only if the asset's value is less than the promised amount. Thus, each contract φ pays

$$\min(v(s), \varphi). \tag{3}$$

I analyze the loan market using a competitive equilibrium notion, *collateral equilibrium*, orig-

⁹There is a potential question of who holds the collateral throughout the term of the loan contract, i.e. should the collateral be locked in a warehouse, held by the lender, or the borrower. In reality (e.g., in mortgages or REPOs), different variants are used intuitively depending on whether the borrower or the lender benefits more from holding the contract during the loan period. A common aspect of all variants of collateralized lending relationships is that the borrower must own the asset at the time of the loan payment. This aspect is necessary because otherwise the borrower would not have any incentive to pay back the loan and collateral would not enforce payment.

In this model, traders receive no utility from holding the collateral in period 0, which implies that the different variants of collateralized lending are essentially equivalent. Therefore, without loss of generality, the borrower is required to own the collateral that she pledges.

¹⁰The assumption that each contract pledges one unit of the asset is a normalization without loss of generality. Note also that, in principle, both the bond B and the asset A could be used as collateral. However, in this model, the assumption that only the asset can be used as collateral is also without loss of generality. More precisely, the equilibrium described below in Theorem 2 continues to be the essentially unique equilibrium in the more general setting in which the bond can also be used as collateral. This is because optimistic borrowers do not hold any bond in equilibrium (except for the corner case in which their wealth is more than sufficient to purchase the entire asset supply).

inally developed by Geanakoplos and Zame (1997, 2009). In particular, each debt contract φ is traded in an anonymous market at a competitive price $q(\varphi)$. Note that the anonymity of the market is ensured by collateral: each lender knows that repayment is only secured by collateral, and that she will get the payment in (3) regardless of the identity of the borrower in the transaction.

I refer to a debt contract $\varphi = v(\bar{s}) \in [v(s^{\min}), v(s^{\max})]$ as a *loan with riskiness* \bar{s} , since this contract defaults if and only if the realized state is below \bar{s} . I refer to the price of the debt contract, $q(v(\bar{s}))$, as the *loan size*, since this is the amount of that the borrower receives by collateralizing one unit of the asset. In equilibrium, $q(v(\bar{s}))$ will be increasing in \bar{s} , hence *larger loans* are also *riskier loans*. Moreover, I define the *interest rate* on the loan as the ratio of the promised interest payment to the loan size:

$$\frac{v(\bar{s}) - q(v(\bar{s}))}{q(v(\bar{s}))}. \quad (4)$$

Given these definitions, an interpretation of the model is that loans of different sizes (and thus different riskiness levels) are being traded in a competitive equilibrium at their corresponding interest rates.

To formalize traders' portfolio choices, I assume that the price function $q(\cdot)$ is Borel measurable. Unlike the asset, debt contracts can be short sold (but subject to a collateral constraint). In particular, taking the price function $q(\cdot)$ as given, type i traders choose a long debt portfolio $\mu_i^+ \in M(\mathbb{R}_+)$ and a short debt portfolio $\mu_i^- \in M(\mathbb{R}_+)$, where $M(\mathbb{R}_+)$ denotes the set of Borel measures over \mathbb{R}_+ .¹¹ In addition, type i traders choose their asset and bond holdings, $x_i = (x_i^A, x_i^B) \in \mathbb{R}_+^2$. Their budget constraint is given by:

$$px_i^A + x_i^B + \int_{\mathbb{R}_+} q(\varphi) d\mu_i^+(\varphi) - \int_{\mathbb{R}_+} q(\varphi) d\mu_i^-(\varphi) \leq w_i + p\alpha_i. \quad (5)$$

Note that short selling debt contracts (borrowing) expands the traders' budget and enables them to invest more on the asset or the bond (or other debt contracts). However, short selling is subject to the following collateral constraint:

$$\int_{\mathbb{R}_+} d\mu_i^-(\varphi) \leq x_i^A. \quad (6)$$

That is, for each unit debt contract traders sell, they need to set aside one unit of the asset they own as collateral.

Type i traders choose their portfolio to maximize their expected utility, i.e., they solve the

¹¹I define traders' portfolios using measures, rather than measurable functions, because the optimal portfolio characterized below will be a Dirac measure (which does not correspond to a measurable function).

problem:

$$\max_{x_i \geq 0; \mu_i^+, \mu_i^- \in M(\mathbb{R}_+)} \left[\begin{array}{l} x_i^A E_i[v(s)] + x_i^B (1+r) + \\ \int_{\mathbb{R}_+} E_i[\min(v(s), \varphi)] d\mu_i^+(\varphi) - \int_{\mathbb{R}_+} E_i[\min(v(s), \varphi)] d\mu_i^-(\varphi) \end{array} \right] \quad (7)$$

subject to (5) and (6).

Market clearing for each unit debt contract φ requires the sum of the long positions to be equal to the sum of the short positions, that is:

$$\sum_{i \in \{1,0\}} \int_C d\mu_i^+(\varphi) = \sum_{i \in \{1,0\}} \int_C d\mu_i^-(\varphi) \text{ for each Borel set } C \subset \mathbb{R}_+. \quad (8)$$

Definition 2 (Collateral Equilibrium). *Given an economy \mathcal{E} with assumptions (O) and (S), a collateral equilibrium is a collection of prices $(p, [q(\cdot)])$ and portfolios $(x_i^A, x_i^B, \mu_i^+, \mu_i^-)_{i \in \{1,0\}}$ such that: the portfolio of type i traders solves Problem (7) for each $i \in \{1,0\}$, the asset market clears, $\sum_{i \in \{1,0\}} x_i^A = 1$, and the debt market clears [cf. Eq. (8)].*

3 Characterization of Collateral Equilibrium

This section provides a characterization of collateral equilibrium and presents the main result which characterizes the effect of belief heterogeneity on the asset price. The equilibrium will intuitively have the form that moderates hold the bond and long positions on collateralized debt contracts (i.e., they lend to optimists), while optimists make leveraged investments in the asset by selling collateralized debt contracts.

To characterize the equilibrium, it is useful to define the notion of a *quasi-equilibrium*, which is a collection of prices $(p, [q(\cdot)])$ and portfolios $(x_i^A, x_i^B, \mu_i^+, \mu_i^-)_{i \in \{1,0\}}$ such that markets clear and the portfolio of type $i \in \{1,0\}$ traders solves Problem (7) with the additional requirement $\mu_1^+ = \mu_0^- = 0$.¹² That is, in a quasi-equilibrium, optimists are restricted not to buy debt contracts, and moderates are restricted not to sell debt contracts. For expositional reasons, I will first construct a quasi-equilibrium. Theorem 2 below establishes that the constructed quasi-equilibrium corresponds to a collateral equilibrium with the same allocations and the same asset price (and with potentially different debt contract prices). The same theorem also establishes that the asset price in a collateral equilibrium is uniquely determined.

To construct a quasi-equilibrium, consider debt contract prices

$$q(\varphi) = \frac{E_0[\min(v(s), \varphi)]}{1+r} \text{ for each } \varphi \in \mathbb{R}_+, \quad (9)$$

that make moderates indifferent between holding the bond and any debt contract $\varphi \in \mathbb{R}_+$. Given the prices in (9) and the asset price $p \geq \frac{E_0[v(s)]}{1+r}$, moderates' optimal decision in a quasi-equilibrium is completely characterized: they are indifferent between holding the bond and any

¹²Here, $\mu_1^+ = 0$ (similarly $\mu_0^- = 0$) denotes the 0 measure, i.e., $\mu_i^+(C) = 0$ for each Borel set $C \subset \mathbb{R}_+$.

debt contract, and they always weakly prefer these options to holding the asset (and strictly so whenever $p > \frac{E_0[v(s)]}{1+r}$). Moreover, these prices ensure that market clearing in debt contracts will be automatic, as moderates will absorb any supply of debt contracts from optimists.

The quasi-equilibrium asset price and allocations are then determined by optimists' portfolio choice. I next analyze optimists' problem for a given asset price p , and I then combine this analysis with asset market clearing to solve for the quasi-equilibrium.

3.1 Main Result: Asymmetric Disciplining of Optimism

The next result, which is also the main result, characterizes optimists' portfolio choice.

Theorem 1 (Optimal Contract Choice and Asymmetric Filtering). *Suppose assumptions (O) and (S) hold, debt prices are given by (9) and the asset price satisfies $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$. In a quasi-equilibrium:*

(i) *There exists $\bar{s} \in \mathcal{S}$ such that μ_i^- is a Dirac measure that puts weight only at the contract $\varphi = v(\bar{s})$, i.e., optimists borrow according to a single loan with riskiness \bar{s} . Optimists' collateral constraint (6) is binding, i.e., they borrow as much as possible according to the optimal loan. Optimists choose $x_1^B = 0$, i.e., they invest all of their leveraged wealth in the asset A.*

(ii) *The riskiness \bar{s} of the optimal loan is characterized as the unique solution to the following equation over \mathcal{S} :*

$$\begin{aligned} p = p^{opt}(\bar{s}) &\equiv \frac{1}{1+r} \left(\int_{s^{\min}}^{\bar{s}} v(s) dF_0 + (1 - F_0(\bar{s})) \int_{\bar{s}}^{s^{\max}} v(s) \frac{dF_1}{1 - F_1(\bar{s})} \right) \\ &= \frac{1}{1+r} (F_0(\bar{s}) E_0[v(s) \mid s < \bar{s}] + (1 - F_0(\bar{s})) E_1[v(s) \mid s \geq \bar{s}]). \end{aligned} \quad (10)$$

The riskiness \bar{s} of the optimal loan is decreasing in the price level p .

If instead the asset price satisfies $p = \frac{E_1[v(s)]}{1+r}$, then optimists are indifferent between making a leveraged investment by selling any safe debt contract $\varphi \leq v(s^{\min})$ or investing in the bond.

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I will shortly provide a sketch proof of this result along with an intuition. Before doing so, I note a couple of important aspects of the function $p^{opt}(\bar{s})$. First, the function $p^{opt}(\bar{s})$ is similar to an inverse demand function: it describes the asset price p for which the riskiness level \bar{s} is optimal. Assumption (O) implies $p^{opt}(\bar{s})$ is strictly decreasing and continuous (cf. Appendix A.1). Since $p^{opt}(s^{\min}) = \frac{E_1[v(s)]}{1+r}$ and $p^{opt}(s^{\max}) = \frac{E_0[v(s)]}{1+r}$, this further implies that there is a unique solution to Eq. (10), and that the solution is strictly decreasing in p .

Second, note that $p^{opt}(\bar{s})$ also describes the equilibrium asset price conditional on the equilibrium loan riskiness \bar{s} . Hence, Theorem 1 is the main result of this paper, as it shows that optimism will be asymmetrically disciplined in equilibrium. In particular, the second

¹³The case $p = \frac{E_0[v(s)]}{1+r}$ is omitted, since the equilibrium asset price always satisfies $p > \frac{E_0[v(s)]}{1+r}$ (cf. Theorem 2).

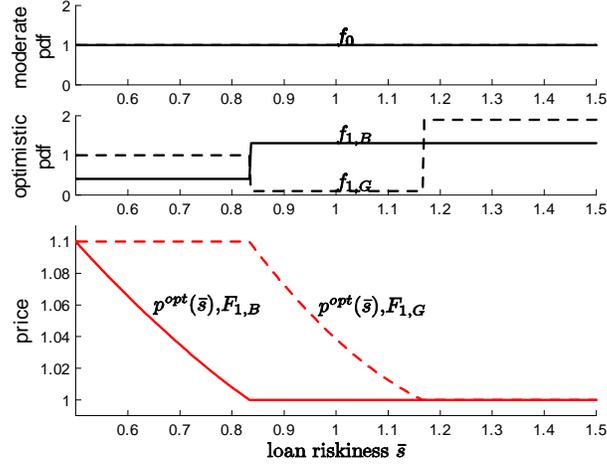


Figure 1: The top two panels display the probability density functions for traders’ priors in the two scenarios of Example 1. The bottom panel displays the corresponding curves $p^{opt}(\bar{s})$, the inverse of which gives the optimal loan riskiness \bar{s} for a given price level p .

line of (10) replicates Eq. (1) from the Introduction and shows that the asset is priced with a mixture of moderate and optimistic beliefs. The moderate belief is used to assess the likelihood of default states $s < \bar{s}$, along with the value of the asset conditional on these states, while the optimistic belief is used to assess the likelihood of non-default states $s > \bar{s}$. Consequently, the function $p^{opt}(\bar{s})$ will “discipline” any optimism about the probability of default states, while “incorporating” any optimism about the relative probability of states conditional on no default. The following example describes two cases that differ about the type of optimism and illustrates the asymmetric disciplining property.

Example 1 (Asymmetric Filtering of Optimism). Consider the state space $\mathcal{S} = [1/2, 3/2]$ and the value function $v(s) = s$. As the first case, suppose moderates and optimists have the prior distributions F_0 and $F_{1,B}$ with density functions:

$$f_0(s) = 1 \text{ for each } s \in \mathcal{S},$$

$$\text{and } f_{1,B}(s) = \begin{cases} 0.4 & \text{if } s \in \mathcal{S}_B \equiv [2/3 - 1/6, 2/3 + 1/6) \\ 1.3 & \text{if } s \in \mathcal{S}_N \equiv [1 - 1/6, 1 + 1/6) \\ 1.3 & \text{if } s \in \mathcal{S}_G \equiv [4/3 - 1/6, 4/3 + 1/6) \end{cases},$$

where \mathcal{S}_B , \mathcal{S}_N , and \mathcal{S}_G intuitively capture bad, normal and good events, respectively. In words, moderates find all states equally likely, while optimists are optimistic because they believe that a bad event, that is, a realization around the bad state $2/3$, is less likely than a normal or a good event (which they find equally likely).¹⁴

¹⁴Note that the belief distributions of this example do not exactly satisfy the regularity assumption (O). In particular, the density functions are not continuous, and $F_{1,B}$ is only weakly more optimistic than F_0 . These

Consider a second case in which moderates have the same prior, but optimists' prior is changed to the distribution $F_{1,G}$ with density function

$$f_{1,G} = \begin{cases} 1 & \text{if } s \in \mathcal{S}_B \\ 0.1 & \text{if } s \in \mathcal{S}_N \\ 1.9 & \text{if } s \in \mathcal{S}_G \end{cases}.$$

That is, optimists are optimistic not because they think the bad event is less likely, but because they believe the good event is more likely than the normal event. Note also that optimists are equally optimistic in both cases, i.e., $E_{1,G}[v(s)] = E_{1,B}[v(s)]$.

The bottom panel of Figure 1 displays the optimality curves, $p^{opt}(\bar{s})$, in both cases. Note that, for any level of loan riskiness \bar{s} , the asset price is higher in the second case than in the first case. Equivalently, for any price p , optimists choose a larger and riskier loan in the second case than in the first case.

I next provide a sketch proof of Theorem 1, which is useful to understand the intuition. The proof in Appendix A.2 shows that optimists borrow using a loan with riskiness $\bar{s} \in \mathcal{S}$ that maximizes the leveraged return:

$$R_1^L(\bar{s}) \equiv \frac{E_1[v(s)] - E_1[\min(v(s), v(\bar{s}))]}{p - \frac{1}{1+r}E_0[\min(v(s), v(\bar{s}))]} \quad (11)$$

This expression is the expected return of optimists who buy one unit of the asset and who finance part of the purchase using a loan with riskiness \bar{s} . The denominator is the *downpayment* optimists make for the leveraged investment: they pay the price p of the asset but they can borrow $q(\bar{s}) = \frac{E_0[\min(v(s), v(\bar{s}))]}{1+r}$ from moderates (given the contract prices (9)). The numerator is optimists' expected payoff from the leveraged investment: they expect to receive $E_1[v(s)]$ from the asset and they also expect to pay $E_1[\min(v(s), v(\bar{s}))]$ on their loan.

The relation $p = p^{opt}(\bar{s})$ is the first order optimality condition corresponding to the maximization of the leveraged return, $R_1^L(\bar{s})$. The leveraged return has a unique maximum over \mathcal{S} , characterized by the first order condition, which completes the sketch proof of Theorem 3.

To understand the intuition for the theorem, it is useful to further break down the leveraged return expression (11) into two components. First consider the left hand side terms in the numerator and the denominator of (11), which constitute the *unleveraged return*:

$$R^U = \frac{E_1[v(s)]}{p}.$$

This expression is the expected return of optimists if they buy the asset with their own wealth (without borrowing). Optimists believe the return on investing in the asset is greater than the

distributions are used for illustration purposes because they provide a clear intuition. The formal results use the stricter assumption (O) for analytical tractability.

benchmark rate, i.e., $R^U > 1 + r$, which creates a force that pushes towards leveraging. In particular, if optimists could borrow at the benchmark rate r without constraints, they would borrow infinitely to leverage this unleveraged return.

However, optimists have to borrow with a collateralized loan with riskiness \bar{s} , which represents a second force that pushes towards deleveraging. This force is related to the right hand side terms in the numerator and the denominator of (11), which constitute optimists' *perceived interest rate* on the loan:

$$1 + r_1^{per}(\bar{s}) \equiv \frac{E_1[\min(v(s), v(\bar{s}))]}{\frac{1}{1+r}E_0[\min(v(s), v(\bar{s}))]} \quad (12)$$

Optimists borrow $\frac{1}{1+r}E_0[\min(v(s), v(\bar{s}))]$ on the loan, but they expect to pay $E_1[\min(v(s), v(\bar{s}))]$, which leads to the perceived interest rate $1 + r_1^{per}(\bar{s})$. Assumption (O) implies that $r_1^{per}(\bar{s})$ is always weakly greater than the benchmark rate r , and that it is increasing in \bar{s} [cf. Appendix (A.1)]. The intuition for this observation is two fold. First, collateralized loans always trade at a *spread* over the benchmark rate (i.e., the interest rate on the loan, (4), is always greater than the benchmark rate), because moderate lenders require compensation for their expected losses in case of default. In particular, since the loan market is competitive, the spread on a loan is just enough to compensate the lenders according to their moderate beliefs. Second, optimists believe that the loan will default less often than moderates believe, hence they think they will end up paying the spread more often. Consequently, optimists believe they will pay a greater interest rate than the benchmark rate, i.e., $r_1^{per}(\bar{s}) > r$. Moreover, for greater levels of \bar{s} , the scope of disagreement for default is greater, which implies that $r_1^{per}(\bar{s})$ is increasing in \bar{s} .

It follows that, while a larger loan with a greater riskiness level \bar{s} enables optimists to leverage the unleveraged return more, it also comes at a greater perceived interest rate, $r_1^{per}(\bar{s})$. Optimists' optimal loan choice balances these two forces, as captured by the maximization of the leveraged return expression (11).

This breakdown of the two forces also provides an intuition for the asymmetric disciplining property of the pricing function $p^{opt}(\bar{s})$. First consider the intuition for the simpler property that $p^{opt}(\bar{s})$ is decreasing in \bar{s} . That is, consider why optimists choose a larger and riskier loan when the price p is lower. This is because a lower asset price increases the unleveraged return, $R^U = \frac{E_1[v(s)]}{p}$, which tilts optimists' trade-off towards larger loans. When the unleveraged return is greater, optimists have a greater incentive to leverage this return by taking a larger (and riskier) loan, agreeing to pay a greater expected interest rate $r_1^{per}(\bar{s})$ at the margin.

To see the intuition for the disciplining property of $p^{opt}(\bar{s})$, fix a loan with riskiness \bar{s} , and consider how much the price should drop (from the optimistic valuation) to entice optimists to take this particular loan. Consider this question in the context of Example 1 for a riskiness level $\bar{s} = 0.8 \in \mathcal{S}_B$. In the first case of Example 1, optimists find the bad event \mathcal{S}_B unlikely. Hence, given a loan with riskiness $\bar{s} \in \mathcal{S}_B$, there is disagreement about the probability of default, which

implies $r_1^{per}(\bar{s}) > r$. As this loan appears expensive to optimists, the asset price should drop considerably to entice optimists to undertake a leveraged investment with this loan. Consider instead the second case of Example 1 in which optimists are optimistic because they find the good event more likely than the normal event. In this case, for a loan with riskiness $\bar{s} \in S_B$, there is no disagreement about the probability of default, which implies $r_1^{per}(\bar{s}) = r$. As the loan appears cheap to optimists, the asset price does not need to fall to entice them to take the loan (see Figure 1).

In other words, the asymmetric disciplining result operates through optimists' perceived financial constraints. Disagreement about default states tightens optimists' financial constraints (captured by a higher $r_1^{per}(\bar{s})$), which lowers their demand for the leveraged investment and leads to an asset price closer to the moderate valuation. In contrast, disagreement about non-default states does not tighten the financial constraints, and leads to an asset price closer to the optimistic valuation.

3.2 Asset Market Clearing and Collateral Equilibrium

Theorem 1 characterizes the riskiness \bar{s} of the optimal contract as a function of the asset price p . I next consider the market clearing price p and solve for the equilibrium.

Suppose optimists choose to borrow using a loan with riskiness \bar{s} and consider the price that clears the asset market. This price depends on the maximum first period consumption good that optimists can obtain:

$$w_1^{\max}(\bar{s}) = w_1 + \frac{1}{1+r} E_0[\min(v(s), v(\bar{s}))]. \quad (13)$$

Optimists are endowed with w_1 units of the consumption good, and if they hold the entire asset supply, they can borrow up to $\frac{1}{1+r} E_0[\min(v(s), v(\bar{s}))]$ units of the consumption good from moderates, leading to the expression in (13). The asset price depends on the comparison of $w_1^{\max}(\bar{s})$ with the value of the asset in the hands of moderates, $\alpha_0 p$, which optimists seek to purchase. In particular:

$$p = p^{mc}(\bar{s}) \equiv \begin{cases} \frac{E_1[v(s)]}{1+r} & \text{if } \frac{w_1^{\max}(\bar{s})}{\alpha_0} > \frac{E_1[v(s)]}{1+r} & \text{[case (i)]} \\ \frac{w_1^{\max}(\bar{s})}{\alpha_0} & \text{if } \frac{w_1^{\max}(\bar{s})}{\alpha_0} \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r} \right] & \text{[case (ii)]} \\ \frac{E_0[v(s)]}{1+r} & \text{if } \frac{w_1^{\max}(\bar{s})}{\alpha_0} \leq \frac{E_0[v(s)]}{1+r} & \text{[case (iii)]} \end{cases} \quad (14)$$

In case (i), optimists have access to a sufficient amount of consumption good in the first period that they purchase all of the asset in the hands of moderate lenders (and they have some consumption good left over, which they invest in the bond). In this case, optimists are marginal holders of the asset and the price is given by their valuation, $\frac{E_1[v(s)]}{1+r}$. In case (ii), optimists still purchase all of the asset from moderate lenders, but they cannot bid up the asset price to their valuation. In this case, the market clearing price is determined by setting optimists' consumption good equal to the value of moderates' assets, i.e., $w_1^{\max}(\bar{s}) = p\alpha_0$. In

case (iii), optimists have access to so little first period consumption good that they cannot purchase all of the asset in the hands of moderate lenders. In this case, moderate lenders hold some of the asset, and the price is given by their valuation, $\frac{E_0[v(s)]}{1+r}$.

