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AXIOMS FOR DEFERRED ACCEPTANCE

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The deferred acceptance algorithm is often used to allocate indivisible objects when monetary transfers are not allowed. We provide two characterizations of agent-proposing deferred acceptance allocation rules. Two new axioms—individually rational monotonicity and weak Maskin monotonicity—are essential to our analysis. An allocation rule is the agent-proposing deferred acceptance rule for some acceptant substitutable priority if and only if it satisfies non-wastefulness and individually rational monotonicity. An alternative characterization is in terms of non-wastefulness, population monotonicity, and weak Maskin monotonicity. We also offer an axiomatization of the deferred acceptance rule generated by an exogenously specified priority structure. We apply our results to characterize efficient deferred acceptance rules.

KEYWORDS: Deferred acceptance algorithm, stable allocations, axioms, individually rational monotonicity, weak Maskin monotonicity, population monotonicity, non-wastefulness.

1. INTRODUCTION

IN AN ASSIGNMENT PROBLEM, a set of indivisible objects that are collectively owned need to be allocated to a number of agents, with each agent being entitled to receive at most one object. Student placement in public schools and university housing allocation are examples of important assignment problems in practice. The agents are assumed to have strict preferences over the objects (and being unassigned). An allocation rule specifies an assignment of the objects to the agents for each preference profile. No monetary transfers are allowed.

In many assignment problems, each object is endowed with a priority over agents. For example, schools in Boston give higher priority to students who live nearby or have siblings already attending. An allocation rule is stable with respect to a given priority profile if there is no agent–object pair (i, a) such that (i) i prefers a to his assigned object and (ii) either i has higher priority for a than some agent who receives a or a is not assigned to other agents up to its quota. In the school choice settings of Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003), priorities represent a social objective. For example, it may be desirable that in Boston students attend high schools within walking distance from their homes or that in Turkey students with excellent achievements in mathematics and science go to the best engineering universities. Stability is regarded as a normative fairness criterion in the following sense. An allocation is stable if no student has justified envy, that is, any school

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that a student prefers to his assigned school is attended (up to capacity) by students who are granted higher priority for it.

The deferred acceptance algorithm of Gale and Shapley (1962) determines a stable allocation which has many appealing properties. The agent-proposing deferred acceptance allocation Pareto dominates any other stable allocation. Moreover, the agent-proposing deferred acceptance rule makes truthful reporting of preferences a dominant strategy for every agent. Consequently, the deferred acceptance rule is used in many practical assignment problems such as student placement in New York City and Boston (Abdulkadiroğlu, Pathak, and Roth (2005), Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005)) and university house allocation at MIT and the Technion (Guillen and Kesten (2008), Perach, Polak, and Rothblum (2007)), to name some concrete examples. There are proposals to apply the rule to other problems such as course allocation in business schools (Sönmez and Ünver (2009)) and assignment of military personnel to positions (Korkmaz, Gökçen, and Çetinyokuş (2008)).

Despite the importance of deferred acceptance rules in both theory and practice, no axiomatization has yet been obtained in an object allocation setting with unspecified priorities. Our first results (Theorems 1 and 2) offer two characterizations of deferred acceptance rules with acceptant substitutable priorities.

For the first characterization, we introduce a new axiom, individually rational (IR) monotonicity. We say that a preference profile R' is an IR monotonic transformation of a preference profile R at an allocation μ if for every agent, any object that is acceptable and preferred to μ under R' is preferred to μ under R. An allocation rule φ satisfies IR monotonicity if every agent weakly prefers the allocation $\varphi(R')$ to the allocation $\varphi(R)$ under R' whenever R' is an IR monotonic transformation of R at $\varphi(R)$. If R' is an IR monotonic transformation of R at $\varphi(R)$, then the interpretation of the change in reported preferences from R to R' is that all agents place fewer claims on objects they cannot receive at R, in the sense that each agent's set of acceptable objects that are preferred to $\varphi(R)$ shrinks. Intuitively, the IR monotonicity axiom requires that all agents be weakly better off when some agents claim fewer objects. The IR label captures the idea that each agent effectively places claims only on acceptable objects; an agent may not be allocated unacceptable objects because he can opt to remain unassigned, so the relevant definition of an upper contour set includes the IR constraint. IR monotonicity requires that allocations be monotonic with respect to the IR constrained upper contour sets. IR monotonicity resembles Maskin monotonicity (Maskin (1999)), but the two axioms are independent.

We also define a weak requirement of efficiency, the non-wastefulness axiom. An allocation rule is non-wasteful if at every preference profile, any object that an agent prefers to his assignment is allocated up to its quota to other agents. Our first characterization states that an allocation rule is the deferred acceptance rule for some acceptant substitutable priority if and only if it satisfies non-wastefulness and IR monotonicity (Theorem 1).

To further understand deferred acceptance rules, we provide a second characterization based on axioms that are mathematically more elementary and tractable than IR monotonicity. An allocation rule is population monotonic if for every preference profile, when some agents deviate to declaring every object unacceptable (which we interpret as leaving the market unassigned), all other agents are weakly better off (Thomson (1983a, 1983b)). Following Maskin (1999), R' is a monotonic transformation of R at μ if for every agent, any object that is preferred to μ under R' is also preferred to μ under R. An allocation rule φ satisfies weak Maskin monotonicity if every agent prefers $\varphi(R')$ to $\varphi(R)$ under R' whenever R' is a monotonic transformation of R at $\varphi(R)$. Our second result shows that an allocation rule is the deferred acceptance rule for some acceptant substitutable priority if and only if it satisfies non-wastefulness, weak Maskin monotonicity, and population monotonicity (Theorem 2).

We also study allocation rules that are stable with respect to an exogenously specified priority profile C (Section 6). We show that the deferred acceptance rule at C is the only stable rule at C that satisfies weak Maskin monotonicity (Theorem 3).

In addition to stability, efficiency is often a goal of the social planner. We apply our axiomatizations to the analysis of efficient deferred acceptance rules. The Maskin monotonicity axiom plays a key role. Recall that an allocation rule φ satisfies Maskin monotonicity if $\varphi(R') = \varphi(R)$ whenever R' is a monotonic transformation of R at $\varphi(R)$ (Maskin (1999)). We prove that an allocation rule is an efficient deferred acceptance rule if and only if it satisfies Maskin monotonicity, along with non-wastefulness and population monotonicity; an equivalent set of conditions consists of Pareto efficiency, weak Maskin monotonicity, and population monotonicity (Theorem 4).

Priorities are not primitive in our model except for Section 6, and our axioms are "priority-free" in the sense that they do not involve priorities. The IR monotonicity axiom conveys the efficiency cost imposed by stability with respect to some priority structure.[2] Whenever some agents withdraw claims for objects that they prefer to their respective assignments, all agents (weakly) benefit. In the context of the deferred acceptance algorithm, the inefficiency is brought about by agents who apply for objects that tentatively accept, but subsequently reject, them. While it is intuitive that deferred acceptance rules satisfy IR monotonicity, it is remarkable that this priority-free axiom fully de-

[2]We do not regard IR monotonicity as a normative (either desirable or undesirable) requirement, but as a positive comprehensive description of the deferred acceptance algorithm. The reason is that priorities often reflect social objectives, and priority-free statements such as IR monotonicity may lack normative implications for priority-based assignment problems. The present welfare analysis disregards the social objectives embedded in the priorities. Nonetheless, it should be reiterated that for a given priority structure, the corresponding deferred acceptance rule attains constrained efficiency subject to stability.

scribes the theoretical contents of the deferred acceptance algorithm (along with the requirement of non-wastefulness).

