MARKOV EQUILIBRIA IN A MODEL OF BARGAINING IN NETWORKS

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Abstract. We study the Markov perfect equilibria (MPEs) of an infinite horizon game in which pairs of players connected in a network are randomly matched to bargain. Players who reach agreement are removed from the network without replacement. We establish the existence of MPEs and show that MPE payoffs are not necessarily unique. A method for constructing pure strategy MPEs for high discount factors is developed. For some networks, we find that all MPEs are asymptotically inefficient as players become patient.

1. Introduction

Many markets involve buyers and sellers of relationship specific products and services. The particularities of these relationships may derive from location, technological compatibility, joint business opportunities, free trade agreements, social contacts, etc. Such markets are naturally modeled as networks and the structure of the network determines the nature of economic interaction between the agents who form the nodes of the network.

For example, imagine a group of employers who have needs for different tasks and a group of workers with distinct sets of skills. The links between workers and employers depend on how skills translate into the necessary tasks and other factors such as physical location and social relationships (Granovetter 1973). In another application, a group of suppliers (for instance, laptop component manufacturers) offer exclusive commitments to a group of upstream producers. Another is the case of licensing arrangements being negotiated between basic technology providers (different platforms for developing computer software or smart

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phone applications, for example) and final product designers. Compatibility issues limit the connections between these groups and lead to non-trivial, incomplete networks.

The setting is as follows. We consider a network where each pair of players connected by a link can jointly produce a unit surplus. The network generates the following infinite horizon discrete time bargaining game. Each period a link is selected according to some probability distribution, and one of the two matched players is randomly chosen to make an offer to the other player specifying a division of the unit surplus between themselves. If the offer is accepted, the two players exit the game with the shares agreed on. If the offer is rejected, the two players remain in the game for the next period. In the next period the game is repeated on the subnetwork induced by the set of remaining players. We assume that all players have perfect information of all the events preceding any of their decision nodes in the game. All players have a common discount factor.

In this environment the following questions arise: How are the relative strengths of the firms affected by the pattern of compatibilities (that is, the network structure)? Which partnerships are possible in equilibrium and on what terms? Is an efficient allocation of the processes achievable in equilibrium? We address these issues in the context of Markov perfect equilibria (MPEs). Fudenberg and Tirole (1991) and Maskin and Tirole (2001) present arguments for the relevance of MPEs. The natural notion of a Markov state in the class of models we consider is the network induced by players who did not reach agreement, along with the selection of a link and a proposer.

We prove that an MPE always exists and demonstrate by example that MPE payoffs are not necessarily unique. Existence of MPEs is established via a fixed-point argument rather than by explicit construction. Finding one MPE for a given network structure is typically a complex exercise due to the simultaneous determination of the pairs of players reaching agreement in equilibrium (if matched to bargain in the first period of the game) and the evolution of the network structure as agreements are realized and play proceeds. We provide a method to construct pure strategy MPEs for high discount factors based on conjectures about the set of links across which agreement may obtain in every subnetwork.

We offer an example where no MPE of the bargaining game is efficient even asymptotically as players become patient. This leads naturally to the question of whether an asymptotically efficient (non-Markovian) subgame perfect equilibrium always exists. In a companion paper, Abreu and Manea (2009), we answer the question affirmatively. Even though MPEs may be inefficient, Markov strategies are still essential to our construction of asymptotically efficient subgame perfect equilibria. The building block of the construction is an MPE of a modified game that differs from the original one primarily in "prohibiting" inefficient agreements.

Manea (2011) assumes that players who reach agreement are replaced by new players at the same positions in the network. The bargaining protocol is identical to the one of the present paper. The two models differ in strategic complexity. In the model of Manea (2011) bargaining opportunities are stationary over time. A player's decisions consist solely in determining who his most favorable bargaining partners are. In effect, each player solves a search problem with prizes endogenously and simultaneously determined by the network structure. In the present model a player's decisions additionally entail anticipating that passing up bargaining opportunities may lead to agreements involving other players which undermine or enhance his position in the network in future bargaining encounters. Technically, this means that we need to compute equilibrium payoffs for every subnetwork that may arise following a series of agreements.

Our modeling strategy has been to allow for full generality of the network structure while keeping other elements of the model relatively simple. Nevertheless, two aspects of the model deserve discussion. One is the assumption that the surplus any pair of players can generate is either zero or one. In fact it would not be difficult to work with a more general and less symmetric model. However, the assumption that all links have the same value is useful in analyzing particular examples and allows us to characterize relative bargaining strengths in terms of the network structure. Another restrictive assumption is that only one link is chosen for bargaining in every round. We provide justification for this assumption below. Nevertheless, the assumption may also be relaxed. Our main results generalize to settings with varying gains from trade and multiple simultaneous matches.

There is an extensive literature on bargaining in markets starting with Rubinstein and Wolinsky (1985). Important subsequent papers include Gale (1987) and Binmore and Herrero (1988) and Rubinstein and Wolinsky (1990). The focus is on the relationship between the equilibrium outcomes of various decentralized bargaining procedures and the competitive equilibrium price as the costs of search and delay become negligible. The various stochastic matching processes considered in this literature treat all buyers and respectively all sellers anonymously. The analogue of this modeling assumption in our setting is the special case of buyer-seller networks in which every buyer is connected to every seller. For such networks, the payoffs in any MPE of our bargaining game converge to the competitive equilibrium outcome, as players become patient. However, in our analysis the network is arbitrary. In particular, some pairs of buyers and sellers are not connected and cannot trade. Since bargaining encounters are restricted by network connections, the competitive equilibrium analysis does not apply.

Polanski (2007) studies a model similar to ours, but with a fundamentally different matching technology. He assumes that a maximum number of pairs of connected players are selected to bargain every period. In that setting Polanski obtains a payoff characterization which is neatly related to the classical Gallai-Edmonds decomposition. As a consequence of the maximum matching assumption, efficiency is not an issue in Polanski's model (in contrast to our work) and furthermore, in equilibrium, all matched pairs reach agreement immediately.

In our completely decentralized matching process a fundamental tension emerges between the global structure of efficient matchings in a network and the local nature of incentives for trade. Even in simple examples, asymptotically inefficient outcomes arise in equilibrium. We also obtain richer dynamics for the evolution of network structure due to the fact that not all matches lead to trade in equilibrium. As mentioned earlier, the tools we develop can be extended to deal with settings where more than one link is chosen for bargaining in every round.

An alternative rationale for the one-match-per-period assumption is as follows. In terms of the essential analytics, what matters is that multiple agreements are not reached at the same instant. If we take the underlying temporal reality to be continuous——and consequently assume that matching takes place in continuous time—then the probability that several matches occur simultaneously is zero. In this view our assumption is indeed natural.

Polanski and Winter (2010) consider a model where buyers and sellers connected by a network are matched to bargain according to a protocol similar to ours. The critical difference is that players do not exit the game upon reaching agreements. Although every player can make several transactions over time, players are assumed to behave as if they derive utility only from their next transaction. Corominas-Bosch (2004) considers a model in which buyers and sellers alternate in making public offers that may be accepted by any of the responders connected to a specific proposer. As in Polanski (2007), the matching process specifies that when there are multiple possibilities to match buyers and sellers (that is, there are multiple agents proposing or accepting identical prices) the maximum number of transactions takes place. Kranton and Minehart (2001) study trade in networks in a model based on centralized simultaneous auctions.

The rest of the paper is organized as follows. In Section 2 we define the model and establish existence of MPEs. Section 3 provides examples of MPEs in some simple networks. Section 4 suggests an approach to computing MPEs. We show that the MPEs are not necessarily payoff equivalent and that asymptotically efficient MPEs do not always exist in Sections 5 and 6, respectively. Section 7 concludes.

2. Framework

Let N denote the set of n **players**, $N = \{1, 2, \ldots, n\}$. A **network** is an **undirected graph** $H = (V, E)$ with set of vertices $V \subset N$ and set of edges (also called **links**) $E \subset \{(i, j) | i \neq j \in V\}$ such that $(j, i) \in E$ whenever $(i, j) \in E$. We identify the links (i, j) and (j, i), and use the shorthand ij or ji instead. We say that player i is **connected** in H to player j if $ij \in E$. We often abuse notation and write $i \in H$ for $i \in V$ and $ij \in H$ for $ij \in E$. A player is **isolated** in H if he has no links in H. A network $H' = (V', E')$ is a **subnetwork** of H if $V' \subset V$ and $E' \subset E$. A network $H' = (V', E')$ is the subnetwork of H **induced** by V' if $E' = E \cap (V' \times V')$. We write $H \ominus V''$ for the subnetwork of H induced by the vertices in $V \setminus V''$. Every network H has an associated probability distribution over links $(p_{ij}(H))_{ij \in H}$ with $p_{ij}(H) > 0, \forall ij \in H$ which describes the matching process.[1]

Let G be a fixed network with vertex set N. A link ij in G is interpreted as the ability of players i and j to jointly generate a unit surplus.[2] Consider the following infinite horizon **bargaining game** generated by the network G. Let $G_0 = G$. Each period $t = 0, 1, \ldots$ a single link ij in G_t is selected with probability $p_{ij}(G_t)$ and one of the players (the proposer) i and j is chosen randomly (with equal conditional probability) to make an offer to the other

[1]Note the flexibility of the matching protocol. In one appealing specification, all links are equally likely to generate a match. In another special case, each player is drawn with equal probability and then one of his links is chosen uniformly at random.

[2]We do not exclude networks in which some players are isolated.

player (the responder) specifying a division of the unit surplus between themselves. If the responder accepts the offer, the two players exit the game with the shares agreed on. If the responder rejects the offer, the two players remain in the game for the next period. In period $t+1$ the game is repeated with the set of players from period t, except for i and j in case period t ends in agreement, on the subnetwork G_{t+1} induced by this set of players in G. Hence $G_{t+1} = G_t \ominus \{i, j\}$ if players i and j reach an agreement in period t, and $G_{t+1} = G_t$ otherwise. We assume that all players have perfect information of all the events preceding any of their decision nodes in the game.[3] All players share a discount factor $\delta \in (0, 1)$. The bargaining game is denoted $\Gamma^\delta(G)$.

There are three types of histories. We denote by h_t a history of the game up to (not including) time t, which is a sequence of $t-1$ pairs of proposers and responders connected in G, with corresponding proposals and responses. We call such histories, and the subgames that follow them, **complete**. A complete history h_t uniquely determines the set of players $N(h_t)$ participating in the game at the beginning of period t; denote by $G(h_t)$ the subnetwork of G induced by $N(h_t)$. Let $\mathcal{G}$ be the set of subnetworks of G induced by the players remaining in any subgame, $\mathcal{G} = \cup_{h_t} G(h_t)$, and define $\mathcal{G}^0 = \mathcal{G} \setminus \{G\}$. We denote by $(h_t; i \to j)$ the history consisting of h_t followed by nature selecting i to propose to j. We denote by $(h_t; i \to j; x)$ the history consisting of $(h_t; i \to j)$ followed by i offering $x \in [0, 1]$ to j.

