BARGAINING IN DYNAMIC MARKETS

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Abstract. We study dynamic markets in which participants are randomly matched to bargain over the price of a heterogeneous good. There is a continuum of players drawn from a finite set of types. Players exogenously enter the market over time and then exit upon trading. At every date, the matching probabilities for each pair of types are endogenously determined by the distribution of traders in the market. A player's bargaining power at any stage depends on intra- and inter-temporal variations in the potential gains from trade, the feasible agreements at future dates, and the induced distribution of bargaining partners. We establish that an equilibrium always exists. Moreover, all equilibria that feature the same evolution of the macroeconomic variables are payoff equivalent. However, we show that multiple self-fulfilling expectations about the trajectory of the economy, generating distinct equilibrium dynamics and payoffs, may coexist. Our analysis extends and complements several models of bargaining in markets.

Keywords: bargaining, decentralized, dynamic markets, random matching, heterogeneous goods, equilibrium existence, multiplicity, iterated conditional dominance.

1. Introduction

We study decentralized dynamic markets in which traders bargain over the price of a heterogeneous good. The surplus that pairs of market participants can generate from trade may differ due to variations in valuations or good quality; cost of transportation between various locations; trade laws (tariffs, trade barriers, quality standards for imports); productivity and disutility of labor. The availability and size of the surplus may also depend on the strength of social relationships, business connections, and exposure to various advertising platforms. Product features that are relevant to customers also lead to match specific values. For instance, buyers of used cars care about the vehicle's make, mileage, manufacturing year, fuel efficiency, and so on. In the market for apartment rentals, search is typically driven by location, the number of bedrooms, and the quality of appliances.

The distribution of bargaining opportunities that market participants face may change over time. The stock of potential trading partners and the amount of surplus available at

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any date depend on the inflows of new players into the market and the outflows of players who complete transactions. Players need to forecast the evolution of the macroeconomy, as determined by the endogenous volume of trade and the relative matching probabilities induced by inflows and outflows, and negotiations should reflect the anticipated market conditions.

We analyze such decentralized markets in the context of an infinite horizon bargaining game played in discrete time. The set of player types is finite, and there is a continuum of players of each type. Players exogenously enter the game over time and leave only upon trading. In every period, a fraction of active traders is matched to bargain in pairs. The surplus available within every match depends on the pair of types involved. The frequency with which matches of each type form at a particular date depends on the distribution of trader types in the underlying market. Every player is involved in at most one match at a time. In any match, one of the two parties is designated to make an offer to the other specifying a division of the surplus created by the pair. If the other player accepts the offer, then the two parties exit the game with the shares agreed on. Otherwise, the match dissolves and the two players resume their search for trading partners in the next period. Players of any given type have a common discount factor.

Our setting encompasses a number of models from the literature on bargaining in markets.[1] The two-type case, in which pairs of players of the same type cannot generate surplus, effectively corresponds to the pioneering model of Rubinstein and Wolinsky (1985). Binmore and Herrero (1988 a,b) developed the study of the two-type case to non-stationary environments. In the dynamic market analyzed by Gale (1987), the heterogeneous reservation values of buyers and sellers determine the amount of surplus available in every buyer-seller match. Surplus functions may also capture network effects, as in the model of Manea (2011), where only pairs of traders linked in a network can engage in exchange.

As the opening remarks suggest, the structure of equilibria in our dynamic setting entails a complex relationship between several objects of infinite dimension. A player's payoff at any point in time incorporates the surplus heterogeneity within and across periods, the bargaining power of his partners, and the set of feasible agreements at future dates. The balance of bargaining power and incentives for agreements depend in turn on the distribution of player types at every stage and the induced path of matching frequencies. We characterize the formal connections between these equilibrium variables and establish that the bargaining game always admits an equilibrium. The proof technique may be useful in other dynamic environments. We note parenthetically that the result complements the analysis of Gale (1987), who explores properties of equilibria abstracting away from existence issues.

[1]Osborne and Rubinstein (1990) provide a survey of the early theoretical research in this area.

We establish a payoff equivalence result for equilibria that generate the same path of market distributions. Restrict attention to equilibria in which no (infinitesimal) player can affect the macroeconomic variables by unilaterally changing his strategy. In such equilibria players take the matching probabilities along the equilibrium path as given. Thus on-path incentives in the benchmark bargaining game are equivalent to those in an alternative model where the matching probabilities are exogenously specified. We show that the latter model can essentially be solved using iterated conditional dominance. Hence all equilibria of the model with exogenous matching probabilities are payoff equivalent. This conclusion generalizes uniqueness results from Binmore and Herrero (1988b) and Manea (2011). We develop a procedure to compute the unique equilibrium payoffs with any degree of accuracy.

Thus the model with exogenous matching probabilities provides a partial equilibrium approach to predicting payoffs for a given evolution of the macroeconomy. The properties of the unique payoffs compatible with a postulated market path play a key role in the proof of equilibrium existence for the benchmark model. Notwithstanding, the alternative model can also be interpreted as a free-standing depiction of situations in which players have stubborn beliefs about the evolution of the macroeconomy.

We show that the benchmark bargaining model does not necessarily have a unique equilibrium. Indeed, we produce an example that accommodates multiple consistent theories about the balance of bargaining power, feasible agreements, and the trajectory of the economy. We interpret the possibility of multiple equilibria as a manifestation of market sentiment. Expectations about future market developments play a crucial role in the dynamics of negotiations and can act as self-fulfilling prophecies.

Rubinstein and Wolinsky (1985), Gale (1987), and Manea (2011) consider stationary bargaining games in which players who reach agreement are replaced by identical, new players in the next round. Their characterizations of equilibrium outcomes are contingent on the economy being at a steady state. It is natural to ask how the distribution of player types is determined in the steady state of an economy with a constant stream of potential market entrants. The stationary bargaining games of the aforementioned papers can be interpreted as special instances of the model with exogenous matching probabilities. The uniqueness result from the latter model implies that any candidate steady state market composition is consistent with a unique payoff profile in equilibrium. Building on this result, Manea (2014) characterizes steady states in a version of the benchmark bargaining game in which new traders face type-dependent entry costs and optimally decide whether to join the market. In a steady state, the measure of players of each type who choose to enter the market needs to match the measure of players of the same type who trade and exit the game. Both inflows and outflows are endogenously determined: entry decisions hinge on how entry costs compare to payoffs in the bargaining game, while the balance of bargaining power and incentives for trade depend on the matching probabilities in the underlying market. Manea (2014) shows that the bargaining model admits a steady state for every configuration of sufficiently small entry costs.[2]

Other related steady state models of random matching and bargaining in addition to the research already mentioned have been considered by Burdett and Coles (1997), Shimer and Smith (2000), Atakan (2006), Satterthwaite and Shneyerov (2007), and Lauermann (2012). The further challenges posed by non-stationary economies are apparent in the work of Jackson and Palfrey (1998) and Shimer and Smith (2001). In the studies cited above each agent trades at most one indivisible good (or accepts a single match) throughout his presence in the marketplace. By contrast, Gale (1986a,b) and McLennan and Sonnenschein (1991) analyzed bargaining games in which players hold bundles of divisible goods and engage in multiple transactions over time.

The rest of the paper is organized as follows. The next section defines the benchmark model and establishes equilibrium existence. In Section 3, we introduce the model with exogenous matching probabilities and show that it is conditional dominance solvable. We discuss equilibrium multiplicity for the benchmark bargaining game in Section 4. Section 5 concludes and the Appendix contains the proofs.

2. The Benchmark Model

We consider dynamic markets with a *finite* set N of *populations* or *player types*. A pair of players from populations i and j can generate a *surplus* $s_{ij} = s_{ji} \geq 0$. In every period $t = 0, 1, \ldots$, an endogenously determined measure $\mu_{it} \geq 0$ of players i participates in the market. Formally, the set of traders of type i active at date t is indexed by the interval $[0, \mu_{it})$, from which it inherits the Borel measure. It is always the case that $\sum_{i \in N} \mu_{it} > 0$. Hence the *market composition* (or *distribution*) at date t is described by a profile of population sizes $\mu_t = (\mu_{it})_{i \in N} \in [0, \infty)^N \setminus \{\mathbf{0}\}$ ($\mathbf{0}$ denotes the zero vector in $\mathbb{R}^N$).

Players encounter bargaining partners, one at a time, according to a random matching process. Types are publicly observable, so players recognize the type of their partners. A *match* is an *ordered pair* $(i, j) \in N \times N$. In the match (i, j), player i assumes the role of the proposer, and j acts as the responder. The matching process is measure preserving, that is, for any measurable set of proposers i engaged in matches of type (i, j), the corresponding set of responders j is measurable and has the same measure. The *matching technology* β specifies the measure $\beta_{ijt}(\mu_t) \geq 0$ of proposers i involved in matches (i, j) in the market μ_t prevailing at date t;[3] since the matching process is measure preserving, the set of traders j

[2]The method of proof is also applied to establish the existence of steady states in a generalization of Shimer and Smith's (2000) search model.

[3]We allow for the possibility that players from the same population i are matched to one another, i.e., $\beta_{iit}(\mu_t) > 0$.

receiving offers from partners of type i also has measure $\beta_{ijt}(\mu_t)$. The function β_{ijt} is required to be continuous on $[0,\infty)^N \setminus \{\mathbf{0}\}$. No player is involved in more than one match (as either proposer or responder) at a time. This leads to the following constraint

$$\mu_{it} \geq \sum_{j\in N} \left(\beta_{ijt}(\mu_t) + \beta_{jit}(\mu_t)\right), \forall i \in N, \forall t \geq 0. \tag{2.1}$$

We assume that a positive measure of players is left unmatched every period, that is, for every date t market μ_t there exists a population i for which the inequality above is strict.

The matching technology treats all players of the same type symmetrically in the following sense. Each player of type i is equally likely to be one of the $\beta_{ijt}(\mu_t)$ proposers i involved in matches (i,j) in the period t market μ_t. Thus a player of type i is selected to make an offer to some trader j with *probability*

$$\pi_{ijt}(\mu_t) = \lim_{\substack{\tilde{\mu}_t \to \mu_t \\ \tilde{\mu}_{it} > 0}} \frac{\beta_{ijt}(\tilde{\mu}_t)}{\tilde{\mu}_{it}}. \tag{2.2}$$

For $\mu_{it} > 0$, the continuity of β_{ijt} implies that the limit above is well-defined and is simply given by $\beta_{ijt}(\mu_t)/\mu_{it}$. We assume that the limit also exists for all $\mu_t \in [0,\infty)^N \setminus \{\mathbf{0}\}$ with $\mu_{it} = 0$. Then the function π_{ijt} is continuous on $[0,\infty)^N \setminus \{\mathbf{0}\}$.