The weak Maskin monotonicity axiom is mathematically similar to and is weaker than (i.e., implied by) Maskin monotonicity. We establish that weak Maskin monotonicity is sufficient, along with non-wastefulness and population monotonicity, to characterize deferred acceptance rules. At the same time, if we replace weak Maskin monotonicity by Maskin monotonicity in the list of axioms above, we obtain a characterization of efficient deferred acceptance rules. The contrast between these two findings demonstrates that the inefficiency of some deferred acceptance rules can be attributed entirely to instances where weak Maskin monotonicity is satisfied while Maskin monotonicity is violated.

Our analysis focuses on substitutable priorities because priorities may be non-responsive but substitutable in applications. Such priorities arise, for example, in school districts concerned with balance in race distribution (Abdulkadiroğlu and Sönmez (2003)) or in academic achievement (Abdulkadiroğlu, Pathak, and Roth (2005)) within each school. A case in point is the New York City school system, where each Educational Option school must allocate 16% of its seats to top performers in a standardized exam, 68% to middle performers, and 16% to bottom performers. In the context of house allocation, some universities impose bounds on the number of rooms or apartments assigned to graduate students in each program (arts and sciences, business, public policy, law, etc.).

Furthermore, substitutability of priorities is an "almost necessary" condition for the non-emptiness of the core.[3] When priorities are substitutable, the core coincides with the set of stable allocations. Since the relevant restrictions on priorities vary across applications, allowing for substitutable priorities is a natural approach.

Special instances of deferred acceptance rules have been characterized in the literature. Svensson (1999) axiomatized the serial dictatorship allocation rules. Ehlers, Klaus, and Papai (2002), Ehlers and Klaus (2004), Ehlers and Klaus (2006), and Kesten (2009) offered various characterizations for the mixed dictator and pairwise-exchange rules. Mixed dictator and pairwise-exchange rules correspond to deferred acceptance rules with acyclic priority structures. For responsive priorities, Ergin (2002) showed that the only deferred acceptance rules that are efficient correspond to acyclic priority structures.

Other allocation mechanisms have been previously characterized. Papai (2000) characterized the hierarchical exchange rules, which generalize the priority-based top trading cycle rules of Abdulkadiroğlu and Sönmez (2003).

[3]Formally, suppose there are at least two proper objects a and b. Fix a non-substitutable priority for a. Then there exist a preference profile for the agents and a responsive priority for b such that, regardless of the priorities for the other objects, the core is empty. The first version of this result, for a slightly different context, appears in Sönmez and Ünver (2009). The present statement is from Hatfield and Kojima (2008).

In the context of housing markets, Ma (1994) characterized the top trading cycle rule of David Gale described by Shapley and Scarf (1974). Kesten (2006) showed that the deferred acceptance rule and the top trading cycle rule for some fixed priority profile are equivalent if and only if the priority profile is acyclic.[4]

When the priority structure is a primitive of the model as in Section 6, alternative characterizations of the corresponding deferred acceptance rule are known. The classic result of Gale and Shapley (1962) implies that the deferred acceptance rule is characterized by constrained efficiency subject to stability. Alcalde and Barbera (1994) characterized the deferred acceptance rule by stability and strategy-proofness. Balinski and Sönmez (1999) considered allocation rules over the domain of pairs of responsive priorities and preferences. An allocation rule respects improvements if an agent is weakly better off when his priority improves for each object. Balinski and Sönmez (1999) showed that the deferred acceptance rule is the only stable rule that respects improvements.

2. FRAMEWORK

Fix a set of *agents* N and a set of (*proper*) *object types* O. There is one *null object* type, denoted $\emptyset$. Each object $a \in O \cup \{\emptyset\}$ has *quota* q_a; $\emptyset$ is not scarce, $q_\emptyset = |N|$. Each agent i is allocated exactly one object in $O \cup \{\emptyset\}$. An *allocation* is a vector $\mu = (\mu_i)_{i \in N}$ that assigns object $\mu_i \in O \cup \{\emptyset\}$ to agent i, with each object a being assigned to at most q_a agents. We write $\mu_a = \{i \in N | \mu_i = a\}$ for the set of agents who receive object a under μ.

Each agent i has a strict (complete, transitive, and antisymmetric) *preference relation* R_i over $O \cup \{\emptyset\}$.[5] We denote by P_i the asymmetric part of R_i, that is, aP_ib if only if aR_ib and $a \neq b$. An object a is *acceptable* (*unacceptable*) to agent i if $aP_i\emptyset$ ($\emptyset P_i a$). Let $R = (R_i)_{i \in N}$ denote the preference profile of all agents. For any $N' \subset N$, we use the notation $R_{N'} = (R_i)_{i \in N'}$.[6] We write $\mu R \mu'$ if and only if $\mu_i R_i \mu'_i$ for all $i \in N$.

We denote by $\mathcal{A}$ and $\mathcal{R}$ the sets of allocations and preference profiles, respectively. An *allocation rule* $\varphi : \mathcal{R} \to \mathcal{A}$ maps preference profiles to allocations. At R, agent i is assigned object $\varphi_i(R)$, and object a is assigned to the set of agents $\varphi_a(R)$.

[4]Kesten's acyclicity condition is stronger than Ergin's.

[5]The null object may represent private schools in the context of student placement in public schools or off-campus housing in the context of university house allocation. Taking into consideration preferences that rank the null object above some proper objects is natural in such applications.

[6]Our analysis carries through if we do not stipulate that preferences rank pairs of unacceptable objects, but alternatively regard as identical all preferences that agree on the ranking of acceptable objects.

3. DEFERRED ACCEPTANCE

A *priority* for a proper object $a \in O$ is a correspondence $C_a : 2^N \to 2^N$, satisfying $C_a(N') \subset N'$ and $|C_a(N')| \leq q_a$ for all $N' \subset N$; $C_a(N')$ is interpreted as the set of high priority agents in N' "chosen" by object a. The priority C_a is *substitutable* if agent i is chosen by object a from a set of agents N' whenever i is chosen by a from a set N'' that includes N'; formally, for all $N' \subset N'' \subset N$, we have $C_a(N'') \cap N' \subset C_a(N')$. The priority C_a is *acceptant* if object a accepts each agent when its quota is not entirely allocated; formally, for all $N' \subset N$, $|C_a(N')| = \min(q_a, |N'|)$.[7] Let $C = (C_a)_{a \in O}$ denote the priority profile; C is substitutable (acceptant) if C_a is substitutable (acceptant) for all $a \in O$.

The allocation μ is *individually rational* at R if $\mu_i R_i \emptyset$ for all $i \in N$. The allocation μ is *blocked* by a pair $(i, a) \in N \times O$ at (R, C) if $a P_i \mu_i$ and $i \in C_a(\mu_a \cup \{i\})$. An allocation μ is *stable* at (R, C) if it is individually rational at R and is not blocked by any pair $(i, a) \in N \times O$ at (R, C). When C is substitutable, the following iterative procedure, called the (agent-proposing) *deferred acceptance algorithm*, produces a stable allocation at (R, C) (Gale and Shapley (1962); extended to the case of substitutable priorities by Roth and Sotomayor (1990)).

Step 1. Every agent applies to his most preferred acceptable object under R (if any). Let $\tilde{N}_a^1$ be the set of agents applying to object a. Object a tentatively accepts the agents in $N_a^1 = C_a(\tilde{N}_a^1)$ and rejects the applicants in $\tilde{N}_a^1 \setminus N_a^1$.