Note that Eq. (14) describes an increasing relation between the asset price and the loan riskiness \bar{s} . Intuitively, when optimists take a larger and riskier loan, they have access to a greater amount of first period consumption good, which enables them to bid up the asset price higher. Combining Theorem 1 and Eq. (14), the equilibrium price and loan riskiness pair, (p, \bar{s}^*) , is determined as the unique intersection of the strictly decreasing function $p^{opt}(\bar{s})$ and the weakly increasing function $p^{mc}(\bar{s})$ (see Figure 2). This analysis completes the characterization of the quasi-equilibrium. The analysis in Appendix A.3 establishes that this quasi-equilibrium is a collateral equilibrium with modified debt contract prices given by:¹⁵

$$q(\varphi) = \max\left(\frac{E_0[\min(v(s), \varphi)]}{1+r}, \frac{E_1[\min(v(s), \varphi)]}{R_1^L(\bar{s}^*)}\right). \quad (15)$$

The following result summarizes this discussion and proves the essential uniqueness of the collateral equilibrium.

Theorem 2 (Existence, Characterization, Essential Uniqueness). *Consider the above described economy with assumptions (O) and (S). There exists a collateral equilibrium in which contract prices are given by (15), moderate types are indifferent between buying bonds and lending to optimists, and optimists make leveraged investments in the asset by borrowing through a single loan with riskiness $\bar{s}^* \in \mathcal{S}$. The asset price p and riskiness \bar{s}^* of loans in this equilibrium are determined as the unique solution to $p = p^{opt}(\bar{s}) = p^{mc}(\bar{s})$ over $\bar{s} \in \mathcal{S}$.*

In any collateral equilibrium, the asset price, p , and the price of the optimal debt contract, $q(v(\bar{s}^))$, are uniquely determined. Except for the corner case in which $p = \frac{E_1[v(s)]}{1+r}$, traders' allocations are also uniquely determined. However, prices of the remaining debt contracts, $q(\varphi)$ for $\varphi \neq v(\bar{s}^*)$, are not uniquely determined.*

In other words, most of the equilibrium is uniquely determined, except for the price of debt contracts that are not traded in equilibrium. Appendix A.3 establishes that, for each contract $\varphi = v(\tilde{s}) \neq v(\bar{s}^*)$, there exists a continuum of prices that can support the equilibrium with no-trade in these contracts. This completes the characterization of the collateral equilibrium.

Figure 2 illustrates the equilibrium, and shows the effect of a decline in optimists' initial endowment of the consumption good. When optimists' wealth declines, the price falls towards the moderate valuation. Note also that the equilibrium loans also become larger and riskier. This is because, as the price falls, optimists see more of a bargain in the asset price which

¹⁵Note that $R_1^L(\bar{s}^*)$ (cf. Eq. (11)) is optimists' expected return on capital in equilibrium. Thus, the expression $\frac{E_1[\min(v(s), \varphi)]}{R_1^L(\bar{s}^*)}$ is optimists' valuation of the debt contract φ in equilibrium. Unlike in a quasi-equilibrium, optimists can demand debt contracts in a collateral equilibrium. Hence, the price of a debt contract is given by the upper-envelope of the moderate and the optimistic valuations, as captured by (15). The analysis in Appendix A.3 establishes that optimists' and moderate lenders' allocations continue to be optimal when the prices are given by (15) and when the constraints $\mu_0^- = 0$ and $\mu_1^+ = 0$ are relaxed.

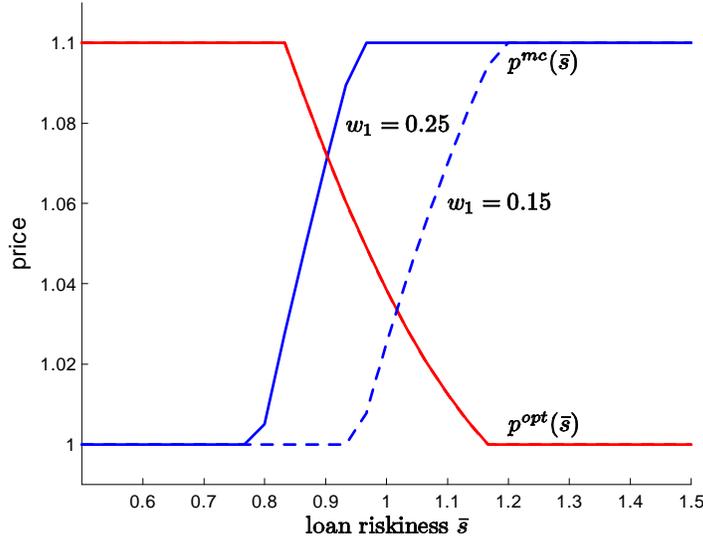


Figure 2: The figure displays the collateral equilibrium, and the response of the equilibrium to a decline in optimists' initial endowment of the consumption good, w_1 .

encourages them to leverage more. Hence, equilibrium leverage responds in a way to ameliorate the drop the initial wealth shock to optimists. These comparative statics are similar to the results in Geanakoplos (2009). I next turn to the focus of this paper, and establish the comparative statics of the equilibrium with respect to the type and the level of belief heterogeneity.

4 Comparative Statics with Respect to Belief Heterogeneity

In addition to the equilibrium loan riskiness \bar{s}^* and the asset price p , I consider the comparative statics of the *leverage ratio* for optimists' asset purchase, denoted by L . Recall that optimists buy one unit of the asset by paying $p - \frac{E_0[v(s), v(\bar{s}^*)]}{1+r}$ out of their wealth and financing the rest of the purchase by borrowing from moderates. Thus, the leverage ratio for the asset purchase is given by

$$L \equiv \frac{p}{p - E_0[\min(v(s), v(\bar{s}^*))]/(1+r)}. \quad (16)$$

The leverage ratio has counterparts in real financial markets: the loan-to-value ratio of a mortgage loan is equal to $1 - \frac{1}{L}$ and the haircut on a REPO loan is equal to $\frac{1}{L}$.

The next definition formalizes the type of belief heterogeneity that is used to state the comparative statics results.

Definition 3 (Skewed Optimism). Consider two probability distributions H, \tilde{H} over $\mathcal{S} = [s^{\min}, s^{\max}]$ with density functions h, \tilde{h} that are continuous and positive over \mathcal{S} , and con-

sider a continuously differentiable and strictly increasing asset value function $v : \mathcal{S} \rightarrow \mathbb{R}_{++}$. The optimism of distribution \tilde{H} about the asset is weakly more right-skewed than H , denoted by $\tilde{H} \succeq_R H$, if and only if:

(a) The distributions yield the same valuation of the asset, that is, $E[v(s) ; \tilde{H}] = E[v(s) ; H]$.

(b) There exists $s^R \in \mathcal{S}$ such that $\frac{1-\tilde{H}(s)}{1-H(s)}$ is weakly decreasing over (s^{\min}, s^R) while it is weakly increasing over (s^R, s^{\max}) , which is the case if and only if the hazard rates of \tilde{H} and H satisfy the (weak) single crossing condition:

$$\begin{cases} \frac{\tilde{h}(s)}{1-\tilde{H}(s)} \geq \frac{h(s)}{1-H(s)} & \text{if } s < s^R, \\ \frac{\tilde{h}(s)}{1-\tilde{H}(s)} \leq \frac{h(s)}{1-H(s)} & \text{if } s > s^R. \end{cases} \quad (17)$$

The optimism of distribution \tilde{H} is weakly more skewed to the right of $\tilde{s} \in \mathcal{S}$ than H , denoted by $\tilde{H} \succeq_{R, \tilde{s}} H$, if the conditions (a)-(b) are satisfied with the additional requirement that $s^R \geq \tilde{s}$.

To interpret this definition, note that the distributions \tilde{H} and H cannot be compared according to the optimism order in Definition 1, since their hazard rates are not ordered. In addition, these distributions lead to the same valuation of the asset, that is, they have the same ‘‘level’’ of optimism. Note also that \tilde{H} has a lower hazard rate than H over the region (s^R, s^{\max}) . Thus, conditional on $s \geq s^R$, \tilde{H} is more optimistic than H in the sense of Definition 1. In contrast, H has a lower hazard rate than \tilde{H} over the region (s^{\min}, s^R) , and thus its optimism is concentrated more on this region. Hence, the optimism of \tilde{H} is right-skewed in the sense that it is concentrated more on relatively good states.

Note that the probability distributions $F_{1,B}$ and $F_{1,G}$ of Example 1 satisfy condition (17). That is, $F_{1,G}$ and $F_{1,B}$ lead to the same valuation for the asset but the optimism of $F_{1,G}$ is weakly more right skewed, as illustrated in Figure 3. The same figure also plots the optimality relation $p^{opt}(\bar{s})$ from Figure 1 together with the market clearing curve $p^{mc}(\bar{s})$, and illustrates that the equilibrium price p and loan riskiness \bar{s}^* are higher when optimists’ optimism is more right-skewed. The next result shows that this observation is generally true.

Theorem 3 (Type of Heterogeneity). *Consider the collateral equilibrium characterized in Theorem 2 and let \bar{s}^* denote the equilibrium loan riskiness.*

(i) *If optimists’ optimism becomes weakly more right-skewed, i.e., if their prior is changed to \tilde{F}_1 that satisfies $\tilde{F}_1 \succeq_R F_1$ and $\tilde{F}_1 \succ_O F_0$ (so that assumption (O) continues to hold), then: the asset price p , the loan riskiness \bar{s}^* , and the leverage ratio L weakly increase.*

(ii) *If moderates’ optimism becomes weakly more skewed to the left of \bar{s}^* , i.e., if their prior is changed to \tilde{F}_0 that satisfies $F_0 \succeq_{R, \bar{s}^*} \tilde{F}_0$ and $F_1 \succ_O \tilde{F}_0$, then: the asset price p weakly increases.*

I provide a sketch proof of this result, which is completed in Appendix A.4. First observe

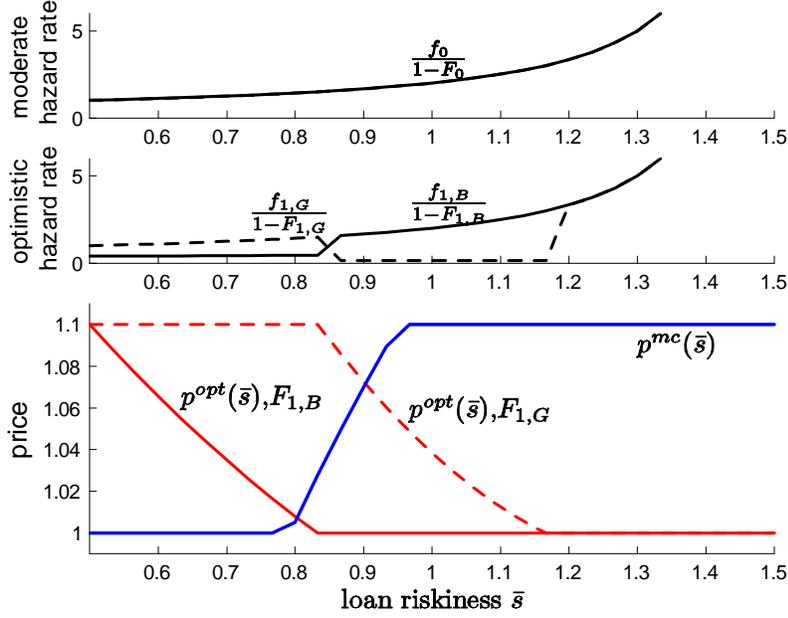


Figure 3: The top two panels display the hazard rates for traders' priors in the two cases analyzed in Example 1. The bottom panel plots the corresponding equilibria.

that Eq. (10) can be written as:

$$p^{opt}(\bar{s}) - \frac{E_0[v(s)]}{1+r} = \frac{1}{1+r} (1 - F_0(\bar{s})) (E_1[v(s) | s \geq \bar{s}] - E_0[v(s) | s \geq \bar{s}]). \quad (18)$$

In view of the asymmetric disciplining result, the difference between the asset price and the moderate valuation depends on the moderate probability of no default, and traders' valuation differences conditional on no default. For part (i), Appendix A.4 shows that

$$\tilde{E}_1[v(s) | s \geq \bar{s}] \geq E_1[v(s) | s \geq \bar{s}] \text{ for each } \bar{s} \in (s^{\min}, s^{\max}), \quad (19)$$

where $\tilde{E}_i[\cdot]$ denotes the expectation operator with respect to distribution \tilde{F}_i . That is, when optimists' optimism becomes more right-skewed, their valuation of the asset conditional on any upper-threshold event increases, even though their unconditional valuation is the same. It follows, by Eq. (19), that the optimality curve $p^{opt}(\bar{s})$ shifts up pointwise. As the market clearing curve $p^{mc}(\bar{s})$ remains constant, the equilibrium asset price p and the loan riskiness \bar{s}^* increase, which further implies the remaining comparative statics. Appendix A.4 uses a similar argument to prove part (ii).

Theorem 3 points to the importance of the *skewness* of belief heterogeneity for the asset price. A related question is whether the *level* of belief heterogeneity has similar robust predictions regarding the price of the asset. The answer is no, as illustrated in the following

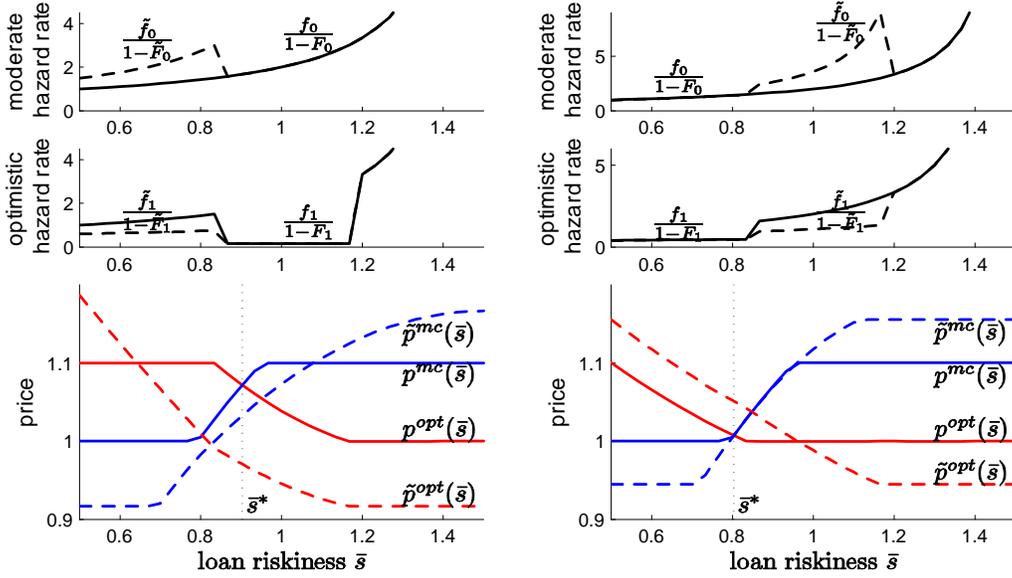


Figure 4: The left panel plots the equilibrium in the first scenario considered in Example 2: the increase in belief heterogeneity is concentrated to the left of state \bar{s}^* and it decreases the asset price. The right panel plots the equilibrium in the second scenario considered in Example 2: the increase in belief heterogeneity is to the right of \bar{s}^* and it increases the asset price.

example.

Example 2 (Ambiguous Price Effect of Increased Belief Heterogeneity). Consider the first case of Example 1 in which optimists are optimistic because they find the bad event unlikely, i.e., they have the prior $F_{1,B}$. Suppose the moderate and the optimistic beliefs are changed to $\tilde{F}_0 = F_{0,G}$ and $\tilde{F}_1 = F_{1,BG}$ with density functions given by

$$f_{0,G} = \begin{cases} 1 & \text{if } s \in \mathcal{S}_B \\ 1 + 0.5 & \text{if } s \in \mathcal{S}_N \\ 1 - 0.5 & \text{if } s \in \mathcal{S}_G \end{cases}, \quad f_{1,BG} = \begin{cases} 0.4 & \text{if } s \in \mathcal{S}_B \\ 1.3 - 0.5 & \text{if } s \in \mathcal{S}_N \\ 1.3 + 0.5 & \text{if } s \in \mathcal{S}_G \end{cases}.$$

That is, moderates' prior probability for the normal event increases and their probability for the good event decreases, while the opposite happens to optimists' prior. As the right panel of Figure 4 shows, in this case, the increase in belief heterogeneity leads to an increase in the asset price.

Consider the second case in Example 1 in which optimists are optimistic because they find the good event more likely than the normal event, i.e., they have the prior $F_{1,G}$. Suppose the moderate and the optimistic beliefs are changed to $\tilde{F}_0 = F_{0,B}$, $\tilde{F}_1 = F_{1,GB}$ with density

functions given by

$$f_{0,B} = \begin{cases} 1 + 0.5 & \text{if } s \in \mathcal{S}_B \\ 1 - 0.25 & \text{if } s \in \mathcal{S}_N \\ 1 - 0.25 & \text{if } s \in \mathcal{S}_G \end{cases}, \quad f_{1,GB} = \begin{cases} 1(1 - 0.5) & \text{if } s \in \mathcal{S}_B \\ 0.1(1 + 0.25) & \text{if } s \in \mathcal{S}_N \\ 1.9(1 + 0.25) & \text{if } s \in \mathcal{S}_G \end{cases}.$$

That is, moderates' prior probability for the bad event increases and their relative probability for good and the normal event remains constant, while optimists' prior probability for the bad event decreases. As Figure 4 shows, in this case, the increase in belief heterogeneity leads to a decrease in the asset price.

Example 2 illustrates that the increase in belief heterogeneity has no robust predictions for the asset price. In particular, the second part provides an example in which optimists become more optimistic but the asset price declines, which is in contrast with the Miller (1977) hypothesis. In view of the asymmetric disciplining property of $p^{opt}(\bar{s})$, both the optimistic and moderate beliefs play a part in the determination of the asset price. While the increase of optimists' optimism tends to increase the asset price, the decrease in moderates' pessimism tends to decrease it by tightening the financial constraints, and the net effect is ambiguous. This observation suggests that the Miller hypothesis may not apply in markets in which optimists finance their purchases by borrowing from less optimistic investors.

I next show that increased belief heterogeneity has robust predictions regarding the asset price if the type of the additional increase is also taken into account. In the first case of Example 2, the belief heterogeneity is concentrated to the left of the default threshold \bar{s}^* , and the asset price decreases. In the second case of the example, the belief heterogeneity is concentrated to the right of the default threshold \bar{s}^* , and the asset price increases (see also Figure 4). The next result shows that these properties are general: belief heterogeneity has an unambiguous effect on the asset price if it is concentrated to the left, or to the right, of the equilibrium default threshold \bar{s}^* .

Theorem 4 (Level of Heterogeneity). *Consider the collateral equilibrium characterized in Theorem 2 and let \bar{s}^* denote the equilibrium loan riskiness, which is also the threshold state below which loans default. Consider a (weak) increase in belief heterogeneity, in the sense that beliefs are changed to \tilde{F}_1 and \tilde{F}_0 that satisfy $\tilde{F}_1 \succeq_O F_1$ and $F_0 \succeq_O \tilde{F}_0$:*

(i) *Suppose the increase in belief heterogeneity is concentrated to the right of \bar{s}^* , that is, suppose $\frac{1-\tilde{F}_1(s)}{1-F_1(s)}$ and $\frac{1-\tilde{F}_0(s)}{1-F_0(s)}$ are constant over the set (s^{\min}, \bar{s}^*) . Then the asset price p , the loan riskiness \bar{s}^* , and the leverage ratio L weakly increase.*

(ii) *Suppose the increase in belief heterogeneity is concentrated to the left of \bar{s}^* , that is, suppose $\frac{1-\tilde{F}_1(s)}{1-F_1(s)}$ and $\frac{1-\tilde{F}_0(s)}{1-F_0(s)}$ are constant over the set (\bar{s}^*, s^{\max}) . Then the asset price p weakly decreases.*

Taken together with the earlier results, this result demonstrates that the type of the belief heterogeneity is a more robust determinant of asset prices than the level of belief heterogeneity.

With endogenous financial constraints, what investors disagree about matters for asset prices, to a greater extent than the level of their disagreement.

5 Collateral Equilibrium with Contingent Contracts

The analysis in the previous sections has concerned the baseline setting in which loans are restricted to be non-contingent and short selling is not allowed. While the baseline model is a good starting point, it is important to verify the robustness of the results to more general settings, especially because allowing for a richer set of contracts introduces new economic forces. The analysis in this section considers an extension in which debt contracts can be fully contingent on the continuation state $s \in \mathcal{S}$, and it establishes three results. First, the optimal contingent contract is not a simple debt contract. Rather it is a threshold contract: optimists promise to make a zero payment in all states above a threshold, but they promise make a payment equal to the asset value in the states below the threshold. Second, this threshold contract is sufficiently similar to a simple debt contract that a version of the asymmetric disciplining result (cf. Theorem 1) also applies in this setting. Third, unlike the case with simple debt contracts, the asset price in this setting may exceed the valuation of even the most optimistic investor.

5.1 Definition of Equilibrium with Contingent Contracts

A unit contingent debt contract, denoted by $\varphi : \mathcal{S} \rightarrow \mathbb{R}_+$, is a collection of promises of $\varphi(s) \geq 0$ units in each state $s \in \mathcal{S}$, collateralized by 1 unit of the asset.¹⁶ The borrower defaults on the contract if and only if the value of the asset is less than the promise on the contract. Thus, the contract pays $\min(v(s), \varphi(s))$ units. Let \mathcal{D} denote the set of all unit debt contracts. As before, each debt contract φ is traded in an anonymous market at a competitive price $q(\varphi)$, where $q(\cdot)$ is a Borel measurable function over \mathcal{D} .

Let $\mu_i^+, \mu_i^- \in M(\mathcal{D})$ respectively denote type i traders' long and short debt portfolios, where $M(\mathcal{D})$ denotes the set of Borel measures over \mathcal{D} . Type i traders solve an analogue of problem (7): they choose their portfolio $(x_i^A, x_i^B, \mu_i^+, \mu_i^-)$ to maximize their expected payoff subject to a budget and a collateral constraint (see problem (A.45) in the appendix). Given this problem and the extended contract space, the equilibrium is defined similarly to Section 2.1.

The characterization of equilibrium closely follows the analysis in Section 3. In particular, consider first a quasi-equilibrium by restricting traders' choices with the constraint $\mu_0^+ = \mu_1^- =$

¹⁶I assume that contracts must make non-negative promises in all continuation states, because a negative promise by the borrower (which is essentially a promise by the lender) would not be enforced by the court system in this economy since lenders do not set aside any collateral. This is without loss of generality, because if they wish, lenders can also make promises by selling a separate collateralized debt contract.

0. To construct a quasi-equilibrium, consider debt contract prices:

$$q(\varphi) = \frac{E_0[\min(v(s), \varphi(s))]}{1+r}, \quad (20)$$

which make moderates indifferent between purchasing the bond and any debt contract φ . The equilibrium will be determined by optimists' portfolio choice given these prices.

