Peaches, Lemons, and Cookies: Designing Auction Markets with Dispersed Information

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Abstract

This paper studies the role of information asymmetries in second price, common value auctions. Motivated by information structures that arise commonly in applications such as online advertising, we seek to understand what types of information asymmetries lead to substantial reductions in revenue for the auctioneer. One application of our results concerns online advertising auctions in the presence of “cookies,” which allow individual advertisers to recognize advertising opportunities for users who, for example, are customers of their websites. Cookies create substantial information asymmetries both ex ante and at the interim stage, when advertisers form their beliefs. The paper proceeds by first introducing a new refinement, which we call “tremble robust equilibrium” (TRE), which overcomes the problem of multiplicity of equilibria in many domains of interest. Second, we consider a special information structure, where only one bidder has access to superior information, and show that the seller’s revenue in the unique TRE is equal to the expected value of the object conditional on the lowest possible signal, no matter how unlikely it is that this signal is realized. Thus, if cookies identify especially good users, revenue may not be affected much, but if cookies can (even occasionally) be used to identify very poor users, the revenue consequences are severe. In the third part of the paper, we study the case where multiple bidders may be informed, providing additional characterizations of the impact of information structure on revenue. Finally, we consider richer market designs that ensure greater revenue for the auctioneer, for example by auctioning the right to participate in the mechanism.

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1 Introduction

At least since Milgrom and Weber (1982a)’s classic paper, economists have studied the role of information revelation in the design of common value auctions. Milgrom and Weber (1982a)’s linkage principle shows that the auctioneer typically benefits by releasing information publicly to all bidders. In many important classes of applications, however, the information revelation problem is more subtle. The auctioneer may not be able to directly observe and release information, but rather has the option to allow bidders to assess information on their own. The auctioneer may not be able to verify whether and how bidders exercise this option, and the content of the information remains the private information of bidders.

This type of problem arises in the classic examples of common value auctions, auctions for natural resources such as oil and timber: in principle, the auctioneer can either limit or facilitate access to bidders seeking to do seismic surveys or cruise tracts of timber. But there are many other applications as well. In used car auctions, the auctioneer has some control over the type and extent of inspections potential buyers may do. In internet car auctions, some buyers may be local and have the ability to inspect a car in person; the seller can choose whether to allow this or not. In auctions for financial assets, some bidders may have access to better information about the assets. In all of these cases, other bidders may not be able to directly verify whether other bidders have access to superior information in a particular auction.

This paper develops new theoretical results about the impact of the information structure on revenue in common value auctions, focusing on situations where there may be strong asymmetries of information at either the ex ante (before bidders observe their signals) or the interim stage (after observing their signals). The primary motivating application for our study is online advertising. In the US, the fast growing online-advertising market is expected to capture 15% of total US ad spending in 2010 ($25Bn) and grow to more than 40% of ad spending by 2014 ($40Bn) (eMarketer 2010). A large part of the success of search advertising ($12Bn of U.S. online advertising 2010 (Morrison 2010)) is undoubtedly due to advertisers’ ability to target advertising to specific audiences by placing ads on keywords that match to search terms entered by users into search engines. Traditionally, display advertising ($9Bn of U.S. online advertising in 2010 (Morrison 2010)) has been less targeted, or targeted based on broad categories of users (e.g. “sports enthusiasts”) identified by the publisher who sells the impression, typically based on the user’s browsing behavior on that publisher. However, a growing trend is that advertisers are targeting display ads with increasing sophistication by tracking web surfers using cookies (Helft and Vega 2010). Cookies placed on users’ computers by specific web sites can be used to match a user with information such as the user’s order history with an online retailer, their recent history of airline searches on a travel website, or their browsing and clicking behavior across a network of online publishers (such as publishers on the same advertising network). A critically important difference between the two cases is the source of information. In the case of search advertising or traditional display advertising, the publisher has as much or more information than the advertiser, and the information is disclosed to all ad buyers symmetrically. In cookie-based display-advertising, however, ad buyers bring their own private information collected via cookies stored on web surfers’ computers.

Although there are a variety of mechanisms for selling display advertising, auctions are a leading method, especially for “remnant” inventory, and it is in these markets (where in the absence of targeting, impressions may have a fairly low value) that cookies potentially play a very important role. For example, Google’s ad exchange is currently described as a second-price auction that takes place in real time: that is, at the moment an internet user views a
page on an internet publisher, a call is made to the ad exchange, bidders on the exchange
instantaneously view information provided by the exchange about the publisher and the user
as well as any cookies they may have for the individual user, and based on that information,
place a bid. The cookie is only meaningful to the bidder if it belongs directly to the bidder
(e.g., Amazon.com may have a cookie on the machines of regular customers), or if the bidder
has purchased access to specific cookies from a third-party information broker. Cookie-based
bidding potentially makes display auctions inherently asymmetric at both the ex ante and the
interim stage. At the ex ante stage, bidders may vary greatly in their likelihood of holding
informative cookies, both because popular websites have more opportunities to track visitors
and because different sites vary in the sophistication of their tracking technologies. At the
interim stage, for a particular impression, a typical bidder may have only a small chance of
having a relevant cookie, but bidders who do have a substantial information advantage relative
to those who do not.

If cookies only provided advertisers private-value information, then increasing sophistica-
tion in the prevalence and use of cookies by advertisers would present ad inventory sellers a
two-way trade-off between better matching of advertisements with impressions and reduced
competition in thinner markets (Levin and Milgrom 2010). In such a private value setting,
Board (2009) shows that irrespective of such asymmetry, more cookies and more targeting
always increase second-price auction revenue as long as the market is sufficiently thick. However,
cookies undoubtedly also contain substantial common value information. (For instance, when
one bidder has a cookie which identifies an impression as due to web-bot rather than a human,
the impression is of zero value to all bidders.) As a result, the inherent asymmetry created
by cookies can lead to cream skimming or lemons-avoidance by informationally advantaged
bidders, with potentially dire consequences for seller revenues.

Thus, a designer of online advertising markets (or other markets with similar informational
issues) faces an interesting set of market design problems. One question is whether the market
should encourage or discourage the use of cookies, and how the performance of the market
will be affected by increases in the prevalence of cookies. This is within the control of the
market designer: in display advertising, it is up to the marketplace to determine how products
are defined. All advertising opportunities from a given publisher can be grouped together, for
example. And Google’s ad exchange reportedly does not support revealing all possible cookies.
A second market design question concerns the allocation problem: if an auction is to be used,
what format performs best? Both first and second price auctions are used in the industry.
There are a number of other design questions, as well, including whether reserve prices, entry
fees, or other modifications to a basic auction should be considered.

In order to understand the market design tradeoffs involved in an environment with these
kinds of information asymmetries, the first part of our paper develops a model of pure common-
value second-price auctions. Perhaps surprisingly, the existing literature leaves a number of
questions open. For example, while it is well known that the presence of an informationally-
advantaged bidder will substantially reduce seller revenues in a sealed-bid first-price auction
(FPA) for an item with common value (Milgrom and Weber 1982b, Engelbrecht-Wiggans,
Milgrom and Weber 1983, Hendricks and Porter 1988), substantially less is known about the
same issue in the context of second-price auctions. One of the main impediments to progress
has been the well known multiplicity of Bayesian Nash Equilibria in second-price common-value
auctions (Milgrom 1981). As a consequence, little is known about what types of information
structures lead to more or less severe reductions in revenue.

In order to address the multiplicity problem, we begin by suggesting a new refinement,
tremble robust equilibrium. Tremble robust equilibrium (TRE) selects only Bayesian Nash
Equilibria that are near to the equilibrium of a perturbed game in which a random bidder enters with vanishingly small probability $\varepsilon$ and then bids smoothly over the support of valuations. In addition to capturing an aspect of the real-world uncertainty faced by bidders in the kinds of applications we are interested in, we argue that this refinement has a number of attractive properties. In many cases, this refinement selects a unique equilibrium. In our setting, when bidders are ex ante symmetric it selects the symmetric equilibrium studied by Milgrom and Weber (1982a) in a setting with continuous signals (we have not yet applied our TRE refinement to Milgrom and Weber's (1982a) model). Moreover it rules out intuitively unappealing equilibria in which uninformed bidders bid aggressively because they can rely on other to set fair prices.

We then proceed to analyze a number of special cases of common value second price auctions using the TRE refinement. To develop some intuition about our main results, consider first a very simple example of an information structure in a common value auction. Only one bidder uses cookie tracking (that is, only one bidder is privately informed), and the bidder can only determine the presence or absence of the cookie: that is, informed bidder has a binary signal which either takes on the value \{no-cookie\} or \{cookie\}. The other bidders cannot assess the existence of the cookie for a particular impression (though they know the overall information structure, including the probability of cookies). Apart from the restriction to a binary rather than continuous signal, this corresponds to the setting of informational advantage studied by Milgrom and Weber (1982b) in first-price auctions.