Step t ($t \geq 2$). Every agent who was rejected at step $t-1$ applies to his next preferred acceptable object under R (if any). Let $\tilde{N}_a^t$ be the new set of agents applying to object a. Object a tentatively accepts the agents in $N_a^t = C_a(N_a^{t-1} \cup \tilde{N}_a^t)$ and rejects the applicants in $(N_a^{t-1} \cup \tilde{N}_a^t) \setminus N_a^t$.

The deferred acceptance algorithm terminates when each agent who is not tentatively accepted by some object has been rejected by every object acceptable to him. Each agent tentatively accepted by a proper object at the last step is assigned that object and all other agents are assigned the null object. The *deferred acceptance rule* φ^C is defined by setting $\varphi^C(R)$ equal to the allocation obtained when the algorithm is applied for (R, C). The allocation $\varphi^C(R)$ is the *agent-optimal stable allocation* at (R, C): it is stable at (R, C) and is weakly preferred under R by every agent to any other stable allocation at (R, C) (Theorem 6.8 in Roth and Sotomayor (1990)).

REMARK 1: It can be easily shown that no two distinct priority profiles induce the same deferred acceptance rule. Therefore, the subsequent characterization results lead to unique representations.

[7] The acceptant *responsive* priority C_a for a linear order $\succ_a$ on N is defined as follows. For all $N' \subset N$, $C_a(N')$ is the set of $\min(q_a, |N'|)$ top ranked agents in N' under $\succ_a$. The class of acceptant responsive priorities is a subset of the class of acceptant substitutable priorities. Studying substitutable priorities is important because priorities may often be non-responsive but substitutable in practice, as discussed in the Introduction.

4. FIRST CHARACTERIZATION OF DEFERRED ACCEPTANCE RULES

We introduce two axioms—non-wastefulness and individually rational (IR) monotonicity—that characterize the set of deferred acceptance rules. Priorities are not primitive in our model except for Section 6, and our axioms are priority-free in the sense that they do not involve priorities.

DEFINITION 1—Non-Wastefulness: An allocation rule φ is *non-wasteful* if

$$aP_i\varphi_i(R) \quad \Rightarrow \quad |\varphi_a(R)| = q_a \quad \forall R \in \mathcal{R}, i \in N, a \in O \cup \{\emptyset\}.$$

Non-wastefulness is a weak efficiency condition. It requires that an object is not assigned to an agent who prefers it to his allocation only if the entire quota of that object is assigned to other agents. Note that if φ is non-wasteful, then $\varphi(R)$ is individually rational at R for every $R \in \mathcal{R}$, as the null object is not scarce.

To introduce the main axiom, we say that R'_i is an *individually rational monotonic transformation* of R_i at $a \in O \cup \{\emptyset\}$ (R'_i i.r.m.t. R_i at a) if any object that is ranked above both a and $\emptyset$ under R'_i is ranked above a under R_i, that is,

$$bP'_i a \& bP'_i \emptyset \quad \Rightarrow \quad bP_i a \quad \forall b \in O.$$

R' is an IR monotonic transformation of R at an allocation μ (R' i.r.m.t. R at μ) if R'_i i.r.m.t. R_i at μ_i for all i.

DEFINITION 2—IR Monotonicity: An allocation rule φ satisfies *individually rational monotonicity* if

$$R' \text{ i.r.m.t. } R \text{ at } \varphi(R) \quad \Rightarrow \quad \varphi(R')R'\varphi(R).$$

In words, φ satisfies IR monotonicity if every agent weakly prefers $\varphi(R')$ to $\varphi(R)$ under R' whenever R' is an IR monotonic transformation of R at $\varphi(R)$. If R' i.r.m.t. R at $\varphi(R)$, then the interpretation of the change in reported preferences from R to R' is that all agents place fewer claims on objects they cannot receive at R, in the sense that each agent's set of acceptable objects that are preferred to $\varphi(R)$ shrinks. Intuitively, the IR monotonicity axiom requires that all agents be weakly better off when some agents claim fewer objects. The IR label captures the idea that each agent effectively places claims only on acceptable objects. An agent may not be allocated unacceptable objects because he can opt to remain unassigned ($\emptyset$ represents the outside option), so the relevant definition of an upper contour set includes the IR constraint. Hence IR monotonicity requires that allocations be monotonic with respect to the IR constrained upper contour sets (ordered according to set inclusion).

THEOREM 1: *An allocation rule* φ *is the deferred acceptance rule for some acceptant substitutable priority* C, *that is*, $\varphi = \varphi^C$, *if and only if* φ *satisfies nonwastefulness and IR monotonicity*.

The proof appears in the Appendix. Example 1 below, borrowed from Ergin (2002), illustrates an instance where a deferred acceptance rule satisfies IR monotonicity and provides some intuition for the "only if" part of the theorem.

IR monotonicity resembles Maskin (1999) monotonicity. R'_i is a *monotonic transformation* of R_i at $a \in O \cup \{\emptyset\}$ (R'_i m.t. R_i at a) if any object that is ranked above a under R'_i is also ranked above a under R_i, that is, $bP'_i a \Rightarrow bP_i a\ \forall b \in O \cup \{\emptyset\}$. R' is a monotonic transformation of R at an allocation μ (R' m.t. R at μ) if R'_i m.t. R_i at μ_i for all i.

DEFINITION 3—Maskin Monotonicity: An allocation rule φ satisfies *Maskin monotonicity* if

$$R' \text{ m.t. } R \text{ at } \varphi(R) \quad \Rightarrow \quad \varphi(R') = \varphi(R).$$

On the one hand, IR monotonicity has implications for a larger set of preference profile pairs (R, R') than Maskin monotonicity, as R' m.t. R at $\varphi(R) \Rightarrow R'$ i.r.m.t. R at $\varphi(R)$. On the other hand, for every preference profile pair (R, R') for which both axioms have implications (i.e., R' m.t. R at $\varphi(R)$), Maskin monotonicity imposes a stronger restriction than IR monotonicity (as $\varphi(R') = \varphi(R) \Rightarrow \varphi(R')R'\varphi(R)$). Example 1 establishes the independence of the IR monotonicity and Maskin monotonicity axioms. The example also shows that deferred acceptance rules do not always satisfy Maskin monotonicity (cf. Kara and Sönmez (1996)) and that some top trading cycle rules violate IR monotonicity, but satisfy Maskin monotonicity.

EXAMPLE 1: Let $N = \{i, j, k\}, O = \{a, b\}$, and $q_a = q_b = 1$. Consider the strict orderings $\succ_a$ and $\succ_b$ specified as

$\succ_a$	$\succ_b$
i	k
j	i
k	j

Let C denote the responsive priorities that correspond to these orderings defined as in footnote 7. Consider the set of preferences for the agents:

R_i	R''_i	R_j	R'_j	R_k
b	$\emptyset$	a	$\emptyset$	a
a	b	$\emptyset$	a	b
$\emptyset$	a	b	b	$\emptyset$

Let $R = (R_i, R_j, R_k)$, $R' = (R_i, R'_j, R_k)$, $R'' = (R''_i, R_j, R_k)$.

In the first step of the deferred acceptance algorithm for (R, C), i applies to b, and j and k apply to a; then k is rejected by a. In the second step, k applies to b and i is rejected by b. At the third step, i applies to a and j is rejected by a. The algorithm terminates after the third step and the final allocation is given by $\varphi^C(R) = (\varphi^C_i(R), \varphi^C_j(R), \varphi^C_k(R)) = (a, \emptyset, b)$. In the first step of the deferred acceptance algorithm for (R', C), i applies to b and k applies to a. The algorithm ends at the first step and $\varphi^C(R') = (b, \emptyset, a)$.