A **strategy** σ_i for player i specifies, for all complete histories h_t and all players j such that $ij \in G(h_t)$, the **offer** $\sigma_i(h_t; i \to j)$ that i makes to j after the history $(h_t; i \to j)$, and the **response** $\sigma_i(h_t; j \to i; x)$ that i gives to j after the history $(h_t; j \to i; x)$. We allow for mixed strategies, hence $\sigma_i(h_t; i \to j)$ and $\sigma_i(h_t; j \to i; x)$ are probability distributions over $[0, 1]$ and {Yes, No}, respectively. A **strategy profile** $\sigma = (\sigma_i)_{i \in N}$ is a **subgame perfect equilibrium** of $\Gamma^\delta(G)$ if it induces Nash equilibria in subgames following every history $(h_t; i \to j)$ and $(h_t; i \to j; x)$.

The equilibrium analysis is simplified if we restrict attention to **Markov strategies**. The state at a certain stage is given by the subnetwork of players who did not reach agreement by that stage, along with the selection of a link and a proposer. Then the only feature of a complete history of past bargaining encounters that is relevant for future behavior is the network induced by the remaining players following that history. That is, for all complete

[3]The requirements on the information structure may be relaxed for the case of Markov perfect equilibria.

histories h_t and all links $ij \in G(h_t)$, the offer $\sigma_i(h_t; i \to j)$ that i makes to j depends only on $G(h_t), i, j$, and i's response $\sigma_i(h_t; j \to i; x)$ to the offer x from j depends only on $G(h_t), i, j, x$.[4] A **Markov perfect equilibrium (MPE)** is a subgame perfect equilibrium in Markov strategies.[5] We first establish existence of MPEs.

Proposition 1. *There exists a Markov perfect equilibrium for the bargaining game* $\Gamma^\delta(G)$.

For the proof, we first provide a characterization of MPE payoffs, and then use it to show that an MPE always exists. Fix $\delta \in (0,1)$. For a set of networks $\mathcal{H}$, a collection of Markov strategy profiles $(\sigma(H))_{H\in\mathcal{H}}$ for the respective games $(\Gamma^\delta(H))_{H\in\mathcal{H}}$ is **subgame consistent** if for every pair of networks $H, H' \in \mathcal{H}$, $\sigma(H)$ and $\sigma(H')$ induce the same behavior in any pair of identical subgames of $\Gamma^\delta(H)$ and $\Gamma^\delta(H')$.[6]

Suppose $\sigma^{*\delta}(G)$ is an MPE of $\Gamma^\delta(G)$. By the definition of an MPE, it must be that $\sigma^{*\delta}(G)$ belongs to a subgame consistent collection of MPEs $(\sigma^{*\delta}(\tilde{G}))_{\tilde{G}\in\mathcal{G}}$ of the respective games $(\Gamma^\delta(\tilde{G}))_{\tilde{G}\in\mathcal{G}}$ with corresponding payoffs $(v^{*\delta}(\tilde{G}))_{\tilde{G}\in\mathcal{G}}$. In particular, when $\Gamma^\delta(G)$ is played according to $\sigma^{*\delta}(G)$, every player k has expected payoffs $v_k^{*\delta}(G)$ at the beginning of any subgame before which no agreement has occurred, and $v_k^{*\delta}(G \ominus \{i,j\})$ at the beginning of any subgame before which only i and j reached an agreement ($k \neq i, j$).

Fix a history $(h_t; i \to j)$ along which no agreement has been reached ($G(h_t) = G$). In the subgame following $(h_t; i \to j)$, it must be that the strategy $\sigma_j^{*\delta}(G)$ specifies that player j accept any offer larger than $\delta v_j^{*\delta}(G)$, and reject any offer smaller than $\delta v_j^{*\delta}(G)$. Then it is not optimal for i to make an offer $x > \delta v_j^{*\delta}(G)$ to j, since i would be better off making some offer in the interval $(\delta v_j^{*\delta}(G), x)$ instead, as j accepts such offers with probability 1. Hence, in equilibrium i has to offer j at most $\delta v_j^{*\delta}(G)$ with probability 1, and j may accept with positive probability only offers of $\delta v_j^{*\delta}(G)$. Let q be the probability (conditional on

[4]Formally, a Markov strategy profile σ satisfies the following conditions

$$\begin{aligned} \sigma_i(h_t; i \to j) &= \sigma_i(h'_{t'}; i \to j) \\ \sigma_i(h_t; j \to i; x) &= \sigma_i(h'_{t'}; j \to i; x) \end{aligned}$$

for all $h_t, h'_{t'}$ with $G(h_t) = G(h'_{t'})$, for every $ij \in G(h_t)$ and $x \in [0,1]$.

[5]In other accounts ([11], [12]), the concepts defined here would be referred to as stationary Markov strategies and stationary Markov perfect equilibrium.

[6]More precisely, subgame consistency of $(\sigma(H))_{H\in\mathcal{H}}$ requires that $\sigma(H)(h_t; i \to j) = \sigma(H')(h'_{t'}; i \to j)$ and $\sigma(H)(h_t; i \to j; x) = \sigma(H')(h'_{t'}; i \to j; x)$ for all pairs of players (i,j), all offers x, all h_t and $h'_{t'}$ such that the players remaining in the subgame h_t of $\Gamma^\delta(H)$ and the subgame $h'_{t'}$ of $\Gamma^\delta(H')$ induce identical networks (which include the link ij), and all $H, H' \in \mathcal{H}$.

$(h_t; i \to j))$ of the joint event that i offers $\delta v_j^{*\delta}(G)$ to j and the offer is accepted. The payoff of any player $k \neq i, j$ at the beginning of the next period is $v_k^{*\delta}(G \ominus \{i, j\})$ in case i and j reach an agreement, and $v_k^{*\delta}(G)$ otherwise. Therefore, the time t expected payoff of k conditional on the history $(h_t; i \to j)$ is $q\delta v_k^{*\delta}(G \ominus \{i, j\}) + (1 - q)\delta v_k^{*\delta}(G)$.

If $\delta(v_i^{*\delta}(G) + v_j^{*\delta}(G)) < 1$, when i is chosen to propose to j, it must be that in equilibrium i offers $\delta v_j^{*\delta}(G)$ and agreement obtains with probability 1, i.e., $q = 1$. For, if $q < 1$ then i's expected payoff conditional on offering $\delta v_j^{*\delta}(G)$ is $q(1-\delta v_j^{*\delta}(G))+(1-q)\delta v_i^{*\delta}(G) < 1-\delta v_j^{*\delta}(G)$, while conditional on offering $\delta v_j^{*\delta}(G)+\varepsilon$ $(\varepsilon > 0)$ is $1 - \delta v_j^{*\delta}(G) - \varepsilon$ (we argued that j accepts offers greater than $\delta v_j^{*\delta}(G)$ with probability 1). But for small $\varepsilon > 0$, $q(1 - \delta v_j^{*\delta}(G)) + (1 - q)\delta v_i^{*\delta}(G) < 1 - \delta v_j^{*\delta}(G) - \varepsilon$. Hence it is not optimal for i to offer $\delta v_j^{*\delta}(G)$ to j. By the same token, offers smaller than $\delta v_j^{*\delta}(G)$ are not optimal for i since they are rejected with probability 1 and yield expected payoff $\delta v_i^{*\delta}(G) < 1-\delta v_j^{*\delta}(G)-\varepsilon$. We already argued that no offer greater than $\delta v_j^{*\delta}(G)$ may be optimal for i either. Therefore, if $\delta(v_i^{*\delta}(G) + v_j^{*\delta}(G)) < 1$ and $q < 1$, then i cannot have a best response to j's equilibrium strategy. We established that if $\delta(v_i^{*\delta}(G) + v_j^{*\delta}(G)) < 1$ then $q = 1$. Similarly, if $\delta(v_i^{*\delta}(G) + v_j^{*\delta}(G)) > 1$ then $q = 0$. If $\delta(v_i^{*\delta}(G) + v_j^{*\delta}(G)) = 1$ then q can be any number in the interval $[0, 1]$.

Consider the correspondence $f^{i\to j} : [0, 1]^n \rightrightarrows [0, 1]^n$ defined by

$$f^{i\to j}(v) = \{q(\underbrace{\delta v^{*\delta}(G \ominus \{i, j\})}_{-i,-j}, \underbrace{1 - \delta v_j}_{i}, \underbrace{\delta v_j}_{j}) + (1 - q)\delta v | \quad q = 1\ (0) \text{ if } \delta(v_i + v_j) < (>)1, \text{ and } q \in [0, 1] \text{ if } \delta(v_i + v_j) = 1\}, \tag{2.1}$$

where $(\underbrace{\delta v^{*\delta}(G \ominus \{i, j\})}_{-i,-j}, \underbrace{1 - \delta v_j}_{i}, \underbrace{\delta v_j}_{j})$ represents the vector in $[0, 1]^n$ with the k $(\neq i, j)$ coordinate equal to $\delta v_k^{*\delta}(G \ominus \{i, j\})$, i coordinate equal to $1 - \delta v_j$, and j coordinate equal to δv_j. Note that $f_k^{i\to j}(v^{*\delta}(G))$ is the set of possible time t expected payoffs for player k conditional on the history $(h_t; i \to j)$, where the behaviors of i and j are constrained by the equilibrium analysis above.

Let $f : [0, 1]^n \rightrightarrows [0, 1]^n$ be the correspondence defined by

$$f(v) = \sum_{\{i\to j | ij\in G\}} \frac{1}{2} p_{ij}(G) f^{i\to j}(v). \tag{2.2}$$

Let h_t be a history along which no agreement has occurred, and consider the resulting period t subgame. Since nature selects player i to make an offer to player j with probability $p_{ij}(G)/2$ for each link $ij \in G$, and conditional on the selection, $f^{i \to j}(v^{*\delta}(G))$ describes the time t expected payoffs constrained by the equilibrium requirements, $f(v^{*\delta}(G))$ is the set of expected payoffs at the beginning of the subgame h_t consistent with our partial equilibrium analysis when players behave according to $\sigma^{*\delta}(G)$. In equilibrium, the time t expected payoff vector conditional on the history h_t is $v^{*\delta}(G)$, hence $v^{*\delta}(G) \in f(v^{*\delta}(G))$. Therefore, $v^{*\delta}(G)$ is a fixed point of f. Conversely, we show in the Appendix that any fixed point of f is an MPE payoff vector.