5.2 Asymmetric Disciplining with Contingent Contracts

Consider optimists' portfolio choice for a given price p . The same analysis for Theorem 1 (cf. Appendix A.2) shows that optimists borrow by selling a contingent debt contract, $[\varphi(s) \in [0, v(s)]]_{s \in \mathcal{S}}$, that maximizes the leveraged return:

$$R_1^{L,cont}(\varphi) = \frac{E_1[v(s)] - E_1[\min(v(s), \varphi(s))]}{p - \frac{1}{1+r}E_0[\min(v(s), \varphi(s))]} \quad (21)$$

The contract that maximizes this expression can be characterized under the following assumption, which is slightly stronger than assumption (O):

Assumption (MLRP). The probability distributions F_1 and F_0 have density functions f_1, f_0 which are continuous and positive over \mathcal{S} , and which satisfy the monotone likelihood ratio property: that is, $\frac{f_1(s)}{f_0(s)}$ is strictly increasing over \mathcal{S} .

The analysis in Appendix A.5 establishes that, under assumption (MLRP), the optimal contract is a *threshold contract*:

$$\varphi_{\bar{s}}(s) \equiv \begin{cases} v(s) & \text{if } s < \bar{s} \\ 0 & \text{if } s \geq \bar{s}, \end{cases} \quad (22)$$

for a threshold state $\bar{s} \in \mathcal{S}$. That is, optimists make as large a promise as possible for states $s < \bar{s}$ (they give up the asset in these states), while promising zero for states $s \geq \bar{s}$ (they keep the asset in these states). Intuitively, optimists find bad states the least likely, and thus they concentrate all of their payments below a threshold state.

The next result, which is the analogue of Theorem 1 for contingent loans, characterizes the threshold state $\bar{s} \in \mathcal{S}$ of the optimal contract given price p . The result also shows that, unlike the case with simple debt contracts, the maximum price at which optimists demand the asset is greater than the optimistic valuation, $\frac{E_1[v(s)]}{1+r}$. This maximum price level is given by:

$$p^{\max} = \frac{1}{1+r} \left(\int_{s^{\min}}^{s^{\text{cross}}} v(s) dF_0 + \int_{s^{\text{cross}}}^{s^{\max}} v(s) dF_1 \right), \quad (23)$$

where $s^{\text{cross}} \in \mathcal{S}$ is the unique state such that $\frac{f_0(s^{\text{cross}})}{f_1(s^{\text{cross}})} = 1$.

Theorem 5 (Asymmetric Disciplining with Contingent Contracts). *Suppose assumptions (MLRP) and (S) hold, debt prices are given by (20) and the asset price satisfies*

$p \in \left(\frac{E_0[v(s)]}{1+r}, p^{\max} \right)$, where p^{\max} is given by (23). In a quasi-equilibrium:

(i) There exists $\bar{s} \in [s^{\text{cross}}, s^{\max}]$ such that μ_i^- is a Dirac measure that puts weight only at the threshold contract $\varphi_{\bar{s}}$ in (22). Optimists' collateral constraint is binding, i.e., they borrow as much as possible according to the optimal contract. Optimists choose $x_1^B = 0$, i.e., they invest all of their leveraged wealth in the asset A .

(ii) The threshold state $\bar{s} \in [s^{\text{cross}}, s^{\max}]$ of the optimal contract is characterized as the unique solution to:

$$p = p^{\text{opt,cont}}(\bar{s}) \equiv \frac{1}{1+r} \left(\int_{s^{\min}}^{\bar{s}} v(s) dF_0 + \frac{f_0(\bar{s})}{f_1(\bar{s})} \int_{\bar{s}}^{s^{\max}} v(s) dF_1 \right). \quad (24)$$

If instead the asset price satisfies $p = p^{\max}$, then optimists are indifferent between making a leveraged investment in the asset by selling the safe debt contract $\varphi_{\bar{s}^{\text{cross}}}$ or investing in the bond.

Note that the function $p^{\text{opt,cont}}(\bar{s})$ is the analogue of the function $p^{\text{opt}}(\bar{s})$: it describes the asset price conditional on optimists' choice of the threshold state \bar{s} . Moreover, the form of $p^{\text{opt,cont}}(\bar{s})$ is very similar to the form of $p^{\text{opt}}(\bar{s})$, which suggests that optimism is asymmetrically disciplined also in this setting. In particular, optimism about the relative likelihood of states above \bar{s} increases the asset price, while the optimism about the relative likelihood of states below \bar{s} does not increase the price. The intuition for this result can be gleaned from the shape of the optimal debt contract $\varphi_{\bar{s}}$. This threshold contract makes the same payment (namely, zero) in all states above the threshold \bar{s} , while it has an increasing payment schedule in the states below the threshold \bar{s} . Hence, any optimism about the relative likelihood of good states does not increase optimists' perceived interest rate, and thus these types of optimism increase the asset price. However, optimism about the relative likelihood of bad states increases optimists' perceived interest rate. Thus, these types of optimism are reflected less in the asset price.¹⁷

Note also that optimists demand the asset even if the price is greater than their valuation, captured by the fact that $p^{\max} > \frac{E_1[v(s)]}{1+r}$. To see the intuition for this result, consider optimists' perceived interest rate on the contract $\varphi_{\bar{s}}$, given by:

$$1 + r_1^{\text{per,cont}}(\bar{s}) = \frac{E_1[\min(v(s), \varphi_{\bar{s}}(s))]}{\frac{1}{1+r} E_0[\min(v(s), \varphi_{\bar{s}}(s))]} = (1+r) \frac{\int_{s^{\min}}^{\bar{s}} v(s) dF_1}{\int_{s^{\min}}^{\bar{s}} v(s) dF_0}.$$

Unlike the case with non-contingent loans, $r_1^{\text{per,cont}}(\bar{s})$ is not necessarily greater than r . In particular, the ability to fine-tune their borrowing enables optimists to take loans which they perceive to be even more favorable than borrowing at the benchmark interest rate. Conse-

¹⁷This intuition also illustrates the limitation of the asymmetric filtering result when loans are fully contingent. Unlike regular debt contracts, a contingent debt contract, $\varphi_{\bar{s}}$, makes a lower payment in states above \bar{s} relative to states below \bar{s} . Hence, if optimists' optimism is changed in a way to assign a lower probability to states below \bar{s} , then the asset price increases (unlike the case with non-contingent contracts).

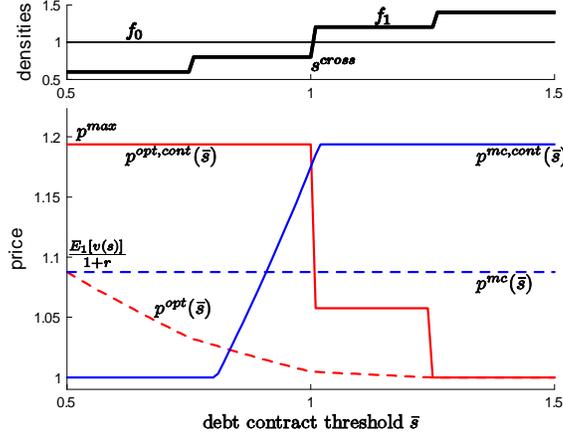


Figure 5: The top panel displays the probability densities. The solid (resp. dashed) lines in the bottom panel illustrate the equilibrium with (resp. without) contingent contracts.

quently, optimists invest in the asset even if the price exceeds their valuation, $\frac{E_1[v(s)]}{1+r}$, because they can finance some of the purchase with these loans which they perceive to be very favorable.

A complementary intuition for this result comes from the form of p^{\max} in (23). The availability of fully contingent loans enables optimists to split the asset in a way that each type traders hold the asset in the states which they assign a greater probability. Consequently, the maximum price at which optimists demand the asset is calculated according to an upper-envelope of the moderate and the optimistic beliefs, which exceeds the optimistic valuation. This result creates a presumption that finer levels of financial engineering of loans can potentially have a large impact on asset prices.

5.3 Equilibrium Asset Price with Contingent Contracts

Similar to Section 5, the equilibrium asset price is determined by combining optimists' optimal contract choice with asset market clearing. The market clearing condition is analogous to Eq. (14), and is given by:

$$p = p^{mc,cont}(\bar{s}) \equiv \begin{cases} p^{\max} & \text{if } \frac{w_1^{\max,cont}(\bar{s})}{\alpha_0} > p^{\max} \\ \frac{w_1^{\max,cont}(\bar{s})}{\alpha_0} & \text{if } \frac{w_1^{\max,cont}(\bar{s})}{\alpha_0} \in \left(\frac{E_0[v(s)]}{1+r}, p^{\max}\right] \\ \frac{E_0[v(s)]}{1+r} & \text{if } \frac{w_1^{\max,cont}(\bar{s})}{\alpha_0} \leq \frac{E_0[v(s)]}{1+r} \end{cases} .$$

Here, $w_1^{\max,cont}(\bar{s}) = w_1 + \frac{1}{1+r} \int_{s^{\min}}^{\bar{s}} v(s) ds$ denotes optimists' maximum first period consumption good given that they choose to borrow with the contingent debt contract \bar{s} . The equilibrium asset price p and the threshold level of the optimal contract \bar{s}^* are characterized by considering the intersection of the strictly decreasing curve $p^{opt,cont}(\bar{s})$ and the weakly increasing curve $p^{mc,cont}(\bar{s})$ over the range $\bar{s} \in [s^{cross}, s^{\max}]$. Figure 5 displays the equilibrium with

contingent and non-contingent contracts. Since $p^{\max} > \frac{E_1[v(s)]}{1+r}$, the equilibrium asset price with contingent contracts exceeds the optimistic valuation whenever the optimistic wealth is sufficiently large.

6 Collateral Equilibrium with Short Selling

This section considers an extension of the baseline setting in which short selling is allowed, which is relevant to understand the data for the fraction of the assets that can be short sold (e.g., for the majority of stocks). The analysis in this section establishes that a version of the asymmetric disciplining result (cf. Theorem 1) applies in this setting. I first generalize the definition of equilibrium to allow for short selling. I then characterize traders' portfolio choices for any given asset price, p . I finally combine this analysis with asset market clearing to solve for the equilibrium price.

A potential short seller of an asset needs to borrow the asset from another trader. But since borrowing in this economy is collateralized, short selling also needs to be collateralized. Formally, a *unit short contract*, denoted by $\psi \in \mathbb{R}_+$, is a promise of $v(s)$ units of the consumption good conditional on state $s \in \mathcal{S}$, collateralized by $\frac{\psi}{1+r}$ units of the bond (so that ψ denotes the value of the collateral in the next period). A trader selling the unit short contract can be interpreted as borrowing the asset from a lender, and posting $\frac{\psi}{1+r}$ units of the bond as collateral in a margin account. In reality, the lender of the security will ask for a short fee.¹⁸ In the model, the short fee is implicitly captured by the price of the short contract, $q^{short}(\psi)$, with the lower price corresponding to a higher short fee.

As in the baseline setting, there are also non-contingent unit debt contracts, $\varphi \in \mathbb{R}_+$, each of which is traded at price $q^{debt}(\varphi)$. I also assume that only a fraction $\gamma^{short} \in [0, 1]$ of traders can sell short contracts, while only a fraction $\gamma^{debt} \in [0, 1]$ can sell debt contracts and leverage. These assumptions are made to simplify the analysis, but they are not unreasonable because short selling in financial markets (and to some extent, leverage) is confined to a small fraction of investors. I denote the short selling ability of a trader with $t^{short} \in \{0, 1\}$, and the leverage ability with $t^{debt} \in \{0, 1\}$. Taking the belief heterogeneity also into account, there are 8 types of traders, where a type is denoted by $T = (i, t^{short}, t^{debt})$.

Let $(\mu_T^{short,+}, \mu_T^{short,-})$ denote measures that represent type T traders' portfolio of short contracts, and define $(\mu_T^{debt,+}, \mu_T^{debt,-})$ similarly for debt contracts. The above restriction is formalized by assuming that

$$\mu_T^{short,-} = 0 \text{ for each } T = (\cdot, t^{short} = 0, \cdot), \text{ and } \mu_T^{debt,-} = 0 \text{ for each } T = (\cdot, \cdot, t^{debt} = 0).$$

The definition of equilibrium follows closely Definition 2 with minor changes that take into

¹⁸For detailed descriptions of the shorting market, see, for example, Jones and Lamont (2001), D'Avolio (2002), and Duffie, Garleanu and Pedersen (2002).

account the additional restriction.

As before, I first consider a quasi-equilibrium in which optimists are restricted to choose $\mu_T^{debt,+} = 0$ (so they are not allowed to buy debt contracts) while moderates are restricted to choose $\mu_T^{short,+} = 0$ (so they are not allowed to buy short contracts). Similar to before, these restrictions will not be binding in equilibrium and the quasi-equilibrium will correspond to a collateral equilibrium. To characterize the quasi-equilibrium, I first conjecture an equilibrium of a particular form in which traders are endogenously matched through competitive markets.

6.1 Matching of Optimists and Moderates in Debt and Short Markets

Under appropriate parametric restrictions there exists a quasi-equilibrium in which traders take the following positions. First, optimists that can leverage, i.e., traders with type $T_1 \equiv (1, \cdot, 1)$, invest all of their wealth in the asset and they leverage as much as possible given their choice of contract φ . Second, optimists that cannot leverage, i.e., traders with type $T_2 \equiv (1, \cdot, 0)$, invest all of their wealth either in the asset or the short contracts sold by moderates that can short sell. Third, moderates that can short sell, i.e., traders with type $T_3 \equiv (0, 1, \cdot)$, invest all of their wealth in the bond and they short sell as much as possible given their choice of contract ψ . Fourth, moderates that cannot short sell, i.e., traders with type $T_4 \equiv (0, 0, \cdot)$, invest all of their wealth either in the bond or the debt contracts sold by type T_1 traders.

In other words, type T_1 optimists borrow from type T_4 moderates that cannot short sell, while type T_3 moderates borrow the asset A from type T_2 optimists that cannot leverage. To see the intuition for this matching, note that type T_3 moderates require a greater interest rate than type T_4 moderates to part with their wealth (i.e., to lend), because, in equilibrium, they receive a greater expected return on their wealth (since they have the ability to short sell). This implies that the debt contracts sold by type T_1 optimists are purchased by type T_4 moderates. A similar reasoning shows that the short contracts sold by type T_3 moderates are bought by type T_2 optimists.

Given this matching, the characterization of the quasi-equilibrium follows closely the analysis in Section 3. In particular, consider debt contract prices given by (9), which corresponds to the valuation of type T_4 moderates. Given these prices and the asset price $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r} \right)$, Theorem 1 continues to apply. That is, type T_1 optimists choose to borrow and leverage with a single loan with riskiness $\bar{s}_{le} \in \mathcal{S}$ that solves $p = p^{opt}(\bar{s}_{le})$.

Similarly, note that type T_2 optimists must be indifferent between investing in the asset and the short contracts. Type T_2 optimists' expected return from investing in the asset is given by $\frac{E_1[v(s)]}{p}$. Thus, consider short contract prices:

$$q^{short}(\psi) = \frac{1}{\frac{E_1[v(s)]}{p}} E_1[\min(\psi, 1)] \text{ for each } \psi \in \mathbb{R}_+. \quad (25)$$

Given the prices in (25), type T_2 optimists absorb any potential supply of short contracts from type T_3 moderates. Hence, the equilibrium in the short contract market is determined by type

T_3 moderates' optimal contract choice. I next characterize the optimal short contract and show that a version of the asymmetric disciplining result applies also in this setting.

6.2 Asymmetric Disciplining with Short Selling

Given the prices in (25) and the asset price $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r} \right)$, type T_3 moderates short sell according to a unit short contract $\psi = v(\bar{s}_{sh})$ that defaults if the realized state is above some threshold state $\bar{s}_{sh} \in \mathcal{S}$. This is because, for sufficiently good states, the value of the promised asset exceeds the value of the posted collateral, and the short seller finds it optimal to default. The next result, which is the counterpart of Theorem 1 for short contracts, characterizes the threshold state \bar{s}_{sh} for the optimal short contract.

Theorem 6 (Asymmetric Disciplining with Short Selling). *Suppose assumption (MLRP) holds, short contract prices are given by (25) and the asset price satisfies $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r} \right)$. In a quasi-equilibrium:*

(i) *There exists $\bar{s}_{sh} \in \mathcal{S}$ such that $\mu_{T_3}^{short,-}$ is a Dirac measure that puts weight only at the contract $\psi = v(\bar{s}_{sh})$, i.e., moderates (that are able) short sell only the unit short contract $\psi = \frac{v(\bar{s}_{sh})}{1+r}$. These moderates invest all of their wealth in the bond and short sell the asset as much as possible subject to the collateral constraint.*

(ii) *The threshold state $\bar{s}_{sh} \in \mathcal{S}$ of the optimal short contract is characterized as the unique solution to:*

$$p = p^{short}(\bar{s}_{sh}) \equiv \frac{E_1[v(s)] / (1+r)}{1 + F_0(\bar{s}_{sh}) \left(\frac{\int_{s^{\min}}^{\bar{s}_{sh}} v(s) dF_1}{\int_{s^{\min}}^{\bar{s}_{sh}} v(s) dF_0} - \frac{\int_{s^{\min}}^{\bar{s}_{sh}} dF_1}{\int_{s^{\min}}^{\bar{s}_{sh}} dF_0} \right)}. \quad (26)$$

Note that $p^{short}(\bar{s}_{sh})$ describes the price for which the short contract with default threshold \bar{s}_{sh} is optimal. Under assumption (MLRP), this curve is strictly decreasing, with $p^{short}(s^{\min}) = \frac{E_1[v(s)]}{1+r}$ and $p^{short}(s^{\max}) = \frac{E_0[v(s)]}{1+r}$. Thus, there is a unique solution to (26).

It can also be seen that the function $p^{short}(\bar{s}_{sh})$ features an asymmetric disciplining property. To see this, suppose the moderate belief, F_0 , is kept constant and the optimistic belief, F_1 , is changed in a way to keep the optimistic valuation $\frac{E_1[v(s)]}{1+r}$ constant. By Eq. (26), the effect of this type of change on $p^{short}(\bar{s}_{sh})$ is characterized by its effect on the expression:

$$\frac{\int_{s^{\min}}^{\bar{s}_{sh}} v(s) dF_1}{\int_{s^{\min}}^{\bar{s}_{sh}} v(s) dF_0} - \frac{\int_{s^{\min}}^{\bar{s}_{sh}} dF_1}{\int_{s^{\min}}^{\bar{s}_{sh}} dF_0}. \quad (27)$$

By assumption (MLRP), this expression is always positive. Intuitively, both terms in the expression can be thought of as an ‘‘average’’ of the likelihood ratios $\left(\frac{f_1(s)}{f_0(s)} \right)_{s \in [s^{\min}, \bar{s}_{sh}]}$, with the term on the left putting relatively greater weight $v(s)$ on the higher likelihood ratios $\frac{f_1(s)}{f_0(s)}$ (corresponding to higher s). This intuition also suggest that a shift of optimism towards the relative likelihood of states above \bar{s}_{sh} decreases the expression in (27). In the most extreme case, if $\frac{f_1(s)}{f_0(s)}$ is constant over $s \in [s^{\min}, \bar{s}_{sh}]$ (so that all the optimism is concentrated on the

relative likelihood of states above \bar{s}_{sh}), then the expression in (27) is equal to zero. By Eq. (26), the function $p^{short}(\bar{s}_{sh})$ negatively depends the expression in (27). It follows that a shift of optimism towards the relative likelihood of states above \bar{s}_{sh} increases $p^{short}(\bar{s}_{sh})$. That is, for any given level of default threshold \bar{s}_{sh} for short contracts, the asset price is higher when optimism is concentrated more on the relative likelihood of good states. This illustrates the asymmetric disciplining property of the optimal short contract.

The proof of Theorem 6 is relegated to Appendix A.6. For an intuition, note that the short contract defaults above the threshold state \bar{s}_{sh} . Thus, they pay the same amount $\psi = v(\bar{s}_{sh})$ in these states. Then, using a short contract with threshold \bar{s}_{sh} , it is impossible for moderates to bet on their pessimism about the relative likelihood of states above \bar{s}_{sh} . Consequently, moderates' pessimism about the relative likelihood of good states is not reflected in the asset price, as suggested by (26). In contrast, moderates can bet on their pessimism about the probability of states below \bar{s}_{sh} by selling the short contract. Thus, this type of pessimism is reflected in the asset price.

Put differently, it is easier for moderates to bet on their pessimism about the probability of bad states than to bet on their pessimism for the relative likelihood of good states. To bet on the latter types of pessimism, moderates need to post a higher level of collateral ψ (equivalently, they need to choose a short contract with a high default threshold \bar{s}_{sh}). Hence, these types of short sales are more difficult to leverage, which leads to the asymmetric disciplining result with short selling. Appendix A.6 provides a more complete intuition that parallels the analysis in Section 3.1.

6.3 Equilibrium Asset Price with Short Selling

The equilibrium is characterized by type T_1 optimists' and type T_3 moderates' optimal contract choice, along with the market clearing condition for the asset, which I derive next. To simplify the analysis, suppose the parameters are such that the equilibrium asset price satisfies $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$. Note that type T_1 optimists spend a total of

$$\gamma_{le} (w_1 + p\alpha_1) \frac{p}{p - \frac{E_0[\min(v(s), v(\bar{s}_{le}))]}{1+r}} \quad (28)$$

units of the consumption good on the asset. Here, recall that γ_{le} is the fraction of investors that are able to leverage, $w_1 + p\alpha_1$ is the total wealth of optimists, and the second term in (28) is the leverage ratio. Next note that type T_2 optimists (that make an unleveraged investment in the asset) spend a total of

$$(1 - \gamma_{le}) (w_1 + p\alpha_1) - W^{short} \quad (29)$$

units of the consumption good on the asset. Here, recall that type T_2 optimists are indifferent between buying the asset and buying the short contracts sold by moderates. Hence, they

invest in the asset all of their wealth net of W^{short} , which represents their expenditure on short contracts.

By market clearing in short contracts, W^{short} is also equal to type T_3 moderates' total revenue from sales of short contracts. The analysis in the appendix shows that this expression has a similar form to the expression in (28), and it is given by:

$$W^{short} = \gamma_{sh} (w_0 + p\alpha_0) \frac{p}{\frac{v(\bar{s}_{sh})}{E_1[\min(v(s), v(\bar{s}_{sh}))]} \frac{E_1(v(s))}{1+r} - p}. \quad (30)$$

Here, $\gamma_{sh} (w_0 + p\alpha_0)$ denotes the wealth of type T_3 moderates, and the second term denotes the *short leverage ratio*, that is, the total value of asset short sold per unit consumption good spending. Market clearing for the asset implies that the total spending on the asset, that is, the sum of the expressions in (28) and (29), is equal to the total value of the asset, p . After substituting for W^{short} from the expression in (30) and rearranging terms, the asset market clearing condition can be written as:

$$\gamma_{le} \frac{w_1 + p\alpha_1}{p - \frac{E_0[\min(v(s), v(\bar{s}_{le}))]}{1+r}} + (1 - \gamma_{le}) \frac{w_1 + p\alpha_1}{p} = 1 + \gamma_{sh} \frac{w_0 + p\alpha_0}{\frac{v(\bar{s}_{sh})}{E_1[\min(v(s), v(\bar{s}_{sh}))]} \frac{E_1(v(s))}{1+r} - p}. \quad (31)$$

This expression shows that short selling effectively expands the supply of the asset, as captured by the second term on the right hand side.