The probability that a type i receives an offer from any trader j in period t can be defined analogously, but is inconsequential in our model. As standard in bargaining with complete information, equilibrium agreements make the responder indifferent between accepting the offer and continuing the search. Hence the rate at which players receive offers does not affect equilibrium payoffs. We do not explicitly model the matching process since the functions π_{ijt} constitute a sufficient statistic for our analysis.[4]

Note that players drawn from populations of measure zero may be matched for bargaining with positive probability and enjoy positive profits. However, the existence of such players does not directly impact matching probabilities and expected payoffs for types represented

[4]We can construct a matching procedure that generates the desired matching probabilities for populations of positive measure by adapting "the roulette method" of Alos-Ferrer (1999). Identify every trader of type i active in the market at date t with some $\tilde{i} \in [0,\mu_{it})$. For every $i \in N$, let $f_i : [0,\mu_{it}) \to \{(i,j)|j \in N\} \cup \{(j,i)|j \in N\} \cup \{0\}$ be an arbitrary measurable function such that the Borel measures of the pre-images of (i,j) and (j,i) for $j \neq i$ are $\beta_{ijt}(\mu_t)$ and $\beta_{jit}(\mu_t)$, respectively, and the measure of the pre-image of (i,i) is $2\beta_{iit}(\mu_t)$. Let $(x_i)_{i\in N}$ be a collection of independent random variables, with x_i uniformly distributed over $[0,\mu_{it})$. For every realization of these variables $(\tilde{x}_i)_{i\in N}$, if $i \neq j$, then the sets of players $\tilde{i} \in [0,\mu_{it})$ and $\tilde{j} \in [0,\mu_{jt})$ satisfying

$$f_i((\tilde{i} + \tilde{x}_i) \bmod \mu_{it}) = (i,j) = f_j((\tilde{j} + \tilde{x}_j) \bmod \mu_{jt})$$

both have measure $\beta_{ijt}(\mu_t)$ (for $b > 0$, we use the notation $a \bmod b$ for the unique $c \in [0,b)$ such that $(a-c)/b$ is an integer). Then there exists a measure-preserving bijection from the former set to the latter, which we use to generate the matches (i,j). Similarly, we can match the mass of $2\beta_{iit}(\mu_t)$ players $\tilde{i}$ satisfying $f_i((\tilde{i} + \tilde{x}_i) \bmod \mu_{it}) = (i,i)$ with one another.

with positive measure in the market. Indeed, $\mu_{it} > 0$ and $\mu_{jt} = 0$, along with (2.1) and (2.2), imply that $\beta_{ijt}(\mu_t) = 0$ and $\pi_{ijt}(\mu_t) = 0$.

A salient matching technology, known in the literature as *linear search* (e.g., Diamond and Maskin (1979), Gale (1987), and Noldeke and Troger (2009)), is obtained by assuming that every player i meets a bargaining partner with a fixed probability p, and the conditional probability of i meeting a type j is given by the proportion of traders j in the market. Each player i is recognized as a proposer in half of the meetings with traders j. The corresponding matching technology is described by

$$\begin{aligned} \beta_{ijt}(\mu_t) &= \frac{p}{2}\frac{\mu_{it}\mu_{jt}}{\sum_{k\in N}\mu_{kt}} \\ \pi_{ijt}(\mu_t) &= \frac{p}{2}\frac{\mu_{jt}}{\sum_{k\in N}\mu_{kt}}, \forall i, j \in N, t \geq 0, \mu_t \in [0,\infty)^N \setminus \{\mathbf{0}\}. \end{aligned} \tag{2.3}$$

While it may be helpful to interpret the results in the context of this simple matching technology, it should be emphasized that our analysis applies for all matching processes that satisfy the minimal regularity conditions above.

The *benchmark bargaining game* is as follows. A measure $\lambda_{i0} \geq 0$ of players of type i is initially present in the game. We assume that $\sum_{i\in N}\lambda_{i0} > 0$ and also use the notation $\mu_0 = \lambda_0$. Every period $t = 0, 1, \ldots$, players are randomly matched to bargain according to $\beta_t(\mu_t)$. In a match (i, j), player i makes an offer to j specifying a division $(x, s_{ij} - x)$ of the surplus s_{ij} with $x \geq 0$.[5] If j accepts the offer, then i and j trade and exit the game with payoffs x and $s_{ij} - x$, respectively. If j rejects the offer, then the match dissolves and the two parties remain in the game for the next period. In period $t+1$, a measure $\lambda_{i(t+1)} \geq 0$ of new players i enters the market, joining the traders from earlier stages who have not previously reached an agreement. The total stock of traders i active at the beginning of period $t+1$ is denoted by $\mu_{i(t+1)}$.[6] Players of type i have a common discount factor $\delta_i \in (0, 1)$.[7]

The model is flexible in terms of the amount of information each player has about other players' past matches and outcomes. One possible treatment assumes perfect information, which entails that all traders observe the entire history of realized matches and ensuing negotiations. Alternatively, players may have partial knowledge of others' past bargaining encounters—e.g., each player observes only the outcomes of his own interactions; additionally, players may be aware of the realized matches, but not the details of each negotiation; or players learn only about the experience of their own population. However, we retain the key

[5]We allow for the possibility that $x > s_{ij}$, so that player i has the option of making offers that j rejects with certainty.

[6]The condition $\sum_{i\in N}\lambda_{i0} > 0$, along with the assumption that a positive measure of traders is left unmatched at every date, implies that $\mu_t \neq \mathbf{0}$ for all $t \geq 0$.

[7]A player who never reaches an agreement obtains a zero payoff.

assumptions that all active players observe the market composition μ_t at the beginning of every date t and that matched players know each other's type.

In the case of perfect information, we use the solution concept of subgame perfect equilibrium. For versions of the game with imperfect information, we introduce the concept of *belief-independent equilibrium*. A strategy profile constitutes a belief-independent equilibrium for an extensive form game if every player's strategy is optimal conditional on each information set with respect to all possible beliefs at that information set. In other words, a strategy profile is a belief-independent equilibrium if it is sequentially rational with respect to every profile of beliefs.[8] For either solution concept, we restrict attention to equilibria that are *robust* in the sense that no (infinitesimal) player can affect the population sizes along the path by unilaterally changing his strategy. Our results apply for all types of information structure discussed above, and henceforth we simply refer to the corresponding solution concept as equilibrium.

Several technical assumptions are necessary to guarantee that the stock of players active in the market at every stage is measurable. The matching process needs to satisfy the measurability requirements above (see footnote 4 for a fully specified example). In order to account for outflows, we need to restrict attention to strategy profiles under which the set of players who trade at every date is measurable. We also need to restrict attention to pure strategies (Aumann 1964). Note that the macroeconomic effects of mixing can be replicated by the idea of distributional strategies (Milgrom and Weber 1985).

The existence of an equilibrium in our dynamic setting is not straightforward because the rate of departures following agreements is endogenously determined in equilibrium, and the matching probabilities at every date depend on the endogenous distribution of trader types in the underlying market. The path of matching probabilities in turn determines the balance of bargaining power and the incentives for agreements. Our first result establishes equilibrium existence. The proof of this and subsequent results may be found in the Appendix.

Theorem 1. *The bargaining game always admits an equilibrium.*

In Section 4 we show that the equilibrium is not necessarily unique. However, a partial uniqueness result holds for robust equilibria that lead to the same path of market distributions. More generally, payoff equivalence is obtained in an environment where the path of matching probabilities is exogenously given. The latter model, which we formally introduce in the next section, can be used to describe behavior on the equilibrium path in the benchmark model—in particular, it provides a building block for the proof of Theorem 1—but is also of independent interest.

[8]Note that in our setting each information set includes knowledge of the current market distribution, so that all agents can correctly asses the matching probabilities.

To sketch the proof of Theorem 1, define the spaces of paths of agreement rates, market distributions, matching probabilities, and feasible payoffs, respectively, as follows

$$\begin{aligned} \mathcal{A} &= \{(a_{ijt})_{i,j \in N, t \geq 0} | a_{ijt} \in [0,1], \forall i,j \in N, t \geq 0\} \\ \mathcal{M} &= \{(\mu_{it})_{i \in N, t \geq 0} | \mu_0 = \lambda_0; \mu_{it} \in [0, \sum_{\tau=0}^{t} \lambda_{i\tau}], \forall i \in N, t \geq 1\} \\ \mathcal{P} &= \{(p_{ijt})_{i,j \in N, t \geq 0} | p_{ijt} \in [0,1], \forall i,j \in N, t \geq 0\} \\ \mathcal{V} &= \{(v_{it})_{i \in N, t \geq 0} | v_{it} \in [0, \max_{j \in N} s_{ij}], \forall i \in N, t \geq 0\}. \end{aligned} \tag{2.4}$$

We construct a correspondence $f : \mathcal{A} \rightrightarrows \mathcal{A}$ by composing the correspondence α and the functions v^*, π, κ, where

$$\mathcal{A} \xrightarrow{\kappa} \mathcal{M} \xrightarrow{\pi} \mathcal{P} \xrightarrow{v^*} \mathcal{V} \overset{\alpha}{\rightrightarrows} \mathcal{A}.$$

These mappings are specified as follows

- $\kappa(a)$ describes the path of the economy under the assumption that a fraction a_{ijt} of the date t matches (i,j) results in agreement
- $\pi(\mu)$ denotes the matching probabilities along the market path μ, as specified by (2.2) (with a minor abuse of notation)
- $v^*(p)$ represents the unique equilibrium payoffs in the model with an exogenous path of matching probabilities p (characterized by Theorem 2 in Section 3)
- $\alpha_{ijt}(v)$ defines the set of agreement rates that are incentive compatible for matches (i,j) at time t if bargaining proceeds assuming that the disagreement payoffs at $t+1$ are given by $v_{i(t+1)}$ and $v_{j(t+1)}$, respectively.

Note that while κ and π stem from the physical constraints of the environment, v^* and α reflect (hypothetical) equilibrium conditions.

We can verify that $f = \alpha \circ v^* \circ \pi \circ \kappa$ satisfies the hypotheses of the Kakutani-Fan-Glicksberg theorem, and thus it has a fixed point a^*. We then construct a robust equilibrium in which agreements arise according to a^*, the economy follows the path $\kappa(a^*)$, and payoffs are given by $v^*(\pi(\kappa(a^*)))$. At stages where the trajectory of the economy diverges from $\kappa(a^*)$, strategies are derived from a fixed point of an appropriately modified correspondence. Departures from the market path expected under the latter strategies are treated analogously, and so on.

The sketch of the proof above provides intuition about how different elements of the game fit together. In particular, it highlights the relationship between payoffs and matching probabilities, which we further explore in the next section.

Remark 1. The structure of agreements may seem an unusual starting point for our fixed point construction. Paths of payoffs and market distributions constitute more natural primitives for describing equilibrium outcomes. These variables suggest the study of the following map compositions

$$\mathcal{V} \overset{\alpha}{\rightrightarrows} \mathcal{A} \overset{\kappa}{\to} \mathcal{M} \overset{\pi}{\to} \mathcal{P} \overset{v^*}{\to} \mathcal{V}$$
$$\mathcal{M} \overset{\pi}{\to} \mathcal{P} \overset{v^*}{\to} \mathcal{V} \overset{\alpha}{\rightrightarrows} \mathcal{A} \overset{\kappa}{\to} \mathcal{M}.$$

However, neither of the compositions above is necessarily convex valued due to generic nonliniarities in κ. Then standard fixed point theorems are not applicable.

3. An Alternative Model

We consider the following *model with exogenous matching probabilities*. Traders with types drawn from the finite set N are active in the market at dates $t = 0, 1, \ldots$ We are agnostic about the composition of the market at each date. Every player of type i gets the opportunity to propose a division of the surplus s_{ij} to a trader j in period t with an exogenous probability $p_{ijt} \geq 0$.[9] [10] Thus the *path of matching probabilities* $p = (p_{ijt})_{i,j \in N, t \geq 0}$ is an element of the set $\mathcal{P}$ defined in the previous section. Players remain in the market until they trade. Discount factors and payoffs are specified as in the benchmark model.

The sketch of the "game" above is purposely vague regarding the set of new traders entering the market at every date, the exact matching procedure, and the information structure. It is conceivable that knowledge of these elements would allow players to make complex inferences about the trajectory of the economy. However, it turns out that the nature of these inferences does not affect equilibrium outcomes in the *class of games* sharing the features outlined above. We are able to make sharp predictions about equilibrium behavior without keeping track of the inflows and outflows, the details of the matching procedure, and the beliefs players hold. Indeed, we show that the matching probabilities p uniquely determine the balance of bargaining power at every date.

Technically, one can imagine that matching probabilities are held fixed under the matching technology from the benchmark model by adjusting the inflows into the market in response to the outflows of traders reaching agreements. As suggested previously, the analysis of the model with exogenous matching probabilities can be alternatively regarded as a partial equilibrium approach to predicting payoffs for a certain evolution of macroeconomic conditions in the benchmark model.

As a free-standing piece, the model describes a market with behavioral participants. Players start with identical beliefs regarding the path of matching probabilities and never revise these expectations in response to information they receive. This is reasonable in a setting where agents rely on public predictions for the macroeconomic variables and ignore evidence

[9] We maintain the assumption that trader types are observable within matches.