Lemma 1. *A vector v is a Markov perfect equilibrium payoff of $\Gamma^\delta(G)$ if and only if there exists a subgame consistent collection of Markov perfect equilibria of the games $(\Gamma^\delta(\tilde{G}))_{\tilde{G} \in \mathcal{G}^0}$ with respective payoffs $(v^{*\delta}(\tilde{G}))_{\tilde{G} \in \mathcal{G}^0}$ such that v is a fixed point of the correspondence f defined by 2.1-2.2.*[7]

In the Appendix, we use a bootstrap approach to construct an MPE for any $\Gamma^\delta(\tilde{G})$ $(\tilde{G} \in \mathcal{G})$ based on a subgame consistent family of MPEs $(\sigma^{*\delta}(\tilde{G} \ominus \{i,j\}))_{ij \in \tilde{G}}$ for the bargaining games $(\Gamma^\delta(\tilde{G} \ominus \{i,j\}))_{ij \in \tilde{G}}$. We establish that the correspondence f derived from the payoffs of the latter family of MPEs has a fixed point, which by Lemma 1 translates into an MPE of $\Gamma^\delta(\tilde{G})$. The proof proceeds by induction on the number of vertices in $\tilde{G}$.

Remark 1. It is straightforward to extend the proof of Proposition 1 to a setting with heterogeneous link values.

Remark 2. We can also generalize the existence result to the case in which multiple pairs of players are matched to bargain simultaneously. In the general specification of the matching protocol, a collection of pairwise disjoint proposer-responder pairs is drawn at each date from a probability distribution which depends only on the underlying network at that date. We assume that a public randomization device is available in this setting. The additional steps necessary for the proof are outlined in the Appendix.

[7] Recall that $\mathcal{G}^0$ denotes the set of subnetworks of G, different from G, induced by the players remaining in any subgame of $\Gamma^\delta(G)$.

3. Examples of MPEs

In this section we provide examples of MPEs for some simple networks. We assume throughout that all links are equally likely to be selected for bargaining in the initial network and in any subnetwork that may arise in subgames. That is, the probability distribution $p(H)$ is **uniform** across the links in H for all networks H. We mainly focus on equilibrium payoffs. Strategies may be constructed as in the proof of Lemma 1.

Consider first a star network, where one player controls the bargaining opportunities of all other players. Formally, in the **star** of n network $G_{star\ n}$ player 1 is connected to each of the players $k = 2, \ldots, n$. Proposition 4(i) in Rubinstein and Wolinsky (1990) shows that the bargaining game $\Gamma^\delta(G_{star\ n})$ has a unique subgame perfect equilibrium (which turns out to be Markovian). In the equilibrium, agreement is obtained in the first match. It is easy to see that the payoffs satisfy the equations

$$\begin{aligned} v_1 &= \frac{1}{2}(1 - \delta v_2 + \delta v_1) \\ v_2 &= \frac{1}{2(n-1)}(1 - \delta v_1 + \delta v_2) \\ v_k &= v_2 \ (k = 3, \ldots, n). \end{aligned}$$

The solution is

$$v_1^{*\delta}(G_{star\ n}) = \frac{n - 1 - \delta}{n(2-\delta) - 2} \text{ and } v_k^{*\delta}(G_{star\ n}) = \frac{1-\delta}{n(2-\delta)-2} \text{ for } k = 2, \ldots, n.$$

As $\delta \to 1$, the equilibrium payoffs converge to $1/2$ for both players if $n = 2$, and to 1 for player 1 and 0 for all other players when $n \geq 3$.

Consider next a line network, in which players are located on a line and can only bargain with their immediate neighbors. Formally, in the **line** of n network $G_{line\ n}$ player k is connected to player $k+1$ for $k = 1, \ldots, n-1$. Computing MPEs of the bargaining game for line networks is feasible for two main reasons. First, all the connected components induced by the players remaining in any subgame are line networks. Second, the number of conjectures about what first period agreements are possible in equilibrium is relatively small because each player has at most 2 neighbors. The networks $G_{line\ 2}$ and $G_{line\ 3}$ are isomorphic[8] to $G_{star\ 2}$ and $G_{star\ 3}$ respectively.

[8]Two networks $H = (V, E)$ and $H' = (V', E')$ are **isomorphic** if there exists a bijection $g : V \to V'$ such that $ij \in E \iff g(i)g(j) \in E'$.

Consider now the bargaining game on the line of 4 network, $\Gamma^\delta(G_{line\ 4})$. If players 2 and 3 reach the first agreement, then 1 and 4 are left disconnected and receive zero payoffs. If players 1 and 2 (3 and 4) reach the first agreement, then 3 and 4 (1 and 2) induce a subnetwork isomorphic to $G_{line\ 2}$ in the ensuing subgame, and obtain expected payoffs of 1/2 in the next period. One can then easily show that in any MPE the pairs of players $(1,2)$ and $(3,4)$ reach agreements with probability 1 when matched to bargain in the first period.

For low δ there is a unique MPE of $\Gamma^\delta(G_{line\ 4})$. In any subgame, every match ends in agreement. By the proof of Proposition 1 and by symmetry, the equilibrium payoffs solve the following system,

$$\begin{aligned} v_1 &= \frac{1}{3}\frac{1}{2}((1-\delta v_2)+\delta v_1)+\frac{1}{3}0+\frac{1}{3}\delta/2 \\ v_2 &= \frac{1}{3}\frac{1}{2}(\delta v_2+(1-\delta v_1))+\frac{1}{3}\frac{1}{2}((1-\delta v_3)+\delta v_2)+\frac{1}{3}\delta/2 \\ v_3 &= v_2,\ v_4 = v_1. \end{aligned}$$

The unique solution is given by

$$v_1^{*\delta}(G_{line\ 4}) = v_4^{*\delta}(G_{line\ 4}) = \frac{6+3\delta-2\delta^2}{12(3-\delta)},\ v_2^{*\delta}(G_{line\ 4}) = v_3^{*\delta}(G_{line\ 4}) = \frac{12+3\delta-2\delta^2}{12(3-\delta)}.$$

There is an MPE with payoffs as above only if the solution satisfies $\delta(v_2^{*\delta}(G_{line\ 4})+v_3^{*\delta}(G_{line\ 4})) \leq 1$. The latter inequality is equivalent to $\delta \leq \underline{\delta} \approx .945$, where $\underline{\delta}$ is the unique root in the interval $[0,1]$ of the polynomial $18-18x-3x^2+2x^3$.

For high δ, there is *no* MPE of $\Gamma^\delta(G_{line\ 4})$ in which players 2 and 3 agree with probability 1 when matched to bargain with each other. In such an equilibrium players 1 and 4 would be weak (receiving zero payoffs in subgames following agreements between 2 and 3), making the patient players 2 and 3 powerful to an extent that prevents them from reaching an agreement with each other. Also, there exists no MPE in which players 2 and 3 disagree with probability 1 when matched to bargain. In such an equilibrium all players would receive payoffs smaller than 1/2, and players 2 and 3 would have incentives to trade.

For $\delta > \underline{\delta}$, there exists an MPE of $\Gamma^\delta(G_{line\ 4})$ in which players 2 and 3 reach agreement with some probability $q^{*\delta} \in (0,1)$ conditional on their link being selected for bargaining.[9]

[9]The probabilities that 2 accepts an offer from 3 and that 3 accepts an offer from 2 are not pinned down by the MPE requirements. Only the average $q^{*\delta}$ of the two conditional probabilities is relevant for MPE payoff computation. There exist multiple MPEs, all payoff equivalent, as explained in footnote 23.

As in the proof of Proposition 1, we need the equilibrium payoffs of players 2 and 3 to satisfy $\delta(v_2^{*\delta}(G_{line\ 4}) + v_3^{*\delta}(G_{line\ 4})) = 1$. By symmetry, the equilibrium payoffs solve the following system,

$$\begin{aligned} v_1 &= \frac{1}{3}\frac{1}{2}((1-\delta v_2)+\delta v_1) + \frac{1}{3}(1-q^{*\delta})\delta v_1 + \frac{1}{3}\delta/2 \\ v_2 &= \frac{1}{3}\frac{1}{2}(\delta v_2 + (1-\delta v_1)) + \frac{1}{3}\delta v_2 + \frac{1}{3}\delta/2 \\ \delta(v_2+v_3) &= 1,\ v_3 = v_2,\ v_4 = v_1. \end{aligned}$$

The unique solution is given by

$$v_1^{*\delta}(G_{line\ 4}) = v_4^{*\delta}(G_{line\ 4}) = \frac{-6+5\delta+2\delta^2}{2\delta^2}, v_2^{*\delta}(G_{line\ 4}) = v_3^{*\delta}(G_{line\ 4}) = \frac{1}{2\delta}$$
$$q^{*\delta} = \frac{2(9-12\delta+\delta^2+2\delta^3)}{\delta(-6+5\delta+2\delta^2)}.$$

Note that, as players become patient, the conditional probability of agreement between players 2 and 3 converges to 0 and the MPE payoffs converge to 1/2 for each player. The intuition is that players 2 and 3 could obtain payoffs greater than 1/2 in the limit only by extorting players 1 and 4 via the threat of an agreement across the link $(2,3)$, which would leave 1 and 4 disconnected. Yet, players 2 and 3 cannot reach an agreement if their limit equilibrium payoffs are larger than 1/2.

Similarly, there exists an MPE of the bargaining game on the line of 6 network, $\Gamma^\delta(G_{line\ 6})$, in which as δ goes to 1, the common probability of first period agreement across the links $(2,3)$ and $(4,5)$ vanishes, while agreement obtains with probability 1 across all other links. All players receive expected payoffs of 1/2 in the limit.

For the line of $5,7,8,9,\ldots$ networks, and other more complex ones, computing MPE payoffs for the bargaining game for every δ may be a difficult task. For such networks, the next section investigates limit MPE payoffs and agreement probabilities as players become patient.

4. Limit Properties of MPEs

Fix a network G. A payoff vector v^* is a **limit MPE payoff** (of $\Gamma^\delta(G)$ as $\delta \to 1$) if there exists a family of MPEs of the games $(\Gamma^\delta(G))_{\delta\in(0,1)}$ with respective payoffs $(v^{*\delta})_{\delta\in(0,1)}$ such that $v^* = \lim_{\delta\to 1} v^{*\delta}$. The **initial agreement probabilities** induced by a Markov strategy σ are described by $(q_{ij})_{ij\in G}$, where q_{ij} is the probability that i and j reach agreement under σ conditional on being matched to bargain in the first period of the game (with either player in the role of the proposer). A collection $(q_{ij})_{ij\in G}$ represents **limit MPE initial agreement probabilities** (for $\Gamma^\delta(G)$ as $\delta \to 1$) if there exists a family of MPEs of the games $(\Gamma^\delta(G))_{\delta\in(0,1)}$ with respective initial agreement probabilities $(q^\delta_{ij})_{ij\in G}$ such that $q_{ij} = \lim_{\delta\to 1} q^\delta_{ij}$ for all $ij \in G$.

For various network structures, we can use a bootstrap approach to directly compute limit MPE payoffs and agreement probabilities as players become patient. We then construct MPEs of $\Gamma^\delta(G)$ for high δ that generate the determined limit payoffs and agreements as $\delta \to 1$. As in Proposition 1, we use known limit MPE payoffs in subgames $\Gamma^\delta(\tilde{G})$ for $\tilde{G} \in \mathcal{G}^0$ in order to characterize equilibrium behavior in $\Gamma^\delta(G)$. Suppose that for every $\delta \in (0,1)$ we specified a subgame consistent family of MPEs for the bargaining games $(\Gamma^\delta(G \ominus \{i,j\}))_{ij\in G}$ with respective payoffs $(v^{*\delta}(G \ominus \{i,j\}))_{ij\in G}$.