All agents prefer $\varphi^C(R')$ to $\varphi^C(R)$ under R' (the preference is weak for j and strict for i and k) due to the fact that R' i.r.m.t. R at $\varphi^C(R)$. Indeed, there is a chain of rejections in the deferred acceptance algorithm for (R, C): k is rejected by a because j claims higher priority to a; next, i is rejected by b because k claims higher priority to b; then j is rejected by a because i claims higher priority for a. Hence j receives the null object in spite of his initial priority claim to a, which starts off the rejection chain. If j does not claim higher priority to a and reports R'_j instead of R_j, then the rejection chain does not occur, weakly benefiting everyone (with respect to R'). Also, note that φ^C violates Maskin monotonicity since R' m.t. R at $\varphi^C(R)$ and $\varphi^C_i(R') \neq \varphi^C_i(R)$.

IR monotonicity is not satisfied by the top trading cycle rule (Abdulkadiroğlu and Sönmez (2003)) associated with the priorities $(\succ_a, \succ_b)$.[8] At R, i and k trade their priorities for a and b; the top trading cycle allocation is $\mu = (b, \emptyset, a)$. At R'', i is assigned the null object and then j receives a, for which he has higher priority than k. The top trading cycle allocation is $\mu'' = (\emptyset, a, b)$. IR monotonicity is violated as R'' i.r.m.t. R at μ and agent k strictly prefers μ to μ'' under R_k. At R'', k does not receive a because he has lower priority than j for a and cannot trade his priority for b with the priority of i for a since i does not place claims for b. The top trading cycle rule considered here satisfies Maskin monotonicity by Papai (2000) and Takamiya (2001).

The following examples show that non-wastefulness and IR monotonicity are independent axioms if $|N|, |O| \geq 2$ and there is at least one scarce object, that is, $q_a < |N|$ for some $a \in O$.

EXAMPLE 2: Consider the rule that allocates the null object to every agent for all preference profiles. This rule trivially satisfies IR monotonicity, but violates non-wastefulness.

EXAMPLE 3: Let $N = \{1, 2, \ldots, n\}$. Suppose that a is one of the scarce objects ($q_a < n$) and b is a proper object different from a (such a and b exist by assumption). Let R denote a (fixed) preference profile at which every agent ranks a first and $\emptyset$ second. Define the following allocation rule:

[8]We follow a definition of top trading cycles from Kesten (2006), which assumes the existence of a null object and allows each agent to consider some objects unacceptable.

(i) At any preference profile where agent q_a reports R_{q_a}, the assignment is according to the serial dictatorship with the ordering of agents $1, 2, \ldots, n$, that is, agent 1 picks his most preferred object, agent 2 picks his most preferred available object, and so on (an object is available for an agent if the number of preceding agents who choose that object is smaller than its quota).

(ii) At any other preference profile, the assignment is specified by the serial dictatorship with the agent ordering $1, 2, \ldots, q_a - 1, q_a + 1, q_a, q_a + 2, \ldots, n$, defined analogously to (i).

The allocation rule described above clearly satisfies non-wastefulness, but violates IR monotonicity. Indeed, let R'_{q_a} be a preference relation for agent q_a that ranks a first and b second. The profile $(R'_{q_a}, R_{N\setminus\{q_a\}})$ i.r.m.t. R at the allocation for R, but agent q_a is assigned a at R and b at $(R'_{q_a}, R_{N\setminus\{q_a\}})$, and $aP'_{q_a}b$.

5. SECOND CHARACTERIZATION OF DEFERRED ACCEPTANCE RULES

We offer an alternative characterization of deferred acceptance rules in terms of more elementary axioms. These axioms are mathematically more tractable and contribute to further understanding of deferred acceptance rules. For instance, in Section 7, we obtain a characterization of Pareto efficient deferred acceptance rules via a simple alteration in the new collection of axioms.

We first define the weak Maskin monotonicity axiom. Recall that R'_i is a *monotonic transformation* of R_i at $a \in O \cup \{\emptyset\}$ (R'_i m.t. R_i at a) if any object that is ranked above a under R'_i is also ranked above a under R_i, that is, $bP'_i a \Rightarrow bP_i a\ \forall b \in O \cup \{\emptyset\}$. R' is a monotonic transformation of R at an allocation μ (R' m.t. R at μ) if R'_i m.t. R_i at μ_i for all i.

DEFINITION 4—Weak Maskin Monotonicity: An allocation rule φ satisfies *weak Maskin monotonicity* if

$$R' \text{ m.t. } R \text{ at } \varphi(R) \quad \Rightarrow \quad \varphi(R')R'\varphi(R).$$

To gain some perspective, note that the implication of R' m.t. R at $\varphi(R)$ is that $\varphi(R') = \varphi(R)$ under Maskin monotonicity, but only that $\varphi(R')R'\varphi(R)$ under weak Maskin monotonicity. Therefore, any allocation rule that satisfies the standard Maskin monotonicity axiom also satisfies weak Maskin monotonicity.

We next define the population monotonicity axiom (Thomson (1983a, 1983b)). As a departure from the original setting, suppose that the collection of all objects (q_a copies of each object type $a \in O \cup \{\emptyset\}$) needs to be allocated to a subset of agents N' or, equivalently, that the agents outside N' receive $\emptyset$ and are removed from the assignment problem. It is convenient to view the new setting as a restriction on the set of preference profiles, whereby the agents in $N \setminus N'$ are constrained to report every object as unacceptable. Specifically, let $R^{\emptyset}$ denote a fixed preference profile that ranks $\emptyset$ first for every agent. For any $R \in \mathcal{R}$, we interpret the profile $(R_{N'}, R^{\emptyset}_{N\setminus N'})$ as a deviation from R generated by restricting the assignment problem to the agents in N'.

DEFINITION 5—Population Monotonicity: An allocation rule φ is *population monotonic* if

$$\varphi_i\big(R_{N'}, R^{\emptyset}_{N\setminus N'}\big) R_i \varphi_i(R) \quad \forall i \in N', \forall N' \subset N, \forall R \in \mathcal{R}.$$

The definitions of weak Maskin monotonicity and population monotonicity are inspired by the connection between IR monotonicity and the deferred acceptance algorithm. IR monotonicity clearly implies both weak Maskin monotonicity and population monotonicity. Building on the intuition for Theorem 1, we prove that the latter two axioms, along with non-wastefulness, are sufficient to characterize deferred acceptance rules (the proof appears in the Appendix).

THEOREM 2: *An allocation rule φ is the deferred acceptance rule for some acceptant substitutable priority C, that is, $\varphi = \varphi^C$, if and only if φ satisfies non-wastefulness, weak Maskin monotonicity, and population monotonicity.*

We show that the three axioms from Theorem 2 are independent if $|N|, |O| \geq 2$ and $q_a < |N| - 1$ for at least one object $a \in O$.[9] The rule described in Example 2 satisfies weak Maskin monotonicity and population monotonicity, and violates non-wastefulness. The rule from Example 3 satisfies non-wastefulness and population monotonicity, but not weak Maskin monotonicity. Last, the following example defines a non-wasteful and weakly Maskin monotonic rule, which is not population monotonic.