[10] As in the benchmark model, the probability with which a player i receives an offer from some trader j is irrelevant to the analysis.

that is inconsistent with their projections. In a large market where mistakes are possible, traders may assume that their own past interactions and observations do not necessarily reflect future trends.

We show that in this strategic environment all belief-independent equilibria are payoff equivalent. A stronger result holds: behavior is essentially pinned down by a process of iterated conditional dominance analogous to the one proposed by Fudenberg and Tirole (1991, Section 4.6) in the context of multi-stage games with observed actions. In our setting, an action a available to some player i at an information set h is *conditionally dominated* if for every belief ν over decision nodes in h, any strategy of i that assigns positive probability to a is strictly dominated by some other strategy when i's payoffs are evaluated from the perspective of information set h based on the beliefs ν. *Iterated conditional dominance* is the process that sequentially eliminates all actions that are conditionally dominated at any information set given opponents' strategies surviving earlier stages of elimination. The following result characterizes the strategies that survive iterated conditional dominance and establishes the existence and payoff equivalence of equilibria.

Theorem 2. *For every $p \in \mathcal{P}$, there exists a unique payoff vector $(v^*_{it}(p))_{i \in N, t \geq 0}$ such that any bargaining game embedded in the model with exogenous matching probabilities p has the following properties.*

(i) The only date t actions that may survive iterated conditional dominance specify that all players of type i reject any offer smaller than $\delta_i v^*_{i(t+1)}(p)$ and accept any offer greater than $\delta_i v^*_{i(t+1)}(p)$.

(ii) In every belief-independent equilibrium, the expected payoff of any trader i active at date t is $v^*_{it}(p)$.

(iii) The equilibrium payoffs $(v^*_{it}(p))_{i \in N, t \geq 0}$ constitute the unique bounded solution $(v_{it})_{i \in N, t \geq 0}$ to the system of equations

$$v_{it} = \sum_{j \in N} p_{ijt} \max\left(s_{ij} - \delta_j v_{j(t+1)}, \delta_i v_{i(t+1)}\right) + \left(1 - \sum_{j \in N} p_{ijt}\right) \delta_i v_{i(t+1)}. \tag{3.1}$$

(iv) A belief-independent equilibrium exists.

(v) For every $i \in N, t \geq 0$, the payoffs $v^*_{it}(p)$ vary continuously in p (with respect to the product topology on $\mathcal{P}$).

Corollary 1. *In any robust equilibrium of the benchmark bargaining game, all traders of the same type active in the market at a given date have identical expected payoffs.*

Corollary 2. *All robust equilibria of the benchmark bargaining game that generate the same path of market distributions are payoff equivalent.*

Corollary 3. *If the matching probabilities p are time-invariant, i.e., $p_{ijt} = p_{ij(t+1)}$ for all $i, j \in N, t \geq 0$, then the equilibrium payoffs $v^*(p)$ are stationary, i.e., $v^*_{it}(p) = v^*_{i(t+1)}(p)$ for all $i \in N, t \geq 0$.*

The latter corollary follows from the finding that $v^*(p)$ is the only bounded solution to the system of equations (3.1) (Theorem 2.ii). For a proof, note that when $p_{ijt} = p_{ij(t+1)}$ for all $i, j \in N, t \geq 0$, the profile v' defined by $v'_{it} = v^*_{i(t+1)}(p)$ also constitutes a bounded solution for the system. Thus $v'_{it} = v^*_{it}(p)$, which means that $v^*_{it}(p) = v^*_{i(t+1)}(p)$, for all $i \in N, t \geq 0$. In particular, the corollary shows that equilibrium payoffs in an economy with a time-invariant matching technology and a steady state market distribution are constant over time. Hence the payoff stationarity standardly postulated in the analysis of steady states can be derived as an equilibrium implication of the underlying assumption that the market composition does not vary over time.

The proof of Theorem 2 can be easily adapted to show uniqueness of security equilibrium payoffs for the model with exogenous matching probabilities. The latter equilibrium concept has been introduced by Binmore and Herrero (1988b).[11] The alternative statement of Theorem 2 asserting payoff equivalence of security equilibria generalizes Theorem 6.3 of Binmore and Herrero (1988b) to settings with more than two types. In turn, the latter result represents an extension of the analysis of Rubinstein and Wolinsky (1985) to non-stationary environments with identical buyers and sellers.

The derivation of bounds for offers (and payoffs) surviving iterated conditional dominance rely on implicit conjectures about which matches lead to trade. To deliver unique payoffs, the bounds need to reflect precise estimates of the best and worst case scenarios for every player and potential bargaining partners. The main difficulty lies in determining whether the best and worst case scenarios for every match involve an agreement.[12]

In general, solving the infinite system of equations (3.1) that characterizes equilibrium payoffs may be intractable. Nonetheless, we can implement the following computational procedure to estimate the equilibrium payoffs. Define the sequences $(m^k_{it})_{i \in N, t \geq 0}$ and $(M^k_{it})_{i \in N, t \geq 0}$

[11]The relationship between iterated conditional dominance and security equilibrium has not been established for general dynamic games (private communication with Ken Binmore).

[12]In the unique equilibrium for the two-type setting of Binmore and Herrero (1988b) all matches result in agreement at every date.

recursively for $k = 0, 1, \ldots$ as follows

$$m^0_{it} = 0, M^0_{it} = \max_{j\in N} s_{ij} \tag{3.2}$$

$$m^{k+1}_{it} = \sum_{j\in N} p_{ijt} \max\left(s_{ij} - \delta_j M^k_{j(t+1)}, \delta_i m^k_{i(t+1)}\right) + \left(1 - \sum_{j\in N} p_{ijt}\right) \delta_i m^k_{i(t+1)} \tag{3.3}$$

$$M^{k+1}_{it} = \sum_{j\in N} p_{ijt} \max\left(s_{ij} - \delta_j m^k_{j(t+1)}, \delta_i M^k_{i(t+1)}\right) + \left(1 - \sum_{j\in N} p_{ijt}\right) \delta_i M^k_{i(t+1)}. \tag{3.4}$$

The proof of Theorem 2 establishes that for all $k \geq 0, i \in N, t \geq 0$, under the strategies that survive iterated conditional dominance, every player of type i rejects any offer smaller than $\delta_i m^k_{i(t+1)}$ and accepts any offer greater than $\delta_i M^k_{i(t+1)}$ in period t regardless of the identity of the proposer. Both sequences $(m^k_{it})_{k\geq 0}$ and $(M^k_{it})_{k\geq 0}$ converge monotonically to $v^*_{it}(p)$ as $k \to \infty$, and $v^*_{it}(p) \in [m^k_{it}, M^k_{it}]$ for all $k \geq 0$. We also show that for every $i \in N, t \geq 0, k \geq 0$,

$$0 \leq M^k_{it} - m^k_{it} \leq (\max_{j\in N} \delta_j)^k \max_{j,j'\in N} s_{jj'}.$$

Therefore, the equilibrium payoffs $v^*_{i0}(p)$ of initial market participants can be approximated by the interval $[m^k_{i0}, M^k_{i0}]$ with precision that improves exponentially in k. Note that the number of steps required to compute m^k_{i0} and M^k_{i0} is linear in k.

4. Equilibrium Multiplicity

In this section we analyze the structure of equilibria of the bargaining game in a two-population setting, $N = \{1, 2\}$. We identify a range of parameters for which multiple equilibria exist. Assume that $s_{11} = a \in (1, 2], s_{12} = s_{22} = 1$ and $\delta_1 = \delta_2 = \delta \in [0, 1)$. Suppose that the initial market distribution is given by $\mu_{10} = x \in [1/2, 1), \mu_{20} = 1 - x$ and that no new players enter the economy after time 0 ($\lambda_{it} = 0$ for all $t \geq 1$).

The possibility of positive value matches within the same population ($s_{11}, s_{22} > 0$) is not crucial to our conclusion. Indeed, the qualitative findings of this section extend to a two-sided setting in which each of the two populations is divided into two subpopulations of equal sizes and only matches between traders from different subpopulations create positive surplus. The example presumes the existence of complementarities among players of type 1. In a labor market application, population 1 would consist of skilled workers and successful companies, while population 2 contains unproductive workers and firms. In the marriage market, players of type 1 could be men and women with liberal views and type 2 would designate the conservative counterpart.

Players are matched to bargain following the protocol from (2.3) with $p = 1/2$. Thus the probability that a player i is selected to make an offer to some player j in the period t market

μ_t is given by

$$\pi_{ijt}(\mu_t) = \frac{\mu_{jt}}{4(\mu_{1t} + \mu_{2t})}.$$

Hence the proportion of players of type 1 present in the market, $\mu_{1t}/(\mu_{1t} + \mu_{2t})$, constitutes a sufficient statistic for the matching probabilities at date t. We refer to this ratio as the *index* of market μ_t.

We inquire into the existence of two classes of robust equilibria. In one class all matches along the equilibrium path result in agreement, while in the other only players of the same type reach agreement at any date. We refer to the former as *hybrid equilibria* and to the latter as *assortative equilibria*. Either type of equilibrium leads to a particular path of market distributions and is consistent with a unique payoff profile in light of Corollary 2. The *total welfare* of these equilibria can be evaluated as x times the (common) expected equilibrium payoff (at $t = 0$) of a player of type 1 plus $1 - x$ times the payoff of a player 2.

Note that under the assumed matching technology exactly a half of each population is matched for bargaining in every market. This means that the market index along the path of the hybrid equilibrium is given by x at every point in time. Then the unique payoffs are stationary (Corollary 3), which makes the computation of payoffs in the candidate equilibrium straightforward. The payoff formulae for an assortative equilibrium are not as tractable. If agreements arise as postulated in the latter equilibrium, play proceeds from a market with index y to one with index $y(2-y)/(1+2y(1-y))$. In particular, the market index declines over time. The non-trivial evolution of the market index (and matching probabilities) over time complicates the estimation of the range of parameters where the two types of equilibria (co)exist and makes welfare comparisons between equilibria technically difficult.

Proposition 1 below shows that the two types of equilibria coexist for an open set of parameter values. When both equilibria exist, players of type 1 are weakly better off in the hybrid equilibrium, while players 2 prefer the assortative one. However, the two types of equilibria cannot be consistently ranked with respect to total welfare.

Proposition 1. *Fix $a \in (1, 2]$.*

(i) For every $x \in [1/2, 1)$, there exist unique $\bar{\delta}(x)$ and $\underline{\delta}(x)$ such that a hybrid equilibrium exists if and only if $\delta \leq \bar{\delta}(x)$, and an assortative equilibrium exists if and only if $\delta \geq \underline{\delta}(x)$.

(ii) If $x \in ((a+1)/4, 1)$, then $\bar{\delta}(x) > \underline{\delta}(x)$ and both equilibria exist for $\delta \in [\underline{\delta}(x), \bar{\delta}(x)]$.

(iii) For every profile of parameters with $x \in [1/2, 1)$ such that both types of equilibria exist, the payoff of a player of type 1 in the hybrid equilibrium is at least as high as in the assortative one. Players of type 2 have the opposite (weak) preferences over the two equilibria.

(iv) *The two types of equilibria are not consistently ranked in terms of total welfare: for every* $a \in (1, 4/3)$*, there exists* $\varepsilon > 0$ *such that the hybrid equilibrium generates greater welfare than the assortative equilibrium for* $x \in ((a+1)/4, (a+1)/4 + \varepsilon)$ *and* $\delta = \bar{\delta}(x)$*, and the comparison is reversed for* $x \in (1 - \varepsilon, 1)$ *and* $\delta = \bar{\delta}(x)$*.*

To gain some intuition into the coexistence of the two equilibria, note first that players of type 1 are intrinsically more powerful because they can generate a surplus $a > 1$ when matched to bargain with one another, while all other pairs of types create only a unit surplus. Moreover, players 1 are given the opportunity to realize the surplus a frequently since population 1 constitutes a proportion $x \geq 1/2$ of the economy. By the same token, players of type 2 are more likely to be matched with players from population 1 than with other players 2. All matches for population 2 generate one unit of surplus, but players 1 are relatively stronger than players 2, so players of type 2 often encounter unfavorable partners. Thus the matching process further boosts the bargaining power of players 1 and undermines the position of players 2. We refer to the impact of the greater amount of surplus available within population 1 on the balance of bargaining power as the *surplus effect*, and to the ramifications of this effect, amplified by the prevalence of population 1 in the economy via matching probabilities, as the *frequency effect*.