Fix a profile of initial agreement probabilities $(q^\delta_{ij})_{ij\in G}$ for every discount factor δ. We set out to construct an MPE for $\Gamma^\delta(G)$ that generates the first period agreement probabilities q^δ and leads to the payoffs $v^{*\delta}(G \ominus \{i,j\})$ in subgames that induce the subnetwork $G \ominus \{i,j\}$. By the proof of Proposition 1, the MPE payoffs solve the $n \times n$ linear system of equations,[10]

$$(4.1) \quad v_k = \sum_{\{i|ik\in G\}} \frac{1}{2} p_{ik} q^\delta_{ik}(1-\delta v_i) + \sum_{ij\in G\ominus\{k\}} p_{ij} q^\delta_{ij} \delta v^{*\delta}_k(G\ominus\{i,j\}) + \left(1 - \sum_{\{i|ik\in G\}} \frac{1}{2} p_{ik} q^\delta_{ik} - \sum_{ij\in G\ominus\{k\}} p_{ij} q^\delta_{ij}\right)\delta v_k, \; k=\overline{1,n}.$$

Contrary to appearances, the equations above do not assume that the probability of an agreement between i and k is split evenly between the events that i or k plays the role of the proposer. The split is not unique only if $q^\delta_{ik} \in (0,1)$, in which case the MPE payoffs should satisfy $1 - \delta v_i = \delta v_k$. Then the exact allocation of the total probability of agreement $p_{ik} q^\delta_{ik}$ between the terms $1-\delta v_i$ and δv_k does not affect the expression on the right hand side. See also footnote 23.

Assume that for all $ij \in G$, q^δ_{ij} and $v^{*\delta}(G \ominus \{i,j\})$ converge to q_{ij} and $v^*(G \ominus \{i,j\})$, respectively, as δ goes to 1. Consider the linear system obtained by taking the limit $\delta \to 1$

[10]To simplify notation, we write p_{ij} for $p_{ij}(G)$.

in 4.1,

$$(4.2)\quad v_k = \sum_{\{i|ik\in G\}} \frac{1}{2} p_{ik} q_{ik} (1 - v_i) + \sum_{ij\in G\ominus\{k\}} p_{ij} q_{ij} v_k^*(G\ominus\{i,j\}) + \left(1 - \sum_{\{i|ik\in G\}} \frac{1}{2} p_{ik} q_{ik} - \sum_{ij\in G\ominus\{k\}} p_{ij} q_{ij}\right) v_k,\ \ k = \overline{1,n}.$$

The next result describes the relationship between the solutions of the two linear systems and provides sufficient conditions under which solutions to the latter system constitute limit MPE payoffs.

Proposition 2. *Suppose that* $\lim_{\delta\to 1} q_{ij}^\delta = q_{ij}$ *and* $\lim_{\delta\to 1} v^{*\delta}(G\ominus\{i,j\}) = v^*(G\ominus\{i,j\})$ *for all* $ij \in G$. *For parts (2)-(4), assume additionally that* $q_{ij} > 0$ *for at least two links* $ij \in G$. *Then the following statements hold.*

1. *The system 4.1 has a unique solution, denoted* v^{δ,q^δ}.
2. *The system 4.2 also has a unique solution, denoted* v^q.
3. *The solutions satisfy* $\lim_{\delta\to 1} v^{\delta,q^\delta} = v^q$.
4. *If* $q_{ij} \in \{0,1\}$ *for all* $ij \in G$ *and* v^q *satisfies the conditions* $v_i^q + v_j^q < 1$ *if* $q_{ij} = 1$ *and* $v_i^q + v_j^q > 1$ *if* $q_{ij} = 0$, *then there exists* $\underline{\delta} < 1$ *such that for every* $\delta \in (\underline{\delta}, 1)$ *there is an MPE of* $\Gamma^\delta(G)$ *with payoffs* $v^{\delta,q}$ *and initial agreement probabilities* q.

The proof appears in the Appendix. Remarks 1 and 2 also apply here. The next section provides an illustration of Proposition 2. We have also applied the result to determine limit MPE payoffs and initial agreement probabilities for the bargaining games on the line of $5, 7, 8, \ldots, 12$ networks. Figure 1 summarizes limit MPE outcomes for all line networks with at most 12 players. In this and subsequent diagrams, for every network, limit MPE payoffs for each player are represented next to the corresponding node. Each link is drawn as a thin,[11] dashed, or thick line segment depending on whether the probability of first period agreement across that link in MPEs for high δ is 0, a number in $(0,1)$ (then the limit probability as $\delta \to 1$ is mentioned next to the link),[12] or 1, respectively.

[11]See, for example, the link $(4,5)$ in the line of 8 network from Figure 1.

[12]For some links the initial agreement probabilities for $\delta < 1$ may be positive, and converge to 0 as $\delta \to 1$, as in the case of the link $(2,3)$ in $G_{line\ 4}$.

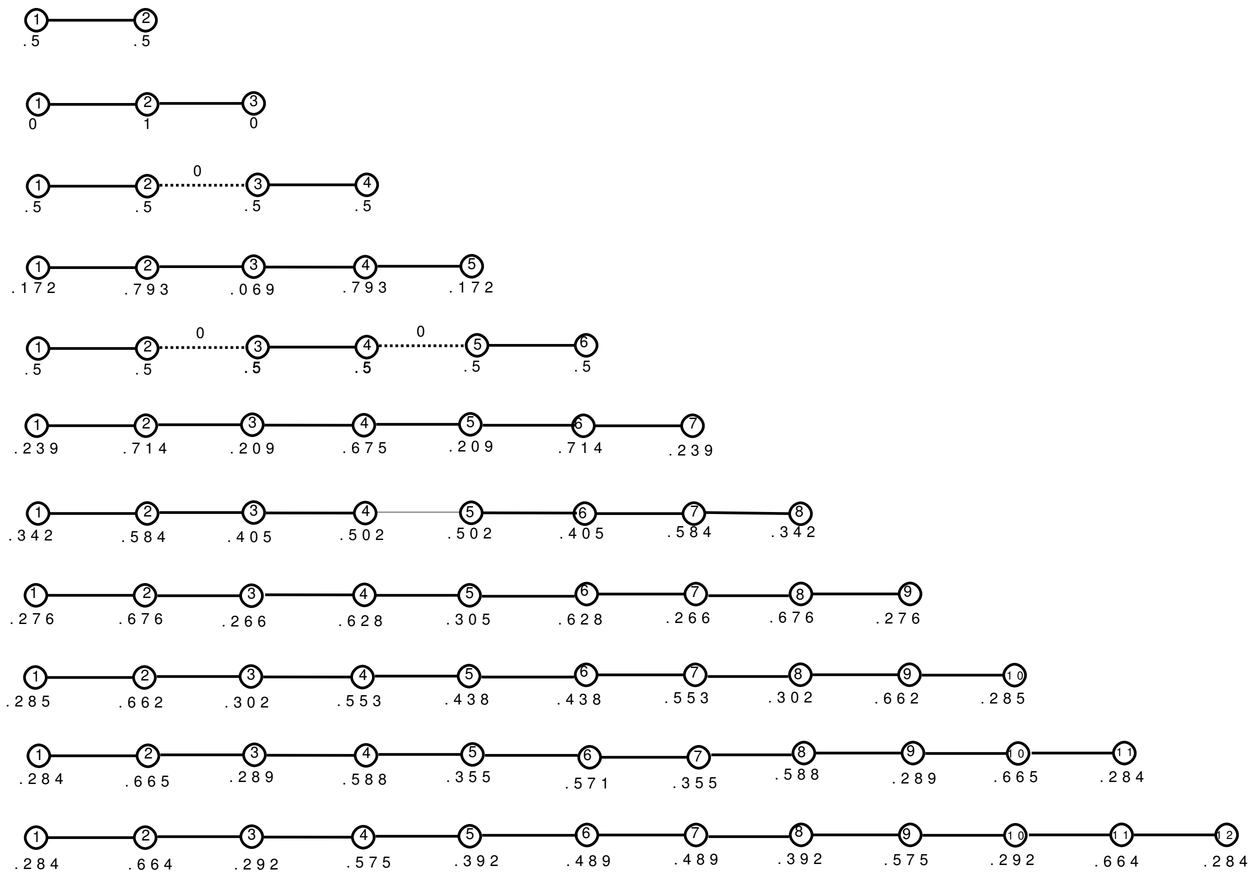

FIGURE 1. Limit MPE payoffs and initial agreements for the bargaining games on the line of $2, 3, \ldots, 12$ networks.

Note that the properties that limit MPE payoffs are 1/2 for all players and that limit probabilities of first period agreement across links $(k, k+1)$ are 0 and 1 for k even and odd, respectively, do not extend to lines with an even number of players greater than 6. In the limit MPE for the line of 8 network illustrated in Figure 1, players 4 and 5 do not reach an agreement when matched to bargain with each other in the first period. However, the bargaining game does not reduce to two independent line of 4 games since players 4 and 5 have incentives to trade in subgames following initial agreements across the links $(2, 3)$ and $(6, 7)$. Indeed, either of the latter agreements leaves 4 and 5 in a subnetwork isomorphic to a line of 5, where all matches result in immediate agreement. A first period agreement between players 2 and 3 (6 and 7) leads to limit continuation payoffs of approximately .172 and .793 for players 4 (5) and 5 (4), respectively, and of .069 for player 6 (3). Player 4 (5) exploits 3 (6)'s vulnerability and obtains an expected limit payoff greater than 1/2. Consequently, for high discount factors, players 4 and 5 do not have incentives to reach an agreement with each other in the first period.

Proposition 2 does not characterize (limit) MPEs in which the probability of an agreement across some links differs from 0 and 1.[13] Relatedly, the result does not cover the possibility that $v_i^q + v_j^q = 1$ for some link $ij \in G$. By part (3) of Proposition 2, $v_i^q + v_j^q = 1$ implies that $\lim_{\delta \to 1} \delta(v_i^{\delta, q^\delta} + v_j^{\delta, q^\delta}) = 1$ for any family $(q^\delta)_\delta$ that converges to q as $\delta \to 1$. The technical challenge is that in general we cannot infer whether $\delta(v_i^{\delta, q^\delta} + v_j^{\delta, q^\delta})$ is smaller than, equal to, or greater than 1. As the proof of Proposition 1 demonstrates, the latter comparison drives the incentives for agreements in MPEs with the structure assumed above.