EXAMPLE 4: Let $N = \{1, 2, \dots, n\}$. Consider the allocation rule defined as follows:

(i) At any preference profile where agent 1 declares every object unacceptable, the assignment is according to the serial dictatorship allocation for the ordering of agents $1, 2, \dots, n-2, n-1, n$.

(ii) Otherwise, the assignment is specified by the serial dictatorship for the ordering $1, 2, \dots, n-2, n, n-1$.

The allocation rule so defined satisfies non-wastefulness and weak Maskin monotonicity, but not population monotonicity. To show that the rule violates population monotonicity, suppose that a is an object with $q_a < n-1$ and $b \neq a$ is a proper object (such a and b exist by assumption). Let R be a preference profile where the first ranked objects are b for agent 1; a for agents $2, 3, \dots, q_a, n-1, n$; and $\emptyset$ for the other agents. Note that agent n receives a at R and some c with aP_nc at $(R^{\emptyset}_1, R_{N\setminus\{1\}})$.

[9]If $q_a \geq |N| - 1$ for all $a \in O$, then non-wastefulness implies population monotonicity. In that case, in any market that excludes at least one agent, every non-wasteful allocation assigns each of the remaining agents his favorite object.

IR monotonicity implies both weak Maskin monotonicity and population monotonicity, and under the assumption of non-wastefulness, by Theorems 1 and 2, is equivalent to the conjunction of the latter two axioms. However, the following example shows that weak Maskin monotonicity and population monotonicity do not imply IR monotonicity if $|N|, |O| \geq 2$.

EXAMPLE 5: Let $N = \{1, 2, \dots, n\}$. Fix two proper objects a and b (such a and b exist by assumption). Consider the allocation rule that, at every preference profile R, specifies the following assignments:

(i) Agent 1 is assigned the higher ranked object between a and $\emptyset$ under R_1.

(ii) Agent 2 is assigned the higher ranked object between b and $\emptyset$ under R_2, except for the case $bP_1\emptyset P_1 a$, when he is assigned $\emptyset$.

(iii) Agents in $N \setminus \{1, 2\}$ are assigned $\emptyset$.

One can check that this allocation rule satisfies weak Maskin monotonicity and population monotonicity. To show that the rule violates IR monotonicity, let R be a preference profile where agent 1 ranks b first and a second, and agent 2 ranks b first, and let R'_1 be a preference for agent 1 that ranks b first and $\emptyset$ second. Then IR monotonicity is violated since $(R'_1, R_{N\setminus\{1\}})$ i.r.m.t. R at the allocation under R, but agent 2 is assigned b at R and $\emptyset$ at $(R'_1, R_{N\setminus\{1\}})$, and $bP_2\emptyset$.

6. AXIOMS FOR STABLE RULES

In this section, we study stable allocation rules with respect to an exogenously specified priority structure C. We say that an allocation rule φ is stable at C if $\varphi(R)$ is stable at (R, C) for all R. We show that the deferred acceptance rule at C is the only allocation rule that is stable at C and satisfies weak Maskin monotonicity.

THEOREM 3: *Let C be an acceptant substitutable priority. Suppose that φ is a stable allocation rule at C. Then φ is the deferred acceptance rule for C, that is, $\varphi = \varphi^C$, if and only if it satisfies weak Maskin monotonicity.*

PROOF: The "only if" part is a consequence of Theorem 2. The "if" part follows from Lemma 2 in the Appendix. *Q.E.D.*

7. EFFICIENT DEFERRED ACCEPTANCE RULES

An allocation μ *Pareto dominates* another allocation μ' at the preference profile R if $\mu_i R_i \mu'_i$ for all $i \in N$ and $\mu_i P_i \mu'_i$ for some $i \in N$. An allocation is *Pareto efficient* at R if no allocation Pareto dominates it at R. An allocation rule φ is Pareto efficient if $\varphi(R)$ is Pareto efficient at R for all $R \in \mathcal{R}$. An allocation rule φ is *group strategy-proof* if there exist no $N' \subset N$ and $R, R' \in \mathcal{R}$ such that $\varphi_i(R'_{N'}, R_{N\setminus N'}) R_i \varphi_i(R)$ for all $i \in N'$ and $\varphi_i(R'_{N'}, R_{N\setminus N'}) P_i \varphi_i(R)$ for some $i \in N'$.

In general, there are deferred acceptance rules that are neither Pareto efficient nor group strategy-proof. Since deferred acceptance rules are often used in resource allocation problems where efficiency is one of the goals of the social planner (besides stability), it is desirable to develop necessary and sufficient conditions for the efficiency of these rules.

PROPOSITION 1: *Let C be an acceptant substitutable priority. The following properties are equivalent.*

(i) *φ^C is Pareto efficient.*
(ii) *φ^C satisfies Maskin monotonicity.*
(iii) *φ^C is group strategy-proof.*

The proof is given in the Appendix.

Proposition 1 generalizes part of Theorem 1 from Ergin (2002). Under the assumption that priorities are responsive, Ergin established that a deferred acceptance rule is Pareto efficient if and only if it is group strategy-proof, and that these properties hold if and only if the priority is acyclic. Takamiya (2001) showed that Maskin monotonicity and group strategy-proofness are equivalent for any allocation rule.

THEOREM 4: *Let φ be an allocation rule. The following conditions are equivalent.*

(i) *φ is the deferred acceptance rule for some acceptant substitutable priority C, that is, $\varphi = \varphi^C$, and φ is Pareto efficient.*
(ii) *φ satisfies non-wastefulness, Maskin monotonicity, and population monotonicity.*
(iii) *φ satisfies Pareto efficiency, weak Maskin monotonicity, and population monotonicity.*

The proof appears in the Appendix.

In view of Proposition 1, two additional characterizations of efficient deferred acceptance rules are obtained by replacing the Pareto efficiency property in condition (i) of Theorem 4 with Maskin monotonicity and, respectively, group strategy-proofness.

Recall from Theorem 2 that weak Maskin monotonicity is sufficient, along with non-wastefulness and population monotonicity, to characterize deferred acceptance rules. Theorem 4 shows that if we replace weak Maskin monotonicity by Maskin monotonicity in the list of axioms above, we obtain a characterization of efficient deferred acceptance rules. The contrast between these two results demonstrates that the inefficiency of some deferred acceptance rules can be attributed entirely to instances where weak Maskin monotonicity is satisfied, but Maskin monotonicity is violated.

8. CONCLUSION

Our axiomatizations provide a comprehensive description of the theoretical contents of deferred acceptance rules. The intuition behind the axioms sheds light on the mechanics of the deferred acceptance algorithm.

The present analysis only restricts the priorities to be acceptant and substitutable. When no additional information about the priority structure is available, our axioms represent the strongest statements satisfied by deferred acceptance rules. The axioms are priority-free and may prove useful in characterizing deferred acceptance rules with restrictions on priorities relevant in applications.

APPENDIX

PROOF OF THEOREM 1: Since IR monotonicity implies weak Maskin monotonicity and population monotonicity, the "if" part of Theorem 1 follows from the "if" part of Theorem 2, which we establish later. We prove the "only if" part here.

We need to show that a deferred acceptance rule φ^C with acceptant substitutable priority C satisfies the non-wastefulness and IR monotonicity axioms. φ^C is non-wasteful since C is acceptant and the deferred acceptance rule is stable.

To prove that φ^C satisfies IR monotonicity, suppose that R' i.r.m.t. R at $\varphi^C(R)$. We need to show that $\varphi^C(R')R'\varphi^C(R) =: \mu^0$. Define μ^1 by assigning each agent i the higher ranked object between μ_i^0 and $\emptyset$ under R'_i.