We show that for this simple information structure, TRE selects a unique equilibrium to the second-price auction, one with intuitive appeal. We are then able to address some interesting comparative statics questions about when, and why, different kinds of information asymmetries can have dramatically different impacts on revenue.

Consider two cases within this simple information structure. In the first case, cookies identify “peaches,” or high-value impressions. This is perhaps the most natural assumption - someone who has been to an advertiser’s website before is more likely to be an active internet shopper than a random web surfer. In the second case, cookies identify “lemons,” or low-value impressions. This might occur if a prior visit indicates the surfer is in fact a web-bot and not a real person. In both cases, information is otherwise symmetric across bidders, and the value of the impression is common to all bidders.

At first, it might seem that for both the “lemons” and “peaches” cases, there could be dire consequences for revenue, due to the extreme adverse selection: one bidder has strictly better information than the others. However, the surprising result is that in the “peaches” case, revenue loss is minimal. In contrast, in the “lemons” case, revenue collapses to the value of the “lemons,” even if the probability that an impression has a cookie is arbitrarily close to zero. This result contrasts with that of Engelbrecht-Wiggans et al. (1983), which shows that the revenue losses from a first-price auction should be proportional to proportion of impressions that have cookies for both lemons and peaches. Putting our results together with Engelbrecht-Wiggans et al. (1983), it follows that a first-price auction will perform substantially better in an environment where one bidder has access to relatively rare cookies for “lemons.”

We next generalize the information structure, allowing for cookies to be more richly informative, and the informed bidder to have a signal drawn from a finite set $S_i$. We show that revenue in the unique TRE of the second-price auction is equal to the object’s expected value.

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1We find that first price auction revenues are always higher than second price auction revenues when only one bidder is informed, but the difference is on the order of $\varepsilon^2$ when the informed bidder has access to relatively rare cookies for “peaches”, that arrive at rate $\varepsilon$. 

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conditional on realizing the worst cookie or the lowest signal \( s_i = \min \{ S_i \} \). Thus the tremble robust equilibrium is one which, like Akerlof’s (1970) classic “market for lemons,” uninformed buyers are not willing to pay more than their lowest possible value. To understand the role of the refinement, we note that there are Bayesian Nash equilibria with higher revenues, in which a single uninformed bidder bids more aggressively and relies on the informed bidder to bid his expected value and set a “fair” price. Our refinement rules these equilibria out because, in a nearby perturbed auction with a random bidder, aggressive bids by an uninformed bidder would sometimes win at a high price set by the random bidder rather than a fair price set by the informed bidder and hence be unprofitable.

Beyond the initial case of a single informed bidder, we have begun to extend our analysis to the general case of multiple informed bidders with richer signal structures. Our progress so far involves a trade-off with respect to relaxing assumptions, where we have pushed the model in two directions. First, we consider all monotonic domains with two bidders, each with a binary-signal (where each signal is either low or high). For any such domain we provide a TRE and prove it is unique. Second, we consider the case of multiple informed bidders with multiple signals, for which we characterize a pure strategy TRE under more restrictive assumptions about the information structure.

In the two-bidder binary-signal model, assume without loss of generality that \( Pr[H_1, L_2](1 - v(H_1, L_2)) \leq Pr[L_1, H_2](1 - v(L_1, H_2)) \). Here \( Pr[H_1, L_2] \) is the probability bidder 1 receives a high signal but bidder 2 receives a low signal and \( v(H_1, L_2) \) is the object’s value in that event. We show that in the unique TRE the following holds. Both bidders bid \( v(L_1, L_2) = 0 \) conditional on receiving a low signal. Conditional on receiving a high signal, bidder 1 (called the strong bidder) always bids aggressively at the objects maximum possible value \( v(H_1, H_2) = 1 \). However, conditional on receiving a high signal, bidder 2 (called the weak bidder) mixes between aggressive and defensive bids, bidding 1 with probability \( \frac{Pr[H_1, L_2](1 - v(H_1, L_2))}{Pr[L_1, H_2](1 - v(L_1, L_2))} \) and \( v(L_1, H_2) \) with the remaining probability.

Note that bidder 1 is called the strong bidder if \( Pr[H_1, L_2](1 - v(H_1, L_2)) \leq Pr[L_1, H_2](1 - v(L_1, H_2)) \). Bidder 1’s strength is the inverse of \( Pr[H_1, L_2](1 - v(H_1, L_2)) \). Conditional on receiving a high signal, \( Pr[H_1, L_2](1 - v(H_1, L_2)) \) captures the "downside risk" faced by bidder 1 that bidder 2 may have a low rather than high signal. It takes into account both the likelihood that bidder 2 has a low rather than a high signal (\( Pr[H_1, L_2] \)) and the size of the value loss \( (v(H_1, H_2) - v(H_1, L_2) = 1 - v(H_1, L_2)) \) if bidder 2 has a low rather than a high signal. The bidder with the smaller downside risk is called the stronger bidder.

Our analysis of the second-price auction between two informed bidders with cookies encompasses two special cases. The first is that in which one bidder never receives a cookie - or that only one bidder is informed. The second is that in which bidders are symmetric ex ante. These are variations of the polar extremes of ex ante asymmetry and symmetry studied respectively by Milgrom and Weber (1982b) (for first price auctions) and Milgrom and Weber (1982a) (for multiple auction formats). As already stated, the first-price auction has higher revenue under extreme asymmetry when only one bidder is informed. Focusing on the symmetric equilibrium of the SPA when bidders are ex ante symmetric, Milgrom and Weber (1982a) show that the SPA has higher revenue than the FPA. Since, in our setting, the TRE refinement selects the symmetric equilibrium when bidders are symmetric ex ante, the same result applies. (Milgrom and Weber’s (1982a) result is proved for continuous signals, but the authors point out in footnote 15 that it is true more generally.) Thus the revenue ranking between first and second price auctions is reversed by sufficient ex ante asymmetry. (We are working on a revenue comparison for the intermediate cases.)
Next, we move to the case of multiple bidders. To make progress for this case, we impose an additional information restriction: we assume that bidders’ information satisfies both a high-signal-is-never-bad-news property and a low-signal-is-never-good-news property. Given this information structure, we can characterize a pure-strategy tremble-robust equilibrium given any number of informed bidders who each receive signals with finite support. (We are working on a uniqueness result in this setting.) The trade-off with respect to assumptions is that the high-signal-is-never-bad-news and low-signal-is-never-good-news properties are restrictive. In a monotonic two-bidder, binary-signal case, the low-signal-is-never-good-news property requires that the bidder with the higher posterior given his high signal has the same posterior given that both agents received their high signals (that is, learning that the other agent also has the high signal does not bring any additional good news). Then, the other agent bids the value conditional on his own signal and conditional on the more informed bidder having the low signal, which potentially has dire consequences to the seller’s revenue. (In such a simple monotonic domain the high-signal-is-never-bad-news property is always satisfied.)

For the case that a number of informed bidders who each receive signals with finite support, the low-signal-is-never-good-news property generalizes the condition stated above for the two-bidder, two-signal case. The property has a recursive nature. First, it says that if the bidder (call her bidder 1) with the highest possible interim value realized her highest signal, she would not revise her expectation upwards upon learning the signals of other bidders. Now suppose it were publicly announced that bidder 1 did not receive her highest signal and all other bidders updated their beliefs and interim valuations. Then the same condition should hold recursively on the new beliefs. If the bidder with the highest-possible revised-interim-value realized his highest feasible remaining signal, he would not revise his expectation upwards upon learning the signals of other bidders. This recursive definition of the low-signal-is-never-good-news property ranks all bidders’ signals from high to low in a single ordering. Suppose that each bidder bids his expected valuation conditional on his own signal being the highest signal according to this ordering. Now, the high-signal-is-never-bad-news property ensures that the resulting bids will be monotonic in the signals, as ordered by the low-signal-is-never-good-news property. When the two properties are satisfied the chosen strategies form a pure-strategy tremble-robust equilibrium in which each bidder bids his expected value conditional on winning. Note that the low-signal-is-never-good-news property limits the extent in which the monotonic domain is strictly monotonic: at any point the agent with the highest interim-value (given his current highest signal) cannot get any ”good news” (increase in his belief about the value) from any of the signals below.

Summarizing our findings for second-price, common value auctions, we show that the nature of information asymmetry has strong implications for revenue. When just one bidder is informed, revenue collapses to the lowest possible expected value a bidder may have after observing his signal, even if that signal is very rarely realized, implying that revenue may be unaffected if cookies identify only peaches, but collapses when they may identify lemons, even if the probability is small. When multiple bidders are informed, revenue need not collapse even when cookies identify lemons. In particular, with two informed bidders who each might receive a cookie that identifies a lemon, revenue is proportional to a ratio that measures the (ex ante) likelihood that the weakly less informed bidder is exclusively informed about the item being a lemon, relative to the likelihood that weakly better informed bidder is exclusively informed about the item being a lemon. When bidders are symmetric ex ante, this ratio is 1 and the seller extracts all surplus.