The following example clarifies that the strict inequalities from part (4) of Proposition 2 cannot be replaced by weak ones. Consider the 4-player network $G_{tr+point}$ from Figure 4 in Section 6, and assume that all links are chosen for bargaining with equal probability. Proposition 4 establishes that for every δ the game $\Gamma^\delta(G_{tr+point})$ has a unique MPE, in which agreement obtains with probability 1 across each link. Let q be the profile of initial agreement probabilities given by $q_{12} = q_{34} = 1$ and $q_{23} = q_{24} = 0$. The corresponding limit system 4.2 (with obvious specifications for limit MPE payoffs in subgames following an agreement) has the unique solution $v_1^q = v_2^q = v_3^q = v_4^q = 1/2$. In particular, $v_i^q + v_j^q = 1$ for all $ij \in G_{tr+point}$. This is the equality case left unaddressed by Proposition 2. Indeed, as Proposition 4 shows, q does not describe limit MPE initial agreement probabilities and v^q does not define limit MPE payoffs for $\Gamma^\delta(G_{tr+point})$.

5. Multiple MPE Payoffs

Multiple MPE payoffs may exist for the bargaining game on some networks for high discount factors. One example is the bargaining game $\Gamma^\delta(G_{sq+line\ 3})$, on the network $G_{sq+line\ 3}$ depicted in Figure 2.

Proposition 3. *There exists $\underline{\delta} < 1$ such that for every $\delta \in (\underline{\delta}, 1)$ the game $\Gamma^\delta(G_{sq+line\ 3})$ has (at least) three MPEs that are pairwise payoff unequivalent.*

[13] In Section 3 we discussed networks in which the MPE probabilities of agreement for high δ are different from 0 and 1, but converge to 0 or 1 as $\delta \to 1$. Section 5 details an example in which for some links even the limit MPE agreement probabilities belong to $(0, 1)$.

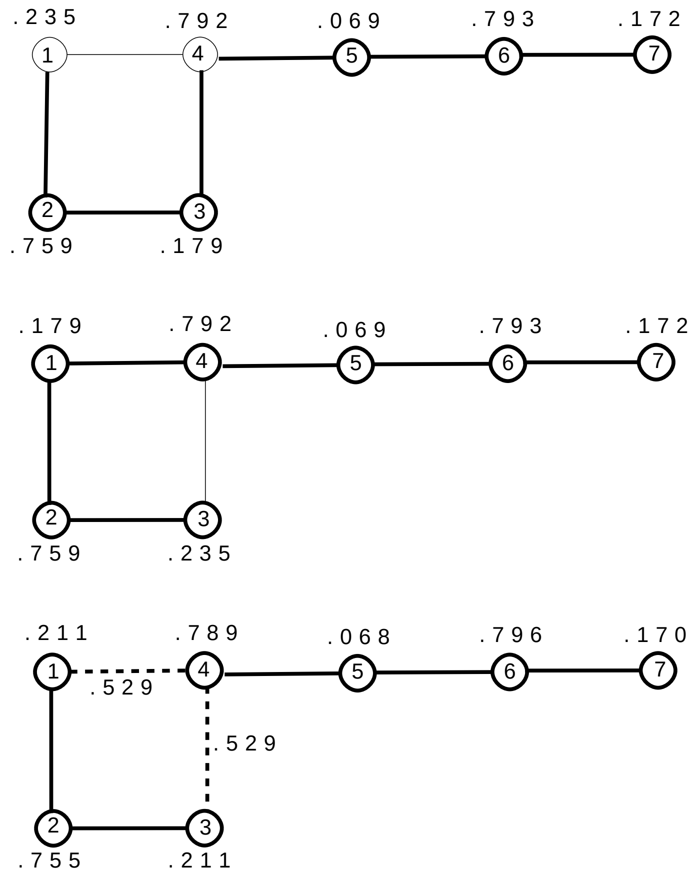

FIGURE 2. Three sets of limit MPE payoffs and initial agreements for $\Gamma^\delta(G_{sq+line\ 3})$

Proof. We intend to use Proposition 2 to show that for high δ the game $\Gamma^\delta(G_{sq+line\ 3})$ admits an MPE in which the (conditional) probability of agreement in the first period is 0 across the link $(1,4)$ and 1 for all other links.

To define the subgame consistent collection of MPEs $(\sigma^{*\delta}(G_{sq+line\ 3} \ominus \{k, k+1\}))_{k=1,2,\ldots,6}$ necessary for 4.1 and 4.2, note that the first agreement may induce the following subgames. If players 1 and 2 (2 and 3) reach the initial agreement, then the remaining players $3,4,5,6,7$ $(1,4,5,6,7)$ induce a subgame on a network isomorphic to the line of 5 network. If players 3 and 4 (4 and 5) reach the first agreement, the induced subnetwork has two connected components, partitioning the set of remaining players into $\{1,2\}$ and $\{5,6,7\}$ ($\{1,2,3\}$ and $\{6,7\}$). Players 1 and 2 (6 and 7) are then involved in a subgame similar to the bargaining game on the line of 2 network, with lower matching frequencies, since they are not matched to bargain when the link $(5,6)$ or $(6,7)$ ($(1,2)$ or $(2,3)$) is selected for bargaining. Similarly players $5,6,7$ $(1,2,3)$ are involved in a version of the bargaining game on the line of 3 network

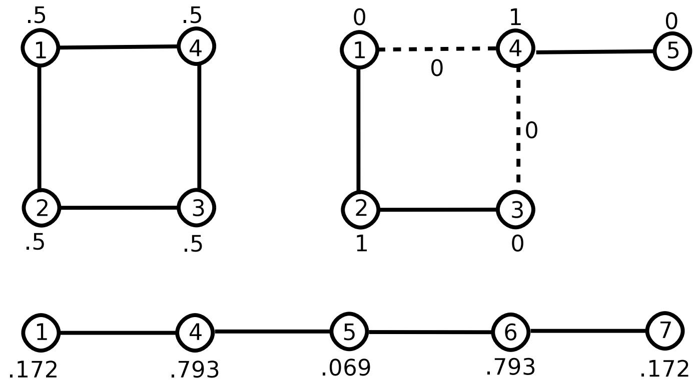

FIGURE 3. Limit MPE payoffs and initial agreements for the bargaining game on G_{sq} (top left), $G_{sq+point}$ (top right), and a network isomorphic to $G_{line\ 5}$

with different matching frequencies. For both variations of the bargaining games on the line of 2 and 3 networks the limit MPE payoffs are identical to those in the respective benchmark versions.

If players 5 and 6 reach the first agreement, then players $1, 2, 3, 4$ induce a subgame equivalent to the bargaining game on the square network, G_{sq}, and player 7 is left disconnected. If players 6 and 7 reach the initial agreement, then players $1, 2, 3, 4, 5$ induce a subgame equivalent to the bargaining game on the square plus point network, $G_{sq+point}$.

The limit MPE payoffs and initial agreements for G_{sq}, $G_{sq+point}$, and a network isomorphic to the line of 5 are summarized in Figure 3. The limit linear system 4.2 for $\Gamma^\delta(G_{sq+line\ 3})$ with the conjectured profile of initial agreement probabilities is as follows,

$$
\begin{aligned}
v_1 &= \frac{1}{7}\frac{1}{2}(v_1 + 1 - v_2) + \frac{1}{7}5/29 + \frac{1}{7}1/2 + \frac{1}{7}0 + \frac{1}{7}1/2 + \frac{1}{7}0 + \frac{1}{7}v_1 \\
v_2 &= \frac{1}{7}\frac{1}{2}(v_2 + 1 - v_1) + \frac{1}{7}\frac{1}{2}(v_2 + 1 - v_3) + \frac{1}{7}1/2 + \frac{1}{7}1 + \frac{1}{7}1/2 + \frac{1}{7}1 + \frac{1}{7}v_2 \\
v_3 &= \frac{1}{7}5/29 + \frac{1}{7}\frac{1}{2}(v_3 + 1 - v_2) + \frac{1}{7}\frac{1}{2}(v_3 + 1 - v_4) + \frac{1}{7}0 + \frac{1}{7}1/2 + \frac{1}{7}0 + \frac{1}{7}v_3 \\
v_4 &= \frac{1}{7}23/29 + \frac{1}{7}23/29 + \frac{1}{7}\frac{1}{2}(v_4 + 1 - v_3) + \frac{1}{7}\frac{1}{2}(v_4 + 1 - v_5) + \frac{1}{7}1/2 + \frac{1}{7}1 + \frac{1}{7}v_4 \\
v_5 &= \frac{1}{7}2/29 + \frac{1}{7}2/29 + \frac{1}{7}0 + \frac{1}{7}\frac{1}{2}(v_5 + 1 - v_4) + \frac{1}{7}\frac{1}{2}(v_5 + 1 - v_6) + \frac{1}{7}0 + \frac{1}{7}v_5 \\
v_6 &= \frac{1}{7}23/29 + \frac{1}{7}23/29 + \frac{1}{7}1 + \frac{1}{7}1/2 + \frac{1}{7}\frac{1}{2}(v_6 + 1 - v_5) + \frac{1}{7}\frac{1}{2}(v_6 + 1 - v_7) + \frac{1}{7}v_6 \\
v_7 &= \frac{1}{7}5/29 + \frac{1}{7}5/29 + \frac{1}{7}0 + \frac{1}{7}1/2 + \frac{1}{7}0 + \frac{1}{7}\frac{1}{2}(v_7 + 1 - v_6) + \frac{1}{7}v_7.
\end{aligned}
$$

In each equation the terms correspond in order to the selection for bargaining of the links $(k, k+1)$ for $k = 1, 2, \ldots, 6$, followed by the link $(1, 4)$. The unique solution is given by[14]

$$v_1 \approx 0.235, v_2 \approx 0.759, v_3 \approx 0.179, v_4 \approx 0.792, v_5 \approx 0.069, v_6 \approx 0.793, v_7 \approx 0.172.$$

The solution satisfies the conditions from Proposition 2 ($v_1 + v_4 > 1$ and $v_i + v_j < 1$ for all links ij different from $(1, 4)$), so for high δ there exists an MPE of $\Gamma^\delta(G_{sq+line\ 3})$ with the assumed agreement structure and payoffs approaching the values above as $\delta \to 1$. The rough intuition for this equilibrium specification is that odd labeled players are relatively weak and even labeled players are relatively strong in the bargaining game $\Gamma^\delta(G_{sq+line\ 3})$ for high δ.[15] However, the asymmetric behavior of players 1 and 3 in first period matches with 4 places player 1 in a better position than 3.

Odd labeled players are relatively weak and even labeled players are relatively strong in the bargaining games on $G_{line\ 3}, G_{line\ 5}$, and $G_{sq+point}$. Players $2, 4$, and 6 occupy the positions corresponding to even labels in the latter networks following some initial equilibrium agreements. The significant difference between the payoff of player 2 and the (almost identical) payoffs of players 4 and 6 is due to the initial agreement between players 3 and 4, which undermines player 2's position. When 3 and 4 reach the first agreement, 2 is left in a bilateral bargaining game with 1, which leads to a limit payoff of $1/2$ for player 2. This diminishes the effect of strong even positions for player 2 in the three types of subnetworks enumerated earlier.