For $t \geq 1$, if μ^t can be blocked at (R', C) we choose an arbitrary object a^t that is part of a blocking pair and define μ^{t+1} by

$$\text{(A.1)} \qquad \mu_i^{t+1} = \begin{cases} a^t, & \text{if } i \in C_{a^t}(\mu_{a^t}^t \cup \{j \in N | a^t P'_j \mu_j^t\}), \\ \mu_i^t, & \text{otherwise.} \end{cases}$$

If μ^t cannot be blocked, we let $\mu^{t+1} = \mu^t$. Part of the next lemma establishes that each μ^t is well defined, that is, μ^t is an allocation for all $t \geq 0$. The sequence $(\mu^t)_{t\geq 0}$ is a variant of the vacancy chain dynamics of Blum, Roth, and Rothblum (1997).[10]

LEMMA 1: *The sequence* $(\mu^t)_{t\geq 0}$ *satisfies*

$$\text{(A.2)} \qquad \mu^t \in \mathcal{A},$$

$$\text{(A.3)} \qquad \mu^t R' \mu^{t-1},$$

$$\text{(A.4)} \qquad \mu_a^t \subset C_a(\mu_a^t \cup \{j \in N | a P'_j \mu_j^t\}) \quad \forall a \in O$$

[10]As in Section 5, the exclusion of agent i with preferences R_i from the market can be modeled as a change in i's reported preferences, making every object unacceptable, which is an IR monotonic transformation of R_i at every object.

for every $t \geq 1$. *The sequence* $(\mu^t)_{t\geq 0}$ *becomes constant in a finite number of steps* T *and the allocation* μ^T *is stable at* (R', C).

PROOF: We prove the claims (A.2)–(A.4) by induction on t.

We first show the induction base case, $t = 1$. The definition of μ^1 immediately implies that $\mu^1 \in \mathcal{A}$ and $\mu^1 R' \mu^0$, proving (A.2) and (A.3) (at $t = 1$). To establish (A.4) (at $t = 1$), fix $a \in O$. We have that

$$\mu_a^0 = C_a(\mu_a^0 \cup \{j \in N | aP_j \mu_j^0\}) \tag{A.5}$$

because μ^0 is stable at (R, C), and C_a is an acceptant and substitutable priority. By construction,

$$\mu_a^1 \subset \mu_a^0. \tag{A.6}$$

Since R' i.r.m.t. R at μ^0, it must be that $\{j \in N | aP'_j \mu_j^1\} \subset \{j \in N | aP_j \mu_j^0\}$.[11] Therefore,

$$\mu_a^1 \cup \{j \in N | aP'_j \mu_j^1\} \subset \mu_a^0 \cup \{j \in N | aP_j \mu_j^0\}. \tag{A.7}$$

C_a's substitutability and (A.5)–(A.7) imply

$$\mu_a^1 \subset C_a(\mu_a^1 \cup \{j \in N | aP'_j \mu_j^1\}).$$

To establish the inductive step, we assume that the conclusion holds for $t \geq 1$ and prove it for $t + 1$. The only nontrivial case is $\mu^t \neq \mu^{t+1}$.

By the inductive hypothesis (A.4) (at t), $\mu_{a^t}^t \subset C_{a^t}(\mu_{a^t}^t \cup \{j \in N | a^t P'_j \mu_j^t\})$. Then the definition of $(\mu^t)_{t\geq 0}$ implies that

$$\mu_{a^t}^{t+1} = C_{a^t}(\mu_{a^t}^t \cup \{j \in N | a^t P'_j \mu_j^t\}). \tag{A.8}$$

To prove (A.2) (at $t + 1$), first note that (A.8) implies $|\mu_{a^t}^{t+1}| = |C_{a^t}(\mu_{a^t}^t \cup \{j \in N | a^t P'_j \mu_j^t\})| \leq q_{a^t}$. If $a \neq a^t$, then by construction, $\mu_a^{t+1} \subset \mu_a^t$, and by (A.2) (at t), we conclude that $|\mu_a^{t+1}| \leq |\mu_a^t| \leq q_a$. Therefore, $\mu^{t+1} \in \mathcal{A}$.

To show (A.3) (at $t + 1$), note that $a^t = \mu_j^{t+1} P'_j \mu_j^t$ for any $j \in \mu_{a^t}^{t+1} \setminus \mu_{a^t}^t$ and that each agent outside $\mu_{a^t}^{t+1} \setminus \mu_{a^t}^t$ is assigned the same object under μ^{t+1} and μ^t. Therefore, $\mu^{t+1} R' \mu^t$.

We show (A.4) (at $t + 1$) separately for the cases $a = a^t$ and $a \neq a^t$.

By construction of μ^{t+1},

$$\mu_{a^t}^{t+1} \cup \{j \in N | a^t P'_j \mu_j^{t+1}\} = \mu_{a^t}^t \cup \{j \in N | a^t P'_j \mu_j^t\}.\text{[12]}$$

[11] Suppose that $aP'_j \mu_j^1$. Then $\mu_j^1 R'_j \emptyset$ implies $aP'_j \emptyset$. By definition, $\mu_j^1 R'_j \mu_j^0$, so $aP'_j \mu_j^0$. The assumption that R'_j i.r.m.t. R_j at μ_j^0, along with $aP'_j \emptyset$ and $aP'_j \mu_j^0$, implies that $aP_j \mu_j^0$.

Then (A.8) implies that

$$\mu_{a^t}^{t+1} = C_{a^t}(\mu_{a^t}^{t+1} \cup \{j \in N | a^t P'_j \mu_j^{t+1}\}).$$

For any $a \neq a^t$, we have $\mu_a^{t+1} \subset \mu_a^t$ by construction, and $\{j \in N | aP'_j\mu_j^{t+1}\} \subset \{j \in N | aP'_j\mu_j^t\}$ since $\mu^{t+1}R'\mu^t$. Therefore,

$$\text{(A.9)} \qquad \mu_a^{t+1} \cup \{j \in N | aP'_j\mu_j^{t+1}\} \subset \mu_a^t \cup \{j \in N | aP'_j\mu_j^t\}.$$

Recall the inductive hypothesis (A.4) (at t): $\mu_a^t \subset C_a(\mu_a^t \cup \{j \in N | aP'_j\mu_j^t\})$. Then (A.9), along with the facts that C_a is substitutable and $\mu_a^{t+1} \subset \mu_a^t$, leads to

$$\mu_a^{t+1} \subset C_a(\mu_a^{t+1} \cup \{j \in N | aP'_j\mu_j^{t+1}\}),$$

completing the proof of the induction step.

By (A.3), the sequence $(\mu^t)_{t\geq 0}$ becomes constant in a finite number of steps T. The final allocation μ^T is individually rational at R' and is not blocked at (R', C), so is stable at (R', C). *Q.E.D.*

To finish the proof of the "only if" part, let μ^T be the stable matching identified in Lemma 1. We have that $\varphi^C(R')R'\mu^T$ because $\varphi^C(R')$ is the agent-optimal stable allocation at (R', C). Therefore, we obtain

$$\varphi^C(R')R'\mu^T R'\mu^{T-1}R' \cdots R'\mu^1 R'\mu^0 = \varphi^C(R),$$

showing that φ^C satisfies IR monotonicity. *Q.E.D.*

PROOF OF THEOREM 2: Since weak Maskin monotonicity and population monotonicity are implied by IR monotonicity, the "only if" part of Theorem 2 follows from the "only if" part of Theorem 1 shown above. We only need to prove the "if" part here.

