ADDENDUM TO: WHAT GOODS DO COUNTRIES TRADE? A QUANTITATIVE EXPLORATION OF RICARDO'S IDEAS

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Abstract. This addendum provides the proofs of Lemma 1, Theorem 1, Lemma 2, and Theorem 2 and the derivation of Equation (16) in Section 4.1.2 of our main paper.

Lemma 1. *Suppose that Assumptions A1-A4 hold. Then for any importer, j, any pair of exporters, i and i', and any pair of goods, k and k',*

$$\ln\left(\frac{x_{ij}^{k}x_{i'j}^{k'}}{x_{ij}^{k'}x_{i'j}^{k}}\right)=\theta\ln\left(\frac{z_{i}^{k}z_{i'}^{k'}}{z_{i}^{k'}z_{i'}^{k}}\right)-\theta\ln\left(\frac{d_{ij}^{k}d_{i'j}^{k'}}{d_{ij}^{k'}d_{i'j}^{k}}\right). \tag{A-1}$$

Proof of Lemma 1. By Assumption A4, we know that bilateral trade flows satisfy

$$x_{ij}^{k}=\frac{\sum_{\omega\in\Omega_{ij}^{k}}\left[p_{j}^{k}(\omega)\right]^{1-\sigma_{j}^{k}}}{\sum_{\omega\in\Omega}p_{j}^{k}(\omega)^{1-\sigma_{j}^{k}}}\cdot\alpha_{j}^{k}\mathrm{w}_{j}L_{j}\,.$$

Since $\Omega_{ij}^{k}\equiv\left\{\omega\in\Omega\,\middle|\,c_{ij}^{k}(\omega)=\min_{1\leq i'\leq I}c_{i'j}^{k}(\omega)\right\}$, this can be rearranged as

$$x_{ij}^{k}=\frac{\sum_{\omega\in\Omega}\left[p_{j}^{k}(\omega)\mathbb{I}\left\{c_{ij}^{k}(\omega)=\min_{1\leq i'\leq I}c_{i'j}^{k}(\omega)\right\}\right]^{1-\sigma_{j}^{k}}}{\sum_{\omega\in\Omega}p_{j}^{k}(\omega)^{1-\sigma_{j}^{k}}}\cdot\alpha_{j}^{k}\mathrm{w}_{j}L_{j}\,,$$

where the function $\mathbb{I}\{\cdot\}$ is the standard indicator function. By Assumption A1, $z_{i}^{k}(\omega)$ is independent and identically distributed (i.i.d.) across varieties so the same holds for $c_{ij}^{k}(\omega)$. In addition, $z_{i}^{k}(\omega)$ is i.i.d. across countries so $\mathbb{I}\left\{c_{ij}^{k}(\omega)=\min_{1\leq i'\leq I}c_{i'j}^{k}(\omega)\right\}$ is i.i.d. across varieties as well. This implies that $p_{j}^{k}(\omega)^{1-\sigma_{j}^{k}}$ and $p_{j}^{k}(\omega)^{1-\sigma_{j}^{k}}\cdot\mathbb{I}\left\{c_{ij}^{k}(\omega)=\min_{1\leq i'\leq I}c_{i'j}^{k}(\omega)\right\}$ are i.i.d. across varieties. Moreover, since $\sigma_{j}^{k}<1+\theta$ we have $E\left[p_{j}^{k}(\omega)^{1-\sigma_{j}^{k}}\right]<\infty$ so we can use the strong law of large numbers for i.i.d. random variables (e.g. Theorem 22.1 in Billingsley, 1995) and the continuous mapping theorem (e.g. Theorem 18.10 (i) in Davidson,

Date: April 18, 2011.

1994) to show that

$$x_{ij}^k = \frac{E\left[p_j^k(\omega)^{1-\sigma_j^k} \cdot \mathbb{1}\left\{c_{ij}^k(\omega) = \min_{1\leq i'\leq I} c_{i'j}^k(\omega)\right\}\right]}{E\left[p_j^k(\omega)^{1-\sigma_j^k}\right]} \cdot \alpha_j^k \mathrm{w}_j L_j \,.$$