Similarly, player 1 is better off than player 3. Although players 1 and 3 have symmetric positions in the network, 1 is at an advantage over 3 since initial agreement obtains across the link $(3, 4)$, but not across $(1, 4)$. Player 7 is slightly weaker than 3 because, as argued above, player 7's only neighbor, player 6, is significantly stronger than one of player 3's neighbors, player 2. Finally, player 5 is the weakest of all odd labeled players because his central position is inferior to the peripheral positions of the other odd players in the subnetworks isomorphic to the line of 5 network induced by initial agreements across the links $(1, 2)$ and $(2, 3)$.

Players 1 and 4 are reluctant to reach the first agreement with each other because each of them can benefit from waiting to be matched with a weaker neighbor. It is possible for

[14]The exact solution involves irreducible fractions with 8-digit denominators.

[15]The words "weak" and "strong" vaguely mean payoffs significantly below and respectively above $1/2$.

all other pairs of payers to trade when matched to bargain in the first period since no other two relatively strong players (in the constructed MPE) with odd and even labels are linked in the network. The conjectured agreement structure is self-enforcing and leads to an MPE for high δ.

Note that players 1 and 3 hold symmetric positions in $G_{sq+line\ 3}$, but they play asymmetric roles in the MPE constructed above. We can obtain another MPE for high δ by simply interchanging the roles of players 1 and 3 in the postulated agreement structure. The payoffs of players 1 and 3 differ between the two pure strategy MPEs for high δ.

For sufficiently high δ, $\Gamma^\delta(G_{sq+line\ 3})$ has a third MPE, in which there is a common probability in the interval $(0, 1)$ of first period agreement across the links $(1, 4)$ and $(3, 4)$.[16] The limit MPE agreement probabilities are $q_{14} = q_{34} \approx$ [17] 0.528758 and $q_{ij} = 1$ for all other links ij. The limit MPE payoffs are

$$v_1 = v_3 \approx 0.211, v_2 \approx 0.755, v_4 \approx 0.789, v_5 \approx 0.068, v_6 \approx 0.796, v_7 \approx 0.170.$$

Therefore, for high δ the bargaining game $\Gamma^\delta(G_{sq+line\ 3})$ has at least three MPEs. Note that the mixed strategy MPE is not payoff equivalent with either of the pure strategy MPEs for any player. □

6. Inefficient MPEs

Let $\mu(G)$ denote the maximum total surplus that can be generated in the network G. That is, $\mu(G)$ is the cardinality of the largest collection of pairwise disjoint links in G.[18] To generate the maximum total surplus $\mu(G)$ in $\Gamma^\delta(G)$ as $\delta \to 1$, pairs of players connected by links that are inefficient in the induced subnetworks in various subgames need to refrain from reaching agreements. However, providing incentives against agreements that are collectively inefficient is difficult. Some players may be concerned that passing up bargaining opportunities can lead to agreements involving their potential bargaining partners which undermine their position in the network in future bargaining encounters. Indeed, one can find networks for which all MPEs of the bargaining game are asymptotically inefficient as players become patient.

[16] The proof is similar to that of Proposition 2.

[17] The value of the limit probability is one of the four roots of an irreducible polynomial of degree 4.

[18] This and related terms are defined formally in Abreu and Manea (2009).

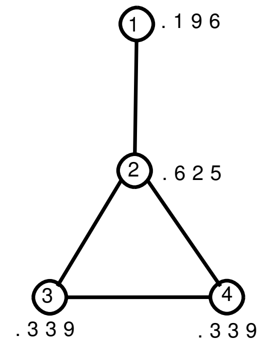

FIGURE 4. Asymptotically inefficient MPEs for the bargaining game on $G_{tr+point}$

Consider the network $G_{tr+point}$ illustrated in Figure 4, with a uniform probability distribution governing the selection of links for bargaining. Assume that δ is close to 1 so that the welfare cost of delay between consecutive matches is negligible. The maximum total surplus in this network is 2 and it can be achieved in the limit as $\delta \to 1$ only if both pairs $(1, 2)$ and $(3, 4)$ reach agreement. It is clearly inefficient for player 2 to trade with either player 3 or 4 because this would leave the remaining players isolated and create only one unit of surplus. Proposition 4 below establishes that for any $\delta \in (0, 1)$, $\Gamma^\delta(G_{tr+point})$ has a unique MPE. In the MPE every pair reaches agreement when matched to bargain. Since $(2, 3)$ or $(2, 4)$ are matched first with probability 1/2, the expected total surplus generated by the MPE approaches $1/2 \times 1 + 1/2 \times 2 = 3/2 < 2 = \mu(G_{tr+point})$ as $\delta \to 1$.

Note that using Proposition 2, we can immediately evaluate the limit MPE payoffs to be $11/56 \approx .196$ for player 1, $5/8 = .625$ for player 2, and $19/56 \approx .339$ for players 3 and 4. One interesting feature of this example is that $G_{tr+point}$ is not unilaterally stable with respect to the limit MPE payoffs.[19] Indeed, if player 4 severed his link with player 2, the line of 4 network would ensue, and player 4's limit MPE payoff would increase from 19/56 to 1/2.[20] Thus player 4 would be better off if he could credibly commit to never trade with player 2.

Proposition 4. *For every $\delta \in (0, 1)$, the game $\Gamma^\delta(G_{tr+point})$ has a unique MPE. In the MPE agreement occurs with probability 1 across every link selected for bargaining in the first period.*

[19]See Jackson and Wolinsky (1996) and Manea (2011) for definitions of stability.

[20]In general, to apply the concept of stability consistently we would need to use an equilibrium selection criterion for networks with multiple (payoff non-equivalent) MPEs (as in Section 5). However, this issue is inconsequential for the current argument, since both $\Gamma^\delta(G_{line\ 4})$ and $\Gamma^\delta(G_{tr+point})$ have unique MPE payoffs for every δ.

Proof. We show that for every $\delta \in (0,1)$, all MPEs of $\Gamma^\delta(G_{tr+point})$ involve agreement with (conditional) probability 1 for every pair of players matched to bargain in the first period. Fix a discount factor $\delta \in (0,1)$ and an MPE σ of $\Gamma^\delta(G_{tr+point})$. Denote by v_i the expected payoff of player i under σ.

We first argue that agreement occurs under σ with probability 1 in the first period if the link $(1,2)$ or $(3,4)$ is selected for bargaining. We only treat the case of the former link, as the latter is similar. The strategy profile σ determines a distribution over joint outcomes for players 1 and 2, where an outcome for a given player specifies the time of an agreement involving that player and the share he receives. For every realization of agreements under σ, the sum of the corresponding discounted payoffs for players 1 and 2 is not greater than 1. Indeed,

- when 2 reaches an agreement with 1, the sum of the undiscounted payoffs of the two players is 1
- when 2 reaches an agreement with 3 or 4, player 2's undiscounted payoff cannot exceed 1 and player 1's is 0 (an agreement between 2 and 3 or 4 isolates 1)
- thus the expected discounted payoffs of 1 and 2 satisfy $v_1 + v_2 \leq 1$.

Therefore, $\delta(v_1 + v_2) < 1$, and hence players 1 and 2 reach an agreement under σ if matched to bargain in the first period of the game.

Let p and q denote the probabilities of first period agreement across the links $(2,3)$ and $(2,4)$, respectively (conditional on the respective link being selected for bargaining). We next show by contradiction that $p = q$. Without loss of generality, assume that $p > q$. It must be that $p > 0, q < 1$. Hence $\delta(v_2 + v_3) \leq 1 \leq \delta(v_2 + v_4)$, so $v_3 \leq v_4$. It can be easily seen that the payoffs satisfy

$$v_3 = \frac{1}{4}(\delta\frac{1}{2} + \frac{1}{2}(\delta v_3 + 1 - \delta v_4) + \frac{1}{2}(1 - \delta v_2 + \delta v_3) + (1-q)\delta v_3) \tag{6.1}$$

$$v_4 = \frac{1}{4}(\delta\frac{1}{2} + \frac{1}{2}(\delta v_4 + 1 - \delta v_3) + \delta v_4 + (1-p)\delta v_4). \tag{6.2}$$

In each of the two sums, the first term represents the continuation payoffs of $1/2$ received by players 3 and 4 conditional on the link $(1,2)$ being selected for bargaining in the first period. The second term corresponds to an agreement between players 3 and 4 when matched to bargain. Here we use the fact that the selection of the links $(1,2)$ and $(3,4)$ leads to trade under σ. The term $(1 - \delta v_2 + \delta v_3)/2$ appears in the evaluation of the payoff of player 3 because under σ,

- if $\delta(v_2 + v_3) < 1$, then player 3 offers δv_2 when selected to make an offer to 2 and player 2 accepts with conditional probability 1
- if $\delta(v_2 + v_3) = 1$, then player 3 obtains a continuation payoff of $1 - \delta v_2 = \delta v_3$ when selected to make an offer to 2 regardless of whether the offer is accepted or rejected.

Similarly, the third term in the expression for v_4 can be explained by the inequality $\delta(v_2 + v_4) \geq 1$. The last terms in the two equations reflect the probabilities of agreements that player 2 reaches with 4 (3), leaving player 3 (4) isolated.

Since $\delta(v_2 + v_3) \leq 1, p > q, v_3 > 0$, we have that

$$\begin{aligned} v_3 &= \frac{1}{4}(\delta\frac{1}{2} + \frac{1}{2}(\delta v_3 + 1 - \delta v_4) + \frac{1}{2}(1 - \delta v_2 + \delta v_3) + (1-q)\delta v_3) \\ &> \frac{1}{4}(\delta\frac{1}{2} + \frac{1}{2}(\delta v_3 + 1 - \delta v_4) + \delta v_3 + (1-p)\delta v_3). \end{aligned} \tag{6.3}$$

Putting together 6.2 and 6.3, we obtain that

$$v_4 - v_3 < \frac{3-p}{4}\delta(v_4 - v_3).$$

This leads to a contradiction, as

$$\delta\frac{3-p}{4} < 1 \text{ and } v_4 - v_3 \geq 0.$$

We have established that $p = q$. It is easy to check that if $p = q = 0$ then $v_1 = v_2 = v_3 = v_4 < 1/2$, and hence $\delta(v_2 + v_3) < 1$, contradicting $p = 0$. Therefore, $p = q > 0$. Assume, by contradiction, that $p = q < 1$. Using arguments similar to those above, it can be argued that the payoffs solve

$$\begin{aligned} v_1 &= \frac{1}{4}(\frac{1}{2}(\delta v_1 + 1 - \delta v_2) + 2(1-p)\delta v_1 + \delta\frac{1}{2}) \\ v_2 &= \frac{1}{4}(\frac{1}{2}(\delta v_2 + 1 - \delta v_1) + 2\delta v_2 + \delta\frac{1}{2}) \\ v_3 &= \frac{1}{4}(\delta\frac{1}{2} + \delta v_3 + (1-p)\delta v_3 + \frac{1}{2}) \\ v_4 &= v_3. \end{aligned}$$

For every $p \in [0,1]$, the unique solution of the system of linear equations above is given by

$$\begin{aligned} v_1(p) &= \frac{4+\delta-3\delta^2}{2(4(1-\delta)(4-\delta)+2\delta^2+\delta p(8-5\delta))} \\ v_2(p) &= \frac{4+\delta-3\delta^2+2\delta p(1+\delta)}{2(4(1-\delta)(4-\delta)+2\delta^2+\delta p(8-5\delta))} \\ v_3(p) &= v_4(p) = \frac{1+\delta}{2(4-\delta(2-p))}. \end{aligned}$$

Note that the expression $v_2(p) + v_3(p) - v_2(0) - v_3(0)$ can be simplified to

$$-\frac{\delta(1+\delta)p(4(1-\delta)(2-\delta)+\delta p(4-3\delta))}{2(2-\delta)(4-\delta(2-p))(4(1-\delta)(4-\delta)+2\delta^2+\delta p(8-5\delta))},$$

which is non-positive. Hence $\delta(v_2(p) + v_3(p)) \leq \delta(v_2(0) + v_3(0)) < 1$ for all $p \in [0,1]$. Therefore, the equilibrium payoffs satisfy $\delta(v_2 + v_3) < 1$, which contradicts $p < 1$. We thus need $p = q = 1$.