With multiple informed bidders, if the information structure satisfies monotonicity and low-signal-is-never-good-news properties, information asymmetry has implications for revenue
that are similar to the case in which only one agent is informed. In the pure strategy TRE, agents bid their expected valuations conditional on winning (conditional on having the highest signal). Thus bids may be significantly below interim valuations that are conditioned only on bidders' own signals. The exact implications for revenue depend on the difference between the two posteriors and the priors over signals.

Taken together, the findings suggest that a market design may be vulnerable to low-revenue outcomes when it uses a second-price auction and enables an information structure that opens the door to certain kinds of asymmetries.

So far, we have focused mainly on the costs of information asymmetry, while suppressing any benefit. As mentioned above, the cost-benefit tradeoff between private value information (which we have suppressed) and market thinness has already received some attention in the literature, so we do not revisit that here. Instead, in the last part of the paper, we extend the model in a slightly different direction, allowing for the possibility that cookies contain action-relevant information. For instance, when placing a display ad on the New York Times website, Zappos can include a picture of the exact running shoes the New York Times reader had previously been looking at on Zappos.com. This means that the information in cookies can directly increase the value of winning to bidders and create a value advantage as well as an informational advantage. Both the presence of private-value and action-relevant information give clear reasons that ad-buyers and sellers alike benefit from incorporating the inherently asymmetric targeting information from cookies into display-ad auctions. Thus, simply banning cookie-based bidding is likely not optimal.

In the final section of our model we discuss a seller’s optimal mechanism design problem when cookies contain both action-relevant and common value information. We assume a particular correlation structure in signals: that one and only one bidder receives an informative cookie. In this setting, although all bidders are symmetric ex ante with equal likelihood of receiving a cookie, because only one becomes informed, revenue collapses to the expected value conditional on the worst possible cookie.

The revenue collapse result follows from the analysis in Section 5.1 of auctions with only one informed bidder. This analysis essentially captures the continuation game that arises at the interim stage once bidders have discovered whether or not they are the single informed bidder. Notice that it does not contradict the result in Section 5.3 that ex ante symmetry leads to full revenue extraction, because that result was specific to the information structure assumed in Section 5.3. The important difference here is that at the interim stage at one case (Section 5.1) at most one bidder has a "high" signal that leads to an interim valuation above the item’s unconditional expected valuation, while in the other case (Section 5.3) more than one bidder might get such a signal.

Because bidders are ex ante symmetric, we show that full revenue can be extracted by charging symmetric entry fees to participate in the SPA. The TRE refinement is crucial for setting entry fees correctly, as it provides a unique prediction of bidder profits in the SPA. Note that this approach is ex ante individually rational, but not interim individually rational.

For bidders with binary signals, we also present an interim individually rational, dominant strategy mechanism that extracts \((1 - 1/n)\)-fraction of the social welfare as revenue, using a variant of Myerson’s optimal auction for the private value setting. The mechanism runs a SPA on all bids above some reserve price \(r\), and if no such bid exists it sells to a random bidder at a given floor price \(f\) (pooling). Loss of revenue arises from lowering the reserve price to satisfy the incentive constraint for bidders with high signals.

Building on this auction we also present a two stage mechanism for the case that the auctioneer has no information regarding the expected values conditional on the two signals
and the extra value that results from taking an advantaged action given the high signal. Agents are bidding in the first stage to set the reserve and floor prices, knowing that each will be set by the lowest bid for it. The bidder that sets the reserve price will not be allowed to bid above it, while the bidder that sets the floor price will be excluded from the pool. Relying on the symmetry between the agents we can show that this mechanism gets at least \((1 - 1/n)\)-fraction of the revenue achieved by the former mechanism that has full priors, thus the revenue achieved is at least \((1 - 1/n)^2\) fraction of the welfare.

## 2 Related Literature

This paper seeks to understand how revenues in a common value second price auction depend on the structure of information held by bidders. A serious challenge to comparing revenues across different information structures is that for any given information structure there are typically many different equilibria with widely different revenues. For instance, consider a setting commonly studied in the literature: There are two bidders 1 and 2 who receive continuously signals \(s_1\) and \(s_2\) which have marginal distributions \(F_1(s_1)\) and \(F_2(s_2)\) and the value of the object conditional on the signals is \(v(s_1, s_2)\). It is well know that there are a continuum of equilibria (Milgrom 1981). In particular, for any increasing function \(h\), the following bidding strategies form an equilibrium (Milgrom (2004) Theorem 5.4.8):

\[
b_1(s_1) = v(s_1, h^{-1}(s_1)), \quad b_2(s_2) = v(h(s_2), s_2).
\]

The bidding strategies described by equation (1) imply that bidders 1 and 2 make the same bid whenever bidder 2 has signal \(s_2\) and bidder 1 has signal \(h(s_2)\). In other words, the function \(h(s_2)\) describes the bidder 1 signal \(s_1\) that ties with \(s_2\) in equilibrium. Because \(h\) can be any increasing function, Nash equilibrium makes no prediction about which bidder 1 signal ties with \(s_2\) in equilibrium and hence no useful prediction about revenue. Similar multiplicity arises in our setting in which bidder signals are drawn from discrete and finite support. For instance, in Section 5.1 where we consider the case of a single informed bidder and \(n - 1\) uninformed bidders, the following is an equilibrium: The informed bidder bids her expected value conditional on her signal, \(n - 2\) uninformed bidders bid to tie the informed bidder’s lowest bid, and the last uninformed bidder bids a bid \(b\) that is weakly higher. This is an equilibrium for any such bid \(b\). Thus for any particular signal of the informed bidder, there exists an equilibrium in which the uninformed bidder bids the same amount. Thus revenue could be anywhere between the informed bidder’s lowest possible valuation and the full surplus. Similarly, in Section 5.3 where 2 bidders each receive binary signals, it is an equilibrium for the informationally advantaged bidder to always bid 1 when her signal is high and zero otherwise, while the other bidder bids zero following a low signal and mixes between bids of 0 and 1 following a high signal. This is an equilibrium for any mixing probability by the weak bidder. Thus in equilibrium the weak bidder with a high signal can tie with the strong bidder with a low signal, the strong bidder with a high signal, or a mixture of the two. In each case our TRE refinement resolves this multiplicity by selecting a unique equilibrium.

A common approach in the literature with symmetric bidders is to focus on the symmetric equilibrium: \(b_i(s_i) = v(s_i, s_i)\) (Milgrom and Weber 1982a, Matthews 1984). As shown by Milgrom and Weber (1982a) and Matthews (1984) others, this selects the equilibrium in which each bidder bids the object’s expected value conditional on the highest signal of competing bidders being equal to her own. This excludes extreme equilibria such as one in which one bidder bids an object’s maximum value and all other bidders bid zero. Unfortunately it is not
clear how the symmetry refinement can be extended to asymmetric environments of the type we are interested in, or why symmetry should be expected in equilibrium.\footnote{Hausch (1987) selects the \( b_i(s_i) = v(s_i, s_j) \) equilibrium in a setting in which asymmetry implies \( v(s_i, s_j) \neq v(s_j, s_i) \) for \( s_i \neq s_j \). The motivation for this choice is unclear.} In fact, Klemperer (1998) argues that with *almost common values* all reasonable equilibria are extremely asymmetric.

Recent work by Parreiras (2006), Cheng and Tan (2007), and Larson (2009) introduce perturbations to select a unique equilibrium in two-bidder auctions with continuously distributed signals. Parreiras (2006) perturbs the auction format by assuming that winning bidders pay their own bid rather than the second highest with probability \( \varepsilon \), and taking the limit as \( \varepsilon \) goes to zero. Cheng and Tan (2007) and Larson (2009) introduce private value perturbations to the common value environment and take the limit as these perturbations go to zero. Cheng and Tan (2007) assume private value perturbations are perfectly correlated with common value signals and are symmetric across bidders. The symmetry of perturbations selects a unique equilibrium. Larson (2009) allows for asymmetric perturbations which are assumed to be independent of common value signals and shows that the equilibrium selected depends on the ratio of the standard deviations of the two bidders’ private value perturbations. Larson (2009) shows that a weakness of Cheng and Tan’s (2007) approach is that it is the assumption of symmetry in perturbations that drives the equilibrium selection. The different choices of perturbations by different authors lead to very different equilibrium selection and conclusions. For instance, Parreiras (2006) show that second-price auctions generate at least as much revenue as first-price auctions even when bidders are asymmetric (given affiliated signals). In contrast, Cheng and Tan (2007) find that first-price auction revenues are strictly higher than second-price auction revenues given any bidder asymmetry ex ante (with independent signals and a submodular value function).