Consider $H_{ij}^k(c_{1j}^k,\ldots,c_{Ij}^k) \equiv E\left[p_j^k(\omega)^{1-\sigma_j^k} \cdot \mathbb{1}\left\{c_{ij}^k(\omega) = \min_{1\leq i'\leq I} c_{i'j}^k(\omega)\right\}\right]$. Assumptions A1, A3 and straightforward computations yield

$$H_{ij}^k(c_{1j}^k,\ldots,c_{Ij}^k) = \Gamma\Big(\frac{\theta+1-\sigma_j^k}{\theta}\Big)\frac{(c_{ij}^k)^{-\theta}}{\left[\sum_{i'=1}^I (c_{i'j}^k)^{-\theta}\right]^{(\theta+1-\sigma_j^k)/\theta}}\,, \tag{A-2}$$

where $\Gamma(\cdot)$ is the Gamma function, $\Gamma(t) \equiv \int_0^{+\infty} v^{t-1}\exp(-v)dv$ for any $t>0$. Note that

$$E\left[p_j^k(\omega)^{1-\sigma_j^k}\right] = \sum_{i=1}^I H_{ij}^k(c_{1j}^k,\ldots,c_{Ij}^k)\,,$$

so that by using Equation (A-2) we get

$$E\left[p_j^k(\omega)^{1-\sigma_j^k}\right] = \Gamma\Big(\frac{\theta+1-\sigma_j^k}{\theta}\Big)\frac{1}{\left[\sum_{i'=1}^I (c_{i'j}^k)^{-\theta}\right]^{(1-\sigma_j^k)/\theta}}\,, \tag{A-3}$$

and hence

$$x_{ij}^k = \frac{(c_{ij}^k)^{-\theta}}{\sum_{i'=1}^I (c_{i'j}^k)^{-\theta}} \cdot \alpha_j^k \mathrm{w}_j L_j \,. \tag{A-4}$$

With iceberg trade costs, Assumption A2, we have $c_{ij}^k = d_{ij}^k \mathrm{w}_i / z_i^k$. Combining the previous expression with Equation (A-4) gives the result of Lemma 1. □

Theorem 1. *Suppose that Assumptions A1-A4 hold. Then for any importer, j, any pair of exporters, i and i', and any pair of goods, k and k',*

$$\ln\left(\frac{\widetilde{x}_{ij}^k \widetilde{x}_{i'j}^{k'}}{\widetilde{x}_{ij}^{k'} \widetilde{x}_{i'j}^k}\right) = \theta \ln\left(\frac{\widetilde{z}_i^k \widetilde{z}_{i'}^{k'}}{\widetilde{z}_i^{k'} \widetilde{z}_{i'}^k}\right) - \theta \ln\left(\frac{d_{ij}^k d_{i'j}^{k'}}{d_{ij}^{k'} d_{i'j}^k}\right), \tag{A-5}$$

where $\widetilde{x}_{ij}^k \equiv x_{ij}^k / \pi_{ii}^k$.

Proof of Theorem 1. We make use of the following Lemma.

Lemma 3. *Suppose that Assumption A2 holds. Then, for all countries i and goods k,*

$$\Omega_i^k = \left\{\omega \,\middle|\, c_{ii}^k(\omega) = \min_{1\leq i'\leq I} c_{i'i}^k(\omega)\right\}. \tag{A-6}$$

Proof of Lemma 3. We proceed by contradiction. Fix an exporter j, and suppose there exists a variety ω_0 of good k and a country $l \neq j$ such that:

$$\begin{cases} c^k_{jl}(\omega_0) = \min_{1\leq i'\leq I} c^k_{i'l}(\omega_0); \\ c^k_{jj}(\omega_0) \neq \min_{1\leq i'\leq I} c^k_{i'j}(\omega_0). \end{cases}$$

Then, there must be an exporter $i \neq j$ such that

$$\begin{cases} d^k_{jl} \cdot \mathrm{w}_j / z^k_j(\omega_0) \leq d^k_{il} \cdot \mathrm{w}_i / z^k_i(\omega_0); \\ d^k_{ij} \cdot \mathrm{w}_i / z^k_i(\omega_0) < d^k_{jj} \cdot \mathrm{w}_j / z^k_j(\omega_0). \end{cases}$$

Since $d^k_{jj} = 1$, multiplying the two inequalities above gives

$$d^k_{ij} \cdot d^k_{jl} < d^k_{il} \, ,$$

which contradicts Assumption A2. This completes the proof of Lemma 3.