The equilibrium tuple $(p, \bar{s}_{le}^*, \bar{s}_{sh}^*)$ is characterized by the optimality conditions $p = p^{opt}(\bar{s}_{le}) = p^{short}(\bar{s}_{sh})$, along with the market clearing condition (31). Note that an increase in the fraction of short sellers, γ_{sh} , decreases the asset price because it increases the effective supply of the asset. Conversely, an increase in the fraction of leveraged investors, γ_{le} , increases the asset price because it increases the demand for the asset, as captured by the left hand side of Eq. (31).

In addition, an increase in the right-skewness of optimism increases the asset price. To illustrate this effect, consider an equilibrium, $(p, \bar{s}_{le}^*, \bar{s}_{sh}^*)$, and suppose optimists' optimism is changed to \tilde{F}_1 that satisfies $\tilde{E}_1[v(s)] = E_1[v(s)]$ and $\tilde{f}_1(s) = f_0(s)$ for each $s \in [0, \bar{s}_{sh}^*]$. That is, the distribution \tilde{F}_1 is "equally" optimistic as the distribution F_1 , but its optimism is concentrated to the right of the current short default threshold \bar{s}_{sh}^* . By Eqs. (10) and (26), this change in the type of optimism leads to an increase in both default thresholds, \bar{s}_{le} and \bar{s}_{sh} , given the old equilibrium price p . Note also that the leverage ratio in (28) is increasing in \bar{s}_{le} , and the short leverage ratio in (30) is decreasing in \bar{s}_{sh} . Hence, at the old equilibrium price, this change increases optimists' leverage ratio, while it decreases moderates' leverage ratio. Consequently, the market clearing condition (31) implies that the equilibrium price increases. That is, an increase in this type of right-skewness of optimism increases the asset price also in the setting with short selling.

Intuitively, when optimism is more right-skewed, optimists leverage more by choosing larger and riskier loans (captured by the increase in \bar{s}_{le}), while short sellers leverage less by posting a

greater amount of collateral $\psi = v(\bar{s}_{sh})$ for each unit short contract (captured by the increase in \bar{s}_{sh}). This increases the demand and decreases the effective supply for the asset (cf. (31)), which leads to a higher equilibrium price.

7 Dynamic Model: Financing Speculative Bubbles

The analysis so far has concerned a two-period economy. However, the asymmetric disciplining result also has dynamic implications. This section considers a dynamic extension of the baseline setting to analyze the interaction of the asymmetric disciplining mechanism with the speculative component of asset prices identified by Harrison and Kreps (1978). The analysis in this section shows that the speculative “bubbles” are also asymmetrically disciplined by endogenous financial constraints. I first describe the basic environment without financial constraints and illustrate that the asset price features a speculative component. I then characterize the dynamic equilibrium with collateral constraints, and analyze the effect of belief heterogeneity on the speculative component.

7.1 Basic Dynamic Environment

Consider an infinite horizon overlapping generations economy in which the periods and generations are denoted by $n \in \{0, 1, \dots\}$. There is a continuum of traders in each generation n , who are born in period n and live in periods n and $n + 1$. Each trader of generation n has an endowment of the consumption good in period n , and consumes only in period $n + 1$. The resources can be transferred between periods by investing either in the bond B or the asset A . Bond B is supplied elastically at a normalized price 1 in every period. Each unit of the bond yields $1 + r$ units of the consumption good in the next period, and then fully depreciates (i.e., the bond pays dividend only once). Asset A is in fixed supply, which is normalized to 1. The asset yields a_n units of dividends in each period n . Suppose that log dividend follows a random walk, that is, the dividend yield follows the process

$$a_{n+1} = a_n s_{n+1}. \tag{32}$$

Here, s_{n+1} is a random variable with distribution F_{true} which has a density function that is continuous and positive over $\mathcal{S} = [s^{\min}, s^{\max}] \subset \mathbb{R}_{++}$. Suppose also that $1 \in \mathcal{S}$ and that the mean of s_{n+1} is normalized to 1. In other words, the next period dividend yield fluctuates around the current dividend yield a_n , with expected value equal to a_n .

All young traders in period n observe all past realizations of the dividend yield and the current realization a_n , but they have heterogeneous priors about the next period realization a_{n+1} . In each period n , similar to the static model, there are two types of young traders, *optimists* and *moderates*, respectively with priors F_1 and F_0 about the next period state s_{n+1} . **Assumption (O_d).** Period n young traders’ belief distributions F_1 and F_0 for the next period

state s_{n+1} have density functions f_1, f_0 that are continuous and positive over \mathcal{S} . The moderate belief distribution is given by $F_0 = F_{true}$ while the optimistic distribution satisfies $F_1 \succ_O F_0$. In addition, traders' beliefs for the random variables s_{n+k} , for $k \geq 2$, are identical and given by the true distribution F_{true} .

One way to interpret this assumption is that all traders know the dividend yield process described in (32), but in every period, some traders (optimists) become optimistic regarding the next period realization.¹⁹ Under assumption (O_d), optimists' expectation for the dividend yields in any future period is given by

$$E_{n,1} [a_{n+k}] = E_{n,1} [a_{n+1}] = E_1 [a_n] (1 + \varepsilon).$$

Here, the parameter

$$\varepsilon \equiv E_{n,1} [s_{n+1}] - 1 > 0$$

controls optimists' level of optimism (recall that the true distribution has mean equal to 1). Consequently, optimists' present discounted value of the future dividends can be calculated as

$$p_1^{pdv} (a_n) \equiv \sum_{k=1}^{\infty} \frac{E_{n,1} [a_{n+k}]}{(1+r)^k} = \frac{a_n (1 + \varepsilon)}{r}.$$

Note that the moderate present discounted value is given by $p_0^{pdv} (a_n) = a_n/r$. Thus, optimists' overvaluation of the asset is given by ε/r . Intuitively, optimists expect the next period realization for the dividend yield to be higher, and they expect future dividend yields to fluctuate around this higher (expected) level. This leads to the valuation difference ε/r .

Similar to the baseline setting, short selling the asset is ruled out by assumption (S). Let $(w_{i,n})_{i \in \{1,0\}}$ denote type i traders' endowment of the consumption good, and suppose

$$w_{i,n} = \omega_i a_n, \text{ where } \omega_i \in \mathbb{R}_{++}. \quad (33)$$

That is, young traders' endowments are proportional to the current dividend yield of the asset. This assumption is not essential for the economic results, but it simplifies the subsequent analysis.²⁰ This completes the description of the basic elements of the dynamic economy. Note that the economy has a recursive structure. This is because the dividend yield process follows a random walk (cf. Eq. (32)), and young traders' beliefs are formed independently of the past dividend yield realizations (cf. assumption (O_d)). This observation leads to the following lemma, which provides a sufficient statistic for the dynamic economy and simplifies

¹⁹There could be a number of explanations for the source of this type of optimism. As in Scheinkman and Xiong (2003), optimists may be overconfident about a signal they receive about the next period shock. Alternatively, optimists may be simply optimistic about the next period shock, thinking that the current period is special. Reinhart and Rogoff (2008) refer to this type of optimism as "this time it is different syndrome."

²⁰I thank Ivan Werning for suggesting this simplification.

the subsequent notation.

Lemma 1. *Given any history $(a_0, \dots, a_{n-1}, a_n)$ of dividend yield realizations, the current dividend yield a_n is a sufficient statistic for the determination of the equilibrium allocations in this economy.*

In view of this lemma, let $a \equiv a_n \in \mathbb{R}_{++}$ denote the current dividend yield, $s \equiv s_{n+1} \in \mathcal{S}$ denote the next period shock, and $p(a)$ denote the current asset price.

7.2 Speculative Bubbles without Financial Constraints

As a benchmark, I first consider the asset price in an economy in which individuals can borrow and lend freely in a competitive loan market at the benchmark rate r . In other words, there exists no limited liability or enforcement problems. In this case, optimists borrow and invest in the asset an infinite amount whenever the asset price is below their valuation. Hence, the equilibrium asset price is equal to the optimistic valuation:

$$p(a) = \frac{1}{1+r} \left(a(1+\varepsilon) + \int_{\mathcal{S}} p(as) dF_1 \right), \text{ for all } a \in \mathbb{R}_{++}. \quad (34)$$

The first term on the right hand side is optimists' expected dividend payoff from the asset, and the second term is their expected payoff from the sale of the asset. Eq. (34) provides a recursive characterization of the asset price which can be solved as

$$p(a) = \frac{a(1+\varepsilon)}{r-\varepsilon}. \quad (35)$$

Note that the asset price $p(a)$ is higher than optimists' present discounted valuation, $p_1^{pdv}(a) = \frac{a(1+\varepsilon)}{r}$. The component of the asset price in excess of the present discounted value of the holder of the asset, $p(a) - p_1^{pdv}(a)$, is what Scheinkman and Xiong (2003) call a speculative "bubble." I also define

$$\lambda = \frac{p(a) - p_1^{pdv}(a)}{p(a)} = \frac{\varepsilon}{r} \quad (36)$$

as *the share of the speculative component*. The asset price features a speculative component because optimists hold the asset not only for the higher expected dividend gains in the next period, but also since they are planning to sell the asset to a trader who will be even more optimistic than them in the next period. In view of these expected speculative capital gains, optimists bid up the asset price higher than the present discounted value of dividends.

The expression in (36) also implies that the speculative component could represent a large fraction of the asset price, even for a relatively small belief disagreement ε (especially when the interest rate is low). The rationale for this observation is related to a powerful amplification effect: *the dynamic multiplier*. Note that optimists in the next period also expect to make speculative capital gains by selling the asset to yet more optimistic traders in the subsequent period, which increases the price in the next period. But this further increases the valuation

of current optimists who are planning to sell to future optimists, increasing the current asset price further. In other words, a high asset price in the next period feeds back into the asset price today, amplifying the effect of heterogeneous beliefs and leading to a large speculative component.

I next incorporate financial constraints into this economy. With financial constraints, the asset price does not necessarily satisfy the recursion in (34). Rather, the asset price lies between the optimistic and the moderate valuations, and the exact recursion (and the share of the speculative component) is determined by the type of financial constraints.

7.3 Financial Frictions and Dynamic Collateral Equilibrium

I model financial constraints using the collateral equilibrium described in Section 3.1. In particular, traders in each generation trade collateralized debt contracts that mature in the next period. As in the baseline setting, debt contracts are non-recourse and non-contingent. Formally, a *unit debt contract*, denoted by $\varphi \in \mathbb{R}_+$, is a promise of φ units of the consumption good in the next period by the borrower collateralized by 1 unit of the asset. Given the current dividend realization a , I define the *value function* as the payoff of the asset in the next period,

$$v(a, s) \equiv as + p(as) \text{ for each } s \in \mathcal{S}. \quad (37)$$

The debt contract φ defaults if and only if $v(a, s) < \varphi$, and thus it pays $\min(v(a, s), \varphi)$. Each debt contract $\varphi \in \mathbb{R}_+$ is traded in an anonymous market at a competitive price $q(a, \varphi)$.

Let $x_i^A(a), x_i^B(a)$ denote type i traders' asset and debt holding, and $\mu_i^+(a), \mu_i^-(a)$ denote their long and short debt portfolios. The traders' problem is given by:

$$\begin{aligned} & \max_{\substack{x_i(a) \geq 0, \\ \mu_i^+(a), \mu_i^-(a)}}} && x_i^A(a) E_i[v(a, s)] + x_i^B(a) (1 + r) + \\ & && \int_{\mathbb{R}_+} E_i[\min(v(a, s), \varphi)] d\mu_i^+(a, \varphi) - \int_{\mathbb{R}_+} E_i[\min(v(a, s), \varphi)] d\mu_i^-(a, \varphi) \quad (38) \\ \text{s.t.} && px_i^A(a) + x_i^B(a) + \int_{\mathbb{R}_+} q(a, \varphi) d\mu_i^+(a, \varphi) - \int_{\mathbb{R}_+} q(a, \varphi) d\mu_i^-(a, \varphi) \leq \omega_i a, \\ && \int_{\mathbb{R}_+} d\mu_i^-(a, \varphi) \leq x_i^A(a). \end{aligned}$$

Definition 4 (Dynamic Collateral Equilibrium). *Under assumptions (O_d) and (S) , and condition $(A.50)$, a dynamic collateral equilibrium is a collection of prices $\left(p(a), [q(a, \varphi)]_{\varphi \in \mathbb{R}_+}\right)_{a \in \mathbb{R}_{++}}$ and allocations $\left((x_i^A(a), x_i^B(a), \mu_i^+(a), \mu_i^-(a))_{i \in \{1, 0\}}\right)_{a \in \mathbb{R}_{++}}$ such that, for each dividend realization $a \in \mathbb{R}_{++}$, the allocation of each trader $i \in \{1, 0\}$ solves problem (38), and asset and unit debt markets clear.*

Note that, given the value function in the next period (cf. Eq. (37)), the economy in the current period is very similar to the static economy analyzed earlier, with the main difference

that the value function, (37), also depends on the price function. Hence, the dynamic equilibrium is characterized with a fixed point argument. The linear homogeneity of endowments (cf. condition (33)) ensures that the price to dividend ratio and the loan riskiness are independent of the current realization of a . The proof of the following theorem is relegated to Appendix A.7.

Theorem 7 (Existence and Characterization of Dynamic Equilibrium). *Under assumptions (O_d) , (S) and the parametric condition (A.50) in Appendix A.7, there exists a recursive collateral equilibrium in which $p(a) = p_d a$ and $\bar{s}^*(a) = \bar{s}_d^*$ for each $a \in \mathbb{R}_{++}$. The price to dividend ratio, p_d , is the unique fixed point of the mapping $P_d : \left[p_d^{\min} \equiv \frac{1}{r}, p_d^{\max} \equiv \frac{1+\varepsilon}{r-\varepsilon} \right] \rightarrow \left[p_d^{\min}, p_d^{\max} \right]$, where $P_d(\tilde{p}_d)$ is the collateral equilibrium price of the static economy*

$$\mathcal{E}(p_d) = \left(\begin{array}{c} \mathcal{S}; v(s \mid p_d) = s(1 + p_d); \\ \{F_i\}_i; \{w_i \equiv \omega_i\}_i; \{\alpha_1 = 0, \alpha_0 = 1\} \end{array} \right). \quad (39)$$

Note that this result reduces the characterization of the dynamic equilibrium to the characterization of the equilibrium for the static economy, $\mathcal{E}(p_d)$, along with a fixed point argument. Intuitively, $P_d(\tilde{p}_d)$ is the price to dividend ratio that would obtain today if the future price to dividend ratio was given by \tilde{p}_d . The upper limit of the fixed point interval, $p_d^{\max} = \frac{1+\varepsilon}{r-\varepsilon}$, is the price to dividend ratio that would obtain if optimists always priced the asset (i.e., it is the price in the unconstrained economy). The lower limit, $p_d^{\min} = \frac{1}{r}$, is the price to dividend ratio that would obtain if moderates always priced the asset (i.e., it is the moderate valuation of the asset). The equilibrium is in the interval $[p_d^{\min}, p_d^{\max}]$. The next example uses this characterization to illustrate the effect of financial constraints on the speculative component of the asset price.

Example 3. *Consider the prior distributions F_0 and $F_{1,G}$ of Example 1 in which the valuation difference for the next period shock is given by $\varepsilon = E_1[s] - E_0[s] = 0.1$. Consider the corresponding dynamic collateral equilibrium with interest rate $r = 0.15$ and optimistic wealth $\omega_1 = 4$. Figure 6 plots the price mapping, $P_d(\cdot)$, and shows that it intersects the 45 degree line exactly once, which corresponds to the equilibrium. The equilibrium price is lower than the unconstrained level, however it is still higher than the present discounted value according to either the moderate or optimistic priors (which are close to each other). In particular, in this example, the price has a large speculative component despite financial constraints.*

The figure also illustrates optimists' balance sheet. Optimists' downpayment is about 1/4 of the asset price, and they borrow the remaining amount from moderates, collateralized against one unit of the asset. In particular, moderate lenders, who correctly know the dividend yield process in (32), agree to finance about 3/4 of the asset purchase despite the fact that the present discounted value of the asset is less than half of its price.

The last feature of this example provides insights for how the price can feature a large speculative component when optimists are financially constrained. In this example, lenders

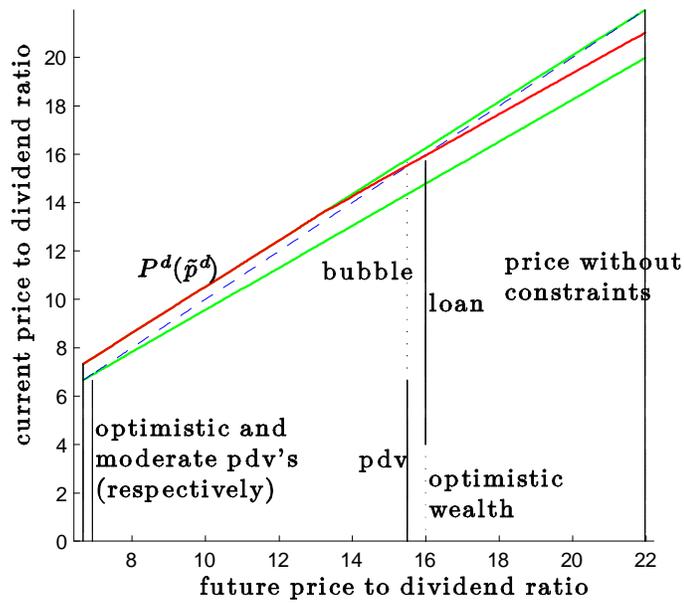


Figure 6: The x axis is the range of possible price to dividend ratios, $[p^{\min}, p^{\max}]$. The lower and higher green curves respectively plot the moderate and the optimistic valuations when the future price to dividend ratio is given by the value at the x axis. The red curve (intermediate to the two green curves) plots the price mapping, $P^d(\tilde{p}^d)$. The equilibrium is the intersection of the red curve with the 45 degree line (dashed blue curve).

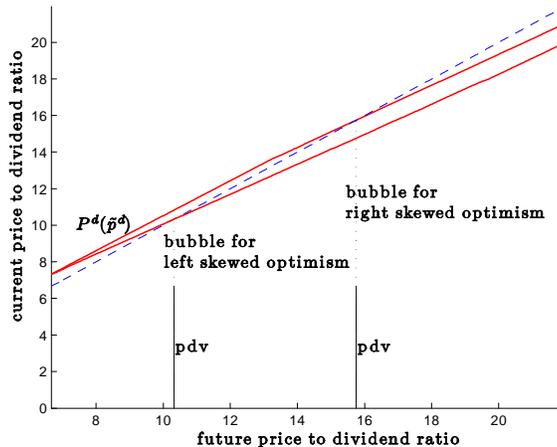


Figure 7: The lower (resp. the higher) red line plots the current price to dividend ratio as a function of the future price to dividend ratio for the belief distributions in the first case (resp. second case) of Example 1. The dynamic equilibrium is the intersection of this curve with the 45 degree line (dashed blue curve).

have correct priors and they know that the asset price is considerably greater than their present discounted valuation. Nonetheless, they agree to extend large loans which are in part collateralized by the speculative component of the price. This is because lenders' valuation of the asset (the lower green line in Figure 6) also contains a speculative component, and thus it is higher than their present discounted valuation. Intuitively, lenders agree to extend large loans because they think that, should the borrower default, they could always sell the collateral to another optimist in the next period.

Put differently, a marked characteristic of this speculative episode is that the bubble raises all boats: both the optimistic and the moderate valuations are greater than their present discounted valuations. Consequently, optimists' and moderates' valuation difference in any period (the difference between the two green lines in Figure 6) is relatively small. As in the unconstrained case, a large speculative bubble forms from the accumulation of small valuation differences through the dynamic multiplier. This is perhaps unfortunate, because a small valuation difference makes the financing of the asset relatively easy, opening the way for large speculative bubbles even when optimists are financially constrained.

Naturally, as the previous sections show, a small valuation difference does not guarantee that financial constraints are lax. Whether financing will actually go through, and the share of the speculative component, also depends on a number of other factors, such as optimists' wealth level and the type of belief heterogeneity. For example, consider the equilibrium in Example 3 with the only difference that the optimistic priors are changed to $F_{1,B}$ (defined in Example 1). This prior leads to the same asset valuation, but it is more left-skewed than $F_{1,G}$. Figure 7 shows that, in response to this change, the speculative component shrinks by about

half.

The next result shows that this is a general property, that is, an increase in the right-skewness of optimism unambiguously increases the asset price and the share of the speculative component. To state the result, I define the overvaluation ratio $\theta_d \in (0, 1]$ as the unique solution to

$$p_d = (1 - \theta_d) \frac{E_0[v_d(\cdot | p_d)]}{1 + r} + \theta_d \frac{E_1[v_d(\cdot | p_d)]}{1 + r}. \quad (40)$$

Intuitively, θ_d captures the fraction of the optimism in prior beliefs that is reflected in the asset price. I generalize the speculative component of the asset price (cf. Eq. (36) to the financially constrained economy) as

$$\lambda_d = \frac{p(a) - p^{pdv}(a)}{p(a)}, \text{ where } p^{pdv}(a) = (1 - \theta_d)p_0^{pdv}(a) + \theta_d p_1^{pdv}(a). \quad (41)$$

Unlike the unconstrained case, the marginal holder of the asset is not necessarily an optimist, hence the relevant present discounted value is defined as an average of optimistic and moderate present discounted values, weighted by the overvaluation ratio θ_d .²¹ The following result establishes that an increase in the right-skewness of belief heterogeneity increases the asset price and the share of the speculative component.

Theorem 8 (Effect of Type of Heterogeneity on the Speculative Component). *Consider the recursive collateral equilibrium characterized in Theorem 7 and let \bar{s}_d^* denote the equilibrium loan riskiness.*

(i) *If optimists' optimism becomes weakly more right-skewed, i.e., if their prior is changed to \tilde{F}_1 that satisfies $\tilde{F}_1 \succeq_R F_1$ and $\tilde{F}_1 \succ_O F_0$ (so that assumption (O_d) continues to hold), then: the price to dividend ratio p_d , the loan riskiness \bar{s}_d^* , and the share of the speculative component λ_d weakly increase.*

(ii) *If moderates' optimism becomes weakly more skewed to the left of \bar{s}_d^* , i.e., if their prior is changed to \tilde{F}_0 that satisfies $F_0 \succeq_{R, \bar{s}_d^*} \tilde{F}_0$ and $F_1 \succ_O \tilde{F}_0$, then: the price to dividend ratio p_d and the share of the speculative component λ_d weakly increase.*

Intuitively, if optimists' optimism becomes more right-skewed, then future optimists will perceive looser financial constraints and they will be able to bid up the asset price higher. This implies that the resale option value to future optimists is higher, which leads to a greater speculative component. Conversely, if optimists' optimism becomes more left-skewed, then the speculative component becomes smaller because the future optimists will perceive tighter financial constraints. This result shows that bubbles can come to an end because of a shift in belief heterogeneity towards the likelihood of bad events.