Consider now a hybrid equilibrium. As explained earlier, the market index is constant along the equilibrium path. Population 2 allows the frequency effect to propagate over time by trading with players of type 1. In effect, population 1 exploits the self-inflicted weakness of population 2. The dynamics is different in the context of an assortative equilibrium. By rejecting agreements with population 1, players 2 secure a market path with declining indices and diminishing frequency effect. The bargaining position of players 2 steadily improves over time, and the prospect of higher future payoffs makes trade with population 1 suboptimal. Therefore, the divergence of market paths in the two equilibria creates differences in the magnitude of the frequency effect and shifts the balance of bargaining power and the incentives for trade between the two populations.

The two equilibria embody contrasting expressions of *market sentiment*. On the one hand, in the hybrid equilibrium players 2 hold the pessimistic belief that mixed matches result in agreement. A persistent frequency effect is expected to emerge. On the other hand, in the assortative equilibrium players 2 optimistically anticipate that mixed agreements do not take place. The frequency effect gradually declines. In both cases, the predicted trajectory of the economy becomes a self-fulfilling prophecy: the anticipated agreements are incentive compatible.

Remark 2. Shimer and Smith (2001) discuss a related two-type example in the context of a continuous-time search model in which players who reach agreements remain matched for an exogenous and stochastic amount of time after which they reenter the search pool. However, their multiplicity result relies on a non-generic specification of the surplus profile s (it is assumed that $s_{11} < 0$ and that the ratio s_{12}/s_{22} is a certain function of other model parameters). By contrast, the multiplicity conclusion of Proposition 1.ii extends to a neighborhood of the chosen s (including instances with $s_{12} \neq s_{22}$). Indeed, the proof reveals that for $\delta \in (\underline{\delta}(x), \bar{\delta}(x))$ players have strict incentives for agreements and disagreements in the two equilibria we identify. Then a continuity argument shows that both types of equilibria are robust to small perturbations in s.

Remark 3. The analysis of this section is also reminiscent of the multiplicity of steady states in a two-type example from the search model of Burdett and Coles (1997). It is important to clarify the differences. Burdett and Coles fix some stationary inflows and restrict attention to steady states. The initial market composition is endogenously determined in their model. The two types of equilibria they construct start with different market compositions and induce distinct paths of constant market indices. By comparison, we allow for non-stationary dynamics in a setting where the initial market distribution is exogenously given. The paths of the market index in our equilibria originate from the same point and diverge gradually. In particular, the assortative equilibrium features a declining path of market indices.

5. Conclusion

We analyzed a general model of bargaining in decentralized dynamic markets. The model features multiple populations that share heterogeneous trading opportunities among them. The inflows of new players are exogenous and possibly non-stationary. The distribution of trading opportunities at any date is determined by the path of inflows and the volume of trade prior to that date. At every point in time, the matching probabilities for any pair of player types are endogenously derived from the underlying market distribution. In this setting, the bargaining power of market participants coevolves over time in relation to the structure of agreements, the path of matching frequencies, and the overall trajectory of the economy. Our comprehensive framework provides insights into rich market dynamics.

We established that an equilibrium always exists. We also proved that all robust equilibria leading to the same evolution of the economy are payoff equivalent. The unique equilibrium payoffs consistent with a given market path can be computed using an iterative method. However, equilibrium outcomes are not necessarily unique. We showed by example that multiple self-fulfilling beliefs about the trajectory of the economy may coexist, giving rise to different equilibrium dynamics.

A significant part of the existing literature on bargaining in markets focuses on the relatively more tractable analysis of steady states. The benchmark bargaining model introduced in this article provides a natural framework for investigating the conditions under which steady states emerge. Manea (2014) builds on results developed here to provide theoretical foundations for steady states.

Appendix A. Proofs

Proof of Theorem 1. We construct a robust equilibrium in which deviations by a single player do not affect the behavior of any trader who does not receive offers from him. Since every player interacts only with a measure zero of other traders, no player can influence the path of market distributions under the constructed strategies. By Corollary 2, the induced market path uniquely pins down the equilibrium payoffs. It is then sufficient to check incentives for the actions specified along the equilibrium path. Strategies following deviations by a positive measure of players can be specified analogously, and so on.

Recall the definition of the sets of paths of possible fractions of agreeing pairs $\mathcal{A}$, market distributions $\mathcal{M}$, matching probabilities $\mathcal{P}$, and feasible payoffs $\mathcal{V}$ from (2.4). Each of the four sets can be regarded as a topological vector space via a natural embedding in the space $\mathbb{R}^{\mathbb{N}}$ (the countable product of the set of real numbers) endowed with the standard product topology. Note that the product topology on $\mathbb{R}^{\mathbb{N}}$ is metrizable, so the characterizations of closed sets and continuous functions in terms of convergent sequences apply for each of the four sets (Theorem 2.40, [1]). The spaces $\mathcal{A}, \mathcal{M}, \mathcal{P}, \mathcal{V}$ are compact by Tychonoff's theorem.

We construct the correspondence $f : \mathcal{A} \rightrightarrows \mathcal{A}$ by composing the correspondence α and the functions v^*, π, κ, where

$$\mathcal{A} \overset{\kappa}{\to} \mathcal{M} \overset{\pi}{\to} \mathcal{P} \overset{v^*}{\to} \mathcal{V} \overset{\alpha}{\rightrightarrows} \mathcal{A}.$$

Thus $f = \alpha \circ v^* \circ \pi \circ \kappa$, where π is given by (2.2)[13] and v^* is derived from Theorem 2, while κ and α are defined below. We will show how fixed points of f can be used to describe an equilibrium path in the benchmark bargaining game.

For any $a \in \mathcal{A}$, the sequence $\kappa(a)$ describes the path of the economy under the assumption that a fraction a_{ijt} of the matches (i, j) result in agreement at time t. Hence $\kappa(a)$ is recursively defined by

$$\begin{aligned} \kappa_{i0}(a) &= \lambda_{i0}, \forall i \in N \\ \kappa_{i(t+1)}(a) &= \kappa_{it}(a) + \lambda_{i(t+1)} - \sum_{j \in N} \left(a_{ijt}\beta_{ijt}(\kappa_t(a)) + a_{jit}\beta_{jit}(\kappa_t(a)) \right), \forall i \in N, t \geq 0. \end{aligned}$$

For any $v \in \mathcal{V}$, the set $\alpha_{ijt}(v)$ consists of the possible rates of agreement among the proposer-responder pairs (i, j) matched at time t, assuming that bargaining proceeds as if expected period $t+1$ payoffs (in case of disagreement) were given by v_{t+1}. In this scenario,

[13] Although $\pi(\mu)$ is not defined if $\mu_t = \mathbf{0}$ for some t, this will not become an issue because $\kappa(\mathcal{A})$ does not contain such μ's.

the fraction of pairs (i, j) that reach agreement is $0, 1$, or any number in $[0, 1]$ depending on whether $\delta_i v_{i(t+1)} + \delta_j v_{j(t+1)}$ is strictly greater, strictly smaller, or equal to s_{ij}, respectively. Thus

$$\alpha_{ijt}(v) = \begin{cases} \{0\} & \text{if } \delta_i v_{i(t+1)} + \delta_j v_{j(t+1)} > s_{ij} \\ [0,1] & \text{if } \delta_i v_{i(t+1)} + \delta_j v_{j(t+1)} = s_{ij} \\ \{1\} & \text{if } \delta_i v_{i(t+1)} + \delta_j v_{j(t+1)} < s_{ij} \end{cases}.$$

Our first goal is to apply the Kakutani-Fan-Glicksberg theorem (Corollary 17.55, [1]) to establish that $f = \alpha \circ v^* \circ \pi \circ \kappa$ has a fixed point. We then show how fixed points of f translate into equilibrium behavior. Note that the definitions of κ and π, along with the continuity of β (assumed) and v^* (Theorem 2), imply that the function $v^* \circ \pi \circ \kappa$ is continuous. Since the correspondence α has a closed graph, it follows that $f = \alpha \circ (v^* \circ \pi \circ \kappa)$ also has a closed graph. Furthermore, f takes non-empty convex values because α does. Clearly, $\mathcal{A}$ is a non-empty compact convex subset of a topological vector space that is linearly homeomorphic to $\mathbb{R}^{\mathbb{N}}$; the latter is a locally convex Hausdorff space (Theorem 16.2, [1]). Thus $f : \mathcal{A} \rightrightarrows \mathcal{A}$ satisfies all the hypotheses of the Kakutani-Fan-Glicksberg theorem, and it must have a fixed point.

We next demonstrate how fixed points of f map into equilibria of the bargaining game. Let a be a fixed point of f. We construct an equilibrium in which the economy follows the path $\kappa(a)$ and payoffs are given by $v^*(\pi(\kappa(a)))$. As long as the market path does not diverge from $\kappa(a)$, strategies are specified as follows. In a match (i, j) at time t, player i offers j the amount $x := \delta_j v^*_{j(t+1)}(\pi(\kappa(a)))$ if $a_{ijt} > 0$ and a negative amount (see footnote 5) otherwise. Player j accepts all offers greater than x and rejects all offers smaller than x. Furthermore, a proportion a_{ijt} of the responders j receiving the offer x from proposers i accepts it.[14] Clearly, if players conform to the prescribed behavior, then the market follows the path $\kappa(a)$.

The description of actions along the equilibrium path is incomplete in that it does not specify which fraction a_{ijt} of responders j must accept the stipulated offer from proposers i at time t in case $a_{ijt} \in (0, 1)$. One may be concerned that any concrete procedure selecting a set of agreements leads to heterogeneity in the expected payoffs of players of the same type at date t, but it turns out that payoffs are not affected by the selection procedure.[15] More specifically, we establish that expected payoffs under the constructed strategies are given by $v^*(\pi(\kappa(a)))$ regardless of the unspecified details. In a brief abuse of notation, we write v^* for $v^*(\pi(\kappa(a)))$ and π for $\pi(\kappa(a))$. Let $\mathcal{U}_{it}$ denote the set of expected payoffs that players of type i may achieve at date t under the collection of strategy profiles with the properties outlined above. We seek to show that $\mathcal{U}_{it} = \{v^*_{it}\}$.

[14] If $a_{ijt} = 1$ then all (as opposed to "almost all") players j accept the offer x from any i at date t.

[15] Note that the "symmetric" treatment whereupon each player j accepts the equilibrium offer from a type i with probability a_{ijt} is not feasible due to the (unavoidable) restriction to pure strategies.

Each value in $\mathcal{U}_{it}$ is obtained as an expectation over several types of payoffs, depending on the *outcome* for the particular *player* i at time t, as follows

- elements of $\delta_i \mathcal{U}_{i(t+1)}$, for situations in which the *player* does not reach an agreement (including events where he is not matched for bargaining at date t)
- $\delta_i v^*_{i(t+1)}$, in instances where the *player* accepts an offer
- $s_{ij} - \delta_j v^*_{j(t+1)}$, for cases in which the *player's* offer to j is accepted.