Proof of Theorem 1 (continued). By definition, we know that $c^k_{ii}(\omega) = d^k_{ii}\mathrm{w}_i / z^k_i(\omega)$. Using Lemma 3 then yields

$$\text{(A-7)} \qquad \tilde{z}^k_i \equiv E\left[z^k_i(\omega)\,|\omega \in \Omega^k_i\right] = \frac{G_{ii}(c^k_{1i}, \ldots, c^k_{Ii})}{\mu^k_{ii}} \cdot d^k_{ii}\mathrm{w}_i \, ,$$

where we have let

$$\begin{aligned} G_{ii}(c^k_{1i}, \ldots, c^k_{Ii}) &\equiv E\left[(c^k_{ii}(\omega))^{-1}\mathbb{I}\left\{c^k_{ii}(\omega) = \min_{1\leq i'\leq I} c^k_{i'i}(\omega)\right\}\right], \\ \mu^k_{ii} &\equiv \Pr\left\{c^k_{ii}(\omega) = \min_{1\leq i'\leq I} c^k_{i'i}(\omega)\right\}. \end{aligned}$$

The expressions for $G_{ii}(c^k_{1i}, \ldots, c^k_{Ii})$ and μ^k_{ii} can easily be computed from the expression for $H^k_{ii}(c^k_{1i}, \ldots, c^k_{Ii})$ in proof of Lemma 1 when the result in Equation (A-2) is evaluated at $\sigma^k_i = 2$ and $\sigma^k_i = 1$, respectively. By Equation (A-2), we formally have

$$\begin{aligned} G_{ii}(c^k_{1i}, \ldots, c^k_{Ii}) &= \Gamma\Big(\frac{\theta-1}{\theta}\Big) \frac{(c^k_{ii})^{-\theta}}{\left[\sum_{i'=1}^I (c^k_{i'i})^{-\theta}\right]^{(\theta-1)/\theta}}, \\ \mu^k_{ii} &= \frac{(c^k_{ii})^{-\theta}}{\sum_{i'=1}^I (c^k_{i'i})^{-\theta}}. \end{aligned}$$

Hence,

$$\text{(A-8)} \qquad \tilde{z}^k_i = \Gamma\Big(\frac{\theta-1}{\theta}\Big) \frac{1}{\left[\sum_{i'=1}^I (c^k_{i'i})^{-\theta}\right]^{-1/\theta}} \cdot d^k_{ii}\mathrm{w}_i = z^k_i \cdot \Gamma\Big(\frac{\theta-1}{\theta}\Big) \left[\frac{(c^k_{ii})^{-\theta}}{\sum_{i'=1}^I (c^k_{i'i})^{-\theta}}\right]^{-1/\theta}.$$

Now, recall that we have defined $\pi^k_{ii} \equiv x^k_{ii}/[\sum_{i'=1}^I x^k_{i'i}]$. Using the expression for x^k_{ij} obtained in (A-4) it then follows that

$$\pi^k_{ii} = \mu^k_{ii} = \frac{(c^k_{ii})^{-\theta}}{\sum_{i'=1}^I (c^k_{i'i})^{-\theta}} \,. \tag{A-9}$$

Combining the two previous equations, we obtain

$$\widetilde{z}^k_i = z^k_i \cdot \Gamma\Big(\frac{\theta-1}{\theta}\Big)\left(\pi^k_{ii}\right)^{-1/\theta} . \tag{A-10}$$

Now, from Equation (A-4), we know that for every i and j,

$$x^k_{ij} = \frac{(d^k_{ij}\mathrm{w}_i/z^k_i)^{-\theta}}{\sum_{i'=1}^I (d^k_{i'j}\mathrm{w}_{i'}/z^k_{i'})^{-\theta}} \cdot \alpha^k_j \mathrm{w}_j L_j \,,$$

so combining with (A-10) and using $\widetilde{x}^k_{ij} = x^k_{ij}/\pi^k_{ii}$ gives

$$\widetilde{x}^k_{ij} = \left[\Gamma\left(\frac{\theta-1}{\theta}\right)\right]^{-\theta} \frac{(d^k_{ij}\mathrm{w}_i/\widetilde{z}^k_i)^{-\theta}}{\sum_{i'=1}^I (d^k_{i'j}\mathrm{w}_{i'}/z^k_{i'})^{-\theta}} \cdot \alpha^k_j \mathrm{w}_j L_j \,.$$

Analogously to Lemma 1, the result of Theorem 1 then follows. □

Lemma 2. *Suppose that Assumptions A1-A5 hold. Adjustments in absolute productivity, $\{Z_i\}_{i \neq i_0}$, can be computed as the solution of the system of equations*

$$\sum_{j=1}^I \sum_{k=1}^K \frac{\pi^k_{ij}\left(z^k_i/Z_i\right)^{-\theta} \alpha^k_j \gamma_j}{\sum_{i'=1}^I \pi^k_{i'j}\left(z^k_{i'}/Z_{i'}\right)^{-\theta}} = \gamma_i, \textit{ for all } i \neq i_0 \,. \tag{A-11}$$