²¹The share of the speculative premium is independent of the state $a \in \mathbb{R}_{++}$ because the functions $p(a)$, $p_0^{pdv}(a)$, and $p_1^{pdv}(a)$ are linearly homogeneous in a .

8 Conclusion

In this paper, I have theoretically analyzed the effect of belief heterogeneity on asset prices. The central feature of the model is that, to take positions in line with their beliefs, investors need to borrow from traders with different beliefs using collateralized contracts. The lenders do not value the collateral as much as the borrowers do, which represents a constraint on investors' ability to borrow and leverage their investments. I have considered the effect of this constraint on asset prices in a variety of settings that differ in the types of collateralized contracts that are available for trade. In the baseline model, I have restricted attention to non-contingent loans and disallowed short selling, and I have relaxed these restrictions in two extensions of the model. In each of these scenarios, my paper has established that optimism is asymmetrically disciplined by endogenous financial constraints. In particular, optimism about the likelihood of bad states has a smaller effect on asset prices than optimism about the relative likelihood of good states. I have also considered a dynamic extension of the model which reveals that the speculative asset price bubbles, identified by Harrison and Kreps (1978), are also asymmetrically disciplined by optimists' financial constraints.

Taken together, my results suggest that certain economic environments that generate uncertainty (and thus belief heterogeneity) about upside returns are conducive to asset price increases and speculative bubbles financed by credit. This prediction is in line with the observations in Kindleberger (1978), who has argued that speculative episodes typically follow a novel event (which arguably generates upside uncertainty), and that the easy availability of credit plays an important role in these episodes.

The asymmetric disciplining characterization of asset prices also emphasizes that what investors disagree about matters for asset prices, to a greater extent than the level of the disagreement. In particular, when optimists are financially constrained, an increase in the level of belief heterogeneity in general has ambiguous effects on asset prices. However, the effect can be characterized once the skewness of the increase is taken into account. Additional belief heterogeneity tends to decrease asset prices when it concerns the likelihood of bad states, but it tends to increase asset prices when it concerns the relative likelihood of good states. A growing empirical literature in finance considers the effect of the level of belief heterogeneity on asset prices and subsequent asset returns (e.g., Chen, Hong and Stein, 2001, Diether, Malloy and Scherbina, 2002, and Ofek and Richardson, 2003). My paper suggests that a fruitful future research direction may be to empirically investigate the effect of the skewness of the belief heterogeneity on asset prices.

A Appendices

A.1 Properties of Optimism Order

This appendix establishes the properties of optimism order (cf. Definition (1)). Consider two probability distributions H, \tilde{H} over $S = [s^{\min}, s^{\max}] \subset R$ with corresponding density functions h, \tilde{h} that are continuous and positive at each $s \in S$.

I first show that $\frac{1-\tilde{H}(s)}{1-H(s)}$ is strictly increasing at some $s \in S$ if and only if the hazard rate inequality in (2) is satisfied. To see this, consider the derivative of $\frac{1-\tilde{H}(s)}{1-H(s)}$

$$\frac{d}{ds} \frac{1-\tilde{H}(s)}{1-H(s)} = \frac{-\tilde{h}(s)(1-H(s)) + h(s)(1-\tilde{H}(s))}{(1-H(s))^2}, \text{ for each } s \in [s^{\min}, s^{\max}],$$

and note that this expression is positive if and only if the hazard rate inequality (2) holds.

I next show that the optimism order is weaker than the monotone likelihood ratio property (MLRP), that is, if $\frac{\tilde{h}(s)}{h(s)}$ is strictly increasing over S , then $\tilde{H} \succ_O H$. To see this, suppose (MLRP) holds and note that this implies, for each $s < s^{\max}$,

$$\frac{\tilde{h}(s)}{h(s)} h(\tilde{s}) < \tilde{h}(\tilde{s}) \text{ for all } \tilde{s} \in (s, s^{\max}).$$

Integrate both sides of this equation over (s, s^{\max}) to get

$$\frac{\tilde{h}(s)}{h(s)} (1-H(s)) < (1-\tilde{H}(s)),$$

which proves the hazard rate inequality (2) and shows that $\tilde{H} \succ_O H$.

I next note the following result, which derives the implications of assumption (O) for the key variables used in the analysis, including the expected payoff of a loan with riskiness \bar{s} , $E_i[\min(v(s), v(\bar{s}))]$, the perceived interest rate, $r_1^{per}(\bar{s})$, and the optimality curve, $p^{opt}(\cdot)$.

Lemma 2. *Consider two probability distributions F_1 and F_0 that satisfy assumption (O).*

- (i) *The expected payoff of a loan with riskiness \bar{s} , $E_i[\min(v(s), v(\bar{s}))]$, is strictly increasing in \bar{s} .*
- (ii) *Optimists' perceived interest rate $r_1^{per}(\bar{s})$ (cf. Eq. (12)) is strictly increasing in \bar{s} . In particular, $r_1^{per}(\bar{s}) > r_1^{per}(s^{\min}) = r$ for each $\bar{s} > s^{\min}$.*
- (iii) *$p^{opt}(\bar{s})$ is continuously differentiable and strictly decreasing, i.e., $\frac{dp^{opt}(\bar{s})}{d\bar{s}} < 0$.*

Proof of Lemma 2. Part (i). Note that the derivative of $E_i[\min(v(s), v(\bar{s}))]$ = $\int_{s^{\min}}^{\bar{s}} v(s) dF_i(s) + v(\bar{s})(1-F_i(\bar{s}))$ is given by

$$\frac{dE_i[\min(v(s), v(\bar{s}))]}{d\bar{s}} = v(\bar{s})f_i(\bar{s}) + v'(\bar{s})(1-F_i(\bar{s})) - v(\bar{s})f(\bar{s}) = v'(\bar{s})(1-F_i(\bar{s})) > 0, \tag{A.1}$$

which completes the proof.

Part (ii). The derivative of $\frac{1+r_1^{per}(\bar{s})}{1+r} = \frac{E_1[\min(v(s),v(\bar{s}))]}{E_0[\min(v(s),v(\bar{s}))]}$ can be calculated as

$$\begin{aligned} \frac{d}{d\bar{s}} \left(\frac{1+r_1^{per}(\bar{s})}{1+r} \right) &= \frac{\frac{dE_1[\min(v(s),v(\bar{s}))]}{d\bar{s}} E_0[\min(v(s),v(\bar{s}))] - E_1[\min(v(s),v(\bar{s}))] \frac{dE_0[\min(v(s),v(\bar{s}))]}{d\bar{s}}}{(E_0[\min(v(s),v(\bar{s}))])^2}} \\ &= \frac{E_0[\min(v(s),v(\bar{s}))](1-F_1(\bar{s})) - E_1[\min(v(s),v(\bar{s}))](1-F_0(\bar{s}))}{(E_0[\min(v(s),v(\bar{s}))])^2}, \end{aligned} \quad (\text{A.2})$$

where the last line uses Eq. (A.1).

I next claim that

$$\frac{E_1[\min(v(s),v(\bar{s}))]}{E_0[\min(v(s),v(\bar{s}))]} < \frac{1-F_1(\bar{s})}{1-F_0(\bar{s})} \text{ for each } \bar{s} \in (s^{\min}, s^{\max}), \quad (\text{A.3})$$

which, in view of Eq. (A.2), proves that the perceived interest rate $1+r_1^{per}(\bar{s})$ is strictly increasing.

To prove the claim, note that for each $\bar{s} \in (s^{\min}, s^{\max})$,

$$\begin{aligned} \frac{E_1[\min(v(s),v(\bar{s}))]}{E_0[\min(v(s),v(\bar{s}))]} &= \frac{\int_{s^{\min}}^{\bar{s}} v(s) dF_1 + v(\bar{s})(1-F_1(\bar{s}))}{\int_{s^{\min}}^{\bar{s}} v(s) dF_0 + v(\bar{s})(1-F_0(\bar{s}))} \\ &< \frac{\int_{s^{\min}}^{\bar{s}} v(s) \frac{1-F_1(s)}{1-F_0(s)} dF_0 + v(\bar{s})(1-F_1(\bar{s}))}{\int_{s^{\min}}^{\bar{s}} v(s) dF_0 + v(\bar{s})(1-F_0(\bar{s}))} \\ &< \frac{\int_{s^{\min}}^{\bar{s}} v(s) dF_0 \frac{1-F_1(\bar{s})}{1-F_0(\bar{s})} + v(\bar{s})(1-F_1(\bar{s}))}{\int_{s^{\min}}^{\bar{s}} v(s) dF_0 + v(\bar{s})(1-F_0(\bar{s}))} = \frac{1-F_1(\bar{s})}{1-F_0(\bar{s})}, \end{aligned}$$

where the first inequality uses the hazard rate inequality (2) and the second inequality uses the fact that $\frac{1-F_1(s)}{1-F_0(s)}$ is strictly increasing. This proves the claim in (A.3) and completes the proof of this part.

Part (iii). Using the definition of $p^{opt}(\bar{s})$ in Eq. (10), note that

$$\begin{aligned} \frac{dp^{opt}(\bar{s})}{d\bar{s}} &\equiv \frac{1}{1+r} \left(v(\bar{s}) f_0(\bar{s}) + \left(-f_0(\bar{s}) + f_1(\bar{s}) \frac{1-F_0(\bar{s})}{1-F_1(\bar{s})} \right) \left(\int_{\bar{s}}^{s^{\max}} v(s) \frac{dF_1}{1-F_1(\bar{s})} \right) \right. \\ &\quad \left. - \frac{1-F_0(\bar{s})}{1-F_1(\bar{s})} v(\bar{s}) f_1(\bar{s}) \right) \\ &= \frac{-1}{1+r} \left(\frac{f_0(\bar{s})}{1-F_0(\bar{s})} - \frac{f_1(\bar{s})}{1-F_1(\bar{s})} \right) (1-F_0(\bar{s})) (E_1[v(s) | s \geq \bar{s}] - v(\bar{s})) \end{aligned} \quad (\text{A.4})$$

where the first line applies the chain rule and the second line substitutes $E_1[v(s) | s \geq \bar{s}]$ and rearranges terms. The term, $\left(\frac{f_0(\bar{s})}{1-F_0(\bar{s})} - \frac{f_1(\bar{s})}{1-F_1(\bar{s})} \right)$, in Eq. (A.4) is positive from the hazard rate inequality (2). Since the terms, $(1-F_0(\bar{s}))$ and $(E_1[v(s) | s \geq \bar{s}] - v(\bar{s}))$, are also positive, it follows that $\frac{dp^{opt}(\bar{s})}{d\bar{s}} < 0$, completing the proof of the lemma.

I next present the final result of this appendix, which uses assumption (O) to derive the effects of an increase in optimists' (moderates') optimism on the curves $p^{opt}(\cdot)$ and $p^{mc}(\cdot)$.

Lemma 3. Consider two probability distributions F_1 and F_0 that satisfy assumption (O).

(i) Suppose optimists become weakly more optimistic, i.e., consider their beliefs are changed to $\tilde{F}_1 \succeq_O F_1$. Then:

(i.1) Conditional expectations increase, that is, $\tilde{E}_1[v(s) | s \geq \bar{s}] \geq E_1[v(s) | s \geq \bar{s}]$ for each $\bar{s} \in [s^{\min}, s^{\max}]$.²²

²²Throughout the appendices, the notation $\tilde{E}_1[s | s \geq \bar{s}]$ corresponds to the conditional expectation

(i.2) The optimality curve $p^{opt}(\bar{s})$ shifts up pointwise, that is,

$$p^{opt}(\bar{s}; \tilde{F}_1) \geq p^{opt}(\bar{s}; F_1) \text{ for each } \bar{s} \in [s^{\min}, s^{\max}].$$

(i.3) The market clearing curve changes as follows:

$$p^{mc}(\bar{s}; \tilde{F}_1) \begin{cases} = p^{mc}(\bar{s}; F_1) & \text{if } p^{mc}(\bar{s}; F_1) < \frac{E_1[v(s)]}{1+r} \\ \geq p^{mc}(\bar{s}; F_1) & \text{if } p^{mc}(\bar{s}; F_1) = \frac{E_1[v(s)]}{1+r} \end{cases}. \quad (\text{A.5})$$

(ii) Suppose moderates become weakly more optimistic, i.e., consider their beliefs are changed to $\tilde{F}_0 \succeq_O F_0$ (which also satisfies $F_1 \succeq_O \tilde{F}_0$ so that assumption (O) continues to hold). Then:

(ii.1) The optimality curve $p^{opt}(\bar{s})$ shifts up pointwise, that is,

$$p^{opt}(\bar{s}; \tilde{F}_0) \geq p^{opt}(\bar{s}; F_0) \text{ for each } \bar{s} \in [s^{\min}, s^{\max}].$$

(ii.2) The market clearing curve $p^{mc}(\bar{s})$ shifts up pointwise, that is,

$$p^{mc}(\bar{s}; \tilde{F}_1) \geq p^{mc}(\bar{s}; F_1) \text{ for each } \bar{s} \in [s^{\min}, s^{\max}].$$

Proof of Lemma 3. Part (i.1). Define the function $g : S \rightarrow R$ with

$$g(\bar{s}) = \tilde{E}_1[v(s) \mid s \geq \bar{s}] - E_1[v(s) \mid s \geq \bar{s}]. \quad (\text{A.6})$$

Note that $g(s^{\max}) = 0$, and note also that the statement in the lemma is equivalent to the following claim:

$$g(\bar{s}) \geq 0 \text{ for each } \bar{s} \in [s^{\min}, s^{\max}]. \quad (\text{A.7})$$

I will first find an upper bound for the derivative of $g(\bar{s})$ which I will then use to prove the claim in (A.7).

To put an upper bound on the derivative of $g(\bar{s})$, consider first the derivative of the conditional expectation $E_1[v(s) \mid s \geq \bar{s}]$ at some $\bar{s} \in [s^{\min}, s^{\max}]$. With some rearrangement, this derivative can be written as

$$\frac{d}{d\bar{s}} E_1[v(s) \mid s \geq \bar{s}] = \frac{f_1(\bar{s})}{1 - F_1(\bar{s})} (E_1[v(s) \mid s \geq \bar{s}] - v(\bar{s})).$$

Using this expression, the derivative of $g(\bar{s})$ can be written as

$$\begin{aligned} g'(\bar{s}) &= \frac{\tilde{f}_1(\bar{s})}{1 - \tilde{F}_1(\bar{s})} \left(\tilde{E}_1[v(s) \mid s \geq \bar{s}] - v(\bar{s}) \right) - \frac{f_1(\bar{s})}{1 - F_1(\bar{s})} (E_1[v(s) \mid s \geq \bar{s}] - v(\bar{s})) \\ &= \left(\frac{\tilde{f}_1(\bar{s})}{1 - \tilde{F}_1(\bar{s})} - \frac{f_1(\bar{s})}{1 - F_1(\bar{s})} \right) \left(\tilde{E}_1[v(s) \mid s \geq \bar{s}] - v(\bar{s}) \right) + \frac{f_1(\bar{s})}{1 - F_1(\bar{s})} g(\bar{s}), \end{aligned} \quad (\text{A.8})$$

where the second line follows by rearranging terms and substituting the definition of $g(\bar{s})$ from Eq. (A.6). Note that $\tilde{E}_1[v(s) \mid s \geq \bar{s}] - v(\bar{s}) > g(\bar{s})$ and the first term in Eq. (A.8) is always non-

of optimists according to the belief distribution \tilde{F}_1 .

positive (since $\tilde{F}_1 \succeq_O F_1$), which provides the following upper bound on the derivative of $g'(\bar{s})$:

$$g'(\bar{s}) \leq \frac{\tilde{f}_1(\bar{s})}{1 - \tilde{F}_1(\bar{s})} g(\bar{s}) \text{ for each } \bar{s} \in [s^{\min}, s^{\max}]. \quad (\text{A.9})$$

Next, to prove the claim in (A.7), suppose the contrary, that is, suppose there exists $\tilde{s} < s^{\max}$ such that $g(\tilde{s}) < 0$. Consider next

$$\hat{s} = \sup \{ \bar{s} \in [\tilde{s}, s^{\max}) \mid g(\bar{s}) \leq g(\tilde{s}) \}.$$

Note that \hat{s} exists and that $g(\hat{s}) = g(\tilde{s}) < 0$ by the continuity of the function $g(\cdot)$. This further implies that $\hat{s} \neq s^{\max}$ since $g(s^{\max}) = 0$. Then, Eq. (A.9) applies for \hat{s} and implies $g'(\hat{s}) \leq \frac{\tilde{f}_1(\hat{s})}{1 - \tilde{F}_1(\hat{s})} g(\hat{s}) < 0$. This further implies that there exists $\bar{s} \in (\hat{s}, s^{\max})$ such that $g(\bar{s}) < g(\hat{s}) = g(\tilde{s})$, which contradicts the definition of \hat{s} . This proves the claim in (A.7) and completes the proof of the first part.

Part (i.2). Note, by Eq. (18), that the optimality curve $p^{opt}(\bar{s})$ can be written as

$$p^{opt}(\bar{s}) = \frac{1}{1+r} (E_0[v(s)] + (1 - F_0(\bar{s})) (E_1[v(s) \mid s \geq \bar{s}] - E_0[v(s) \mid s \geq \bar{s}])).$$

Then, using part (i.1) shows that $p^{opt}(\bar{s})$ shifts up pointwise, completing the proof.

Part (i.3). Consider the definition of $w^{\max}(\bar{s})$ in (13) and note that $w^{\max}(\bar{s})$ does not depend on F_1 , as it depends on moderates' valuation of debt contracts. Eq. (A.5) then follows by the definition of $p^{mc}(\bar{s})$ in (14). Intuitively, the change, $\tilde{F}_1 \succeq F_1$, only affects $p^{mc}(\bar{s})$ by increasing optimists' valuation. Thus, it only shifts the $p^{mc}(\bar{s})$ curve in case (ii) region of Eq. (14), while it leaves it constant in other cases.

Part (ii.1). Similar to part (i.1) of the lemma, define the function $g_{mix} : S \rightarrow R$ with

$$g_{mix}(\bar{s}) = p^{opt}(\bar{s}; \tilde{F}_0) - p^{opt}(\bar{s}; F_0).$$

Note that the statement in the lemma is equivalent to the claim:

$$g_{mix}(\bar{s}) \geq 0 \text{ for each } \bar{s} \in (s^{\min}, s^{\max}). \quad (\text{A.10})$$

I will prove a stronger claim, that

$$\frac{dg_{mix}(\bar{s})}{d\bar{s}} \geq 0 \text{ for each } \bar{s} \in (s^{\min}, s^{\max}), \quad (\text{A.11})$$

which implies the claim in (A.10) since $g_{mix}(s^{\min}) = 0$.

To prove the claim in (A.11), note that using Eq. (A.4) and rearranging terms, the derivative of $g_{mix}(\bar{s})$ can be written as

$$\frac{dg_{mix}(\bar{s})}{d\bar{s}} = \frac{1}{1+r} \left[\begin{array}{c} f_0(\bar{s}) - \tilde{f}_0(\bar{s}) \\ -\frac{f_1(\bar{s})}{1-F_1(\bar{s})} (\tilde{F}_0(\bar{s}) - F_0(\bar{s})) \end{array} \right] (E_1[v(s) \mid s \geq \bar{s}] - v(\bar{s})). \quad (\text{A.12})$$

Next note that $F_1 \succeq_O \tilde{F}_0 \succeq_O F_0$ implies $\frac{f_0(\bar{s})}{1-F_0(\bar{s})} \geq \frac{\tilde{f}_0(\bar{s})}{1-\tilde{F}_0(\bar{s})} \geq \frac{f_1(\bar{s})}{1-F_1(\bar{s})}$. After rearranging terms, this further implies

$$\frac{f_0(\bar{s}) - \tilde{f}_0(\bar{s})}{\tilde{F}_0(\bar{s}) - F_0(\bar{s})} \geq \frac{\tilde{f}_0(\bar{s})}{1 - \tilde{F}_0(\bar{s})} \geq \frac{f_1(\bar{s})}{1 - F_1(\bar{s})}.$$

Using this inequality in Eq. (A.12) and noting that $E_1[v(s) \mid s \geq \bar{s}] - v(\bar{s}) \geq 0$ proves the claim in (A.11), completing the proof of the lemma.

Part (ii.2). First note that, applying the argument in part (iii) of Lemma 2 for the distributions $\tilde{F}_0 \succeq_{\mathcal{O}} F_0$ implies

$$\tilde{E}_0[\min(v(s), v(\bar{s}))] \geq E_0[\min(v(s), v(\bar{s}))] \text{ for each } \bar{s} \in \mathcal{S}.$$

By Eq. (13), this further implies $w_1^{\max}(\bar{s}; \tilde{F}_0) \geq w_1^{\max}(\bar{s}; F_0)$. Using this inequality and the fact that $\tilde{E}_0[v(s)] \geq E_0[v(s)]$, Eq. (14) implies that $p^{mc}(\bar{s})$ shifts up pointwise, completing the proof.

A.2 Characterization of Quasi-equilibrium

This section completes the analysis of the quasi-equilibrium, by providing the proofs for Theorem 1 and Eq. (14).

Proof of Theorem 1. I prove the theorem in two steps. I first show that $\varphi = v(\bar{s}) \in \text{supp}(\mu_1^-)$ only if \bar{s} maximizes the leveraged return expression in (11). I then show that the problem has a unique solution characterized as the solution to Eq. (10). This establishes that μ_1^- is a Dirac measure at the contract $v(\bar{s})$, completing the sketch proof provided after the theorem statement.