The term $s_{ij} - \delta_j v^*_{j(t+1)}$ appears in the expectation with positive probability only if $a_{ijt} > 0$. Since $a \in f(a) = \alpha(v^*)$ by definition, the condition $a_{ijt} > 0$ implies that $s_{ij} - \delta_j v^*_{j(t+1)} \geq \delta_i v^*_{i(t+1)}$. If the latter constraint holds with equality, then $s_{ij} - \delta_j v^*_{j(t+1)}$ simply enters the expectation as $\delta_i v^*_{i(t+1)}$. Otherwise, we have $s_{ij} - \delta_j v^*_{j(t+1)} > \delta_i v^*_{i(t+1)}$, so $a_{ijt} = 1$, which implies that all players j accept the offer $\delta_j v^*_{j(t+1)}$ from any i at date t (see footnote 14). In this case, the value $s_{ij} - \delta_j v^*_{j(t+1)}$ is weighted in the expectation by the probability π_{ijt}. To sum up, any payoff in $\mathcal{U}_{it}$ can be represented as a convex combination of elements of $\delta_i \mathcal{U}_{i(t+1)}$ as well as terms $\delta_i v^*_{i(t+1)}$ and $s_{ij} - \delta_j v^*_{j(t+1)}$, where the latter receives positive weight—equal to π_{ijt}—only if $s_{ij} - \delta_j v^*_{j(t+1)} > \delta_i v^*_{i(t+1)}$. Formally, for all $u \in \mathcal{U}_{it}$, there exist $w \in \text{co}(\mathcal{U}_{i(t+1)})$ and $q \in [0,1]$ such that

$$u = \sum_{\{j \in N | s_{ij} - \delta_j v^*_{j(t+1)} > \delta_i v^*_{i(t+1)}\}} \pi_{ijt} \left(s_{ij} - \delta_j v^*_{j(t+1)}\right) + \left(1 - q - \sum_{\{j \in N | s_{ij} - \delta_j v^*_{j(t+1)} > \delta_i v^*_{i(t+1)}\}} \pi_{ijt}\right) \delta_i v^*_{i(t+1)} + q \delta_i w.$$

By Theorem 2,

$$v^*_{it} = \sum_{j \in N} \pi_{ijt} \max\left(s_{ij} - \delta_j v^*_{j(t+1)}, \delta_i v^*_{i(t+1)}\right) + \left(1 - \sum_{j \in N} \pi_{ijt}\right) \delta_i v^*_{i(t+1)}, \tag{A.1}$$

which can be rewritten as

$$v^*_{it} = \sum_{\{j \in N | s_{ij} - \delta_j v^*_{j(t+1)} > \delta_i v^*_{i(t+1)}\}} \pi_{ijt} \left(s_{ij} - \delta_j v^*_{j(t+1)}\right) + \left(1 - \sum_{\{j \in N | s_{ij} - \delta_j v^*_{j(t+1)} > \delta_i v^*_{i(t+1)}\}} \pi_{ijt}\right) \delta_i v^*_{i(t+1)}.$$

We immediately obtain that

$$\sup_{u \in \mathcal{U}_{it}} |u - v^*_{it}| \leq \sup_{w \in \text{co}(\mathcal{U}_{i(t+1)}), q \in [0,1]} q \delta_i |w - v^*_{i(t+1)}| \leq \delta_i \sup_{u \in \mathcal{U}_{i(t+1)}} |u - v^*_{i(t+1)}|.$$

Iterating the inequalities above, and observing that the sequence $(v^*_{i\tau})_{\tau \geq 0}$ and the sets $(\mathcal{U}_{i\tau})_{\tau \geq 0}$ are uniformly bounded, we conclude that $\sup_{u \in \mathcal{U}_{it}} |u - v^*_{it}| = 0$, which means that $\mathcal{U}_{it} = \{v^*_{it}\}$, for all t. Therefore, the constructed strategies yield an expected payoff of v^*_{it} for all players i active in the market at date t.

We can finally prove that players do not have incentives to deviate from the prescribed behavior as long as the economy follows the trajectory $\kappa(a)$. Note that the single deviation principle applies to our setting. Since under the constructed strategies players cannot unilaterally influence the path of the economy and the expected payoffs v^* satisfy (A.1), we can easily check that no player has a profitable one-shot deviation from the specified equilibrium play. The construction of strategies and the verification of incentives following deviations by a positive measure of traders from the path $\kappa(a)$ proceeds similarly, using a fixed point of an appropriately modified correspondence (the set $\mathcal{M}$ and the function κ need to be redefined taking into account the market composition at the first stage where divergence occurs). □

Proof of Theorem 2. (i) We refer to strategies that assign positive probability only to actions that survive iterated conditional dominance as "surviving strategies." Recall the definition of the sequences $(m^k_{it})_{i\in N, t\geq 0}$ and $(M^k_{it})_{i\in N, t\geq 0}$ from (3.2)-(3.4). We simultaneously establish the following claims by induction on k. Under all surviving strategies, in period t every player of type i

(1) rejects any offer smaller than $\delta_i m^k_{i(t+1)}$ (regardless of the identity of the proposer)
(2) has an expected payoff (at the beginning of the period) of at most M^k_{it}
(3) accepts any offer greater than $\delta_i M^k_{i(t+1)}$ (regardless of the identity of the proposer)
(4) does not make offers greater than $\delta_j M^k_{j(t+1)}$ in matches (i, j).

For the base case $k = 0$, claims (1) and (2) hold trivially. We also note at this stage that claims (3) and (4) follow from (2) for all k. Indeed, suppose that claim (2) holds for some k. Fix a period t information set where i receives some offer $x > \delta_i M^k_{i(t+1)}$. Any strategy under which i rejects the offer x in period t leads to a period $t+1$ expected payoff of at most $M^k_{i(t+1)}$ under the surviving strategies. Hence such strategies are conditionally dominated by accepting x at the information set under consideration. We now show that claim (3) implies (4). Let $y > \delta_j M^k_{j(t+1)}$, and consider all strategies under which i offers y to some j in period t at a particular information set. If, as per claim (3), j accepts every offer greater than $\delta_j M^k_{j(t+1)}$, then each of the latter strategies is conditionally dominated by any strategy that prescribes an offer in the interval $(\delta_j M^k_{j(t+1)}, y)$ at the given information set.

Therefore, we only need to prove the induction hypotheses (1) and (2) for step $k+1$, assuming that the four claims hold for all earlier steps. Consider a period t information set where some player i has to respond to an offer $x < \delta_i m^{k+1}_{i(t+1)}$. We argue that accepting the offer x is conditionally dominated for player i by the following plan of action for sufficiently small $\varepsilon > 0$. Player i rejects any offer received at dates $t' \geq t$. When selected to make an offer to some j at time $t' = t+1, t+2, \ldots, t+k+1$, player i offers $\delta_j M^{k+t+1-t'}_{j(t'+1)} + \varepsilon$ if $s_{ij} - \delta_j M^{k+t+1-t'}_{j(t'+1)} > \delta_i m^{k+t+1-t'}_{i(t'+1)}$; otherwise i makes an unacceptable offer (e.g., specifying a negative payoff for j). Player i makes unacceptable offers when selected as a proposer after date $t+k+1$. By the induction hypothesis, all players j accept the offers $\delta_j M^{k+t+1-t'}_{j(t'+1)} + \varepsilon$ at time $t' = t+1, t+2, \ldots, t+k+1$. Note that

$$m^{k+1}_{i(t+1)} = \sum_{\{j \in N | s_{ij} - \delta_j M^k_{j(t+2)} > \delta_i m^k_{i(t+2)}\}} p_{ijt}(s_{ij} - \delta_j M^k_{j(t+2)}) + \left(1 - \sum_{\{j \in N | s_{ij} - \delta_j M^k_{j(t+2)} > \delta_i m^k_{i(t+2)}\}} p_{ijt}\right) \delta_i m^k_{i(t+2)},$$

and we can use an analogous equation to expand the term $m^k_{i(t+2)}$ in the expression above, and then $m^{k-1}_{i(t+3)}$ in the resulting equation, and so on until we reach the variable $m^0_{i(t+k+2)} (= 0)$. The resulting formula for $m^{k+1}_{i(t+1)}$ proves that the strategy constructed above generates an expected period t payoff for i of $\delta_i m^{k+1}_{i(t+1)}$ as $\varepsilon \to 0$ under the surviving strategies for the opponents. Hence this strategy conditionally dominates accepting x in period t if $\varepsilon > 0$ is sufficiently small.

We now show that all surviving strategies deliver expected payoffs of at most M^{k+1}_{it} at the beginning of period t to the players of type i present in the game at that time. Consider a period t information set where i is given the opportunity to make an offer to j. By the induction hypothesis, player j rejects any offer lower than $\delta_j m^k_{j(t+1)}$. When j rejects an offer, i can expect a period $t+1$ payoff of at most $M^k_{i(t+1)}$ under the surviving strategies. Hence i cannot make an offer that generates an expected payoff greater than $\max(s_{ij} - \delta_j m^k_{j(t+1)}, \delta_i M^k_{i(t+1)})$. By the induction hypothesis, any period t action of some player j specifying an offer greater than $\delta_i M^k_{i(t+1)}$ for i is eliminated in the process of iterated conditional dominance. Also by the induction hypothesis, in all cases where i does not reach an agreement in period t, he enjoys a period $t+1$ expected payoff of at most $M^k_{i(t+1)}$. Therefore, i's date t payoff under the surviving strategies cannot exceed the expression on the right-hand side of (3.4), which defines M^{k+1}_{it}.

Our next goal is to show that the sequences $(m^k_{it})_{k \geq 0}$ and $(M^k_{it})_{k \geq 0}$ converge to a common limit. One can easily demonstrate by induction that for all $i \in N, t \geq 0$,

- the sequence $(m^k_{it})_{k \geq 0}$ is increasing in k
- the sequence $(M^k_{it})_{k \geq 0}$ is decreasing in k
- $\max_{j \in N} s_{ij} \geq M^k_{it} \geq m^k_{it} \geq 0$ for all $k \geq 0$.

Hence the sequences $(m^k_{it})_{k \geq 0}$ and $(M^k_{it})_{k \geq 0}$ convergent. We now prove that they have the same limit.

Let $D^k = \sup_{i\in N, t\geq 0}[M^k_{it} - m^k_{it}]$. We have

$$\begin{aligned}
D^{k+1} &= \sup_{i\in N,t\geq 0}[M^{k+1}_{it} - m^{k+1}_{it}] \\
&= \sup_{i\in N,t\geq 0}\left[\sum_{j\in N} p_{ijt}\max\left(s_{ij} - \delta_j m^k_{j(t+1)}, \delta_i M^k_{i(t+1)}\right) + \Big(1 - \sum_{j\in N} p_{ijt}\Big)\delta_i M^k_{i(t+1)}\right. \\
&- \left.\sum_{j\in N} p_{ijt}\max\left(s_{ij} - \delta_j M^k_{j(t+1)}, \delta_i m^k_{i(t+1)}\right) - \Big(1 - \sum_{j\in N} p_{ijt}\Big)\delta_i m^k_{i(t+1)}\right] \\
&= \sup_{i\in N,t\geq 0}\left[\sum_{j\in N} p_{ijt}\left(\max\left(s_{ij} - \delta_j m^k_{j(t+1)}, \delta_i M^k_{i(t+1)}\right) - \max\left(s_{ij} - \delta_j M^k_{j(t+1)}, \delta_i m^k_{i(t+1)}\right)\right)\right. \\
&+ \left.\Big(1 - \sum_{j\in N} p_{ijt}\Big)\delta_i\left(M^k_{i(t+1)} - m^k_{i(t+1)}\right)\right] \\
&\leq \sup_{i\in N,t\geq 0}\left[\sum_{j\in N} p_{ijt}\max\left(\delta_j(M^k_{j(t+1)} - m^k_{j(t+1)}), \delta_i(M^k_{i(t+1)} - m^k_{i(t+1)})\right) + \Big(1 - \sum_{j\in N} p_{ijt}\Big)\delta_i D^k\right] \\
&\leq \max_{j\in N}\delta_j D^k,
\end{aligned}$$

where the first inequality is a consequence of the following observation.