An alternative approach taken in the literature that has been applied to auctions with more than two bidders is to select equilibria that survive iterated deletion of dominated strategies. Harstad and Levin (1985) consider the case in which the first order-statistic of bidders’ signals is a sufficient statistic for the object’s value in the Milgrom and Weber (1982a) setting with symmetric bidders and continuously distributed signals. For this case, Harstad and Levin (1985) shows that iterated deletion of dominated strategies uniquely selects the symmetric Milgrom and Weber (1982a) equilibrium. Einy, Haimanko, Orzach and Sela (2002) consider the case of asymmetric bidders and discrete signals with finite support. Einy et al. (2002) show that if the information structure is *connected* the iterated deletion of dominated strategies selects a set of equilibria with a unique pareto-dominant (from bidders’ perspective) equilibrium. Malueg and Orzach (2009) apply Einy et al.’s (2002) refinement in two examples and Malueg and Orzach (2011) apply Einy et al.’s (2002) refinement to the special case of two-bidder auctions with *connected* and *overlapping* information partitions. For a particular one-parameter family of common-value distributions, Malueg and Orzach (2011) find that distributions with sufficiently thin left tails yield lower revenues in second-price auctions than in first-price auctions. The primary drawback to Einy et al.’s (2002) approach is that the required assumptions on the information structure are very restrictive. For instance, we show in Appendix \( C.2 \) that Einy et al.’s (2002) connectedness property is strictly more restrictive than our *low-signal-is-never-good-news* property. Thus connectedness rules out many interesting settings such as our model of two bidders with binary signals in which neither bidder is perfectly informed. Even when one bidder is perfectly informed, iterated deletion of dominated strategies is unhelpful on its own: Any bid by the uninformed bidder between the informed bidder’s low (0) and
high (1) interim valuations survives, so revenue may be anywhere between 0 and the object’s expected value.

An alternative literature on almost-common-value auctions perturbs the common value framework by assuming that one bidder has a small value-advantage and is known ex ante to value the object slightly more than other bidders. The common wisdom from early papers which modeled two-bidder auctions is that a slight value advantage causes: (1) the strong bidder to win almost all the time, (2) for revenues to collapse in second-price auctions, and (3) for first-price auctions to generate higher revenue (Bikhchandani 1988, Avery and Kagel 1997, Klemperer 1998, Bulow, Huang and Klemperer 1999). However, more recently Levin and Kagel (2005) show that dramatic revenue losses from small asymmetries rely on the two-bidder assumption and that revenue losses are proportional to the value advantage when there are three or more bidders.

Our approach focuses on the pure common value model where no bidder has a value advantage. We do not wish to restrict the information structure to be symmetric or connected, so cannot focus on symmetric equilibria as in Milgrom and Weber (1982a) or use iterated deletion of dominated strategies as in Einy et al. (2002). Instead we introduce a new refinement, TRE, which selects equilibria near those of a perturbed game with an additional random bidder. This is similar in spirit to Parreiras (2006), Cheng and Tan (2007), and Larson (2009). Unlike Cheng and Tan’s (2007) and Larson’s (2009) private value perturbation refinements, TRE typically does not need further refinement to select among perturbations since there is often a unique TRE. For instance, the TRE refinement selects the symmetric equilibrium when bidders are ex ante symmetric (providing an additional justification for focusing on such symmetric equilibria). Cheng and Tan’s (2007) and Larson’s (2009) private value perturbation refinements do the same with the additional assumption that the perturbations be symmetric across bidders. Our finding that sufficient ex ante asymmetry favors first price auctions over second price auctions (reversing Milgrom and Weber’s (1982a) result from the symmetric case) is similar to Cheng and Tan’s (2007) result that ex ante asymmetry favors first-price auctions but contrasts with Parreiras’s (2006) finding that Milgrom and Weber’s (1982a) first and second-price auction revenue ranking result is robust to asymmetry. Our finding that revenue losses in second price auctions due to informationally advantaged bidders are much larger when the information advantage concerns “lemons” rather than ”peaches” mirrors the cost of private seller information in Akerlof’s (1970) market for lemons. However the result is novel as it depends on the tremble robust equilibrium refinement - alternative equilibrium selection rules would lead to a different result.

In the context of analyzing the generalized second price (GSP) auction for sponsored search with independent valuations and complete information, Hashimoto (2010) proposes to refine the set of equilibria by adding a non-strategic random bidder that participates in the auction with small probability. Edelman, Ostrovsky and Schwarz (2007) and Varian (2007) have shown that GSP has an envy-free efficient equilibrium, the main result of Hashimoto (2010) is that this equilibrium does not survive the refinement.

### 3 The Solution Concept

Consider the following simple scenario. We run a second price auction (with random tie breaking) for a common value good. Assume that the good has only two possible values, $P$ (Peach) and $L$ (Lemon) and it holds that $P > L = 0$. Each value is realized with probability $1/2$. There are two agents, one is perfectly informed about the value of the good, while the
other only knows the prior. Agents must submit non-negative bids. What bidding strategies and revenues should we expect?

Nash equilibrium provides no prediction about revenue beyond an upper bound of the full surplus \((L + P)/2\). It is an equilibrium for the informed bidder to bid his value and the uninformed bidder to bid \(P\), which results in full surplus extraction. However, it is also an equilibrium for the uninformed bidder to bid \(10P\) and the informed bidder to bid 0, earning 0 revenue.

A natural refinement is to restrict attention to Nash equilibria in which bidders to use undominated strategies. Notice that unlike in the private value model, agents do not necessarily have a dominant strategy in a common value second price auction. Indeed, in the scenario described above the informed agent has a dominant strategy (to bid the value given his signal), while the uninformed agent does not. To see that, observe that for any two bids \(b_1\) and \(b_2\) such that \(P \geq b_1 > b_2 \geq L\) there exist two strategies of the informed agent such that for one strategy the utility from \(b_1\) is higher, while for the other strategy the utility from \(b_2\) is higher. Bidding \(b_1\) is superior to bidding \(b_2\) when the informed is bidding \((b_1 + b_2)/2\) when the value is \(P\), and bidding \(L\) when the value is \(L\). On the other hand bidding \(b_2\) is superior to bidding \(b_1\) when the informed is bidding \((b_1 + b_2)/2\) when the value is \(L\), and bidding \(L\) when the value is \(P\) (handing out the good items to the other bidder).

Thus ruling out dominated strategies restricts the informed bidder to use her dominant strategy and bid her value. However the only restriction placed on the uninformed bidder is that he not bid less than \(L\) or more than \(P\). Revenue could be anywhere between \(L\) and the full surplus.

A common approach to the multiplicity problem in the literature is to focus on settings in which bidders are ex ante symmetric and assume bidders bid symmetrically (e.g. Milgrom and Weber (1982a)). Unfortunately this is not applicable when we are studying situations in which bidders are known to be substantially different ex ante. Einy et al. (2002) restrict attention to sophisticated equilibria which survive iterative simultaneous maximal elimination of weakly dominated strategies. However, this refinement by itself does not identify a unique outcome.

In the current example, for instance, the set of sophisticated equilibria are the same as the set of Nash equilibria in undominated strategies. Thus, Einy et al. (2002) further refine the set of sophisticated equilibria by focusing on strategies that guarantee a payoff of zero, which does identify a unique outcome in the restricted information structure ("connected domains") they study. Einy et al. (2002) also show that in connected domains uniqueness can be achieved by introducing an additional rational uninformed bidder. Unfortunately, focusing on strategies that guarantee a payoff of zero or introducing another rational uninformed bidder are not sufficient to derive a unique outcome even in simple domains such as the one we analyze in which two informed bidders each receive a binary signal.

We also believe that for the example in discussion the natural outcome is that the informed bidder bid her posterior value and the uninformed bidder bid \(L\). As observed by Einy et al. (2002), introducing another rational uninformed bidder provides the needed refinement for some domains. Yet, as this refinement does not provide unique outcome even in some rather simple domains we are interested in, we suggest a different approach that achieves the same outcome for the example in discussion. In our refinement, with some small probability another uninformed bidder enters the auction and bids somewhere between \(L\) and \(P\). That "random"
bidder is not assumed to act rationally and his sole propose is to make the game "noisy" in order to remove unreasonable equilibria. Indeed, in the example discussed, in the presence of such a bidder if the uninformed bidders bids higher than \( L \) she risks overpaying for a low value item without ever winning the high value item at a discount.

We formalize this intuition by considering a perturbation of the game in which with some small probability \( \epsilon > 0 \) there is an additional bidder that comes to the auction and bids a random value drawn from some distribution which is "nice" (satisfying some simple assumptions: support on the relevant values, differentiability and density that is continuous and positive on the interval). We want to consider only Nash equilibria that are nearby to Nash equilibria (in undominated strategies) of such perturbed games.