Proof of Lemma 2. Throughout this proof, we use labor in country i_0 as our numeraire in the initial and counterfactual trade equilibrium: $\mathrm{w}_{i_0} = (\mathrm{w}_{i_0})' = 1$. By definition, we know that Z_i is chosen for any $i \neq i_0$ such that the value of the relative wage $(\mathrm{w}_i/\mathrm{w}_{i_0})'$ in the counterfactual equilibrium is the same as in the initial equilibrium $(\mathrm{w}_i/\mathrm{w}_{i_0})$. Thus Assumption A5 implies

$$\sum_{j=1}^I \sum_{k=1}^K (\pi^k_{ij})' \alpha^k_j \mathrm{w}_j L_j = \mathrm{w}_i L_i \,, \tag{A-12}$$

where $(\pi^k_{ij})'$ is the share of exports from country i in country j and industry k in the counterfactual equilibrium. Using Equation (A-4), one can easily check that

$$\pi^k_{ij} \equiv \frac{x^k_{ij}}{\sum_{i'=1}^I x^k_{i'j}} = \frac{\left(\mathrm{w}_i d^k_{ij}/z^k_i\right)^{-\theta}}{\sum_{i'=1}^I \left(\mathrm{w}_{i'} d^k_{i'j}/z^k_{i'}\right)^{-\theta}} \,, \tag{A-13}$$

and similarly that

$$(\pi_{ij}^k)' = \frac{\left[(\mathrm{w}_i)' d_{ij}^k / (z_i^k)'\right]^{-\theta}}{\sum_{i'=1}^I \left[(\mathrm{w}_{i'})' d_{i'j}^k / (z_{i'}^k)'\right]^{-\theta}} . \tag{A-14}$$

Combining Equations (A-13) and (A-14) and using the fact that the relative wages remain unchanged in the counterfactual equilibrium, we get after rearrangements

$$(\pi_{ij}^k)' = \frac{\pi_{ij}^k \left[z_i^k/(z_i^k)'\right]^{-\theta}}{\sum_{i'=1}^I \pi_{i'j}^k \left[z_{i'}^k/(z_{i'}^k)'\right]^{-\theta}} = \frac{\pi_{ij}^k (z_i^k/Z_i)^{-\theta}}{\sum_{i'=1}^I \pi_{i'j}^k (z_{i'}^k/Z_{i'})^{-\theta}} , \tag{A-15}$$

where the second equality uses $(z_i^k)' \equiv Z_i \cdot z_{i_0}^k$. Equations (A-12) and (A-15) imply

$$\sum_{j=1}^I \sum_{k=1}^K \frac{\pi_{ij}^k (z_i^k/Z_i)^{-\theta} \alpha_j^k \gamma_j}{\sum_{i'=1}^I \pi_{i'j}^k (z_{i'}^k/Z_{i'})^{-\theta}} = \gamma_i ,$$

where $\gamma_i \equiv \mathrm{w}_i L_i / \sum_{j=1}^I \mathrm{w}_j L_j$ is the share of country i in world income. □

Theorem 2. *Suppose that Assumptions A1-A5 hold. If we remove country i_0's Ricardian comparative advantage, then:*

(1) *Counterfactual changes in bilateral trade flows, x_{ij}^k, satisfy*

$$\widehat{x}_{ij}^k = \frac{\left(z_i^k/Z_i\right)^{-\theta}}{\sum_{i'=1}^I \pi_{i'j}^k \left(z_{i'}^k/Z_{i'}\right)^{-\theta}}, \text{ for all } i,\ j,\ k. \tag{A-16}$$

(2) *Counterfactual changes in country i_0's welfare, $W_{i_0} \equiv w_{i_0}/p_{i_0}$, satisfy*

$$\widehat{W}_{i_0} = \prod_{k=1}^K \left[\sum_{i=1}^I \pi_{ii_0}^k \left(\frac{z_i^k}{z_{i_0}^k Z_i}\right)^{-\theta}\right]^{\alpha_{i_0}^k/\theta} . \tag{A-17}$$

Proof of Theorem 2. Similar to previously and throughout this proof, we use labor in country i_0 as our numeraire in the initial and counterfactual trade equilibrium: $\mathrm{w}_{i_0} = (\mathrm{w}_{i_0})' = 1$.