To prove the first step, first note that optimists' debt contract choice can be restricted to $\varphi \in [v(s^{\min}), v(s^{\max})]$ without loss of generality, i.e., suppose $\mu_1^-(C) = 0$ for each $C \subset \mathbb{R}_+ \setminus [v(s^{\min}), v(s^{\max})]$.²³ Consider the change of notation $\tilde{s} = v^{-1}(\varphi)$ and let η denote the pushforward measure of μ_1^- over $S = [s^{\min}, s^{\max}]$, i.e.,

$$\eta(\tilde{S}) = \mu_1^-(v(\tilde{S})) \text{ for each Borel set } \tilde{S} \subset S. \quad (\text{A.13})$$

Using this notation, and after substituting the debt prices from Eq. (9), optimists' problem in a quasi-equilibrium can be written as:

$$\begin{aligned} \max_{x_1^A \geq 0, \eta \in M([s^{\min}, s^{\max}])} \quad & x_1^A E_1[v(s)] - \int_{[s^{\min}, s^{\max}]} E_1[\min(v(s), v(\tilde{s}))] d\eta(\tilde{s}), \\ \text{s.t.} \quad & px_1^A - \int_{[s^{\min}, s^{\max}]} \frac{E_0[\min(v(s), v(\tilde{s}))]}{1+r} d\eta(\tilde{s}) \leq w_1 + p\alpha_1, \\ & \int_{[s^{\min}, s^{\max}]} d\eta(\tilde{s}) \leq x_1^A. \end{aligned}$$

Optimists solve a linear optimization problem. At the optimum, the budget constraint binds. The collateral constraint also binds because (since $p > \frac{E_1[v(s)]}{1+r}$) optimists always prefer borrowing and investing in the asset to not borrowing. Then, letting λ denote the Lagrange multiplier for the budget constraint and γ the Lagrange multiplier for the collateral constraint, the first order conditions are given by:

²³This is because any safe contract with $\varphi < v(s^{\min})$ can be replicated by the alternative safe contract $v(s^{\min})$ (which has the additional benefit of using less collateral), and any contract $\varphi > v(s^{\max})$ that defaults in all states can be replicated by the contract $v(s^{\max})$.

$$\lambda \frac{E_0 [\min (v (s), v (\tilde{s}))]}{1+r} \leq E_1 [\min (v (s), v (\tilde{s}))] + \gamma \quad (\text{A.14})$$

with equality if $\tilde{s} \in \text{supp}(\eta)$.

Moreover, the first order condition with respect to x_1^A leads to

$$\gamma = \lambda p - E_1 [v (s)].$$

Plugging this expression for γ into (A.14) yields the following first order condition:

$$R_1^L (\tilde{s}) = \frac{E_1 [v (s)] - E_1 [\min (v (s), v (\tilde{s}))]}{p - E_0 [\min (v (s), v (\tilde{s}))] / (1+r)} \leq \lambda, \quad (\text{A.15})$$

with strict inequality only if $\tilde{s} \in \text{supp}(\eta)$.

This equation implies that any $\bar{s} \in \text{supp}(\eta)$ maximizes $R_1^L (\tilde{s})$, completing the first step of the proof.

As the second step, I show that problem (11) has a unique solution, and I characterize the solution. To this end, consider the derivative of $R_1^L (\tilde{s})$, which can be written as

$$\frac{d}{d\tilde{s}} R_1^L (\tilde{s}) = \frac{1}{p - \frac{E_0 [\min (v (s), v (\tilde{s}))]}{1+r}} \left(\frac{R_1^L (\tilde{s})}{1+r} (1 - F_0 (\tilde{s})) - (1 - F_1 (\tilde{s})) \right). \quad (\text{A.16})$$

Note that

$$R_1^L (s^{\min}) = \frac{E_1 [v (s)] - v (s^{\min})}{p - v (s^{\min}) / (1+r)} > 0 \text{ and } R_1^L (s^{\max}) = \frac{E_1 [v (s)] - E_1 [v (s)]}{p - E_0 [v (s)] / (1+r)} = 0.$$

Thus, the derivative in (A.16) satisfies the boundary conditions

$$\frac{d}{d\tilde{s}} R_1^L (\tilde{s}) \Big|_{\tilde{s}=s^{\min}} > 0 \text{ and } \frac{d}{d\tilde{s}} R_1^L (\tilde{s}) \Big|_{\tilde{s}=s^{\max}} < 0. \quad (\text{A.17})$$

Eq. (A.16) also leads to the first order condition

$$\frac{d}{d\tilde{s}} R_1^L (\tilde{s}) = 0 \text{ for } \tilde{s} \in [s^{\min}, s^{\max}) \text{ iff } \frac{R_1^L (\tilde{s})}{1+r} = \frac{1 - F_1 (\tilde{s})}{1 - F_0 (\tilde{s})}. \quad (\text{A.18})$$

Plugging this first order condition into (11) and rearranging terms yields $p = p^{opt} (\tilde{s})$. By Lemma 2, $p^{opt} (\tilde{s})$ is strictly decreasing, which implies that there exists exactly one $\bar{s} \in S$ (the solution to $p = p^{opt} (\bar{s})$) that satisfies the first order condition in (A.18). By the boundary conditions in (A.17) and the continuity of $\frac{d}{d\tilde{s}} R_1^L (\tilde{s})$, it follows that $R_1^L (\tilde{s})$ has a unique maximum characterized as the solution to Eq. (10). This establishes the second step, and completes the proof of Theorem 1.

Proof of Eq. (14). Consider optimists' budget constraint (5) and note that:

$$\begin{aligned}
w_1 &= p(x_1^A - \alpha_1) + x_1^B + \int_{\varphi} q(\varphi) d\mu_1^-(\varphi) \\
&= p(x_1^A - \alpha_1) + x_1^B + q(\bar{s}) \int_{\varphi} d\mu_1^-(\varphi) \\
&= p(x_1^A - \alpha_1) + x_1^B + q(\bar{s})x_1^A,
\end{aligned}$$

where the second line uses the fact that μ^- is a Dirac measure at \bar{s} , and the last line uses the fact that optimists' collateral constraint (6) binds. Substituting contract prices from Eq. (15), the previous displayed equation implies the period 1 flow of funds constraint:

$$p(x_1^A - \alpha_1) + x_1^B = w_1 + \frac{E_0[\min(v(s), v(\bar{s}))]}{1+r}x_1^A. \quad (\text{A.19})$$

Next note that optimists choose $x_1^B = 0$ except for the corner case $p = \frac{E_1[v(s)]}{1+r}$. In view of this observation, Eq. (A.19) characterizes optimists' demand for the asset. Recall also that moderates choose $x_1^A = 0$ except for the corner case $p = \frac{E_0[v(s)]}{1+r}$, which characterizes moderates' demand for the asset. Finally, recall that the asset market clearing condition is given by $x_1^A + x_0^A = 1$.

There are three cases to consider. First consider case (ii), i.e., suppose the market clearing price is given by some $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$. In this case, moderates demand x_0^A , which implies that optimists' demand must be given by $x_1^A = 1$. Using this in Eq. (A.19) (along with $x_1^B = 0$), the market clearing price is solved as:

$$p = \frac{1}{\alpha_0} \left(w_1 + \frac{E_0[\min(v(s), v(\bar{s}))]}{1+r} \right) = \frac{w_1^{\max}(\bar{s})}{\alpha_0}. \quad (\text{A.20})$$

As long as the expression $\frac{w_1^{\max}(\bar{s})}{\alpha_0}$ lies inside $\left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$, p is indeed the market clearing price, proving case (ii). If $\frac{w_1^{\max}(\bar{s})}{\alpha_0} \geq \frac{E_1[v(s)]}{1+r}$, then the market clearing price is $p = \frac{E_1[v(s)]}{1+r}$ (along with allocations $x_1^A = 1$ and $x_1^B \geq 0$), proving case (iii). Finally, if $\frac{w_1^{\max}(\bar{s})}{\alpha_0} \leq \frac{E_0[v(s)]}{1+r}$, then the market clearing price is $p = \frac{E_0[v(s)]}{1+r}$ (along with allocations $x_1^A < 1$ and $x_1^B = 0$), proving case (i). This completes the proof of Eq. (14).

Analytical Characterization of Equilibrium I next provide an analytical characterization of the quasi-equilibrium described by $p = p^{opt}(\bar{s}) = p^{mc}(\bar{s})$, which will be useful for some of the subsequent proofs. Note that if optimists' wealth is not too large, in particular, if

$$w_1 < \frac{\alpha_0 E_1[v(s)] - v(s^{\min})}{1+r}, \quad (\text{A.21})$$

then the two curves intersect in the case (ii) region of Eq. (14) and the equilibrium pair (p, \bar{s}^*) is characterized as follows:

$$p = p^{mc}(\bar{s}^*) = \frac{1}{\alpha_0} \left(w_1 + \frac{1}{1+r} E_0[\min(v(s), v(\bar{s}^*))] \right), \quad (\text{A.22})$$

where \bar{s}^* is the unique solution to

$$G(\bar{s}^*) \equiv \alpha_0 \frac{1 - F_0(\bar{s}^*)}{1 - F_1(\bar{s}^*)} \int_{\bar{s}^*}^{s^{\max}} (v(s) - v(\bar{s}^*)) dF_1 - \alpha_1 E_0[\min(v(s), v(\bar{s}^*))] = w_1(1+r). \quad (\text{A.23})$$

In this case, optimists take loans with riskiness $\bar{s}^* \in (s^{\min}, s^{\max})$ and the price satisfies $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$. Note also that the function $G(\cdot)$ in Eq. (A.23) is differentiable and strictly decreasing, which implies that there is a unique solution to Eq. (A.23).

If the opposite of condition (A.21) holds, then the two curves intersect in the case (i) region of Eq. (14). In this case, optimists' financial constraints are not binding, they borrow with a safe loan (with riskiness $\bar{s}^* = s^{\min}$) and they bid up the asset price to the optimistic valuation, i.e., $p = \frac{E_1[v(s)]}{1+r}$. This analysis also verifies that the two curves never intersect in case (iii) region of Eq. (14), which implies that the equilibrium price satisfies $p > \frac{E_0[v(s)]}{1+r}$. This completes the analytical characterization of equilibrium.

A.3 Characterization of Collateral Equilibrium

This section completes the characterization of the collateral equilibrium by providing the proof of Theorem 2.

Proof of Theorem 2. As the first step, I show that the prices and allocations in Theorem 2 constitute a collateral equilibrium. I next prove the essential uniqueness of the collateral equilibrium.

Existence of the Collateral Equilibrium. I claim that the allocation in Theorem 2 constitutes a collateral equilibrium. The analysis for the corner price $p = \frac{E_1[v(s)]}{1+r}$ is straightforward. Therefore, suppose that the asset price satisfies $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$. Eq. (14) ensures that the asset market clearing condition is satisfied. Hence, all that remains to check is that loan market is in equilibrium. This amounts to checking that debt contract choices are optimal for traders after relaxing the restrictions $\mu_0^- = 0$ and $\mu_1^+ = 0$, and that debt contract prices clear the market.

I next establish an easy-to-check condition for equilibrium in the loan market. Recall that moderates' rate of return on capital is given by $1+r$, while optimists' rate of return on capital is given by $R_1^L(\bar{s}^*) > 1+r$ (cf. Eq. (11)), where the inequality follows since $p < \frac{E_1[v(s)]}{1+r}$. Given the rates of return $1+r$ and $R_1^L(\bar{s}^*)$, consider traders' bid prices for each debt contract $\varphi \in R_+$, defined as:

$$q_0^{\text{bid}}(\varphi) = \frac{E_0[\min(v(s), \varphi)]}{1+r} \quad \text{and} \quad q_1^{\text{bid}}(\varphi) = \frac{E_1[\min(v(s), \varphi)]}{R_1^L(\bar{s}^*)}. \quad (\text{A.24})$$

Note that these are the prices that would make moderates (resp. optimists) indifferent between holding a debt contract φ and holding their equilibrium portfolio.

Similarly, consider the ask prices for a debt contract φ that would make the traders indifferent between selling the debt contract φ and holding their equilibrium portfolio. There is a slight complication because, to be able to short sell the contract φ , the trader must also hold 1 unit of the asset. Hence, consider the cross investment strategy of short selling one unit of contract φ and buying one unit of the asset. Let $q_i^{\text{ask}}(\varphi)$ denote the price that makes type i traders indifferent between pursuing this strategy and holding their equilibrium portfolio. Note that $q_0^{\text{ask}}(\varphi)$ and $q_1^{\text{ask}}(\varphi)$ are respectively

defined as the solutions to:

$$\frac{E_0[v(s)] - E_0[\min(v(s), \varphi)]}{p - q_0^{ask}(\varphi)} = 1 + r \text{ and } \frac{E_1[v(s)] - E_1[\min(v(s), \varphi)]}{p - q_1^{ask}(\varphi)} = R_1^L(\bar{s}^*). \quad (\text{A.25})$$

The bid and ask prices in (A.24) and (A.25) can also be used to define the aggregate bid and ask price for the contract φ , given by:

$$q^{bid}(\varphi) = \max_i q_i^{bid}(\varphi) \text{ and } q^{ask}(\varphi) = \min_i q_i^{ask}(\varphi).$$

Note that, if the price of a contract φ is below $q^{bid}(\varphi)$, a trader would demand infinite units of the contract, which would violate market clearing. Similarly, if the price is above $q^{ask}(\varphi)$, a trader would sell infinite units, which would again violate market clearing. Moreover, non-zero trade in a contract requires at least one type of trader to buy the contract and another type of trader to sell, which can happen only if $q^{bid}(\varphi) = q^{ask}(\varphi)$. It follows that the loan market is in equilibrium if and only if debt contract prices and allocations satisfy the following condition:

$$\left\{ \begin{array}{l} q^{bid}(\varphi) \leq q(\varphi) \leq q^{ask}(\varphi), \text{ and} \\ q(\varphi) = q^{bid}(\varphi) = q^{ask}(\varphi) \text{ whenever } \varphi \in \text{supp}(\mu_i^-) \text{ for some } i. \end{array} \right. \quad (\text{A.26})$$

I next show that the loan market allocation of Theorem 2 satisfies the loan market equilibrium condition (A.26). In particular, I claim:

$$q^{bid}(\varphi) \leq q^{ask}(\varphi) \text{ with equality iff } \varphi = v(\bar{s}^*). \quad (\text{A.27})$$

Note that the debt contract prices of Theorem 2 (cf. Eq. (15)) are chosen such that $q(\varphi) = q^{bid}(\varphi)$. Moreover, the allocations are such that there is trade only for contract $\varphi = v(\bar{s}^*)$. Hence, the claim in (A.27) implies (A.26), which ensures that the loan market is indeed in equilibrium.

Note that the claim in (A.27) is true for all $\varphi \in R_+$, if it is true for all $\varphi \in [v(s^{\min}), v(s^{\max})]$. To prove the claim for the relevant set of debt contracts, $\varphi = v(\tilde{s})$ for some $\tilde{s} \in \mathcal{S}$, first note that

$$q_i^{bid}(v(\tilde{s})) < q_i^{ask}(v(\tilde{s})) \text{ for each } \tilde{s} \in \mathcal{S} \text{ and } i. \quad (\text{A.28})$$

which is straightforward to check. There is a wedge between each type traders' bid and ask prices, intuitively because buying the debt contract has no collateral requirements while selling the debt contract requires the trader to pledge collateral (and thus, the traders' ask price to sell a contract is higher).

Second, note that \bar{s}^* is the unique solution to problem (7) by definition, and thus

$$R_1^L(\bar{s}^*) = \frac{E_1[v(s)] - E_1[\min(v(s), v(\bar{s}^*))]}{p - E_0[\min(v(s), v(\bar{s}^*))]/(1+r)} > \frac{E_1[v(s)] - E_1[\min(v(s), v(\tilde{s}))]}{p - E_0[\min(v(s), v(\tilde{s}))]/(1+r)} \text{ for each } \tilde{s} \neq \bar{s}^*.$$

Using this inequality and the definition of $q_1^{ask}(v(\tilde{s}))$ in (A.25) shows

$$q_1^{ask}(v(\tilde{s})) \geq \frac{E_0[\min(v(s), v(\tilde{s}))]}{1+r} = q_0^{bid}(v(\tilde{s})) \text{ with equality iff } \tilde{s} = \bar{s}^*. \quad (\text{A.29})$$

Third, recall that $\frac{E_1[\min(v(s), v(\tilde{s}))]}{E_0[\min(v(s), v(\tilde{s}))]}$ is equal to 1 for $\tilde{s} = s^{\min}$, and is strictly increasing in \tilde{s} . By Eq. (A.24), it follows that $\frac{q_1^{bid}(v(s^{\min}))}{q_0^{bid}(v(s^{\min}))} = \frac{1+r}{R_L(\bar{s})} < 1$, and that $\frac{q_1^{bid}(v(\tilde{s}))}{q_0^{bid}(v(\tilde{s}))}$ is strictly increasing in \tilde{s} . Then, there are two cases to consider. As the first case, $\frac{q_1^{bid}(v(\tilde{s}))}{q_0^{bid}(v(\tilde{s}))}$ may never exceed 1, that is, it may be the

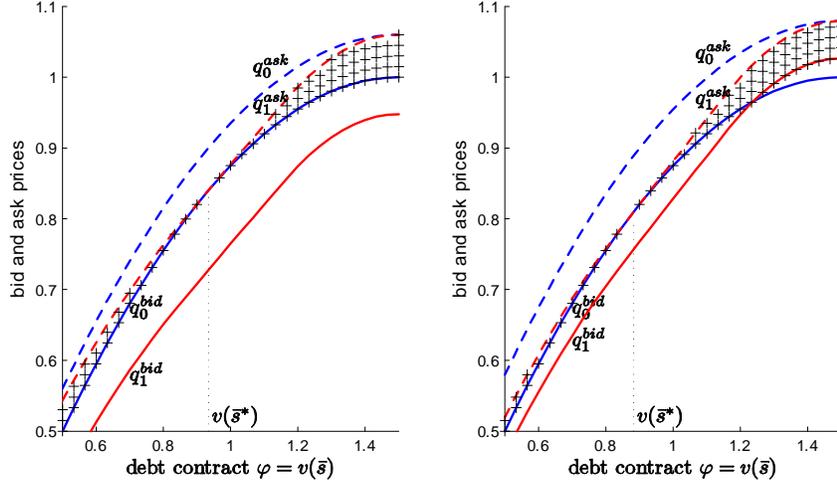


Figure 8: The left panel displays the bid and ask prices for the case in which the inequality in (A.30) holds, and the right panel displays the case in which the inequality in (A.30) fails. The shaded areas display the set of all possible equilibrium debt contract prices in each case.

case that

$$q_1^{bid}(v(\tilde{s})) < q_0^{bid}(v(\tilde{s})) \text{ for each } \tilde{s} \in \mathcal{S}. \quad (\text{A.30})$$

In this case, combining Eqs. (A.28),(A.29) and (A.30) proves the claim in (A.27). The left panel of Figure 8 plots the bid and ask prices in this first case. The figure illustrates that, in this case, the quasi-equilibrium debt prices in (9) and the collateral equilibrium debt prices in (9) are identical.

As the second case, $\frac{q_1^{bid}(v(\tilde{s}))}{q_0^{bid}(v(\tilde{s}))}$ may exceed 1 for sufficiently large \tilde{s} . That is, it may be the case that there exists \tilde{s}^{cross} such that

$$\begin{cases} q_1^{bid}(v(\tilde{s})) < q_0^{bid}(v(\tilde{s})) \text{ for all } \tilde{s} < \tilde{s}^{cross}, \\ q_1^{bid}(v(\tilde{s})) \geq q_0^{bid}(v(\tilde{s})) \text{ for all } \tilde{s} \geq \tilde{s}^{cross}. \end{cases} \quad (\text{A.31})$$

Note that, in this case, \tilde{s}^{cross} is uniquely defined as the solution to

$$\frac{E_1[\min(v(s), v(\tilde{s}^{cross}))]}{E_0[\min(v(s), v(\tilde{s}^{cross}))]} = \frac{R_L(\tilde{s}^*)}{1+r}.$$

Moreover, it can be checked that $\frac{E_1[\min(v(s), v(\tilde{s}^*))]}{E_0[\min(v(s), v(\tilde{s}^*))]} < \frac{R_L(\tilde{s}^*)}{1+r}$,²⁴ which implies $\tilde{s}^{cross} > \tilde{s}^*$. It can also

²⁴To see this, consider the leveraged return expression (11), which can be rewritten as

$$p - \frac{E_0[\min(v(s), v(\tilde{s}^*))]}{1+r} = \frac{E_1[s]}{R_1^L(\tilde{s}^*)} - \frac{E_1[\min(v(s), v(\tilde{s}^*))]}{R_1^L(\tilde{s}^*)}.$$

Note that $R_1^L(\tilde{s}^*) > \frac{E_1[v(s)]}{p}$ because optimists always have the option of buying the asset without borrowing. Hence, the previous inequality implies $\frac{E_1[\min(v(s), v(\tilde{s}^*))]}{1+r} < \frac{E_0[\min(v(s), v(\tilde{s}^*))]}{R_1^L(\tilde{s}^*)}$, which can be rewritten as $\frac{E_1[\min(v(s), v(\tilde{s}^*))]}{E_0[\min(v(s), v(\tilde{s}^*))]} < \frac{1+r}{R_1^L(\tilde{s}^*)}$.

be seen that²⁵

$$q_0^{ask}(v(\tilde{s})) \geq q_1^{ask}(v(\tilde{s})) \text{ for each } \tilde{s} \geq \tilde{s}^{cross}. \quad (\text{A.33})$$

Then, using Eqs. (A.28), (A.29) and (A.31), it follows that $q^{bid}(v(\tilde{s})) \leq q^{ask}(v(\tilde{s}))$ for each $\tilde{s} \leq \tilde{s}^{cross}$, with equality iff $\tilde{s} = \bar{s}^*$. Moreover, using Eqs. (A.28), (A.29), (A.33) and (A.31), it also follows that $q^{bid}(\tilde{s}) < q^{ask}(\tilde{s})$ for each $\tilde{s} \geq \tilde{s}^{cross}$. This completes the proof of claim (A.27), and establishes that the allocation characterized in Theorem 2 is indeed a collateral equilibrium. The right panel of Figure 8 plots the bid and ask prices in this second case. This figure illustrates that, in this case, the quasi-equilibrium debt prices in (9) and the collateral equilibrium debt prices in (9) are not the same, but the difference in prices does not overturn the optimality of the debt contract \bar{s}^* .

Figure 8 also illustrates that the debt contract prices are not uniquely determined in equilibrium (except for the price of the optimal contract $v(\bar{s}^*)$, which is uniquely determined). In particular, any price function $q(\cdot)$ such that $q(v(\tilde{s})) \in [q^{bid}(v(\tilde{s})), q^{ask}(v(\tilde{s}))]$ can support the equilibrium allocation in equilibrium. However, the equilibrium allocations in the loan market and the equilibrium asset price p is uniquely determined, as I next prove.

Essential Uniqueness of Collateral Equilibrium. I first prove that the equilibrium asset price p is uniquely determined. Let R_0 and R_1 denote traders' equilibrium rates of return on capital (in the above equilibrium, $R_0 = 1 + r$ and $R_1 = R_1^L(\bar{s})$). Since traders always have the option to invest in the bond, R_0 and R_1 are always weakly greater than $1 + r$. Moreover, in equilibrium some investors must agree to hold the bond, which implies that either R_0 or R_1 must be equal to $1 + r$. Since optimists have a greater valuation of the asset, the equilibrium rates of return satisfy $R_1 \geq R_0 = 1 + r$.