Returning to the example with one informed bidder and one uninformed bidder, recall that bidding her interim value was the only undominated strategy for the informed bidder. Given that the informed bidder bids her posterior value, the presence of a random bidder means that \( L \) is the only undominated bid for an uninformed bidder. The informed bidder ensures that the uninformed bidder can never win the object at a discount below value. However the random bidder ensures that any bid above \( L \) risks overpaying for a low value object when the random bidder sets the price. Thus bidding above \( L \) leads to a strictly negative payoff. We observe that by adding noise a unique strategy profile and revenue is predicted.

Motivated by the above we suggest the following refinement of (mixed) Nash Equilibrium for an auction scenario. The solution concept picks a (mixed) Nash equilibrium that is the limit, as \( \epsilon \) goes to zero, of a series of mixed Nash equilibrium of each modification of the original game in which another "random" bidder is added with small probability \( \epsilon \). The random bidder is bidding a random value drawn from some distribution with support over the "relevant" values, is differentiable and has density that is continuous and positive on the interval. We call such a profile of strategies a Tremble Robust Equilibrium (TRE). The formal definition of this new refinement is presented in Section 4.2. Moreover, if there is a profile of strategies that is (mixed) Nash Equilibrium in any such small perturbation of the original game, we call it a strong Tremble Robust Equilibrium.

Our TRE refinement similar in spirit to other perturbation based refinements discussed in Section 2 (Parreiras 2006, Cheng and Tan 2007, Larson 2009). One can naturally ask whether instead of using the new refinement of TRE one can use the classical refinement of Tremble Hand Perfect Equilibrium (PE) by Selten (1975). It turns out that the PE solution concept (adjusted to games with infinite sets of actions and incomplete information) is too permissive and does not provide the natural unique prediction one would expect in the most basic setting with two agents discussed above: the setting with one informed agent with a binary signal, and one uninformed agent. In Appendix A we show that two extensions by Simon and Stinchcombe (1995) of PE to infinite games (which we adjust to incomplete information) do not provide unique prediction in the above setting. On the other hand, in the same setting, if we restrict the perturbation of the informed agent to be independent of his signal then in the unique equilibrium the uninformed is bidding the unconditional expected value of the item, contrary to our expectation.

4 Auctions where each Agent has Finitely Many Signals

We start by presenting out model followed by the refinement.
4.1 The Model

An auctioneer is offering an indivisible good to a set $N$ of $n$ potential buyers. The value that an agent derives from the good is a function of the state of the world. Agents only get a signal about the state of the world. We next formally define the model.

Let $\Omega$ be set of states of the world, and $\omega \in \Omega$ be a state of the world. There is a prior distribution $H \in \Delta(\Omega)$ over the states, this prior is commonly known, but the exact state $\omega$ is not.

Each buyer gets a signal about the state of the world. For each bidder $i$ there is a set of signals $S_i$. For every state $\omega \in \Omega$ and buyer $i$ there is a distribution over signals $d_i(\omega) \in \Delta(S_i)$ which is commonly known; and buyer $i$ gets a private signal $s_i$ sampled from $d_i(\omega)$. We assume that every signal $s_i \in S_i$ has positive probability of being sampled (when the quality ranges over $\Omega$).\footnote{This is without loss of generality as we can define $S_i$ to be the set of signal with positive probability of being sampled.}

The value of the item to agent $i$ when the state of the world is $\omega$ is $v_i(\omega)$.

When buyer $i$’s signal $s_i$ was realized to $s_i'$ he updates his belief about his expected value of the good, his posterior belief is that the expected value is $E[v_i(\omega)|s_i = s_i']$, where the expectation is taken both over the randomness $H$ that generated $\omega$ and the randomness $d_i(\omega)$ that generated the realized signal $s_i'$. Similarly, we denote by $v_i(s')$ the posterior expected value given that each agent $i$ receives signal $s_i'$ and $s'$ is the vector of received signals, that is $v_i(s') = E[v_i(\omega)|s_i = s_i' \forall i]$.

Let $T_j$ be a set of signals for buyer $j$, and let $T = (T_1, T_2, ..., T_n)$ be a vector of such subsets, one for each buyer. The vector $T$ is feasible if there exists $\omega$ which will generate some vector of signals $(s_1, s_2, ..., s_n)$ such that for every $j$ it holds that $s_j \in T_j$. For $T$ that is feasible let $v_i(T)$ be the expected value that agent $i$ has for the good, conditional on the signal $s_j$ of each buyer $j$ being from $T_j$. A strategy $\mu_i$ for agent $i$ is a mapping from his signal to his bid: $\mu_i : S_i \rightarrow \mathbb{R}_+$, that is $\mu_i(s_i) \in \mathbb{R}_+$.

**Definition 1.** A domain is a monotonic domain if for each agent $j$ there exists a linear order over his set of signals $S_j$, and for every agent $i$ and two feasible vectors of signals $s$ and $s'$ such that $s \leq s'$ (that is, for every $j$ it holds that $s_j \leq s_j'$ according to the linear order on $S_j$) it holds that $v_i(s) \leq v_i(s')$.

4.2 The Refinement

We present the following refinement with the goal of pointing out a unique outcome of the game defined by an auction (specifically we use it for the Second Price Auction (SPA)) in our model. The refinement is defined for every game induced by an auction.

The refinement is based on a random bidder that bids according to a distribution that satisfy some properties.

Let $v_{\min} = \min_i \inf_T v_i(T)$ and $v_{\max} = \max_i \sup_T v_i(T)$.

**Definition 2.** We say that a distribution $R$ is standard if the support of $R$ is $[v_{\min}, v_{\max}]$ (the "relevant" values), $R$ is continuous, strictly increasing and differentiable, and its density $r$ is continuous and positive on the interval.

Consider an auction and the game $\lambda$ that is induced by the auction. We next define the game with the random bidder added to it.

\footnote{If the vector $s'$ can never be realized we define $v_i(s') = 0$.}
Definition 3. For a standard distribution $R$ and $\epsilon > 0$ define $\lambda(\epsilon, R)$ to be the game induced by $\lambda$ with the following modification: with probability $\epsilon$ there is an additional bidder submitting a bid $b$ sampled according to $R$. We call $\lambda(\epsilon, R)$ an $(\epsilon, R)$-tremble of the game $\lambda$.

Alternatively, one can think of the $(\epsilon, R)$-tremble of the game $\lambda$ as a game with 3 agents, the 2 original agents and a random bidder, that random bidder bids 0 with probability $1 - \epsilon$ and bid according to $R$ with probability $\epsilon$. The unconditional distribution according to which the random bidder is bidding is denoted by $\hat{R}$ and is defined as $\hat{R}(x) = 1 - \epsilon + \epsilon \cdot R(x)$. The density of $\hat{R}(x)$ for every $x > 0$ is $\hat{r}(x) = \epsilon \cdot r(x)$.

Let $\mu_i$ be a strategy of agent $i$. A strategy maps the signal of the agent to distribution over bids. The strategy is a pure strategy if for every signal the mapping is to a single bid. Let $\mu$ be a vector of strategies, one for each agent.

Definition 4. (i) A (pure or mixed) Nash equilibrium $\mu$ is a Tremble Robust Equilibrium (TRE) of the game $\lambda$, if for every standard distribution $R$ the following holds:

1. $\lim_{j \to \infty} \epsilon_j = 0$.
2. $\mu^{\epsilon_j}$ is a (pure or mixed) Nash equilibrium of the game $\lambda(\epsilon_j, R)$, the $(\epsilon_j, R)$-tremble of the game $\lambda$.

it holds that for every agent $i$ and signal $s_i$, $\{\mu^{\epsilon_j}_i(s_i)\}_{j=1}^{\infty}$ converges in distribution to $\mu_i(s_i)$.

(ii) $\mu$ is a strong Tremble Robust Equilibrium if it is a TRE and, in addition, for every decreasing sequence $\{\epsilon_j\}_{j=1}^{\infty}$ satisfying (1) and (2) above, there exists $k$ such that for every $j > k$ in $[\mathbb{R}]$ it holds that $\mu^{\epsilon_j} = \mu$.

5 Common Value SPA Auction

In this section we consider the restriction of the above model to the common value case and study the SPA. When we talk about the Second Price Auction (SPA) game we refer to the game induced by a Second Price Auction (SPA) with random tie breaking rule. In the common value model the state of the world determines the quality of the good, and thus determines its value. Thus, in the common value model, there exists a value function $v$ such that when the state of the world is $\omega \in \Omega$ the value of the good to any of the agents is $v_i(\omega) = v(\omega)$.

In this paper we focus on the case that for each agent $i$ the set of signals $S_i$ is finite.