1. *Counterfactual changes in bilateral trade flows, x_{ij}^k.*

Since the relative wages are unchanged in the counterfactual equilibrium, we must have

$$\hat{x}_{ij}^k = (x_{ij}^k)'/x_{ij}^k = (\pi_{ij}^k)'/\pi_{ij}^k .$$

Combining this observation with Equation (A-15), we obtain

$$\hat{x}_{ij}^k = \frac{(z_i^k/Z_i)^{-\theta}}{\sum_{i'=1}^I \pi_{i'j}^k (z_{i'}^k/Z_{i'})^{-\theta}} .$$

2. *Counterfactual changes in country i_0's welfare, $W_{i_0} \equiv \mathrm{w}_{i_0}/\,p_{i_0}$.*

By definition, we know that

$$\widehat{p}^k_{i_0} = \left(p^k_{i_0}\right)'/p^k_{i_0} = \left[\frac{\sum_{\omega\in\Omega}\left[p^k_{i_0}(\omega)\right]'^{\left(1-\sigma^k_{i_0}\right)}}{\sum_{\omega\in\Omega} p^k_{i_0}(\omega)^{1-\sigma^k_{i_0}}}\right]^{1/(1-\sigma^k_{i_0})}.$$

By invoking the strong law of large numbers for i.i.d. random variables and the continuous mapping theorem as we did in Theorem 1, then using Equation (A-3), we can rearrange the previous expression as

$$\widehat{p}^k_{i_0} = \left[\frac{\sum_{i=1}^{I}\left[(\mathrm{w}_i)'\,d^k_{ii_0}/(z^k_i)'\right]^{-\theta}}{\sum_{i=1}^{I}\left(\mathrm{w}_i d^k_{ii_0}/z^k_i\right)^{-\theta}}\right]^{-1/\theta}. \tag{A-18}$$

Combining Equations (A-13) and (A-18) and using the fact that the relative wages remain unchanged in the counterfactual equilibrium, we get after rearrangements

$$\widehat{p}^k_{i_0} = \left[\sum_{i=1}^{I}\pi^k_{ii_0}\left(\frac{z^k_i}{z^k_{i_0}Z_i}\right)^{-\theta}\right]^{-1/\theta}.$$

By definition of $p_{i_0} \equiv \prod_{k=1}^{K}\left(p^k_{i_0}\right)^{\alpha^k_{i_0}}$, we therefore have

$$\widehat{p}_{i_0} = \prod_{k=1}^{K}\left[\sum_{i=1}^{I}\pi^k_{ii_0}\left(\frac{z^k_i}{z^k_{i_0}Z_i}\right)^{-\theta}\right]^{-\alpha^k_{i_0}/\theta},$$

which immediately implies Equation (A-17). □

We conclude this online appendix by showing that for any pair of goods, k and k', and any pair of countries, i and i', Assumptions A1-A3 imply

$$\frac{\widetilde{z}^k_i\widetilde{z}^{k'}_{i'}}{\widetilde{z}^k_{i'}\widetilde{z}^{k'}_i} = \frac{E\left[p^k_{i'}(\omega)\middle|\,\Omega^k_{i'}\right]E\left[p^{k'}_i(\omega)\middle|\,\Omega^{k'}_i\right]}{E\left[p^k_i(\omega)\middle|\,\Omega^k_i\right]E\left[p^{k'}_{i'}(\omega)\middle|\,\Omega^{k'}_{i'}\right]},$$

as stated in Equation (16) of Section 4.1.2 in our main paper. $E\left[p^k_i(\omega)\middle|\,\Omega^k_i\right]$ can be readily computed from the expression for $H^k_{ii}(c^k_{1i},\ldots,c^k_{Ii})$ in the proof of Lemma 1 when the result in Equation (A-2) is evaluated at $\sigma^k_i = 0$ and $\sigma^k_i = 1$, respectively. Specifically, we have

$$E\left[p^k_i(\omega)\middle|\,\Omega^k_i\right] = \Gamma\left(\frac{\theta+1}{\theta}\right)\frac{1}{\left[\sum_{i'=1}^{I}(c^k_{i'i})^{-\theta}\right]^{1/\theta}}.$$

By Equation (A-8), we also know that

$$\tilde{z}_i^k = \Gamma\Big(\frac{\theta-1}{\theta}\Big)\frac{1}{\big[\sum_{i'=1}^{I}(c_{i'i}^k)^{-\theta}\big]^{-1/\theta}}\cdot \mathrm{w}_i,$$

where we have used the fact that $d_{ii}^k = 1$. Combining the two previous expressions for any pair of goods, k and k', we obtain

$$\frac{\tilde{z}_i^k}{\tilde{z}_i^{k'}} = \frac{E\left[p_i^{k'}(\omega)\middle|\,\Omega_i^{k'}\right]}{E\left[p_i^{k}(\omega)\middle|\,\Omega_i^{k}\right]},$$

from which the desired result follows.

References

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