Lemma 1. *For all $w_1, w_2, w_3, w_4 \in \mathbb{R}$,*

$$|\max(w_1, w_2) - \max(w_3, w_4)| \leq \max(|w_1 - w_3|, |w_2 - w_4|).$$

Proof of Lemma 1. Suppose $w_1 = \max(w_1, w_2, w_3, w_4)$. Then

$$|\max(w_1, w_2) - \max(w_3, w_4)| = w_1 - \max(w_3, w_4) \leq w_1 - w_3 \leq \max(|w_1 - w_3|, |w_2 - w_4|).$$

The proof is similar for the cases when w_2, w_3, or w_4 is equal to $\max(w_1, w_2, w_3, w_4)$. □

It follows that $D^k \leq (\max_{j\in N}\delta_j)^k D^0 = (\max_{j\in N}\delta_j)^k \max_{j,j'\in N} s_{jj'}$ for all $k \geq 0$. Therefore, for every $i \in N, t \geq 0$, we have

$$0 \leq M^k_{it} - m^k_{it} \leq (\max_{j\in N}\delta_j)^k \max_{j,j'\in N} s_{jj'}, \forall k \geq 0,$$

which implies that the sequences $(m^k_{it})_{k\geq 0}$ and $(M^k_{it})_{k\geq 0}$ have the same limit, which we denote by $v^*_{it}(p)$. We omit the parameter p in $v^*(p)$ until we address the issue of continuity with respect to p.

Recall that iterated conditional dominance predicts that in period t every player of type i rejects offers smaller than $\delta_i m^k_{i(t+1)}$ and accepts offers greater than $\delta_i M^k_{i(t+1)}$. Since

$$\lim_{k\to\infty} m^k_{i(t+1)} = \lim_{k\to\infty} M^k_{i(t+1)} = v^*_{i(t+1)},$$

it follows that only actions specifying that i reject offers smaller than $\delta_i v^*_{i(t+1)}$ and accept offers greater than $\delta_i v^*_{i(t+1)}$ at time t can survive iterated conditional dominance.

(ii) Note that all actions used with positive probability in any belief-independent equilibrium must survive iterated conditional dominance. Then step (2) in the proof by induction from part (i) demonstrates that each player i obtains an expected payoff of at most M^k_{it} at the beginning of period t in every equilibrium. In the inductive argument we also constructed a sequence of strategies for i that, under the surviving actions of the opponents, generates a limit payoff for i of $m^{k+1}_{i(t+1)}$ at the beginning of period $t+1$. A reindexing of that construction leads to strategies that deliver a limit period t payoff of m^k_{it} to i. In every equilibrium, i must not find it profitable to deviate to any of the latter strategies, so his period t expected payoff should be at least m^k_{it}. Since $\lim_{k\to\infty} m^k_{it} = \lim_{k\to\infty} M^k_{it} = v^*_{it}$, the arguments above establish that in every belief-independent equilibrium any player i present in the game at the beginning of period t has an expected payoff of v^*_{it}.

(iii) Taking the limit $k \to \infty$ in (3.3), we obtain the following system of equations for v^*

$$v^*_{it} = \sum_{j\in N} p_{ijt} \max\left(s_{ij} - \delta_j v^*_{j(t+1)}, \delta_i v^*_{i(t+1)}\right) + \left(1 - \sum_{j\in N} p_{ijt}\right)\delta_i v^*_{i(t+1)}, \forall i \in N, t \geq 0. \tag{A.2}$$

Thus we showed indirectly that the system (3.1) has a bounded solution. Inequalities similar to those from the inductive proof demonstrate that any two payoff vectors v and v' that solve (3.1) must satisfy

$$\max_{i\in N} |v_{it} - v'_{it}| \leq \max_{j\in N} \delta_j \max_{i\in N} |v_{i(t+1)} - v'_{i(t+1)}|.$$

If the components of v and v' are uniformly bounded, then we can immediately conclude that $v = v'$. Therefore, v^* is the unique bounded solution for the system of equations (3.1).

(iv) We claim that the following strategy profile constitutes an equilibrium. In a match (i, j) formed at date t, player i offers $\delta_j v^*_{j(t+1)}$ to j if $\delta_i v^*_{i(t+1)} + \delta_j v^*_{j(t+1)} \leq s_{ij}$ and proposes a negative payoff for j otherwise. At time t, any player j accepts all offers greater than or equal to $\delta_j v^*_{j(t+1)}$ and rejects all offers smaller than that amount. In what follows, we show that the strategies above generate expected payoffs of v^*_{it} for all players of type i active at date t in the game. Then one can easily check that the constructed strategies constitute an equilibrium (the single-deviation principle extends straightforwardly to the present setting).

Fix a trader of type i participating in the market at time t. Let $q_{ijt'}$ denote the probability that this player accepts an offer at date $t' \geq t$ from proposers of type j under the strategies constructed above.[16] We rewrite equation (A.2) as follows

$$v_{it}^* = \sum_{\{j\in N|\delta_i v_{i(t+1)}^* + \delta_j v_{j(t+1)}^* \le s_{ij}\}} \left(p_{ijt}(s_{ij} - \delta_j v_{j(t+1)}^*) + q_{ijt}\delta_i v_{i(t+1)}^*\right) + \left(1 - \sum_{\{j\in N|\delta_i v_{i(t+1)}^* + \delta_j v_{j(t+1)}^* \le s_{ij}\}} (p_{ijt} + q_{ijt})\right) \delta_i v_{i(t+1)}^*.$$

Substituting an analogous formula for $v_{i(t+1)}^*$ in the last term of the equation for v_{it}^*, then a similar formula for $v_{i(t+2)}^*$ in the last term of the proxy for $v_{i(t+1)}^*$, and so on, we find that v_{it}^* represents the expected value—evaluated at date t, using discount factor δ_i—of a stochastic prize generated as follows. At each date $t' \ge t$, conditional on not having received a prize by that time, for every $j \in N$ with $\delta_i v_{i(t'+1)}^* + \delta_j v_{j(t'+1)}^* \le s_{ij}$, the prizes $s_{ij} - \delta_j v_{j(t'+1)}^*$ and $\delta_i v_{i(t'+1)}^*$ are realized with respective probabilities $p_{ijt'}$ and $q_{ijt'}$ (all events are mutually exclusive; a prize is not awarded in period t' with conditional probability $1 - \sum_{\{j\in N|\delta_i v_{i(t'+1)}^* + \delta_j v_{j(t'+1)}^* \le s_{ij}\}} (p_{ijt'} + q_{ijt'})$). Note that the strategies constructed above lead to the same distribution over outcomes for the fixed player i at dates $t' \ge t$ as the stochastic prize. Hence the constructed strategies yield expected payoffs of v_{it}^* for all players of type i active in period t, as claimed.

(v) Fix $i \in N$ and $t \ge 0$. To show that $v_{it}^*(p)$ varies continuously in p, fix $\varepsilon > 0$ and let k be such that

$$(\max_{j\in N} \delta_j)^k \max_{j,j'\in N} s_{jj'} < \varepsilon/3.$$

The definition of M_{it}^k relies on the matching probabilities p, and we instate the notation $M_{it}^k(p)$ to highlight this dependence. The resulting function M_{it}^k is obviously continuous in p. Then any given p has a neighborhood P such that

$$|M_{it}^k(p) - M_{it}^k(p')| < \varepsilon/3, \forall p' \in P.$$

Earlier arguments show that for all $p' \in P$,

$$v_{it}^*(p') \in [m_{it}^k(p'), M_{it}^k(p')]$$
$$0 \le M_{it}^k(p') - v_{it}^*(p') \le M_{it}^k(p') - m_{it}^k(p') \le (\max_{j\in N} \delta_j)^k \max_{j,j'\in N} s_{jj'} < \varepsilon/3.$$

It follows that

$$|v_{it}^*(p) - v_{it}^*(p')| \le |v_{it}^*(p) - M_{it}^k(p)| + |M_{it}^k(p) - M_{it}^k(p')| + |M_{it}^k(p') - v_{it}^*(p')| < \varepsilon, \forall p' \in P,$$

which completes the proof of continuity. □

[16]As footnote 10 asserts, the model with exogenous matching probabilities does not impose any restrictions on the frequencies at which players receive offers. Hence, for a *given* player i, the probability $q_{ijt'}$ is derived from the underlying matching procedure in the particular game form under consideration and the constructed strategies. The argument applies independently for every trader of type i.

Proof of Proposition 1. It is useful to first explore properties of the two types of equilibria for a given δ and varying x, and then apply the findings in the context of a fixed x and changing δ.

Equilibrium analysis for fixed δ and variable x.

Hybrid equilibria. We first inquire into the existence of hybrid equilibria for economies with initial market index $x \in [1/2, 1)$. As argued in Section 4, the market index must be constantly given by x along the equilibrium path. By Corollary 3, payoffs in a hybrid equilibrium are unique and stationary. The payoffs $(u_1(x), u_2(x))$ for the two player types solve the linear system

$$\begin{aligned} u_1(x) &= \frac{x}{4}(a - \delta u_1(x)) + \frac{1-x}{4}(1 - \delta u_2(x)) + \frac{3}{4}\delta u_1(x) \\ u_2(x) &= \frac{x}{4}(1 - \delta u_1(x)) + \frac{1-x}{4}(1 - \delta u_2(x)) + \frac{3}{4}\delta u_2(x). \end{aligned}$$

The unique solution to the system is

$$\begin{aligned} u_1(x) &= \frac{1}{2(2-\delta)} - \frac{\delta x^2(a-1)}{2(2-\delta)(4-3\delta)} + \frac{x(a-1)}{4-3\delta} \\ u_2(x) &= \frac{1}{2(2-\delta)} - \frac{\delta x^2(a-1)}{2(2-\delta)(4-3\delta)}. \end{aligned}$$

Incentives for all matched pairs to trade, as assumed in a hybrid equilibrium, require that $u_1(x) \geq 0, u_2(x) \geq 0, 2\delta u_1(x) \leq a, \delta(u_1(x) + u_2(x)) \leq 1, 2\delta u_2(x) \leq 1$. One can show that for every $x \in [1/2, 1)$, the inequalities $u_2(x) \geq 0$ and $\delta(u_1(x) + u_2(x)) \leq 1$ imply that all other incentive constraints are satisfied. Indeed, since

$$u_1(x) - u_2(x) = \frac{x(a-1)}{4-3\delta} > 0,$$

the following conditions hold:

$$\begin{aligned} u_2(x) \geq 0 &\Rightarrow u_1(x) \geq 0 \\ \delta(u_1(x) + u_2(x)) \leq 1 &\Rightarrow 2\delta u_2(x) \leq 1. \end{aligned}$$

To see that $\delta(u_1(x) + u_2(x)) \leq 1$ implies $2\delta u_1(x) \leq a$, note that the former inequality leads to

$$2\delta u_1(x) \leq 1 + \delta(u_1(x) - u_2(x)) = 1 + \delta\frac{x(a-1)}{4-3\delta} < a.$$

The last inequality is equivalent to $\delta(x+3) < 4$, which holds for all $\delta < 1, x < 1$.

Note that $u_2(x)$ is decreasing in x, so

$$u_2(x) \geq \lim_{y \to 1} u_2(y) = \frac{4 - (2+a)\delta}{2(2-\delta)(4-3\delta)} > 0,$$

as by assumption, $\delta < 1, a \leq 2$. Thus a hybrid equilibrium exists if and only if $\delta(u_1(x) + u_2(x)) \leq 1$. To study the latter inequality, define $f : [1/2, 1) \to \mathbb{R}$ by $f(x) = 1 - \delta(u_1(x) + u_2(x))$.

We have

$$\lim_{y \to 1} f(y) = \frac{2(1-\delta)(4-(2+a)\delta)}{(2-\delta)(4-3\delta)} > 0$$

because $(2+a)\delta < 4$ for $\delta < 1, a \leq 2$. If we additionally assume that $\delta > 8/(7+a)$, then

$$f(1/2) = \frac{8-(7+a)\delta}{4(2-\delta)} < 0.$$

Since f is a quadratic function with a positive leading coefficient, for any $\delta > 8/(7+a)$ there exists a unique $\underline{x} \in (1/2, 1)$ such that $f(\underline{x}) = 0$, $f(x) > 0$ for $x \in (\underline{x}, 1)$ and $f(x) < 0$ for $x \in [1/2, \underline{x})$. Therefore, for $\delta > 8/(7+a)$ a hybrid equilibrium exists if and only if $x \in [\underline{x}, 1)$.