5.1 Only One Informed Bidder

We first describe the important special case that only one agent is fully informed about the state, while all others are completely uninformed. Now, all agents but agent 1 are completely uninformed about the state of the world. We call buyer 1 the informed buyer and the rest of the buyers are called the uninformed buyers. When the informed buyer’s signal $s$ was realized to $s'$ he updates his belief about his expected value of the good, his posterior belief is that the expected value is $E[v(\omega)|s = s']$, where the expectation is taken both over the randomness $H$ that generated $\omega$ and the randomness $d(\omega)$ that generated the realized signal $s'$. We are interested in predicting the equilibrium and the revenue of the second price auction in this model.

The next theorem present a strong TRE in this game and shows that it is the unique TRE in undominated strategies.
Theorem 5. In any domain with one informed agent and any number of uninformed agents a strong TRE (in pure strategies) of the SPA game is the profile of strategies \( \mu \) in which:

1. the informed buyer with signal \( s \) realized to \( s' \) bids \( b_I(s') = E[v(\omega)|s = s'] \).
2. each of the uniformed buyers bids \( b_U = \min_{\hat{s} \in S} E[v(\omega)|s = \hat{s}] \).

Moreover, this profile is the unique TRE in undominated strategies.

Proof. To show that \( \mu \) is a strong TRE of the SPA game it is sufficient to show that it is a pure NE in any \((\epsilon, R)\)-tremble of the game. This is indeed true as the strategy of the informed bidder is dominant, thus is a best response to any strategy of the uninformed bidder. Additionally, the strategy of the uninformed is a best response to the dominant strategy played by the informed bidder (it gives 0 utility and no strategy give positive utility). Finally, \( \mu \) is trivially a pure strategy profile.

Next we show that it is the unique TRE in undominated strategies. Clearly the strategy of the informed bidder is the unique undominated strategy (even among mixed strategies) as for any signal his bid is the unique bid that dominates any other bid. For the uninformed, bidding below \( b_U \) is dominated by bidding \( b_I \), while due to the random bidder, bidding above \( b_U \) cannot be a best response to the unique undominated strategy of the informed bidder in any \((\epsilon, R)\)-tremble of the game (thus will not be a NE in any \((\epsilon, R)\)-tremble of the game).

Note that there are multiple strong TRE in this game, as any strategy profile in which the informed is bidding according to \( \mu \) while the uninformed is bidding below his bid in \( \mu \) is also a strong TRE. Yet under the natural refinement of undominated strategies we have a uniqueness result.

We stress that the strategy of the uniformed is independent of the probability of the informed buyer receiving the signal that generates the lowest expectation: even a tiny (but positive) probability of receiving a signal is sufficient to cause the uniformed buyers to bid so low.

The following corollary is immediate from Theorem 5; it shows that the revenue of the SPA with only one informed bidder is very low.

Corollary 6. In the unique TRE in undominated strategies of the SPA game with 1 informed agent and any number of uninformed agents, the expected revenue of the auctioneer is \( \min_{\hat{s} \in S} E[v(\omega)|s = \hat{s}] \). Moreover, in any TRE the revenue is at most \( \min_{\hat{s} \in S} E[v(\omega)|s = \hat{s}] \).

We point out the connection to the Lemon Market problem (Akerlof 1970): similar Adverse Selection phenomena derive both results. Yet, we note that in the SPA with common value, Adverse Selection by itself does not necessarily imply revenue collapse in any Nash Equilibrium: it is a Nash Equilibrium for the informed agent to bid according to his signal while the uninformed agent bids any value \( X \) (as any bid results with 0 utility to the uninformed agent). In particular, the uninformed agent is able to win high quality items in NE (unlike in the Market for Lemons). Thus, multiplicity of Nash equilibria as well as the ability of the uninformed party to win high quality items in NE make the common value SPA somewhat different than the Markets for Lemons. Our TRE refinement enables a result in the spirit of Markets for Lemons, by predicting a unique NE for which there is indeed revenue collapse.

We next discuss the implications of these results to the revenue of the seller in display advertisement common-value SPA with asymmetric information.

\( ^6 \)Alternatively, in any TRE the bid of any agent given a signal is at most the bid he submits at \( \mu \). Thus, \( \mu \) exhibits the most "aggressive" bidding in any TRE.
Example 7. Impressions in display ads auction have various qualities (values) dependent on the likelihood of the user to be influenced by the ad to buy some product. Assume that there are two qualities (common value for an impression), low (L for Lemon) and high (P for Peach), that is $\Omega = \{L, P\}$. A peach is more valuable than a lemon, that is $v(P) > v(L)$. The commonly known prior is that with probability $p \in (0, 1)$ the impression is a peach, and with probability $1 - p$ it is a lemon. Fix small $\epsilon > 0$. We consider the follow two possible information structures.

In the case that the informed buyer is $\epsilon$-informed about peaches the set of signals for the informed is $S = \{\emptyset, S_P\}$. Conditional on the quality being high ($\omega = P$) the informed buyer gets signal $S_P$ with probability $\epsilon$, otherwise he gets the signal $\emptyset$. 

In the case that the informed buyer is $\epsilon$-informed about lemons the set of signals for the informed is $S = \{\emptyset, S_L\}$. Conditional on the quality being low ($\omega = L$) the informed buyer gets signal $S_L$ with probability $\epsilon$, otherwise he gets the signal $\emptyset$.

Let us now examine the revenue for the two cases. Although in both cases the informed buyer has a tiny probability ($\epsilon$) of knowing the exact quality, there is a substantial difference in the revenue the seller gets in the SPA.

When the informed buyer is $\epsilon$-informed about peaches the revenue is

$$\min_{\hat{s} \in S} E[v(\omega)|s = \hat{s}] = \min\{E[v(\omega)|s = \emptyset], E[v(\omega)|s = S_P]\}$$

Now, $E[v(\omega)|s = S_P] = v(P)$. To compute $E[v(\omega)|s = \emptyset]$ we observe that

$$\Pr[s = \emptyset] = \Pr[s = \emptyset|\omega = L]\Pr[\omega = L] + \Pr[s = \emptyset|\omega = P]\Pr[\omega = P]$$

$$= (1 - p) + (1 - \epsilon)p = 1 - \epsilon p$$

We now use Bayes rule to compute $E[v(\omega)|s = \emptyset]$.

$$E[v(\omega)|s = \emptyset] = \sum_{\omega \in \Omega} v(\omega)\Pr[\omega|s = \emptyset] = \sum_{\omega \in \Omega} v(\omega) \frac{\Pr[s = \emptyset|\omega]\Pr[\omega]}{\Pr[s = \emptyset]}$$

$$= v(L) \frac{1 - p}{1 - \epsilon p} + v(P) \frac{(1 - \epsilon)p}{1 - \epsilon p} = v(L)(1 - p) + v(P)(1 - \epsilon)p$$

This is clearly smaller than $E[v(\omega)|s = S_P] = v(P)$, thus this is the expected revenue of the auction, that is the revenue is $R_{peach}^{SPA} = \frac{E[v(\omega)] - \epsilon v(P)p}{1 - \epsilon p}$. We observe that the revenue continuously converges to the unconditional expectation $E[v(\omega)]$ when $\epsilon$ converges to 0.

Next we contrasts the above with the case that the informed buyer is $\epsilon$-informed about lemons. Now the revenue is

$$\min_{\hat{s} \in S} E[v(\omega)|s = \hat{s}] = \min\{E[v(\omega)|s = \emptyset], E[v(\omega)|s = S_L]\}$$

Now, $E[v(\omega)|s = S_L] = v(L)$ while $E[v(\omega)|s = \emptyset] > v(L)$. Thus the revenue is $R_{lemon}^{SPA} = v(L)$, a complete collapse, independent of how small $\epsilon$ is! The revenue of the seller is discontinuous in $\epsilon$ at $\epsilon = 0$. Note that this revenue collapse result extends to the case that the informed bidders also sometimes gets a signal about a peach, as long as he has positive probability of getting a signal about a lemon.
5.1.1 FPA vs. SPA

By comparing the SPA revenue result in Corollary 6 with the FPA revenue result in Theorem 4 of Engelbrecht-Wiggans et al. (1983), it is straightforward to show that FPA revenues are always higher than SPA revenues when only one bidder is informed.

**Corollary 8.** Consider any common value domain with $n$ agents, $n-1$ of them are uninformed, and the last agent is informed with any information structure. In any such domain the revenue of the first price auction is strictly higher than the revenue in any TRE of the second price auction game.