It can be immediately verified that the strategies in which player i offers $\delta v_j(1)$ when chosen to make an offer to j and player j accepts offers greater than or equal to $\delta v_j(1)$ and rejects smaller offers define an MPE. The arguments above establish that this constitutes the unique MPE. □

Proposition 4 leaves open the possibility that efficiency might be attainable as $\delta \to 1$ when non-Markovian strategies are considered. A companion paper (Abreu and Manea 2009) shows that this is indeed the case. The canonical specification of asymptotically efficient equilibria for arbitrary networks is delicate. Interestingly, a key simplification is achieved by defining MPEs of a *modified* bargaining game in which agreements that would lead to inefficiency are prohibited by *fiat*. The overall strategies involve non-Markovian threats and rewards to sustain the artificial prohibitions, within a completely non-cooperative (and subgame perfect) equilibrium construction. Thus the current analysis of MPEs plays an unexpectedly critical role in our construction of asymptotically efficient equilibria.

7. Conclusion

Networks are ubiquitous and have been the subject of much scholarly attention in recent years (Jackson (2008) offers an excellent overview). However, there has been limited analysis of decentralized trade in a network setting. Models of decentralized bargaining in networks provide a natural framework to investigate the connection between network structure, feasible agreements, and the division of the gains from trade.

In the model introduced here we establish the existence of MPEs and show that MPEs are not necessarily unique. We relate the properties of MPEs to features of the underlying network and provide a method to construct MPEs. Finally, we demonstrate that in some networks MPEs are incompatible with efficient trade even asymptotically as players become patient (or the time between matchings goes to zero). This robust finding motivates our companion paper (Abreu and Manea 2009), which focuses on the construction of asymptotically efficient (hence, in general, non-Markovian) equilibria. Nevertheless, that equilibrium construction has a strong Markovian flavor as it relies on MPEs of a modified bargaining game, the existence of which is premised on arguments developed here.

Many open questions remain, including the analysis of network structures which lead to multiplicity or inefficiency of MPEs. It is unclear at this stage whether useful characterizations are attainable. Another interesting direction is to endogenize the matching process.[21] The latter undertaking entails qualitative changes in the model structure. These are intriguing topics for future research.

Appendix A

Proof of Lemma 1. We established the "only if" part after the statement of Proposition 1. To prove the "if" part, suppose that the subgame consistent collection of MPEs $(\sigma^{*\delta}(\tilde{G}))_{\tilde{G}\in\mathcal{G}^0}$ of the games $(\Gamma^\delta(\tilde{G}))_{\tilde{G}\in\mathcal{G}^0}$ with respective payoffs $(v^{*\delta}(\tilde{G}))_{\tilde{G}\in\mathcal{G}^0}$ defines the correspondence f by 2.1-2.2, and that $v \in f(v)$. It follows that

$$v = \sum_{\{i\to j|ij\in G\}} \frac{1}{2} p_{ij}(G) z^{i\to j},$$

where $z^{i\to j} \in f^{i\to j}(v)$. Then, there exists $q^{i\to j}$ such that

$$z^{i\to j} = q^{i\to j}(\underbrace{\delta v^{*\delta}(G\ominus\{i,j\})}_{-i,-j}, \underbrace{1-\delta v_j}_{i}, \underbrace{\delta v_j}_{j}) + (1-q^{i\to j})\delta v,$$

with $q^{i\to j} = 1$ (0) if $\delta(v_i+v_j) < (>) 1$ and $q^{i\to j} \in [0,1]$ if $\delta(v_i+v_j) = 1$.

[21] For instance players might be able to expend resources to increase the likelihood of bargaining encounters and perhaps to direct the search at specific partners.

The strategy profile $\sigma^{*\delta}(G)$ defined below constitutes an MPE with payoffs v. We first define the strategies for histories h_t along which at least one agreement occurred. Recall that $G(h_t)$ denotes the network induced by the players remaining in the subgame h_t. Construct the time t strategy of each player according to the date 0 behavior specified by $\sigma^{*\delta}(G(h_t))$.[22] For histories along which no agreement has occurred, $\sigma^{*\delta}(G)$ specifies that when i is chosen to propose to j he offers $\min(1-\delta v_i, \delta v_j)$, and when i has to respond to an offer from j he accepts with probability 1 any offer greater than δv_i, accepts with probability $q^{j\to i}$ an offer of δv_i, and rejects with probability 1 any smaller offers.[23]

The subgame consistency of the collection $(\sigma^{*\delta}(\tilde{G}))_{\tilde{G}\in\mathcal{G}^0}$ guarantees that under the constructed $\sigma^{*\delta}(G)$ the expected payoffs in any subgame of $\Gamma^\delta(G)$ with induced network $\tilde{G}\in\mathcal{G}^0$ are $v^{*\delta}(\tilde{G})$, and that $\sigma^{*\delta}(G)$ is an MPE with expected payoffs v. □

Continuation of the Proof of Proposition 1. We use Lemma 1 to show the existence of MPEs. We prove more generally that there exists a subgame consistent collection of MPEs for the games $(\Gamma^\delta(\tilde{G}))_{\tilde{G}\in\mathcal{G}(n')}$, where $\mathcal{G}(n')$ denotes the subset of subnetworks in $\mathcal{G}$ that have at most n' vertices. We proceed by induction on n'. For $n'=0,1$, the statement is trivially satisfied since the corresponding games are eventless.

Suppose we established the statement for all lower values, and we proceed to proving it for n' $(2\le n'\le n)$. By the induction hypothesis, there exists a subgame consistent collection of MPEs $(\sigma^{*\delta}(\tilde{G}))_{\tilde{G}\in\mathcal{G}(n'-1)}$ of the corresponding games $(\Gamma^\delta(\tilde{G}))_{\tilde{G}\in\mathcal{G}(n'-1)}$. Fix a network $G'\in\mathcal{G}$ with n' vertices, and let $\mathcal{G}'^0$ be the set of all subnetworks of G', excluding G', induced in all subgames of $\Gamma^\delta(G')$. Then $\mathcal{G}'^0$ is a subset of $\mathcal{G}(n'-1)$. Therefore, the collection of MPEs $(\sigma^{*\delta}(\tilde{G}))_{\tilde{G}\in\mathcal{G}'^0}$ for the games $(\Gamma^\delta(\tilde{G}))_{G\in\mathcal{G}'^0}$ is subgame consistent, and we can use their payoffs to define $f^{i\to j}$ and f as in 2.1-2.2 for the game $\Gamma^\delta(G')$.

Note that each $f^{i\to j}:[0,1]^n\to[0,1]^n$ is an upper hemi-continuous correspondence with non-empty convex images. The correspondence $f:[0,1]^n\to[0,1]^n$ is a convex combination

[22]Formally, $\sigma_i^{*\delta}(G)(h_t; i\to j) = \sigma_i^{*\delta}(G(h_t))(h_0; i\to j)$ and $\sigma_i^{*\delta}(G)(h_t; j\to i; x) = \sigma_i^{*\delta}(G(h_t))(h_0; j\to i; x)$ for all $ij\in G(h_t)$ and $x\in[0,1]$, where h_0 denotes an empty history.

[23]Payoff irrelevant MPE multiplicity may arise for two reasons. First, if $\delta(v_i+v_j)>1$, when i is selected to propose to j, in the construction above i offers $\min(1-\delta v_i,\delta v_j)=1-\delta v_i$ to j and the offer is rejected. The strategies may be modified so that i offers j any (mixed) offer $x<\delta v_j$, as rejection obtains regardless (if we specify that j reject offers of δv_j with probability 1 the constraint becomes $x\le\delta v_j$). Second, when $\delta(v_i+v_j)=1$, we stipulated that i's offer to j is accepted with probability $q^{i\to j}$, and j's offer to i is accepted with probability $q^{j\to i}$. If $q^{i\to j}+q^{j\to i}\neq 0,2$ then the equilibrium construction may be modified so that the two agreement probabilities become $q^{i\to j}+\varepsilon$ and $q^{j\to i}-\varepsilon$, respectively, for a range of values of ε.

of the correspondences $(f^{i\to j})_{\{i\to j|ij\in G'\}}$, hence it is upper hemi-continuous with non-empty convex images as well. By Kakutani's fixed point theorem, f has a fixed point.

We can use the steps from Lemma 1 to construct an MPE $\sigma^{*\delta}(G')$ of $\Gamma^\delta(G')$ so that the collection of MPEs $(\sigma^{*\delta}(\tilde{G}))_{\tilde{G}\in\mathcal{G}(n'-1)\cup\{G'\}}$ is subgame consistent. If we append the MPEs $\sigma^{*\delta}(G')$ for all subnetworks $G'\in\mathcal{G}$ with n' vertices to the subgame consistent collection of MPEs $(\sigma^{*\delta}(\tilde{G}))_{\tilde{G}\in\mathcal{G}(n'-1)}$, the resulting collection of MPEs $(\sigma^{*\delta}(\tilde{G}))_{\tilde{G}\in\mathcal{G}(n')}$ for the respective games $(\Gamma^\delta(\tilde{G}))_{\tilde{G}\in\mathcal{G}(n')}$ is subgame consistent. This completes the proof of the induction step. □

Proof of Remark 2. We assume the existence of a public randomization device. As in the proof of Proposition 1, consider a subgame consistent collection of MPEs $(\sigma^{*\delta}(\tilde{G}))_{\tilde{G}\in\mathcal{G}}$ of the respective games $(\Gamma^\delta(\tilde{G}))_{\tilde{G}\in\mathcal{G}}$ with corresponding payoffs $(v^{*\delta}(\tilde{G}))_{\tilde{G}\in\mathcal{G}}$. Fix a history of length t along which no agreement has been reached and a realization of the randomization device at date t. Suppose that the pairs $(i_1,j_1),(i_2,j_2),\ldots,(i_m,j_m)$ are matched to bargain at time t, with i_l in the role of the proposer for $l\in\{1,2,\ldots,m\}=:M$.