Assortative equilibria. We next look for assortative equilibria. If the period t market distribution is μ_t, with a corresponding index $x = \mu_{1t}/(\mu_{1t} + \mu_{2t})$, and agreements arise as desired in an assortative equilibrium, then the next period market is given by

$$\mu_{i(t+1)} = \mu_{it}\left(1 - 2\frac{\mu_{it}}{4(\mu_{1t} + \mu_{2t})}\right) \ (i = 1, 2),$$

with an index

$$\frac{\mu_{1(t+1)}}{\mu_{1(t+1)} + \mu_{2(t+1)}} = \frac{x(2-x)}{1+2x(1-x)} =: \tau(x).$$

One can easily check that $\tau(x) \in [1/2, 1)$ and $\tau(x) \leq x$ for all $x \in [1/2, 1)$. The function $\tau : [1/2, 1) \to [1/2, 1)$ has the following properties:

- τ is strictly increasing and continuous on $[1/2, 1)$ and has an inverse $\tau^{-1} : [1/2, 1) \to [1/2, 1)$
- for every $x \in [1/2, 1)$, the sequence $(\tau^k(x))_{k \geq 0}$ is decreasing and converges to $1/2$, which is the unique fixed point of τ on $[1/2, 1)$
- for every $x \in (1/2, 1)$, the sequence $(\tau^{-k}(x))_{k \geq 0}$ is increasing and converges to 1.[17]

We will show that for $x \in [1/2, \tau^{-1}(\underline{x})]$ (with $\underline{x}$ defined in the analysis of hybrid equilibria) there exists an assortative equilibrium. The market index along the path of such an equilibrium is given by $(\tau^t(x))_{t \geq 0}$. Then by Theorem 2, the expected equilibrium payoffs $(v_{1t}, v_{2t})_{t \geq 0}$ solve

$$\begin{aligned} v_{1t} &= \frac{\tau^t(x)}{4}\left(a - \delta v_{1(t+1)}\right) + \left(1 - \frac{\tau^t(x)}{4}\right)\delta v_{1(t+1)} \\ v_{2t} &= \frac{1-\tau^t(x)}{4}\left(1 - \delta v_{2(t+1)}\right) + \frac{3+\tau^t(x)}{4}\delta v_{2(t+1)}. \end{aligned}$$

[17] τ^k (τ^{-k}) denotes τ's (τ^{-1}'s) composition with itself k times (by convention, τ^0 is the identity function).

The unique bounded solution of the equations above is immediately found to be

$$v_{1t} = w_1(\tau^t(x)) \;\&\; v_{2t} = w_2(\tau^t(x)), \forall t \geq 0,$$

where the functions w_1 and w_2 are defined by

$$\begin{aligned} w_1(x) &= a \sum_{t\geq 0} \delta^t \left(1 - \frac{x}{2}\right)\left(1 - \frac{\tau(x)}{2}\right)\cdots\left(1 - \frac{\tau^{t-1}(x)}{2}\right)\frac{\tau^t(x)}{4} \\ w_2(x) &= \sum_{t\geq 0} \delta^t \frac{1+x}{2}\,\frac{1+\tau(x)}{2}\cdots\frac{1+\tau^{t-1}(x)}{2}\,\frac{1-\tau^t(x)}{4}. \end{aligned}$$

The conjectured structure of agreements and disagreements is incentive compatible if

$$2\delta v_{1(t+1)} \leq a,\; 2\delta v_{2(t+1)} \leq 1,\; \delta\left(v_{1(t+1)} + v_{2(t+1)}\right) \geq 1, \forall t \geq 0,$$

or equivalently

$$2\delta w_1(\tau^t(x)) \leq a,\; 2\delta w_2(\tau^t(x)) \leq 1,\; \delta\left(w_1(\tau^t(x)) + w_2(\tau^t(x))\right) \geq 1, \forall t \geq 1.$$

For $x \in [1/2, \tau^{-1}(\underline{x})]$, we have $\underline{x} \geq \tau^1(x) \geq \tau^2(x) \geq \ldots$, so it sufficies to show that

(A.3) $$\forall x \in [1/2, \underline{x}]:\; 2\delta w_1(x) \leq a,\; 2\delta w_2(x) \leq 1,\; \delta\left(w_1(x) + w_2(x)\right) \geq 1.$$

A range of x where an assortative equilibrium exists. The first inequality in (A.3) holds because

$$\begin{aligned} w_1(x) &= a \sum_{t\geq 0} \delta^t \left(1 - \frac{x}{2}\right)\left(1 - \frac{\tau(x)}{2}\right)\cdots\left(1 - \frac{\tau^{t-1}(x)}{2}\right)\frac{\tau^t(x)}{4} \\ &\leq a/2 \sum_{t\geq 0} \left(1 - \frac{x}{2}\right)\left(1 - \frac{\tau(x)}{2}\right)\cdots\left(1 - \frac{\tau^{t-1}(x)}{2}\right)\frac{\tau^t(x)}{2} \\ &= a/2 \sum_{t\geq 0} \left[\left(1 - \frac{x}{2}\right)\left(1 - \frac{\tau(x)}{2}\right)\cdots\left(1 - \frac{\tau^{t-1}(x)}{2}\right) - \left(1 - \frac{x}{2}\right)\left(1 - \frac{\tau(x)}{2}\right)\cdots\left(1 - \frac{\tau^{t}(x)}{2}\right)\right] \\ &= a/2. \end{aligned}$$

The second inequality from (A.3) can be proven analogously.

We are left to establish that $\delta(w_1(x) + w_2(x)) \geq 1$ for all $x \in [1/2, \underline{x}]$. Note that $\tau^t(1/2) = 1/2$ for all $t \geq 0$. Then $w_1(1/2) = a\sum_{t\geq 0} \delta^t(3/4)^t(1/8) = a/(8-6\delta)$, and analogously $w_2(1/2) = 1/(8-6\delta)$. Hence $\delta(w_1(1/2) + w_2(1/2)) = \delta(a+1)/(8-6\delta) > 1$ for $\delta > 8/(7+a)$, which we assume for the rest of this subsection.[18] Clearly, $w_1(x)$ and $w_2(x)$ vary continuously in x, so there exists $x_0 \in (1/2, \underline{x})$ such that $\delta(w_1(x) + w_2(x)) > 1$ for all $x \in [1/2, x_0]$.

[18]The resurrection of the condition $\delta > 8/(7+a)$ is not coincidental. We previously found that a hybrid equilibrium does not exist for $x = 1/2$ if $\delta > 8/(7+a)$. For these parameter values, we expect an assortative equilibrium to emerge.

Define $x_k = \tau^{-k}(x_0)$ for $k \geq 1$. As stated earlier, the sequence $(x_k)_{k\geq 0}$ is increasing and converges to 1 as $k \to \infty$. We prove by induction on k that $\delta(w_1(x) + w_2(x)) > 1$ for all $x \in [1/2, \min(x_k, \underline{x})]$. In particular, this implies that

$$\delta(w_1(\underline{x}) + w_2(\underline{x})) > 1 \text{ whenever } \delta > 8/(7+a), \tag{A.4}$$

a fact which we exploit in the main proof. Note that we have already established the induction hypothesis for the base case $k = 0$. We now assume that the hypothesis is true over the interval $[1/2, \min(x_{k-1}, \underline{x})]$ and show that it holds over $[1/2, \min(x_k, \underline{x})]$.

Fix $x \in [1/2, \min(x_k, \underline{x})]$. For the purposes of proving the induction step, we abuse notation and write w_i for $w_i(x)$, w_i' for $w_i(\tau(x))$, and u_i for $u_i(\underline{x})$ $(i = 1, 2)$. The goal is thus to show that $\delta(w_1 + w_2) > 1$.

Since $x \in [1/2, \min(x_k, \underline{x})]$, we have that $\tau(x) \leq \tau(\min(x_k, \underline{x})) = \min(x_{k-1}, \tau(\underline{x})) \leq \min(x_{k-1}, \underline{x})$. Hence the induction hypothesis implies that $\delta(w_1' + w_2') > 1$.

The earlier payoff equations can be rewritten as follows

$$\begin{aligned} w_1 &= \frac{x}{4}a + \left(1 - \frac{x}{2}\right)\delta w_1' \\ w_2 &= \frac{1-x}{4} + \frac{1+x}{2}\delta w_2' \\ u_1 &= \frac{\underline{x}}{4}a + \left(1 - \frac{\underline{x}}{2}\right)\delta u_1 \\ u_2 &= \frac{1-\underline{x}}{4} + \frac{1+\underline{x}}{2}\delta u_2. \end{aligned}$$

The last pair of formulae reflect the fact that $f(\underline{x}) = 1 - \delta(u_1 + u_2) = 0$ (recall the definition of f from the analysis of hybrid equilibria).

We set out to show that $\delta(w_1+w_2) > \delta(u_1+u_2) = 1$, or equivalently that $w_1+w_2-u_1-u_2 > 0$. Manipulating the identities above, we obtain

$$\begin{aligned} &w_1 + w_2 - u_1 - u_2 \\ &= \frac{x - \underline{x}}{4}(a-1) + \left(1 - \frac{x}{2}\right)\delta w_1' - \left(1 - \frac{\underline{x}}{2}\right)\delta u_1 + \frac{1+x}{2}\delta w_2' - \frac{1+\underline{x}}{2}\delta u_2 \\ &= \frac{x - \underline{x}}{4}(a-1) + \left(1 - \frac{x}{2}\right)\delta(w_1' - u_1) + \frac{\underline{x} - x}{2}\delta u_1 + \frac{1+x}{2}\delta(w_2' - u_2) + \frac{x - \underline{x}}{2}\delta u_2 \\ &= \frac{\underline{x} - x}{4}\left(2\delta(u_1 - u_2) - (a-1)\right) + \left(1 - \frac{x}{2}\right)\delta(w_1' + w_2' - u_1 - u_2) + \left(x - \frac{1}{2}\right)\delta(w_2' - u_2). \end{aligned} \tag{A.5}$$

We show that every term in the last sum is non-negative, with the second one being positive. Since $x \in [1/2, \min(x_k, \underline{x})]$ and $\underline{x} < 1$, the coefficients satisfy the following inequalities $\underline{x} - x \geq 0, 1 - x/2 > 0, x - 1/2 \geq 0$. The second term is positive since we argued that $\delta(w_1' + w_2') > 1 = \delta(u_1 + u_2)$.

To show that the first term is non-negative, we need to prove that $2\delta(u_1-u_2)-(a-1)\geq 0$, which can be rewritten as $u_1-u_2\geq (a-1)/(2\delta)$, or

$$\frac{\underline{x}(a-1)}{4-3\delta}\geq \frac{a-1}{2\delta}.$$

The latter inequality is equivalent to $\underline{x}\geq 2/\delta-3/2$. Recall the assumption that $\delta > 8/(7+a)$. Since $2/\delta-3/2>1/2$, using the properties of the function f discussed earlier, the condition $\underline{x}\geq 2/\delta-3/2$ is equivalent to $f(2/\delta-3/2)\leq 0$. We find that

$$f(2/\delta-3/2)=\frac{8-\delta(7+a)}{4(2-\delta)}<0.$$

The third term is non-negative because

(A.6)
$$w_2'=\sum_{t\geq 0}\delta^t\frac{1+\tau(x)}{2}\,\frac{1+\tau^2(x)}{2}\cdots\frac{1+\tau^t(x)}{2}\,\frac{1-\tau^{t+1}(x)}{4}\geq\sum_{t\geq 0}\delta^t\left(\frac{1+\underline{x}}{2}\right)^t\frac{1-\underline{x}}{4}=u_2.$$

For a proof, note that the first sum represents the expected value of a random variable generated as follows. A coin is tossed at every date $t=0,1,\dots$ until a heads outcome is observed. The conditional probability of heads turning up at time t is $(1-\tau^{t+1}(x))/2$. In the event that the first heads appears at date t, the realized discounted payoff is $\delta^t/2$. Similarly, the second sum can be interpreted as the present value of an analogous process where heads is obtained with probability $(1-\underline{x})/2$ at each date. The inequality follows from the fact that the distribution of the former random variable first-order stochastically dominates that of the latter ($\tau^{t+1}(x)\leq x\leq\underline{x}$ for $x\in[1/2,\min(x_t,\underline{x})]$ and $t\geq 0$).