**Proof.** To make the comparison, define the informed bidder’s interim expected value conditional on receiving signal $s$ as $h(s) = E[v(\omega)|s]$ and the minimum such value as $\underline{h} = \min_{s \in S} E[v(\omega)|s]$. Further, let $F$ be the cumulative distribution function of $h$. According to Corollary 6, SPA revenue is at most $\underline{h}$. According to Theorem 4 of Engelbrecht-Wiggans et al. (1983), FPA revenue is

$$\int_{0}^{\infty} (1 - F(h))^2 dh$$

which can be re-written as $\underline{h} + \int_{\underline{h}}^{\infty} (1 - F(h))^2 dh$. For an informed bidder, $F(\underline{h}) < 1$ so this is clearly strictly more than $\underline{h}$. \qed

This result clearly implies that for both information structures we consider in Example 7 the revenue of the FPA is larger than the revenue of any TRE of the SPA game. For that example we can compute the revenue differences exactly. In Appendix B.1 we use the Engelbrecht-Wiggans et al.’s (1983) revenue result for FPA and show that in both the case that the informed is $\epsilon$-informed about lemons and the case he is $\epsilon$-informed about peaches, the revenue of the FPA is

$$R_{FPA} = E[v(\omega)] - \epsilon p (v(P) - v(L))$$

Notice that the revenue loss is proportional to $\epsilon$, the arrival rate of cookies, regardless of whether cookies contain information about lemons or about peaches. Thus while FPA revenues are always higher than SPA revenues, the difference is substantial only when cookies identify lemons. In particular, loss in revenue from using a SPA rather than a FPA is proportional to $\epsilon^2$ when cookies identify peaches:

$$R_{FPA \text{peaches}} - R_{SPA \text{peaches}} = \epsilon^2 p^2 (1 - p) \frac{v(P) - v(L)}{1 - \epsilon p}.$$  

However, when cookies identify lemons, the loss is

$$R_{FPA \text{lemons}} - R_{SPA \text{lemons}} = (1 - \epsilon(1 - p)) p (v(P) - v(L)),$$

or approximately $p (v(P) - v(L))$.

We have seen that for both the case that the informed agent is $\epsilon$-informed about lemons and the case that the informed agent is $\epsilon$-informed about peaches, revenue of FPA does not collapse (does not tend to zero with epsilon). We next show that this is implied by a much more general observation. We observe that the revenue of FPA can be bounded from below, independent of the information structure. In Appendix B.2 we prove the following proposition.

**Proposition 9.** Consider any common value domain with items of value in $[0, 1]$ and expected value of $E$. Assume that there are $n$ agents, $n-1$ of them are uninformed. For any information structure for the informed agent the revenue of the FPA is at least $E^2$. 

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Consider the case that items can have very low value (say 0) and that the expected value is some positive constant $E$. This observation, in particular, says that the revenue of the FPA does not collapse to zero no matter what the information structure is, in contrast to the revenue of SPA in any strong TRE, which can be arbitrarily small if the informed agent has positive probability of getting a signal with posterior close to zero (like in the case he is $\epsilon$-informed about lemons).

We also observed that the revenue of the FPA is continuous in $F(h)$, thus a small change in the information structure of the informed agent implies a small change in the revenue of the seller. This is in contrast to the SPA revenue, which by our result can change dramatically due to a small change in the information structure. This is exactly the case when all agents are uninformed and one of them becomes $\epsilon$-informed about lemons. This small change in information structure have major implication on the revenue of the SPA.

5.2 Many Agents, each with Finitely Many Signals

In this section we present conditions on the information structure that allow us to generalize the result for a single informed agent to the case that multiple agents have informative signals. The conditions are sufficient to ensure existence of a strong TRE in pure strategies. We define a procedure that finds a set of strategies for the agents by iteratively determines the bids of the agents for their signals in a decreasing order. If the information structure satisfy two properties which we call "low-signal-is-never-good-news" and the "high-signal-is-never-bad-news" then the resulting profile of strategies is a strong TRE in pure strategies.

The procedure works as follows. For each agent $i$ it maintains the set of signals $T_i$ for which the bid was not yet determined, $T_i$ is initialized to all the signals of agent $i$. It iteratively sets the bid for the maximal element $s_i$ of $T_i$ for some agent $i$ for which the following holds. The vector $(\{s_i\},T_{-i})$ is feasible and the posterior value given signal $s_i$ and the set of signals of all other agents $T_{-i}$ (signals of others for which the bid was not set yet), is maximal. Finally, for every agent $i$ and signal $s_i$ for which the bid was not set yet, it bids the minimal feasible value.

We next formally present the procedure:

1. Initialization: For each $i$ let $T_i = S_i$ be the set of signals for which the bid of agent $i$ was not yet determined.

2. Iterate: For each agent $i$ and any signal $s_i \in T_i$ such that $(\{s_i\},T_{-i})$ is feasible, compute $v(\{s_i\},T_{-i})$. If no such feasible vector exists, break. Let $j$ and $s_j$ be a pair of a bidder and the corresponding signal with maximal such value. Set the bid for bidder $j$ when the signal is $s_j$ to be $v(\{s_j\},T_{-j})$: $\mu_j(s_j) = v(\{s_j\},T_{-j})$. Remove $s_j$ from $T_j$. Iterate.

3. Now for every agent $i$ and signal $s_i \in T_i$, the vector $(\{s_i\},T_{-i})$ is not feasible. Set the bid $\mu_i(s_i)$ to be $v(\{s_i\},S_{-i})$ for $s_{-i} \in S_{-i}$ such that $(\{s_i\},s_{-i})$ is feasible and $v(\{s_i\},s_{-i})$ is minimal over all such feasible vectors.

For domains that satisfy following properties about the expected value given a set of signals we would be able to prove that a strong TRE in pure strategies exists (Observation 13 shows that if one is violated existence of a strong TRE in pure strategies is not guaranteed).

\footnote{In case of a tie only one agent-signal pair is picked and the bid for that signal is set at this iteration. Any other pair with maximal value is only considered at the next iteration, after updating the set $T_j$ and recomputing the values. This in particular means that bids for symmetric signals might be different.}
Definition 10. Assume that for agent $j$ and signal $s_j$ the bid $\mu_j(s_j)$ was set at stage $2$ of the procedure. It holds that $\mu_j(s_j) = v(\{s_j\}, T_{-j})$ where for any $i \neq j$ the set $T_i$ is the set of signals of agent $i$ for which the bids were not set yet.

- A domain satisfies the low-signal-is-never-good-news property if for any such agent $j$ and signal $s_j$ with bid $\mu_j(s_j)$, and for any other agent $k \neq j$ and signal $s_k \in T_k$ (the bid $\mu_k(s_k)$ was not set yet), if $(\{s_j\}, \{s_k\}, T_{-(j,k)})$ is feasible then
  \[ v(\{s_j\}, \{s_k\}, T_{-(j,k)}) \leq v(\{s_j\}, T_k, T_{-(j,k)}) = \mu_j(s_j) \]

- A domain satisfies the high-signal-is-never-bad-news property if for any such agent $j$ and signal $s_j$ with bid $\mu_j(s_j)$, and for any other agent $k \neq j$ and signal $s_k \notin T_k$ (the bid $\mu_k(s_k)$ was set before the bid $\mu_j(s_j)$) if $(\{s_j\}, \{s_k\}, T_{-(j,k)})$ is feasible then
  \[ v(\{s_j\}, \{s_k\}, T_{-(j,k)}) \geq v(\{s_j\}, T_k, T_{-(j,k)}) = \mu_j(s_j) \]

Let us present the low-signal-is-never-good-news property informally. Once the procedure sets a bid for some agent $j$ and signal $s_j$, there is still some set $T_k$ of signals for agent $k \neq j$ for which the bid was not set yet. Any such signal $s_k \in T_k$ cannot be a good news in the sense that knowing that the signal is $s_k$ instead of the current set $T_k$ of left over signals of agent $k$, will not increase the expected value of the item. This condition limits the extent in which the domain is strictly monotonic because, in particular, it implies that knowing that $k$ got the signal that is the maximal element of $T_k$ does not increase the expected value. An example for this is the case of 2 agents with 2 signals each that was discussed at Observation 19. When $v(H_1, H_2) = v(H_1) > v(H_2)$ it holds that when agent 1 get signal $H_1$ he knows for sure that the value is $v(H_1, H_2)$, so the “low signal” $H_2$ of agent 2 is not good news: the value given both $H_2$ and $H_1$ is the same as the value given that agent 1 got signal $H_1$.

Similarly, the high-signal-is-never-bad-news property ensures that if we condition on one of the signals for which the bid was already set (“higher” signal), the value does not decrease.

We next state the main result of this section, its proof appears in Section C.1.

Theorem 11. Consider any domain with multiple bidders and multiple signals each, in which the low-signal-is-never-good-news and the high-signal-is-never-bad-news properties hold. For such a domain the above procedure computes a strong TRE of the SPA game, in pure strategies.

In the next version of this working paper we plan to provide a generalization of this theorem to every monotonic domain (removing the assumptions about the properties satisfied by the domain), in the spirit of Theorem 11. Such a generalization will also provide a uniqueness result.

The procedure and and Theorem 11 apply to the case that only one agent is informed, in this case the procedure outputs the strong TRE in pure strategies that is exactly the one described by Theorem 5. They also apply to the case of monotonic domain with two agents with binary signals when $v(H_1, H_2) = v(H_1) > v(H_2)$. In this case the procedure outputs the strong TRE in pure strategies that is exactly the one described by Observation 19 for this special case.