I next claim that, for any given price $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$, optimists' rate of return is uniquely determined as $R_1 = R_1^L(\bar{s})$, and the loan market allocations are uniquely determined by Theorem 1. To prove this claim, consider optimists' bid and ask prices in (A.24) and (A.25) for an arbitrary price level $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$ and an arbitrary required rate of return R_1 (i.e., replace $R_1^L(\bar{s}^*)$ in these expressions with R_1). Eq. (A.25) shows that $q_1^{ask}(v(\bar{s}))$ increases in the required rate of return (and Eq. (A.24) shows $q_1^{bid}(v(\bar{s}))$ decreases in the required rate of return). It follows that the loan market is at equilibrium for a unique required rate of return such that $q_1^{ask}(v(\tilde{s})) \geq q_0^{bid}(v(\tilde{s}))$ for all \tilde{s} with equality for exactly one state \bar{s} (as displayed in Figure (8)). Then, this rate of return R_1 satisfies:

$$\begin{aligned} \frac{E_1[v(s)] - E_1[\min(v(s), v(\bar{s}))]}{p - q_0^{bid}(v(\bar{s}))} &= R_1 = \frac{E_1[v(s)] - E_1[\min(v(s), v(\tilde{s}))]}{p - q_1^{ask}(v(\tilde{s}))} \\ &\geq \frac{E_1[v(s)] - E_1[\min(v(s), v(\bar{s}))]}{p - q_0^{bid}(v(\bar{s}))}, \end{aligned} \quad (\text{A.34})$$

²⁵Note that from the definition of $q_i^{ask}(v(\tilde{s}))$ in (A.25), the inequality in (A.33) is equivalent to

$$\frac{E_0[v(s)] - E_0[\min(v(s), v(\tilde{s}))]}{1+r} \leq \frac{E_1[v(s)] - E_1[\min(v(s), v(\tilde{s}))]}{R_1^L(\bar{s}^*)} \text{ for each } \tilde{s} \geq \tilde{s}^{cross}. \quad (\text{A.32})$$

Recall that $\frac{E_1[v(s)]}{E_0[v(s)]} \geq \frac{E_1[\min(v(s), v(\bar{s}))]}{E_0[\min(v(s), v(\bar{s}))]}$, which implies that

$$\frac{E_1[s] - E_1[\min(v(s), v(\bar{s}))]}{E_0[s] - E_0[\min(v(s), v(\bar{s}))]} \geq \frac{E_1[\min(v(s), v(\bar{s}))]}{E_0[\min(v(s), v(\bar{s}))]} \geq \frac{R_1^L(\bar{s}^*)}{1+r},$$

where the last inequality holds for each $\tilde{s} \geq \tilde{s}^{cross}$ in view of the definition of \tilde{s}^{cross} . This proves the inequality in (A.33), which in turn shows the inequality in (A.33).

where the first equality uses the definition of $q_1^{ask}(v(\bar{s}))$ and the fact that $q_1^{ask}(v(\bar{s})) = q_0^{bid}(v(\bar{s}))$, the second equality uses the definition of $q_1^{ask}(v(\tilde{s}))$, and the last inequality follows from the inequality $q_1^{ask}(v(\tilde{s})) \geq q_0^{bid}(v(\tilde{s}))$. The comparison between the first and the last terms in (A.34) shows that $\bar{s} \in S$ solves problem (11). In particular, for any price $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$, the unique loan market allocation is the same as the quasi-equilibrium loan market allocation characterized by Theorem 1, and the unique rate of return that equilibrates the loan market is given by $R_1 = R_1^L(\bar{s})$.

I next prove the uniqueness of the collateral equilibrium price p . For any price $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$, optimists loan market allocation is uniquely determined. This implies that optimists' leveraged investment on the asset (i.e., their demand for the asset) is uniquely determined. Combining this with the asset market clearing condition, Eq. (14), the equilibrium price p is also uniquely determined.

The above analysis also establishes that the price of the optimal contract, $q(v(\bar{s}^*))$, and the equilibrium allocations are uniquely determined for any price $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$. For the corner price $p = \frac{E_1[v(s)]}{1+r}$ (which corresponds to the optimal contract $\bar{s}^* = s^{\min}$), the price of the optimal contract $q(v(s^{\min}))$ is still uniquely determined. However, in this case, the equilibrium allocations are not necessarily unique since optimists may be indifferent among some of the safe contracts $\varphi \leq v(s^{\min})$. This establishes the essential uniqueness of the collateral equilibrium and completes the proof of Theorem 2.

A.4 Comparative Statics with Respect to Belief Heterogeneity

Proof of Theorem 3. Part (i). Similar to the proof of Lemma 3, define the function $g : S \rightarrow R$ with

$$g(\bar{s}) = \tilde{E}_1[v(s) \mid s \geq \bar{s}] - E_1[v(s) \mid s \geq \bar{s}].$$

Note that $g(s^{\min}) = 0$ since $\tilde{E}_1[v(s)] = E_1[v(s)]$, and note also that $g(s^{\max}) = 0$. I claim that

$$g(\bar{s}) \geq 0 \text{ for all } \bar{s} \in (s^{\min}, s^{\max}), \quad (\text{A.35})$$

which implies Eq. (19) in the main text. Most of the proof then follows by the argument provided after Theorem 3. For the comparative statics of the leverage ratio, substitute Eq. (A.22) into (16) to get:

$$L = \frac{p}{p - (p\alpha_0 - w_1)} = \frac{1}{1 - \alpha_0 + \frac{w_1}{p}}.$$

Since p weakly increases, the leverage ratio also weakly increases, completing the proof conditional on the claim in (A.35).

To prove the claim in (A.35), first note that Eq. (A.8) applies also in this setting. Since $\frac{\tilde{f}_1(\bar{s})}{1 - \tilde{F}_1(\bar{s})} \leq \frac{f_1(\bar{s})}{1 - F_1(\bar{s})}$ over the range $\bar{s} \in (s^R, s^{\max})$, the same argument used in the proof of part (i) of Lemma 3 shows that

$$g(\bar{s}) \geq 0 \text{ for all } \bar{s} \in [s^R, s^{\max}). \quad (\text{A.36})$$

Second, suppose, to reach a contradiction, that there exists $\tilde{s} \in [s^{\min}, s^R)$ with $g(\tilde{s}) < 0$. Since $g(s^{\min}) = 0$, this further implies that there exists $\hat{s} \in [s^{\min}, \tilde{s})$ such that $g(\hat{s}) = 0$ and $g'(\hat{s}) < 0$, since otherwise, the differentiable function $g(\cdot)$ could not become negative over the range $[s^{\min}, \tilde{s})$.

Considering Eq. (A.8) for $\bar{s} = \hat{s}$ and using $g(\hat{s}) = 0$ implies

$$g'(\hat{s}) = \left(\frac{\tilde{f}_1(\hat{s})}{1 - \tilde{F}_1(\hat{s})} - \frac{f_1(\hat{s})}{1 - F_1(\hat{s})} \right) \left(\tilde{E}_1[v(s) \mid s \geq \hat{s}] - v(\hat{s}) \right) \geq 0,$$

where the inequality follows since $\frac{\tilde{f}_1(\hat{s})}{1 - \tilde{F}_1(\hat{s})} \geq \frac{f_1(\hat{s})}{1 - F_1(\hat{s})}$ (as $\hat{s} < s^R$). Since $g'(\hat{s}) < 0$ by the choice of \hat{s} , the previous displayed inequality yields a contradiction, completing the proof.

Part (ii). Applying the proof of part (i) for distributions $F_0 \succeq_R \tilde{F}_0$ shows that

$$E_0[v(s) \mid s \geq \bar{s}] \geq \tilde{E}_0[v(s) \mid s \geq \bar{s}] \text{ for each } \bar{s} \in \mathcal{S}. \quad (\text{A.37})$$

Note also that $F_0 \succeq_{R, \bar{s}^*} \tilde{F}_0$ implies $\frac{1 - \tilde{F}_0(\bar{s})}{1 - F_0(\bar{s})}$ is weakly increasing for $\bar{s} \in (s^{\min}, \bar{s}^*)$, which further implies $\tilde{F}_0(\bar{s}) \leq F_0(\bar{s})$ over this range. In view of this observation and Eq. (A.37), Eq. (18) implies that

$$p^{opt}(\bar{s}) \text{ weakly increases for each } \bar{s} \in (s^{\min}, \bar{s}^*). \quad (\text{A.38})$$

Next consider the effect on the market clearing curve $p^{mc}(\bar{s})$. Note that since $F_0 \succeq_{R, \bar{s}^*} \tilde{F}_0$,

$$\frac{\tilde{f}_0(\bar{s})}{1 - \tilde{F}_0(\bar{s})} \leq \frac{f_0(\bar{s})}{1 - F_0(\bar{s})} \text{ for each } \bar{s} \in (s^{\min}, \bar{s}^*).$$

Then, the same steps as in the proof of part (ii.2) of Lemma 3 applies in this case and shows

$$p^{mc}(\bar{s}) \text{ weakly increases for each } \bar{s} \in (s^{\min}, \bar{s}^*). \quad (\text{A.39})$$

Using Eqs. (A.38) and (A.39) along with the fact that $p^{opt}(\bar{s})$ is a decreasing relation and $p^{mc}(\bar{s})$ is a weakly increasing relation, it follows that p is weakly greater at the new intersection point, completing the proof.²⁶

Proof of Theorem 4. Part (i). The fact that \tilde{F}_0 and F_0 are equally optimistic over (s^{\min}, \bar{s}^*) implies that $\frac{1 - \tilde{F}_0(\bar{s})}{1 - F_0(\bar{s})} = \frac{1 - \tilde{F}_0(s^{\min})}{1 - F_0(s^{\min})} = 1$, that is, $\tilde{F}_0(\bar{s}) = F_0(\bar{s})$ for each $\bar{s} \in (s^{\min}, \bar{s}^*)$. Then, Eq. (10) implies

$$p^{opt}(\bar{s}; \tilde{F}_0, \tilde{F}_1) = p^{opt}(\bar{s}; F_0, \tilde{F}_1) \text{ for each } \bar{s} \in (s^{\min}, \bar{s}^*).$$

By part (i) of Lemma 3, $\tilde{F}_1 \succeq_O F_1$ implies $p^{opt}(\bar{s}; F_0, \tilde{F}_1) \geq p^{opt}(\bar{s}; F_0, F_1)$. Using this along with the previous displayed equation shows Eq. (A.38), that is, $p^{opt}(\bar{s})$ weakly increases for each $\bar{s} \in (s^{\min}, \bar{s}^*)$.

Note also that $\tilde{E}_0[v(s)] \leq E_0[v(s)]$. Combining this with the fact that $\tilde{F}_0(\bar{s}) = F_0(\bar{s})$ for each $\bar{s} \in (s^{\min}, \bar{s}^*)$ implies that $p^{mc}(\bar{s})$ curve remains constant over (s^{\min}, \bar{s}^*) except for the fact that its

²⁶ On the other hand, the effect on \bar{s}^* is ambiguous. The effect on \bar{s}^* depends on how much the $p^{mc}(\bar{s})$ curve shifts up. If the effect on the $p^{mc}(\bar{s})$ curve is strong, perhaps because α_0 is small, then \bar{s}^* may decrease. To see the intuition for this result, suppose many units of the asset are already endowed to optimists, i.e. α_1 is high. Then the increase in lenders' valuation of debt contracts acts similar to a positive wealth shock to optimists (cf. Eq. (13)), because optimists can borrow more against the units they already own. This effect tends to lower the leverage ratio, and if sufficiently strong, it can overcome the effect from the shift of the $p^{opt}(\bar{s})$ curve which (similar to part (i)) tends to increase the leverage ratio.

lower bound “extends,” that is,

$$p^{mc}(\bar{s}; \tilde{F}_0, \tilde{F}_1) \begin{cases} = p^{mc}(\bar{s}) & \text{if } p^{mc}(\bar{s}) > \frac{E_0[v(s)]}{1+r} \\ \leq p^{mc}(\bar{s}) & \text{if } p^{mc}(\bar{s}) = \frac{E_0[v(s)]}{1+r} \end{cases} \quad \text{for each } \bar{s} \in (s^{\min}, \bar{s}^*).$$

Note also that the proof of Theorem 2 establishes that $p = p^{mc}(\bar{s}^*) > \frac{E_0[v(s)]}{1+r}$. Using this in the previous displayed equation implies that there exists $\varepsilon > 0$ such that $p^{mc}(\bar{s})$ remains constant in a neighborhood $\bar{s} \in (\bar{s}^* - \varepsilon, \bar{s}^*)$. Combining this with Eq. (A.38) and using the facts that $p^{opt}(\cdot)$ is a decreasing curve and $p^{mc}(\cdot)$ is an increasing curve shows that the new intersection point of these curves is weakly to the right of \bar{s}^* , which further implies p and \bar{s}^* weakly increase. The comparative statics for the leverage ratio follows by the same argument as in part (i) of Theorem 3.

Part (ii). First, I claim that F_1 and \tilde{F}_1 have the same distribution conditional on any upper-threshold event $[\bar{s}, s^{\max}]$ with $\bar{s} > \bar{s}^*$. That is, for any $\bar{s} \in (\bar{s}^*, s^{\max})$,

$$\frac{f_1(s)}{1 - F_1(\bar{s})} = \frac{\tilde{f}_1(s)}{1 - \tilde{F}_1(\bar{s})} \quad \text{for each } s \in [\bar{s}, s^{\max}]. \quad (\text{A.40})$$

To see this, note that by assumption

$$\frac{1 - \tilde{F}_1(s)}{1 - F_1(s)} = \frac{1 - \tilde{F}_1(\bar{s})}{1 - F_1(\bar{s})} \quad \text{for each } s \in [\bar{s}, s^{\max}]. \quad (\text{A.41})$$

Moreover, taking the derivative of this equation with respect to s implies

$$\frac{f_1(s)}{1 - F_1(s)} = \frac{\tilde{f}_1(s)}{1 - \tilde{F}_1(s)} \quad \text{for each } s \in [\bar{s}, s^{\max}]. \quad (\text{A.42})$$

Using Eqs. (A.41) and (A.42), it follows that, for each $s \in (\bar{s}, s^{\max})$,

$$\frac{f_1(s)}{1 - F_1(\bar{s})} = \frac{f_1(s)}{1 - F_1(s)} \frac{1 - F_1(s)}{1 - F_1(\bar{s})} = \frac{\tilde{f}_1(s)}{1 - \tilde{F}_1(s)} \frac{1 - \tilde{F}_1(s)}{1 - \tilde{F}_1(\bar{s})} = \frac{\tilde{f}_1(s)}{1 - \tilde{F}_1(\bar{s})},$$

proving Eq. (A.40).

Next consider the case in which condition (A.21) is violated so that $\bar{s}^* = s^{\min}$. Using Eq. (A.40) for $\bar{s} = \bar{s}^* + \varepsilon$ and taking $\varepsilon \rightarrow 0$ shows that, in this case, F_1 and \tilde{F}_1 are the same distribution. By the same argument, F_0 and \tilde{F}_0 are also the same distribution. It follows that the asset price p remains constant for this case.

Consider next the case in which condition (A.21) holds so that $\bar{s}^* > s^{\min}$. Consider respectively the shift in the $p^{opt}(\bar{s})$ and $p^{mc}(\bar{s})$ curves. First consider the optimality curve $p^{opt}(\bar{s})$ and note that Eq. (A.40) in Eq. (10) implies

$$p^{opt}(\bar{s}; \tilde{F}_0, \tilde{F}_1) = p^{opt}(\bar{s}; \tilde{F}_0, F_1) \quad \text{for each } \bar{s} \in (\bar{s}^*, s^{\max}).$$

By part (ii) of Lemma 3, $F_0 \succeq_O \tilde{F}_0$ implies $p^{opt}(\bar{s}; \tilde{F}_0, F_1) \leq p^{opt}(\bar{s}; F_0, F_1)$. Using this in the previous displayed equation shows

$$p^{opt}(\bar{s}) \text{ weakly decreases for each } \bar{s} \in (\bar{s}^*, s^{\max}). \quad (\text{A.43})$$

Next consider the market clearing curve $p^{mc}(\bar{s})$. Note that $\bar{s}^* > s^{\min}$ implies $p^{opt}(\bar{s}^*) = p^{mc}(\bar{s}^*) <$

$\frac{E_1[v(s)]}{1+r}$, thus by Eq. (A.5) (cf. part (ii) of Lemma 3), the increase in optimism of F_1 leaves $p^{mc}(\bar{s})$ unchanged in a neighborhood $(\bar{s}^*, \bar{s}^* + \varepsilon)$. Note also that the decrease in optimism of F_0 weakly decreases $p^{mc}(\bar{s})$ downwards pointwise. It follows that

$$p^{mc}(\bar{s}) \text{ weakly decreases for each } \bar{s} \in (\bar{s}^*, \bar{s}^* + \varepsilon). \quad (\text{A.44})$$

Combining Eqs. (A.43) and (A.44) and using the fact that $p^{opt}(\bar{s})$ is a decreasing curve while $p^{mc}(\bar{s})$ is an increasing curve, the asset price p is weakly lower at the new intersection point. This completes the proof of Theorem 4.

A.5 Collateral Equilibrium with Contingent Contracts

With contingent loans, traders solve the following analogue of problem (7):

$$\begin{aligned} \max_{x_i \geq 0; \mu_i^+, \mu_i^- \in M(\mathcal{D})} & \left[\begin{array}{l} x_i^A E_i[v(s)] + x_i^B (1+r) + \\ \int_{\mathcal{D}} E_i[\min(v(s), \varphi(s))] d\mu_i^+(\varphi) - \int_{\mathcal{D}} E_i[\min(v(s), \varphi(s))] d\mu_i^-(\varphi) \end{array} \right] \\ \text{s.t.} & \quad p x_i^A + x_i^B + \int_{\mathcal{D}} q(\varphi) d\mu_i^+(\varphi) - \int_{\mathcal{D}} q(\varphi) d\mu_i^-(\varphi) \leq w_i + p\alpha_i. \\ & \quad \int_{\mathcal{D}} d\mu_i^-(\varphi) \leq x_i^A. \end{aligned} \quad (\text{A.45})$$

The same analysis as in the proof of Theorem 1 applies and shows that optimists choose a debt contract, $[\varphi(s) \in [0, v(s)]]_{s \in S}$, that maximizes the leveraged return (21).

Proof of Theorem 5. I first claim that the leveraged return in (21) is maximized by the solution to equation $p = p^{opt, cont}(\bar{s})$. Second, I claim that the optimal leveraged return, $R_1^{L, cont}(\varphi_{\bar{s}} | p)$, is greater than $1+r$ if and only if $p < p^{\max}$. This implies that optimists make a leveraged investment in the asset if and only if $p < p^{\max}$, which completes the proof of the theorem.

To prove the first claim, note that the debt contracts can be restricted to the set such that $\varphi(s) \in [0, v(s)]$ for each $s \in S$. The same steps in the proof of Theorem 1 show that $\varphi \in \text{supp}(\mu_1^-)$ if and only if φ solves Problem (21). To characterize the solution to this problem, note that the derivative of $R_1^{L, general}(\varphi)$ with respect to $\varphi(s)$ is given by:

$$\frac{\partial R_1^{L, cont}(\varphi)}{\partial \varphi(s)} = \frac{-f_0(s)}{p - \frac{1}{1+r} \int_{s^{\min}}^{s^{\max}} \varphi(s) dF_0} \left(\frac{f_1(s)}{f_0(s)} - \frac{R_1^{L, cont}(\varphi)}{1+r} \right).$$

From this expression and assumption (MLRP), it follows that the derivative $\frac{\partial R_1^{L, cont}(\varphi)}{\partial \varphi(s)}$ satisfies a cutoff property. In particular, there exists a threshold state $\bar{s} \in S$ such that

$$\frac{\partial R_1^{L, cont}(\varphi)}{\partial \varphi(s)} \begin{cases} > 0 \text{ for each } s < \bar{s} \\ < 0 \text{ for each } s > \bar{s} \end{cases}.$$

Consequently, the optimal level of promise for each state $s \in S \setminus \{\bar{s}\}$ has a corner solution. Hence, the solution φ to Problem (21) has the form in Eq. (22), except potentially a Lebesgue measure zero of states. In particular, the contract specified in Eq. (22) is one optimal solution to Problem (21).

Moreover, the threshold state $\bar{s} \in S$ is characterized as the solution to $\frac{f_1(\bar{s})}{f_0(\bar{s})} = \frac{R_1^{L, cont}(\varphi)}{1+r}$, which after

using the form in Eq. (22) can be written as

$$\frac{\frac{1}{1+r} \int_{\bar{s}}^{s^{\max}} v(s) dF_1}{p - \frac{1}{1+r} \int_{s^{\min}}^{\bar{s}} v(s) dF_0} = \frac{f_1(\bar{s})}{f_0(\bar{s})}.$$

Rearranging this expression shows that the threshold state is characterized as the unique solution to $p = p^{opt,cont}(\bar{s})$, completing the proof of the first claim.

To prove the second claim, fix a price level p , consider the corresponding optimal threshold $\bar{s}(p)$, and note that the optimal leveraged return is given by:

$$R_1^{L,cont}(\varphi_{\bar{s}(p)} | p) = \frac{E_1[v(s)] - \int_{s^{\min}}^{\bar{s}(p)} v(s) dF_1}{p^{opt,cont}(\bar{s}(p)) - \frac{1}{1+r} \int_{s^{\min}}^{\bar{s}(p)} v(s) dF_0} = (1+r) \frac{f_1(\bar{s}(p))}{f_0(\bar{s}(p))}. \quad (\text{A.46})$$

Here, the first equality uses the fact that $p = p^{opt,cont}(\bar{s}(p))$ (since $\bar{s}(p)$ is optimal), and the second equality uses Eq. (24). Next, note that $p^{\max} = p^{opt,cont}(s^{cross})$ (cf. Eqs. (23) and (24)), and thus s^{cross} is the optimal threshold corresponding to price level p^{\max} . Using Eq. (A.46), this further implies that $R_1^{L,cont}(\varphi_{s^{cross}} | p^{\max}) = 1+r$. Since $R_1^{L,cont}(\varphi_{\bar{s}(p)} | p)$ is decreasing in p , it follows that the optimal leveraged return is greater than $1+r$ if and only if $p < p^{\max}$. Hence, optimists borrow and invest in the asset if $p < p^{\max}$, but they are indifferent between investing in the asset and the bond if $p > p^{\max}$. This completes the proof of the theorem.

A.6 Collateral Equilibrium with Short Selling

This appendix provides a proof of Theorem 6 in Section 6. It then uses the proof to provide an intuition for the asymmetric disciplining property (which is more complete than the intuition provided in the main text). The appendix ends by deriving the total expenditure on short sales, Eq. (30), which is used in the main text.

Similar steps as in the derivation of Theorem 1 show that the default threshold \bar{s}_{sh} for short contracts maximizes:

$$R_0^{short}(\tilde{s}) = \frac{v(\tilde{s}) - E_0[\min(v(s), v(\tilde{s}))]}{\frac{v(\tilde{s})}{1+r} - E_1[\min(v(s), v(\tilde{s}))] / \frac{E_1[v(s)]}{p}}. \quad (\text{A.47})$$

Eq. (26) corresponds to the first order condition for this maximization problem. Under assumption (MLRP), the unique solution to this equation maximizes the return in (A.47), completing the sketch proof of Theorem 6.