We proved the existence of the two types of equilibria for the bargaining game when $\delta>8/(7+a)$ and $x\in[\underline{x},1)\cap[1/2,\tau^{-1}(\underline{x})]=[\underline{x},\tau^{-1}(\underline{x})]$ (note that $\underline{x}$ depends on δ). The expected payoffs at $t=0$ are $(u_1(x),u_2(x))$ in the hybrid equilibrium and $(w_1(x),w_2(x))$ in the assortative one.

Equilibrium analysis for fixed x and variable δ. We next explore the existence of the two types of equilibria for a given $x\in[1/2,1)$, as we vary $\delta\in[0,1)$, to prove each part of Proposition 1. We revise the notation to recognize that $u_i(x),w_i(x),f(x),\underline{x}$ depend on δ and write $u_i(x,\delta),w_i(x,\delta),f(x,\delta),\underline{x}(\delta)$ instead.

Part (i). As already argued, a hybrid equilibrium exists if and only if

$$f(x,\delta)=\frac{8-2\delta(7+(a-1)x)+\delta^2(6+(a-1)x(x+1))}{(2-\delta)(4-3\delta)}\geq 0.$$

The inequality above is equivalent to

$$g(x,\delta):=8-2\delta(7+(a-1)x)+\delta^2(6+(a-1)x(x+1))\geq 0.$$

Note that g is a quadratic function in the second variable with a positive leading coefficient and $g(x,0) = 8 > 0 > -(a-1)x(1-x) = g(x,1)$. It follows that for every $x \in [1/2, 1)$ there exists a unique $\bar{\delta}(x) \in (0,1)$ such that $g(x, \bar{\delta}(x)) = 0$. Moreover, $g(x,\delta) > 0$ for $\delta \in [0, \bar{\delta}(x))$ and $g(x,\delta) < 0$ for $\delta \in (\bar{\delta}(x), 1)$. Hence a hybrid equilibrium exists if and only if $\delta \leq \bar{\delta}(x)$.

The assortative equilibrium exits if and only if

$$h(x,\delta) := 1 - \delta(w_1(x,\delta) + w_2(x,\delta)) \leq 0.$$

Since $w_1(x,\delta)$ and $w_2(x,\delta)$ are continuous and strictly increasing in δ, the function h is continuous and strictly decreasing in the second argument. Then $h(x,0) = 1 > 0 > h(x,1) = (1-a)/2$ implies the existence of a unique $\underline{\delta}(x) \in (0,1)$ such that $h(x, \underline{\delta}(x)) = 0$, with $h(x,\delta) > 0$ for $\delta \in [0, \underline{\delta}(x))$ and $h(x,\delta) < 0$ for $\delta \in (\underline{\delta}(x), 1)$. Thus an assortative equilibrium exits if and only if $\delta \geq \underline{\delta}(x)$.

Part (ii). Suppose that $x \in ((a+1)/4, 1)$. Then

$$g\left(x, \frac{8}{7+a}\right) = \frac{32(a-1)(2x-1)}{(7+a)^2}\left(x - \frac{a+1}{4}\right) > 0,$$

which means that $\bar{\delta}(x) > 8/(7+a)$. In current notation, (A.4) shows that $\delta(w_1(\underline{x}(\delta), \delta) + w_2(\underline{x}(\delta), \delta)) > 1$ for $\delta > 8/(7+a)$. Since $\bar{\delta}(x) > 8/(7+a)$, it follows that

$$\bar{\delta}(x)(w_1(\underline{x}(\bar{\delta}(x)), \bar{\delta}(x)) + w_2(\underline{x}(\bar{\delta}(x)), \bar{\delta}(x))) > 1. \tag{A.7}$$

However, note that $f(x, \bar{\delta}(x)) = 0$ by definition, which leads to $\underline{x}(\bar{\delta}(x)) = x$. Then (A.7) becomes

$$\bar{\delta}(x)(w_1(x, \bar{\delta}(x)) + w_2(x, \bar{\delta}(x))) > 1,$$

which is equivalent to $h(x, \bar{\delta}(x)) < 0$. The latter inequality implies that $\bar{\delta}(x) > \underline{\delta}(x)$, as desired. Thus the two equilibria co-exist for $\delta \in [0, \bar{\delta}(x)] \cap [\underline{\delta}(x), 1] = [\underline{\delta}(x), \bar{\delta}(x)]$.

Part (iii). Consider a pair (x,δ) with $x \in [1/2, 1)$ for which both types of equilibria exist. As argued earlier, the unique expected payoffs (u_1, u_2) for the two types in the hybrid equilibrium at $t = 0$ satisfy the conditions

$$\begin{aligned} u_1 &= \frac{x}{4}(a - \delta u_1) + \frac{1-x}{4}(1 - \delta u_2) + \frac{3}{4}\delta u_1 \\ u_2 &= \frac{x}{4}(1 - \delta u_1) + \frac{1-x}{4}(1 - \delta u_2) + \frac{3}{4}\delta u_2 \\ \delta(u_1 + u_2) &\leq 1. \end{aligned}$$

Since $1 - \delta u_2 \geq \delta u_1$, we have

$$u_1 \geq \frac{x}{4}(a - \delta u_1) + \left(1 - \frac{x}{4}\right)\delta u_1 = \frac{x}{4}a + \left(1 - \frac{x}{2}\right)\delta u_1,$$

which leads to

$$u_1 \geq a \sum_{t \geq 0} \delta^t \left(1 - \frac{x}{2}\right)^t \frac{x}{4}.$$

On the other hand, the expected period 0 payoffs (w_1, w_2) in the assortative equilibrium are given by

$$\begin{aligned} w_1 &= a \sum_{t \geq 0} \delta^t \left(1 - \frac{x}{2}\right)\left(1 - \frac{\tau(x)}{2}\right) \ldots \left(1 - \frac{\tau^{t-1}(x)}{2}\right) \frac{\tau^t(x)}{4} \\ w_2 &= \sum_{t \geq 0} \delta^t \frac{1+x}{2} \frac{1+\tau(x)}{2} \ldots \frac{1+\tau^{t-1}(x)}{2} \frac{1-\tau^t(x)}{4} \end{aligned}$$

and satisfy $\delta(w_1 + w_2) \geq 1$. The inequalities $x = \tau^0(x) \geq \tau^1(x) \geq \tau^2(x) \geq \ldots$ coupled with an argument similar to the one for (A.6) establish that

$$\sum_{t \geq 0} \delta^t \left(1 - \frac{x}{2}\right)^t \frac{x}{4} \geq \sum_{t \geq 0} \delta^t \left(1 - \frac{x}{2}\right)\left(1 - \frac{\tau(x)}{2}\right) \ldots \left(1 - \frac{\tau^{t-1}(x)}{2}\right) \frac{\tau^t(x)}{4},$$

and hence $u_1 \geq w_1$. Then the inequalities $\delta(u_1 + u_2) \leq 1 \leq \delta(w_1 + w_2)$ imply that $u_2 \leq w_2$.

Part (iv). Let $U(x, \delta)$ and $W(x, \delta)$ denote the total welfare attained in the bargaining game with an initial measure $x \in (1/2, 1)$ of players 1 and $1 - x$ of players 2, sharing the discount factor δ, if agreements arise as in the hybrid and assortative equilibria, respectively. U solves the following equation[19]

$$U(x, \delta) = a\frac{x^2}{4} + \frac{x(1-x)}{2} + \frac{(1-x)^2}{4} + \frac{1}{2}\delta U(x, \delta).$$

Thus

$$U(x, \delta) = \frac{(a-1)x^2 + 1}{2(2-\delta)}.$$

Similarly, W satisfies the formula

$$W(x, \delta) = a\frac{x^2}{4} + \frac{(1-x)^2}{4} + \left(\frac{1}{2} + x(1-x)\right)\delta W(\tau(x), \delta).$$

To obtain bounds on $W(x, \delta)$, note that if the expression

$$D(y, \delta) := W(y, \delta) - \left(\frac{1}{2} + y(1-y)\right)\delta W(\tau(y), \delta) - \left(U(y, \delta) - \left(\frac{1}{2} + y(1-y)\right)\delta U(\tau(y), \delta)\right)$$

is positive (negative) for all $y \in (1/2, x]$, then we can immediately conclude that $W(x, \delta)$ is greater (smaller) than $U(x, \delta)$.

[19] In a market with x players of type 1 and $1 - x$ players of type 2, there is a mass of $x^2/4$ pairs of players 1 matched to bargain with one another, $2 \times x(1-x)/4$ pairs of players of types 1 and 2, and $(1-x)^2/4$ pairs of players 2. The measures of players of type 1 and 2 left unmatched in the first period are $x - \left(2 \times x^2/4 + x(1-x)/2\right) = x/2$ and $1 - x - \left(2 \times (1-x)^2/4 + x(1-x)/2\right) = (1-x)/2$, respectively. If all first period matches result in agreement, the second period market contains half of the players in each population and contributes to welfare with a surplus of $\delta U(x, \delta)/2$.

Using the formula for $U(\cdot,\delta)$ and the recursion for $W(\cdot,\delta)$, we compute

$$D(y,\delta) = \frac{y(1-y)(4+(5+3a)y(1-y))}{4(2-\delta)(1+2y(1-y))}\left(\delta - \frac{4+8y(1-y)}{4+(5+3a)y(1-y)}\right)$$

Hence $D(y,\delta)$ is positive (negative) for all $y \in (1/2,x]$ if

$$\delta > \ (<) \ \frac{4+8y(1-y)}{4+(5+3a)y(1-y)} =: d(y), \forall y \in (1/2,x].$$

Since $d(y)$ is strictly increasing in y for $y \in (1/2,x]$, we have that

$$\delta > d(x) \quad \Rightarrow \quad W(x,\delta) > U(x,\delta)$$
$$\delta \le \lim_{y \to 1/2} d(y) = \frac{8}{7+a} \quad \Rightarrow \quad W(x,\delta) < U(x,\delta).$$

The arguments above show that if $\bar{\delta}(x) > d(x)$ then $W(x,\bar{\delta}(x)) > U(x,\bar{\delta}(x))$. The inequality $\bar{\delta}(x) > d(x)$ is equivalent to

$$g(x,d(x)) = \frac{24(a-1)x^2(2-x)(1-x)(2x-1)}{(4+(5+3a)x(1-x))^2}\left(\frac{(x+1)(7x-4x^2-1)}{3x(2-x)} - a\right) > 0.$$

Thus $W(x,\bar{\delta}(x)) > U(x,\bar{\delta}(x))$ whenever

$$\frac{(x+1)(7x-4x^2-1)}{3x(2-x)} > a.$$

For every $a \in (1,4/3)$, there exists $\varepsilon > 0$ such that the inequality above holds for all $x \in (1-\varepsilon,1)$, as

$$\lim_{x \to 1} \frac{(x+1)(7x-4x^2-1)}{3x(2-x)} = 4/3.$$

Consider now $\tilde{x} = (a+1)/4$. We have $\bar{\delta}(\tilde{x}) = 8/(7+a)$, and the discussion above proves that $W(\tilde{x},\bar{\delta}(\tilde{x})) < U(\tilde{x},\bar{\delta}(\tilde{x}))$. Since $U, W, \bar{\delta}$ are continuous functions on their respective domains, it follows that there exists $\varepsilon > 0$ such that $W(x,\bar{\delta}(x)) < U(x,\bar{\delta}(x))$ for all $x \in ((a+1)/4,(a+1)/4+\varepsilon)$. □

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