Fix a rule φ that satisfies the non-wastefulness, weak Maskin monotonicity, and population monotonicity axioms. To show that φ is a deferred acceptance rule for some acceptant substitutable priority, we proceed in three steps. First, we construct a priority profile C and verify that it is acceptant and substitutable. Second, we show that for every $R \in \mathcal{R}$, $\varphi(R)$ is a stable allocation at (R, C). Third, we prove that $\varphi(R)$ is the agent-optimal stable allocation at (R, C).

For $a \in O \cup \{\emptyset\}$, let R^a be a fixed preference profile which ranks a as the most preferred object for every agent. For each $a \in O$, $N' \subset N$, define

$$C_a(N') = \varphi_a(R^a_{N'}, R^{\emptyset}_{N\setminus N'}).$$

[12]We have $\mu_{a^t}^t \subset \mu_{a^t}^{t+1}$ by construction and $\{j \in N | a^t P'_j \mu_j^{t+1}\} \subset \{j \in N | a^t P'_j \mu_j^t\}$ since $\mu_j^{t+1} R'_j \mu_j^t$ for every $j \in N$. At the same time, an inspection of (A.1) reveals that $\mu_{a^t}^{t+1} \setminus \mu_{a^t}^t = \{j \in N | a^t P'_j \mu_j^t\} \setminus \{j \in N | a^t P'_j \mu_j^{t+1}\}$.

We have that $C_a(N') \subset N'$ because φ is non-wasteful and the null object is not scarce.

Step 1—*C_a is an acceptant and substitutable priority for all objects $a \in O$.*

C_a is an acceptant priority because φ is non-wasteful.

To show that C_a is substitutable, consider $N' \subset N'' \subset N$. Assume that $i \in C_a(N'') \cap N'$. By definition, $\varphi_i(R^a_{N''}, R^{\emptyset}_{N\setminus N''}) = a$. Since $i \in N' \subset N''$, population monotonicity for the subset of agents N' and the preference profile $(R^a_{N''}, R^{\emptyset}_{N\setminus N''})$ implies that $\varphi_i(R^a_{N'}, R^{\emptyset}_{N\setminus N'}) R^a_i \varphi_i(R^a_{N''}, R^{\emptyset}_{N\setminus N''}) = a$. Hence $\varphi_i(R^a_{N'}, R^{\emptyset}_{N\setminus N'}) = a$, which by definition means that $i \in C_a(N')$. This shows $C_a(N'') \cap N' \subset C_a(N')$.

Step 2—*$\varphi(R)$ is a stable allocation at (R, C) for all $R \in \mathcal{R}$.*

For all R, $\varphi(R)$ is individually rational because φ is non-wasteful and the null object is not scarce.

To show that no blocking pair exists, we proceed by contradiction. Assume that $(i, a) \in N \times O$ blocks $\varphi(R)$, that is,

$$aP_i\varphi_i(R), \tag{A.10}$$

$$i \in C_a(\varphi_a(R) \cup \{i\}). \tag{A.11}$$

Let $N' = \varphi_a(R)$. N' has q_a elements by non-wastefulness of φ and (A.10). Fix a preference $R^{a\varphi_i(R)}_i$ for agent i, which ranks a first and $\varphi_i(R)$ second. Note that $(R^{a\varphi_i(R)}_i, R^a_{N'}, R_{N\setminus(N'\cup\{i\})})$ m.t. R at $\varphi(R)$ ($R^{a\varphi_i(R)}_i$ m.t. R_i at $\varphi_i(R)$ by (A.10), R^a_j m.t. R_j at $\varphi_j(R)$ for $j \in N'$ because $\varphi_j(R) = a$ by the definition of N', and the preferences of the agents outside $N' \cup \{i\}$ are identical under the two preference profiles). As φ satisfies weak Maskin monotonicity, it follows that for all $j \in N'$,

$$\varphi_j\big(R^{a\varphi_i(R)}_i, R^a_{N'}, R_{N\setminus(N'\cup\{i\})}\big) R^a_j \varphi_j(R) = a, \quad \text{hence}$$
$$\varphi_j\big(R^{a\varphi_i(R)}_i, R^a_{N'}, R_{N\setminus(N'\cup\{i\})}\big) = a.$$

Using φ's population monotonicity for the subset of agents $N' \cup \{i\}$ and the preference profile $(R^{a\varphi_i(R)}_i, R^a_{N'}, R_{N\setminus(N'\cup\{i\})})$, we obtain

$$\forall j \in N', \quad \varphi_j\big(R^{a\varphi_i(R)}_i, R^a_{N'}, R^{\emptyset}_{N\setminus(N'\cup\{i\})}\big) R^a_j \varphi_j\big(R^{a\varphi_i(R)}_i, R^a_{N'}, R_{N\setminus(N'\cup\{i\})}\big) = a.$$

From the definition of R^a, it follows that

$$\forall j \in N', \quad \varphi_j\big(R^{a\varphi_i(R)}_i, R^a_{N'}, R^{\emptyset}_{N\setminus(N'\cup\{i\})}\big) = a. \tag{A.12}$$

From the construction of C_a, (A.11) is equivalent to $\varphi_i(R^a_{N'\cup\{i\}}, R^{\emptyset}_{N\setminus(N'\cup\{i\})}) = a$. Note that $(R^{a\varphi_i(R)}_i, R^a_{N'}, R^{\emptyset}_{N\setminus(N'\cup\{i\})})$ m.t. $(R^a_{N'\cup\{i\}}, R^{\emptyset}_{N\setminus(N'\cup\{i\})})$ at $\varphi(R^a_{N'\cup\{i\}}, R^{\emptyset}_{N\setminus(N'\cup\{i\})})$ ($R^{a\varphi_i(R)}_i$ m.t. R^a_i at $\varphi_i(R^a_{N'\cup\{i\}}, R^{\emptyset}_{N\setminus(N'\cup\{i\})}) = a$ and the preferences of all other agents are identical under the two preference profiles). As φ satisfies weak Maskin monotonicity, it follows that

$$\varphi_i\big(R_i^{a\varphi_i(R)}, R_{N'}^a, R_{N\setminus(N'\cup\{i\})}^{\emptyset}\big) R_i^{a\varphi_i(R)} \varphi_i\big(R_{N'\cup\{i\}}^a, R_{N\setminus(N'\cup\{i\})}^{\emptyset}\big) = a, \quad \text{hence} \tag{A.13}$$

$$\varphi_i\big(R_i^{a\varphi_i(R)}, R_{N'}^a, R_{N\setminus(N'\cup\{i\})}^{\emptyset}\big) = a. \tag{A.14}$$

By (A.12) and (A.14), $\varphi_a(R_i^{a\varphi_i(R)}, R_{N'}^a, R_{N\setminus(N'\cup\{i\})}^{\emptyset}) \supset N' \cup \{i\}$, hence $\varphi(R_i^{a\varphi_i(R)}, R_{N'}^a, R_{N\setminus(N'\cup\{i\})}^{\emptyset})$ allocates a to at least $|N'| + 1 = q_a + 1$ agents, which is a contradiction with the feasibility of φ.

Step 3—*$\varphi(R) = \varphi^C(R)$ for all $R \in \mathcal{R}$.*

We state and prove the main part of this step as a separate lemma so that we can use it in the proof of Theorem 3 as well.