To interpret problem (A.47), note that $R_0^{short}(\tilde{s})$ is the return of short sellers from selling one unit of the short contract $\psi = v(\tilde{s})$. More specifically, short sellers receive

$$q^{short}(v(\tilde{s})) = E_1[\min(v(s), v(\tilde{s}))] / \frac{E_1[v(s)]}{p} \quad (\text{A.48})$$

from the sale of this contract, and they use this amount towards meeting the collateral requirement. However, they need to post a total of $\frac{v(\tilde{s})}{1+r}$ units of the consumption good as collateral. Thus, they pay the difference (the denominator of (A.47)) out of their wealth. In the next period, short sellers receive $v(\tilde{s})$ from the collateral that they have posted, and they expect to pay $E_0[\min(v(s), v(\tilde{s}))]$ on the promises they have made. This is because, short sellers return the asset if the realized state is below \tilde{s} , but they default on the short contract if the realized state is above \tilde{s} . In the latter scenario,

short sellers lose only the collateral that they have posted, which is worth $v(\tilde{s})$. Hence, short sellers' expected payment is given by $E_0[\min(v(s), v(\tilde{s}))]$.

Problem (A.47) captures the essential trade-off that short sellers are facing. Note that moderates' perceived interest rate on a short contract is given by:

$$1 + r_0^{per}(\tilde{s}) \equiv \frac{E_0[\min(v(s), v(\tilde{s}))]}{E_1[\min(v(s), v(\tilde{s}))]/\frac{E_1[v(s)]}{p}} = \frac{E_0[\min(v(s), v(\tilde{s}))]}{E_1[\min(v(s), v(\tilde{s}))]} \frac{E_1[v(s)]}{p}. \quad (\text{A.49})$$

This expression further implies that $r_0^{per}(\tilde{s}) < r$ for the equilibrium short contract $\tilde{s} = \bar{s}_{sh}$. Intuitively, short sellers expect to make a net positive return, $r - r_0^{per}(\bar{s}_{sh})$, by selling the short contract and buying the bond with the proceeds. Moreover, under assumption (MLRP), this return is increasing in the short threshold \tilde{s} . This is because, the higher \tilde{s} , the less often the short contract defaults, and the greater portion of the asset the short sellers effectively sell. On the other hand, problem (A.47) shows that a higher threshold \tilde{s} requires short sellers to post a greater amount of collateral, $\frac{v(\tilde{s})}{1+r}$. This restricts short sellers' ability to leverage the net return $r - r_0^{per}(\bar{s}_{sh})$. It follows that, when choosing \bar{s}_{sh} , short sellers trade off greater leverage against a lower net return. This trade-off is resolved by problem (A.47), and leads to the optimal short contract characterized by (26).

I next provide the intuition for why the function $p^{short}(\bar{s}_{sh})$ is decreasing in the default threshold \bar{s}_{sh} , and why it has the asymmetric disciplining property. Consider first the former statement, which is equal to saying that the default threshold \bar{s}_{sh} for the optimal short contract is decreasing in the asset price. Note that, by Eq. (A.49), a higher price p increases the wedge $r - r_0^{per}(\bar{s}_{sh})$ that short sellers expect to make. This incentivizes short sellers to leverage more, by choosing a lower default threshold \bar{s}_{sh} . Intuitively, as prices are higher, short sellers see a greater bargain in short selling and they leverage their short sales more.

Consider next the intuition for the asymmetric disciplining property of $p^{short}(\bar{s}_{sh})$. To understand this property, suppose the equilibrium default threshold is given by \bar{s}_{sh} , and consider how high the asset price should be (relative to the moderate valuation) to entice moderates to choose a short contract with this default threshold. If the belief heterogeneity is concentrated on states below \bar{s}_{sh} , then Eq. (A.49) reveals that the return wedge $r - r_0^{per}(\bar{s}_{sh})$ perceived by moderates is higher. Thus, the asset price does not need to be too high to entice moderates to choose the short contract with threshold \bar{s}_{sh} . Consequently, with this type of belief heterogeneity, the asset price is closer to the moderate valuation. In contrast, suppose the belief heterogeneity is concentrated more on the relative likelihood of states above \bar{s}_{sh} . In this case, Eq. (A.49) implies that the perceived return wedge $r - r_0^{per}(\bar{s}_{sh})$ is lower. Then, moderates are enticed to choose the threshold level \bar{s}_{sh} only if prices are sufficiently higher than the moderate valuation. Hence, optimism about the probability of states below \bar{s}_{sh} is disciplined more than optimism about the relative likelihood of states above \bar{s}_{sh} , as suggested by the form of $p^{short}(\bar{s}_{sh})$.

Finally, consider the derivation of the total expenditure on short sales, denoted by W^{short} . Note that $\frac{v(\bar{s}_{sh})}{1+r} - q^{short}\left(\frac{v(\tilde{s})}{1+r}\right)$ is the amount of wealth moderates need to allocate to sell one unit of the short contract $\psi = \frac{v(\bar{s}_{sh})}{1+r}$. Type T_3 moderates (that are able to short sell) have a total wealth of $\gamma_{sh}(w_0 + p\alpha_0)$. Thus, the total number of short contracts $\frac{v(\bar{s}_{sh})}{1+r}$ sold by moderates is given by $\frac{\gamma_{sh}(w_0 + p\alpha_0)}{\frac{v(\bar{s}_{sh})}{1+r} - q^{short}\left(\frac{v(\tilde{s})}{1+r}\right)}$. The total expenditure on short sales is then given by:

$$W^{short} = \frac{\gamma_{sh}(w_0 + p\alpha_0)}{\frac{v(\bar{s}_{sh})}{1+r} - q^{short}\left(\frac{v(\tilde{s})}{1+r}\right)} q^{short}\left(\frac{v(\tilde{s})}{1+r}\right).$$

Substituting for $q^{short} \left(\frac{v(\bar{s})}{1+r} \right)$ from Eq. (A.48), and rearranging terms yields the expression (30) for W^{short} .

A.7 Characterization of Dynamic Equilibrium

This section completes the characterization of the dynamic equilibrium analyzed in Section 7, by providing a proof of Theorem 7. I first note a preliminary lemma which is necessary for the proof of the theorem.

Note that the problem of traders in the dynamic economy (cf. (38)) is similar to their problem in the static economy (cf. (7)), with the only difference that the asset is not endowed to the current young generation. The next lemma uses this observation to show that a recursive collateral equilibrium can be constructed based on the analysis in Section 3. The result requires the condition

$$\omega_0 \geq \frac{1 + \varepsilon}{r - \varepsilon}, \quad (\text{A.50})$$

which ensures that young traders' endowment is sufficient to purchase the entire asset supply. Recall that a loan with riskiness \bar{s} is a debt contract $\varphi = v(a, \bar{s})$ that defaults if and only if the next period state is below the threshold level $\bar{s} \in S$.

Lemma 4. *Consider a dynamic economy with condition (A.50), and suppose there exists a collection of price and loan riskiness pairs, $(p(a) \in \mathbb{R}_+, \bar{s}^*(a) \in \mathcal{S})_{a \in \mathbb{R}_{++}}$, such that for each $a \in \mathbb{R}_{++}$, the pair $(p(a), \bar{s}^*(a))$ corresponds to the collateral equilibrium characterized in Theorem 2 for the static economy*

$$(\mathcal{S}; v(a, \cdot); \{F_i\}_i; \{w_i \equiv \omega_i a\}_i; \{\alpha_1 = 0, \alpha_0 = 1\}). \quad (\text{A.51})$$

Then, there exists a recursive collateral equilibrium in which, for each $a \in \mathbb{R}_{++}$, optimists make leveraged investments in the asset by borrowing through a single loan with riskiness $\bar{s}^(a)$ and the asset price is $p(a)$.*

Proof of Lemma 4. Let the tuple $\left(p(a), [q(a, \varphi)]_{\varphi \in \mathbb{R}}, (x_i^A(a), x_i^B(a), \mu_i^+(a), \mu_i^-(a))_{i \in \{1, 0\}} \right)_{a \in \mathbb{R}_{++}}$ be such that, the prices and allocations for each a correspond to the collateral equilibrium of the static economy in (A.51). I claim that this tuple corresponds to a dynamic equilibrium with a modified bond allocation for moderates, $\tilde{x}_0^B(a)$.

Note that optimists' problem (38) is equivalent to their problem (7) in this static economy, given prices

$$p = p(a) \text{ and } q(\varphi) = q(a, \varphi) \text{ for each } \varphi \in \mathbb{R}_+. \quad (\text{A.52})$$

Hence optimists' allocations are also optimal in the dynamic economy. Moderates' problem is slightly different since $\alpha_0 = 1$ in the static economy whereas $\alpha_0 = 0$ in the dynamic economy (as the asset is held by the old generation). Because of this difference, the allocation $(x_0^A(a), x_0^B(a), \mu_0^+(a), \mu_0^-(a))$ violates the budget constraint of moderates in the dynamic economy by an amount $p(a)$. Consider instead the modified bond allocation

$$\tilde{x}_0^B(a) \equiv x_0^B(a) - p(a) \geq 0 \text{ for each } a \in \mathbb{R}_{++}. \quad (\text{A.53})$$

Note that the allocation $(x_0^A(a), \tilde{x}_0^B(a), \mu_0^+(a), \mu_0^-(a))$ satisfies moderates' budget constraint. When this is the case, it can also be seen that this allocation solves Problem (38) given the prices in (A.52).²⁷

²⁷To see this, note that Eq. (A.53) implies $x_0^B(a) > p(a) > 0$ for the static economy in (A.51),

Hence, if the inequality in (A.53) is satisfied, then the conjectured tuple with the modified $\tilde{x}_0^B(a)$ constitutes an equilibrium of the dynamic economy. To verify the inequality in (A.53), consider moderates' budget constraint in the static economy

$$x_0^B(a) + \int_{\mathbb{R}_+} q(a, \varphi) d\mu_0^+(a, \varphi) = \omega_0 a + p(a), \quad (\text{A.54})$$

where the equality holds since $\alpha_0 = 1$ and $x_0^A(a) = 0$. Next note that

$$\begin{aligned} \int_{\mathbb{R}_+} q(a, \varphi) d\mu_0^+(a, \varphi) &\leq p(a) \int_{\mathbb{R}_+} d\mu_0^+(a, \varphi) \\ &= p(a) \int_{\mathbb{R}_+} d\mu_1^-(a, \varphi) \\ &\leq p(a) x_1^A(a) = p(a), \end{aligned}$$

where the first line follows from the inequality $q(a, \varphi) \leq p(a)$ (which follows from Eq. (9)), the second line uses the debt market clearing condition (8), the third line uses the collateral constraint (6), and the last line uses the asset market clearing condition $x_1^A(a) = 1$. Using the last displayed inequality, the budget constraint (A.54) implies

$$x_0^B(a) \geq \omega_0 a \geq \frac{1 + \varepsilon}{r - \varepsilon} a \geq p(a),$$

where the second inequality follows from condition (A.50), and the third inequality follows from the fact that $p(a)$ is weakly less than the unconstrained level $\frac{1 + \varepsilon}{r - \varepsilon} a$. It follows that $\tilde{x}_0^B(a)$ in (A.53) is positive, completing the proof of Lemma 4.

Lemma 4 reduces the characterization of the dynamic equilibrium to the static case, along with a fixed point argument (since the value function $v(a, \cdot)$ depends on the price function). I next use this characterization to prove Theorem 7.

Proof of Theorem 7. Plugging the conjecture, $p(a) = p_d a$ and $\bar{s}^*(a) = \bar{s}_d^* \in S$, into (37) implies that the value function is also linearly homogeneous. In particular, $v(a, s) = v_d(s \mid p_d) a$, where

$$v_d(s \mid p_d) = s(1 + p_d). \quad (\text{A.55})$$

Next note that using the conjecture in the characterization of the static equilibrium (cf. Eqs. (10) and (14)) and using linear homogeneity in a , the constants p_d, \bar{s}_d^* are characterized as the collateral equilibrium of the static economy $\mathcal{E}(p_d)$ in (39). In particular, (p_d, \bar{s}_d^*) is the unique solution to the following equations:²⁸

$$p_d = p^{opt}(\bar{s}_d; v_d(\cdot \mid p_d)) = p^{mc}(\bar{s}_d; v_d(\cdot \mid p_d)), \quad (\text{A.56})$$

where the notation $p^{opt}(\cdot; v_d(\cdot \mid p_d))$ denotes the function $p^{opt}(\cdot)$ evaluated with the value function $v_d(\cdot \mid p_d)$. Given a pair, (p_d, \bar{s}_d^*) , that solves (A.56), Lemma 4 implies that the conjectured allocation is an equilibrium.

which further implies that moderate traders are indifferent between holding bonds and debt contracts. As their budgets and bond holdings are reduced by the same amount $p(a)$, the allocation $(x_0^A(a), \tilde{x}_0^B(a), \mu_0^+(a), \mu_0^-(a))$ is optimal for moderates in the dynamic economy.

²⁸The notations $p^{opt}(\cdot; v), p^{mc}(\cdot; v)$ respectively denote the functions $p^{opt}(\cdot), p^{mc}(\cdot)$ evaluated with the particular value function $v(\cdot)$.

The remaining step is to characterize the solution to the fixed point equation (A.56). To this end, let $(P_d(\tilde{p}_d), S_d(\tilde{p}_d))$ denote the solution to (A.56) when the value function is given by $v_d(\cdot | \tilde{p}_d)$ (i.e., when the future price to dividend ratio is given by \tilde{p}_d). Then, the solution to (A.56) is a fixed point of the mapping $P_d(\cdot)$ over the interval $\left[p_d^{\min} = \frac{1}{r}, p_d^{\max} = \frac{1+\varepsilon}{r-\varepsilon}\right]$. I next claim that $P_d(\cdot)$ is strictly increasing over this interval, and it satisfies the boundary conditions

$$P_d(p_d^{\min}) > p_d^{\min} \text{ and } P_d(p_d^{\max}) \leq p_d^{\max}. \quad (\text{A.57})$$

This claim implies that that $P_d(\cdot)$ has a unique fixed point $p_d \in (p_d^{\min}, p_d^{\max}]$, which characterizes the dynamic equilibrium.

To prove the claim, I first show that the loan riskiness $S_d(\tilde{p}_d) \in [s^{\min}, s^{\max})$ is weakly increasing in \tilde{p}_d . There are two cases depending on condition (A.21). Using the value function $v_d(s | \tilde{p}_d) = s(1 + \tilde{p}_d)$ (cf. Eq. (A.55)), $\alpha_0 = 1$, and $E_1[v(s)] = 1 + \varepsilon$, condition (A.21) can be written as

$$\omega_1 < (1 + \tilde{p}_d) \frac{1 + \varepsilon - s^{\min}}{1 + r}. \quad (\text{A.58})$$

First suppose \tilde{p}_d is sufficiently large that this condition is violated. In this case, by the characterization in the proof of Theorem 2, the loan riskiness $S_d(\tilde{p}_d) = s^{\min}$ is constant. Second, suppose condition (A.58) is satisfied, and thus $S_d(\tilde{p}_d) \in (s^{\min}, s^{\max})$ is determined as the unique solution to Eq. (A.23). This equation can be simplified to

$$\frac{1 - F_0(\bar{s})}{1 - F_1(\bar{s})} \int_{\bar{s}}^{s^{\max}} (v(s) - v(\bar{s})) dF_1 = w_1 \frac{1 + r}{1 + \tilde{p}_d}.$$

The proof of Theorem 2 shows that the left hand side of this expression is a strictly decreasing function of \bar{s} . Since the right hand side is decreasing in \tilde{p}_d , it follows that in this case $S_d(\tilde{p}_d)$ is increasing. Combining the two cases, $S_d(\tilde{p}_d)$ is weakly increasing in \tilde{p}_d .

Next, to show that $P_d(\tilde{p}_d)$ is strictly increasing in \tilde{p}_d , note that

$$\begin{aligned} P_d(\tilde{p}_d) &= p^{mc}(S_d(\tilde{p}_d); v_d(\cdot | \tilde{p}_d)) \\ &= \min\left(\frac{E_1[v_d(s | \tilde{p}_d)]}{1 + r}, \omega_1 + \frac{E_0[\min(v_d(s | \tilde{p}_d), v_d(S_d(\tilde{p}_d) | \tilde{p}_d))]}{1 + r}\right), \end{aligned}$$

where the second equality combines cases (i) and (ii) of Eq. (14) and uses $\alpha_0 = 1$. Substituting the value function $v_d(s | \tilde{p}_d) = s(1 + \tilde{p}_d)$ (cf. Eq. (A.55)) and using $E_1[s] = 1 + \varepsilon$, the previous displayed equation can be written as

$$P_d(\tilde{p}_d) = \min\left((1 + \tilde{p}_d) \frac{1 + \varepsilon}{1 + r}, \omega_1 + (1 + \tilde{p}_d) \frac{E_0[\min(s, S_d(\tilde{p}_d))]}{1 + r}\right).$$

Since $S_d(\tilde{p}_d)$ is weakly increasing in \tilde{p}_d , this equation implies that $P_d(\tilde{p}_d)$ is strictly increasing in \tilde{p}_d .

Finally, to show that $P_d(\tilde{p}_d)$ satisfies the boundary conditions in (A.57), note that Eq. (10) implies

$$P_d(\tilde{p}_d) = p^{opt}(S_d(\tilde{p}_d); v_d(\cdot | \tilde{p}_d)).$$

Using the definition of $p^{opt}(\cdot)$ from Eq. (10) and substituting $v_d(s | \tilde{p}_d) = s(1 + \tilde{p}_d)$ (cf. Eq. (A.55)), the previous displayed equation can be written as

$$P_d(\tilde{p}_d) = \frac{1 + \tilde{p}_d}{1 + r} \left(\int_{s^{\min}}^{S_d(\tilde{p}_d)} s dF_0 + \frac{1 - F_0(S_d(\tilde{p}_d))}{1 - F_1(S_d(\tilde{p}_d))} \int_{S_d(\tilde{p}_d)}^{s^{\max}} s dF_1 \right).$$

Next, consider this expression for $\tilde{p}_d = \frac{1}{r}$ and note that

$$\begin{aligned} P_d\left(\frac{1}{r}\right) &= \frac{1 + \frac{1}{r}}{1 + r} \left(\int_{s^{\min}}^{S_d\left(\frac{1}{r}\right)} s dF_0 + \frac{1 - F_0\left(S_d\left(\frac{1}{r}\right)\right)}{1 - F_1\left(S_d\left(\frac{1}{r}\right)\right)} \int_{S_d\left(\frac{1}{r}\right)}^{s^{\max}} s dF_1 \right) \\ &> \frac{1 + \frac{1}{r}}{1 + r} \left(\int_{s^{\min}}^{s^{\max}} s dF_0 + \frac{1 - F_0\left(s^{\max}\right)}{1 - F_1\left(s^{\max}\right)} \int_{s^{\max}}^{s^{\max}} s dF_1 \right) = \frac{1 + \frac{1}{r}}{1 + r} E_0[s] = \frac{1}{r}. \end{aligned}$$

Here, the second line replaces $S_d\left(\frac{1}{r}\right)$ in the first line with $s^{\max} > S_d\left(\frac{1}{r}\right)$, and the inequality follows since the expression in the first line is a decreasing function of $S_d\left(\frac{1}{r}\right)$. Similarly,

$$\begin{aligned} P_d\left(\frac{1 + \varepsilon}{r - \varepsilon}\right) &= \frac{1 + \frac{1 + \varepsilon}{r - \varepsilon}}{1 + r} \left(\int_{s^{\min}}^{S_d\left(\frac{1 + \varepsilon}{r - \varepsilon}\right)} s dF_0 + \frac{1 - F_0\left(S_d\left(\frac{1 + \varepsilon}{r - \varepsilon}\right)\right)}{1 - F_1\left(S_d\left(\frac{1 + \varepsilon}{r - \varepsilon}\right)\right)} \int_{S_d\left(\frac{1 + \varepsilon}{r - \varepsilon}\right)}^{s^{\max}} s dF_1 \right) \\ &\leq \frac{1}{r - \varepsilon} \left(\int_{s^{\min}}^{s^{\min}} s dF_0 + \frac{1 - F_0\left(s^{\min}\right)}{1 - F_1\left(s^{\min}\right)} \int_{s^{\min}}^{s^{\max}} s dF_1 \right) = \frac{1}{r - \varepsilon} E_1[s] = \frac{1 + \varepsilon}{r - \varepsilon}, \end{aligned}$$

where the second line replaces $S_d\left(\frac{1 + \varepsilon}{r - \varepsilon}\right)$ in the first line with $s^{\min} \leq S_d\left(\frac{1 + \varepsilon}{r - \varepsilon}\right)$. It follows that $P_d(\tilde{p}_d)$ satisfies the boundary conditions in (A.57), completing the proof of Theorem 7.

Proof of Theorem 8. Part (i). Since $(P_d(\tilde{p}_d), S_d(\tilde{p}_d))$ is the static equilibrium for the economy $\mathcal{E}(\tilde{p}_d)$ and since optimists' optimism becomes weakly more right-skewed, Theorem 3 applies and shows that $P_d(\tilde{p}_d)$ and $S_d(\tilde{p}_d)$ weakly increase for each \tilde{p}_d . Since p_d is the fixed point of the strictly increasing mapping $P_d(\tilde{p}_d)$ and since $P_d(\cdot)$ shifts up, it follows that p_d weakly increases.

Next note that $S_d(\cdot)$ is a weakly increasing function (by the proof of Theorem 7) and that the equilibrium price to dividend ratio, p_d , weakly increases. Since $S_d(\tilde{p}_d)$ also weakly increases for each \tilde{p}_d , it follows that the equilibrium loan riskiness, $\bar{s}_d^* = S_d(p_d)$, weakly increases.

Next consider the share of the speculative component, λ_d . Plugging in the value function, $v_d(\cdot | p_d) = s(1 + p_d)$ (cf. Eq. (A.55)), and using $E_0[s] = 1$ and $E_1[s] = 1 + \varepsilon$, Eq. (40) can be rewritten as

$$\frac{p_d}{1 + p_d} = (1 - \theta_d) \frac{1}{1 + r} + \theta_d \frac{1 + \varepsilon}{1 + r}.$$

Note also that Eq. (41) can be written as

$$1 - \lambda_d = \frac{p^{pdv}(a)}{p(a)} = \frac{1}{p_d} \left((1 - \theta_d) \frac{1}{r} + \theta_d \frac{1 + \varepsilon}{r} \right).$$

Combining the last two displayed equalities, the share of the speculative component is given by:

$$\lambda_d = 1 - \frac{1 + 1/r}{1 + p_d}$$

Since p_d weakly increases, λ_d also weakly increases, completing the proof for the first part.

Part (ii). Since $(P_d(\tilde{p}_d), S_d(\tilde{p}_d))$ is the static equilibrium for the economy $\mathcal{E}(\tilde{p}_d)$ and since optimists' optimism becomes weakly more skewed to the left of \bar{s}_d^* , Theorem 3 applies and shows that $P_d(\tilde{p}_d)$ and $S_d(\tilde{p}_d)$ weakly increase for each \tilde{p}_d . Since p_d is the fixed point of the strictly increasing mapping $P_d(\tilde{p}_d)$ and since $P_d(\cdot)$ shifts up, it follows that p_d weakly increases. The same steps for part (i) show that λ_d also weakly increases, completing the proof of Theorem 8.

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