Let q_l denote the conditional probability of agreement between i_l and j_l at date t under $\sigma^{*\delta}(G)$ $(l\in M)$. For $l\notin A\subset M$, let $\mathbb{P}_l(A)$ be the probability that the pairs $(i_h,j_h)_{h\in A}$ reach agreement, while $(i_h,j_h)_{h\in M\setminus(\{l\}\cup A)}$ do not, at time t,

$$\mathbb{P}_l(A)=\prod_{h\in A}q_h\prod_{h\in M\setminus(\{l\}\cup A)}(1-q_h).$$

Player j_l's (discounted) expected continuation payoff conditional on rejecting i_l's offer at time t is

$$\sum_{A\subset M\setminus\{l\}}\mathbb{P}_l(A)\delta v^{*\delta}_{j_l}(G\ominus\cup_{h\in A}\{i_h,j_h\}).$$

Player i_l's continuation payoff conditional on his offer being rejected is obtained by simply replacing the subscript j_l with i_l in the expression above. Thus player i_l prefers to make an acceptable (unacceptable) offer to j_l if

$$1-\sum_{A\subset M\setminus\{l\}}\mathbb{P}_l(A)\delta v^{*\delta}_{j_l}(G\ominus\cup_{h\in A}\{i_h,j_h\})$$

is greater (smaller) than

$$\sum_{A\subset M\setminus\{l\}}\mathbb{P}_l(A)\delta v^{*\delta}_{i_l}(G\ominus\cup_{h\in A}\{i_h,j_h\}).$$

Clearly, the simultaneous decisions of players i_l ($l \in M$) concerning whether to make acceptable offers to their respective partners are interdependent. We next define an auxiliary normal form game which captures how the agreement probabilities $(q_l)_{l\in M}$ feed into the expected payoffs of all players in the network. The player set for the auxiliary game consists of all n players. However, only players $i_1, \ldots, i_m$ make non-trivial decisions—each of these m players has a strategy space $\{agree, disagree\}$; all other players have a single action available. If the subset of players $(i_h)_{h\in A}$ chooses the action "agree" and the players $(i_h)_{h\in M\setminus A}$ decide to "disagree," then the payoff to a player $k \notin \cup_{h\in A}\{i_h, j_h\}$ is given by $\delta v_k^{*\delta}(G \ominus \cup_{h\in A}\{i_h, j_h\})$. For $l \in A$, the payoffs for i_l and j_l are $1-\delta v_{j_l}^{*\delta}(G \ominus \cup_{h\in A\setminus\{l\}}\{i_h, j_h\})$ and $\delta v_{j_l}^{*\delta}(G \ominus \cup_{h\in A\setminus\{l\}}\{i_h, j_h\})$, respectively.

One can immediately check that the payoffs obtained if each player i_l chooses the action "agree" with probability q_l in the auxiliary game coincide with the expected period t payoffs in the bargaining game (conditional on the realizations of the randomization device and the match). Moreover, the incentives for agreements in the considered subgame of the bargaining game map to the Nash equilibrium conditions for the auxiliary game. Therefore, $(q_l)_{l\in M}$ describes a Nash equilibrium for the auxiliary game.

Consider now versions of the auxiliary game in which the payoff entries involving $v^{*\delta}(G)$ are replaced by the corresponding components of a variable vector $v \in [0,1]^n$.[24] Let $f^{i_1\to j_1,\ldots,i_m\to j_m}(v)$ denote the convex hull of the set of mixed strategy Nash equilibrium payoffs of the modified auxiliary game for a given v. By standard properties of Nash equilibria, the correspondence defined by $f^{i_1\to j_1,\ldots,i_m\to j_m}$ is non-empty valued and has a closed graph. By construction, $f^{i_1\to j_1,\ldots,i_m\to j_m}$ is convex valued.

We can then define a correspondence f as the sum of the correspondences $f^{i_1\to j_1,\ldots,i_m\to j_m}$ weighted by the probability that the proposer-responder pairs matched in period t are $(i_h, j_h)_{h\in M}$ (analogously to formula 2.2). The constructed f has a fixed point by Kakutani's theorem.

We can extend Lemma 1 to the current setting to establish the relationship between fixed points of f and MPEs of $\Gamma^\delta(G)$. The public randomization device plays the following role in the proof. For a fixed point v of f, the $f^{i_1\to j_1,\ldots,i_m\to j_m}$ component of $f(v)$ may involve a convex

[24]In the modified game, v affects only the payoffs for pure strategy profiles with $|A| \leq 1$ in the earlier description.

combination of Nash equilibria of the corresponding auxiliary game. In that case, play is coordinated on each of the latter equilibria using the randomization device to match their weights in the convex combination. Then the equilibrium construction proceeds inductively as in the proof of Proposition 1. □

Proof of Proposition 2. We first show that the system 4.1 has a unique solution v^{δ,q^δ}. The solutions to 4.1 are fixed points of the function $h : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$h_k(v) = \sum_{\{i|ik\in G\}} \frac{1}{2} p_{ik} q_{ik}^\delta (1 - \delta v_i) + \sum_{ij \in G \ominus \{k\}} p_{ij} q_{ij}^\delta \delta v_k^{*\delta}(G \ominus \{i,j\}) + \left(1 - \sum_{\{i|ik\in G\}} \frac{1}{2} p_{ik} q_{ik}^\delta - \sum_{ij\in G\ominus\{k\}} p_{ij} q_{ij}^\delta \right) \delta v_k.$$

It can be easily checked that h is a contraction with respect to the sup norm on $\mathbb{R}^n$, mapping $[0,1]^n$ into itself, hence it has a unique fixed point, denoted v^{δ,q^δ}, which belongs to $[0,1]^n$. Therefore, for all $\delta \in (0,1)$, v^{δ,q^δ} is the unique solution to 4.1. In particular, the linear system 4.1 is non-singular.

A more involved contraction argument establishes that the linear system 4.2 is non-singular when $q_{ij} > 0$ for at least two links $ij \in G$. Redefine the function $h : \mathbb{R}^n \to \mathbb{R}^n$ by

$$h_k(v) = \sum_{\{i|ik\in G\}} \frac{1}{2} p_{ik} q_{ik} (1 - v_i) + \sum_{ij \in G \ominus \{k\}} p_{ij} q_{ij} v_k^*(G \ominus \{i,j\}) + \left(1 - \sum_{\{i|ik\in G\}} \frac{1}{2} p_{ik} q_{ik} - \sum_{ij\in G\ominus\{k\}} p_{ij} q_{ij} \right) v_k.$$

If $q_{ij} > 0$ for at least two links $ij \in G$, then one can prove that $h \circ h$ is a contraction with respect to the sup norm on $\mathbb{R}^n$ that maps $[0,1]^n$ into itself.[25] Hence $h \circ h$ has a unique fixed point v^q, which belongs to $[0,1]^n$. But if v^q is a fixed point of $h \circ h$, so is $h(v^q)$. Then the fact that v^q is the unique fixed point of $h \circ h$ implies that $h(v^q) = v^q$, i.e., v^q is a fixed point of h. However, h cannot have any fixed points distinct from v^q, since any fixed point of h is also a fixed point of $h \circ h$. Hence v^q is the unique fixed point of h.

We next extablish that $\lim_{\delta\to 1} v_l^{\delta,q^\delta} = v_l^q$ for all $l \in N$. Consider the linear system 4.1. All entries in the coefficient matrix and the augmented matrix are polynomial functions of

[25]Note that $h \circ h$ is a linear function. The hypothesis that $q_{ij} > 0$ for at least two links $ij \in G$ guarantees that the absolute values of the coefficients of v's components in $h_k(h(v))$ sum to less than 1 for every k.

δ, q_{ij}^δ for $ij \in G$, and $v_k^{*\delta}(G \ominus \{i,j\})$ for triplets of players (i,j,k) with $ij \in G$ and $k \neq i,j$ (henceforth, ijk refers to any such triplet). Then v_l^{δ,q^δ} is computed by Cramer's rule, as the ratio of two determinants,

$$v_l^{\delta,q^\delta} = \bar{D}_l \left(\delta, (q_{ij}^\delta)_{ij}, (v_k^{*\delta}(G \ominus \{i,j\}))_{ijk}\right) / D \left(\delta, (q_{ij}^\delta)_{ij}\right),$$

where $\bar{D}_l$ and D are polynomials in several variables. $D\left(\delta, (q_{ij}^\delta)_{ij}\right) \neq 0$ for all $\delta \in (0,1)$ since the corresponding linear systems 4.1 are non-singular. We can also compute v_l^q by Cramer's rule,

$$v_l^q = \bar{D}_l \left(1, (q_{ij})_{ij}, (v_k^*(G \ominus \{i,j\}))_{ijk}\right) / D(1, (q_{ij})_{ij}).$$

Note that $D(1, (q_{ij})_{ij}) \neq 0$ since the linear system 4.2 is non-singular. Because $\bar{D}_l$ and D are polynomial functions, they are continuous in their arguments, hence

$$\begin{aligned} \lim_{\delta \to 1} \bar{D}_l \left(\delta, (q_{ij}^\delta)_{ij} (v_k^{*\delta}(G \ominus \{i,j\}))_{ijk}\right) &= \bar{D}_l \left(1, (q_{ij})_{ij}, (v_k^*(G \ominus \{i,j\}))_{ijk}\right) \\ \lim_{\delta \to 1} D\left(\delta, (q_{ij}^\delta)_{ij}\right) &= D(1, (q_{ij})_{ij}). \end{aligned}$$

Therefore, $\lim_{\delta \to 1} v_l^{\delta,q^\delta} = v_l^q$.

Suppose now that $q_{ij} \in \{0,1\}$ for all $ij \in G$ and that v^q satisfies the conditions $v_i^q + v_j^q < 1$ if $q_{ij} = 1$ and $v_i^q + v_j^q > 1$ if $q_{ij} = 0$. Since $\lim_{\delta \to 1} v^{\delta,q} = v^q$ it follows that there exists $\underline{\delta}$ such that for every $\delta \in (\underline{\delta}, 1)$, we have $\delta(v_i^{\delta,q} + v_j^{\delta,q}) < 1$ if $q_{ij} = 1$ and $\delta(v_i^{\delta,q} + v_j^{\delta,q}) > 1$ if $q_{ij} = 0$. For $\delta \in (\underline{\delta}, 1)$, since $v^{\delta,q}$ solves 4.1, $v^{\delta,q}$ is a fixed point of the correspondence f defined using $(v_k^{*\delta}(G \ominus \{i,j\}))_{ijk}$ as in the proof of Proposition 1. Lemma 1 then implies that, for every $\delta \in (\underline{\delta}, 1)$, $\Gamma^\delta(G)$ has an MPE with payoffs $v^{\delta,q}$. $\square$

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