LEMMA 2: *Let C be an acceptant substitutable priority and suppose that φ is a stable allocation rule at C that satisfies weak Maskin monotonicity. Then φ is the deferred acceptance rule for C, that is, $\varphi = \varphi^C$.*

PROOF: Fix a preference profile R. For each $i \in N$, let R'_i be the truncation of R_i at $\varphi_i^C(R)$, that is, R_i and R'_i agree on the ranking of all proper objects, and a proper object is unacceptable under R'_i if and only if it is less preferred than $\varphi_i^C(R)$ under R_i.

We first establish that $\varphi^C(R)$ is the unique stable allocation at (R', C). Since $\varphi^C(R)$ is stable at (R, C), it is also stable at (R', C). By definition, $\varphi^C(R')$ is the agent-optimal stable allocation at (R', C), thus $\varphi^C(R')R'\varphi^C(R)$. This leads to

$$\varphi^C(R')R\varphi^C(R),$$

as R'_i is the truncation of R_i at $\varphi_i^C(R)$ for all $i \in N$. Then the stability of $\varphi^C(R')$ at (R', C) implies its stability at (R, C). But $\varphi^C(R)$ is the agent-optimal stable allocation at (R, C), so it must be that

$$\varphi^C(R)R\varphi^C(R').$$

The series of arguments above establishes that

$$\varphi^C(R) = \varphi^C(R').$$

Thus $\varphi^C(R)$ is the agent-optimal stable allocation at (R', C).

Let μ be a stable allocation at (R', C). We argue that $\mu = \varphi^C(R)$. Since $\varphi^C(R)$ is the agent-optimal stable allocation at (R', C), we have that $\varphi_i^C(R)R'_i\mu_i$ for all $i \in N$. Since μ is stable at (R', C) and R'_i is the truncation of R_i at $\varphi_i^C(R)$, it follows that $\mu_i \in \{\varphi_i^C(R), \emptyset\}$ for all $i \in N$. If $\mu_i \neq \varphi_i^C(R)$ for some agent $i \in N$, then $\varphi_i^C(R)P'_i\mu_i = \emptyset$ and $|\mu_{\varphi_i^C(R)}| < |\varphi^C_{\varphi_i^C(R)}(R)| \le q_{\varphi_i^C(R)}$, which is a contradiction with the stability of μ at (R', C) (as $C_{\varphi_i^C(R)}$ is acceptant). It follows that $\mu = \varphi^C(R)$, hence $\varphi^C(R)$ is the unique stable allocation at (R', C).

By hypothesis, φ is a stable allocation rule at C, thus $\varphi(R')$ is a stable allocation at (R', C). As $\varphi^C(R)$ is the unique stable allocation at (R', C), we need

$$\varphi(R') = \varphi^C(R).$$

We have that R m.t. R' at $\varphi(R')$ because R'_i is the truncation of R_i at $\varphi_i(R') = \varphi_i^C(R)$ for all $i \in N$. As φ satisfies weak Maskin monotonicity, it follows that $\varphi(R) R \varphi(R') = \varphi^C(R)$. Since $\varphi(R)$ is a stable allocation at (R, C) and $\varphi^C(R)$ is the agent-optimal stable allocation at (R, C), we obtain that $\varphi(R) = \varphi^C(R)$, finishing the proof of the lemma. *Q.E.D.*

We resume the proof of Step 3. By assumption, φ satisfies weak Maskin monotonicity. Step 1 shows that C is an acceptant substitutable priority and Step 2 proves that φ is a stable allocation rule at C, so φ satisfies all the hypotheses of Lemma 2. Therefore, $\varphi = \varphi^C$, which completes the proof of Step 3 and of the "if" part of the theorem. *Q.E.D.*

PROOF OF PROPOSITION 1: We prove each of the three implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) by contradiction.

To show (i) $\Rightarrow$ (ii), assume that φ^C is Pareto efficient, but not Maskin monotonic. Then there exist preference profiles R, R' such that R' m.t. R at $\varphi^C(R)$ and $\varphi^C(R') \neq \varphi^C(R)$. As φ^C satisfies weak Maskin monotonicity by Theorem 2, it follows that $\varphi^C(R')$ Pareto dominates $\varphi^C(R)$ at R'. Since R' m.t. R at $\varphi^C(R)$, this implies that $\varphi^C(R')$ Pareto dominates $\varphi^C(R)$ at R, which contradicts the assumption that φ^C is Pareto efficient.

To show (ii) $\Rightarrow$ (iii), assume that φ^C is Maskin monotonic, but not group strategy-proof. Then there exist $N' \subset N$ and preference profiles R, R' such that $\varphi_i^C(R'_{N'}, R_{N\setminus N'}) R_i \varphi_i^C(R)$ for all $i \in N'$, with strict preference for some i. For every $i \in N'$, let R''_i be a preference relation that ranks $\varphi_i^C(R'_{N'}, R_{N\setminus N'})$ first and $\varphi_i^C(R)$ second.[13] Clearly, $(R''_{N'}, R_{N\setminus N'})$ m.t. $(R'_{N'}, R_{N\setminus N'})$ at $\varphi^C(R'_{N'}, R_{N\setminus N'})$ and $(R''_{N'}, R_{N\setminus N'})$ m.t. R at $\varphi^C(R)$. Then the assumption that φ^C is Maskin monotonic leads to

$$\varphi^C(R'_{N'}, R_{N\setminus N'}) = \varphi^C(R''_{N'}, R_{N\setminus N'}) = \varphi^C(R),$$

which is a contradiction with $\varphi_i^C(R'_{N'}, R_{N\setminus N'}) P_i \varphi_i^C(R)$ for some $i \in N'$.

To show (iii) $\Rightarrow$ (i), suppose that φ^C is group strategy-proof, but not Pareto efficient. Then there exist a preference profile R and an allocation μ such that μ Pareto dominates $\varphi^C(R)$ at R. For every $i \in N$, let R'_i be a preference that ranks μ_i as the most preferred object. Clearly, μ is the agent-optimal stable allocation at (R', C), hence $\varphi^C(R') = \mu$. The deviation for all agents

[13]If $\varphi_i^C(R'_{N'}, R_{N\setminus N'}) = \varphi_i^C(R)$, then we simply require that R''_i rank $\varphi_i^C(R'_{N'}, R_{N\setminus N'})$ first.

in N to report R' rather than R leads to a violation of group strategy-proofness of φ^C. *Q.E.D.*

PROOF OF THEOREM 4: We prove the three implications (i) ⇒ (ii) ⇒ (iii) ⇒ (i).

To show (i) ⇒ (ii), assume that $\varphi = \varphi^C$ for some acceptant substitutable priority C and that φ is Pareto efficient. By the equivalence of properties (i) and (ii) in Proposition 1, φ satisfies Maskin monotonicity. By Theorem 2, φ satisfies non-wastefulness and population monotonicity.

To show (ii) ⇒ (iii), suppose that φ satisfies non-wastefulness, Maskin monotonicity, and population monotonicity. Since Maskin monotonicity implies weak Maskin monotonicity, Theorem 2 shows that $\varphi = \varphi^C$ for some acceptant substitutable priority C. As φ^C satisfies Maskin monotonicity by assumption, the equivalence of conditions (i) and (ii) in Proposition 1 implies that φ is Pareto efficient.

To show (iii) ⇒ (i), assume that φ satisfies Pareto efficiency, weak Maskin monotonicity, and population monotonicity. As Pareto efficiency implies non-wastefulness, by Theorem 2 we obtain that $\varphi = \varphi^C$ for some acceptant substitutable priority C. *Q.E.D.*

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