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8The signal $s_k$ is a “low signal” as $s_k \in T_k$ and its bid was not yet determined (it is lower in the order of bids set by the procedure).

9The signal $s_k$ is a “higher signal” as $s_k \notin T_k$ and its bid was already determined (it is higher in the linear order of bids set in stage 2).
Another family of domains for which the procedure and Theorem 11 apply is the family of connected domains which are studied by Einy et al. (2002). Connected domains are defined as follows.

**Definition 12.** A domain is a connected domain if the following hold. Each agent \( i \) has a partition \( \Pi_i \) of the state of nature and his signal is the element of the partition that include the realized state. The information partition \( \Pi_i \) of bidder \( i \) is connected (with respect to the common value \( v \)) if every \( \pi_i \in \Pi_i \) has the property that, when \( \omega_1, \omega_2 \in \pi_i \) and \( v(\omega_1) \leq v(\omega_2) \) then every \( \omega \in \Omega \) with \( v(\omega_1) \leq v(\omega) \leq v(\omega_2) \) is necessarily in \( \pi_i \). A common-value domain is connected (with respect to the common value) if for every agent \( i \) his information partition \( \Pi_i \) is connected.

In Appendix C.2 we show that any connected domain satisfies the low-signal-is-never-good-news and the high-signal-is-never-bad-news properties, thus Theorem 11 can be applied. Moreover, we observe that for connected domains the TRE picked by the procedure is exactly the profile of strategies pointed out by Einy et al. (2002) (the single "sophisticated equilibrium" that Pareto-dominates the rest in terms of bidders resulting utilities). We note that while connected domains allow multiple agents to have multiple signals each, there are some simple domains, even ones with a single informed bidder, that are not connected. In Appendix C.2 we also present a simple domain that is not connected and also is not equivalent to any connect domain (thus the result of Einy et al. does not apply) and for which our result apply.

For the case of a single informed bidder, Section 5.1 presents a unique TRE in undominated strategies (Theorem 5), in that equilibrium the revenue is low, as any uninformed bidder is bidding as if the informed bidder got the worse signal (Corollary 6). The procedure we have presented for the case of many bidders each having many signals works also in the case of a single informed bidder, and it predicts the same TRE (as the properties are trivially satisfied). In this more general case, when the low-signal-is-never-good-news condition is satisfied, each bidder with a given signal is bidding taking into account that for any item he wins, none of the other bidders got a signal that results in a higher bid (and these signals are some prefix of the highest signals of each of the other agents). Thus, his bid, which equals to his posterior belief about the expected value of the items he wins with that bid, can be significantly smaller than his posterior belief given only his own signal. Similar to the single informed agent case, this can have significant implications on the revenue of the SPA, this is so as payments are determined by the second highest bidder, and thus can be significantly lower than the posterior expected value of the winning bidder.

### The low-signal-is-never-good-news Property

The following observation follows form the uniqueness result presented in Theorem 20 for any domain covered by that theorem. It implies that the low-signal-is-never-good-news property is necessary to derive the result presented in Theorem 11.

**Observation 13.** There exists a domain for which the low-signal-is-never-good-news property does not hold, and for which there does not exist any strong TRE in pure strategies.

#### 5.2.1 Generalizing "Lemons and Peaches" to \( n \) agents

In this section we generalize the result about lemons and peaches information structures when there is only one informed agent, to many agents.
Consider a common value domain with items of expected value \(E[v(\omega)]\) which we normalize to 1. Assume that there are \(n\) agents, each receiving a binary signal. Agent \(i\) receives a signal in \(\{L_i, H_i\}\). Assume that the domain satisfies the conditions of Theorem 11 and thus a strong TRE exists. We next consider the revenue of the seller in such a strong TRE with various information structures. We will assume without loss of generality that for every agent \(i\) the bid in that strong TRE for signal \(H_i\) is at least the bid for signal \(L_i\) (equivalently, \(v(L_1, L_2, \ldots, L_{i-1}, H_i) \geq v(L_1, L_2, \ldots, L_{i-1}, L_i)\)). We also assume without loss of generality that the order of the agents is such that in that strong TRE the bid of agent \(i\) with signal \(H_i\) is at least the bid of agent \(i + 1\) for signal \(H_{i+1}\), for every \(i\).

An example for such a domain is a connected domain in which each agent \(i\) has a threshold \(t_i\) and he receives the signal \(H_i\) if the value of the item is at least \(t_i\), otherwise he receives the signal \(L_i\). Note that for such a domain, for every \(i\) it holds that \(v(H_i) > v(L_i)\) (\(H_i\) is the higher signal) and that \(v(H_i) > v(H_{i+1})\) (agent \(i\)’s high signal is better news than agent \(i + 1\)’s high signal).

We first define what it means for an agent to be informed about peaches.

**Definition 14.** For \(\epsilon > 0\), agent \(i\) is \(\epsilon\)-informed about peaches if

- \(\Pr[L_1, L_2, \ldots, L_{i-1}, H_i] < \epsilon\).
- \(v(L_1, L_2, \ldots, L_{i-1}) - v(L_1, L_2, \ldots, L_{i-1}, L_i) < \epsilon\). (for \(i = 1\) this means that \(E[v(\omega)] - v(L_1) = 1 - v(L_1) < \epsilon\)).

Informally, the first condition states that given \(L_1, L_2, \ldots, L_{i-1}\), the signal \(H_i\), which is the peaches signal for agent \(i\), is only rarely received (with probability at most \(\epsilon\)). Note that this does not mean that signal \(H_i\) is rarely received (only that it is rarely received given \(L_1, L_2, \ldots, L_{i-1}\)). The second condition states that by removing these peaches from the pool of items with signals \(L_1, L_2, \ldots, L_{i-1}\), the value of the leftover pool (signals \(L_1, L_2, \ldots, L_{i-1}, L_i\)) does not decrease too much (by at most \(\epsilon\)).

We next define what it means for an agent to be informed about lemons.

**Definition 15.** For \(\epsilon > 0\), agent \(i\) is \(\epsilon\)-informed about lemons if

- \(\Pr[L_1, L_2, \ldots, L_{i-1}, L_i] < \epsilon^{[10]}\)
- \(\sup\{v(\omega)\mid \omega\text{ implies }L_1, L_2, \ldots, L_{i-1}, L_i\} < \epsilon^{[11]}\)

Informally, the first condition states that lemon signal \(L_i\) for agent \(i\) is rare, happens with probability of at most \(\epsilon\). The second condition states that when \(i\) receives the lemons signal \(L_i\) it actually indicates that the value of the item, even in the best case, is very low (at most \(\epsilon\)).

We first show that if all agents are \(\epsilon\)-informed about peaches, and \(\epsilon\) is small, then the revenue of the SPA in the strong TRE will be high (close to the social welfare which equals to \(E[v(\omega)] = 1\)).

**Proposition 16.** Consider the strong TRE that is the outcome of the procedure defined in Section 5.2 for the above connected domain.

If every agent \(i\) is \(\epsilon_i\)-informed about peaches then the revenue of the SPA is at least \(1 - \sum_{j=1}^{n} \epsilon_j\). In particular, if \(\sum_{j=1}^{n} \epsilon_j\) tends to 0 the revenue will tend to 1.

---

\(^{10}\)Note that in a connected domain with binary signals the assumption \(v(H_i) > v(H_{i+1})\) implies that \(H_{i+1}\) only if \(H_i\), or alternatively, \(L_i\) implies \(L_{i+1}\) (for every \(i\)). Thus, in such domains \(\Pr[L_1, L_2, \ldots, L_{i-1}, L_i] = \Pr[L_i]\).

\(^{11}\)This condition can be replaced by the following weaker condition while preserving the correctness of Proposition 17. The alternative condition is that \(v(L_1, L_2, \ldots, L_{i-1}, L_i, S_{i+1}, \ldots, S_n) < \epsilon\) for any signals \(S_{i+1}, \ldots, S_n\) for agents \(i + 1\) to \(n\).
Proof. As all bids are at least \(v(L_1, L_2, \ldots, L_n)\), the revenue is at least \(v(L_1, L_2, \ldots, L_n)\) of the unconditional expectation. Thus it is sufficient to show that \(v(L_1, L_2, \ldots, L_n) \geq 1 - \sum_{j=1}^{n} \epsilon_j\).

As agent 1 is \(\epsilon_1\)-informed about peaches it holds that \(1 - \epsilon_1 < v(L_1)\). Since agent 2 is \(\epsilon_2\)-informed about peaches it holds that \(v(L_1) - \epsilon_2 < v(L_1, L_2)\). By chaining the two inequalities we derive that \(1 - (\epsilon_1 + \epsilon_2) < v(L_1, L_2)\). This argument generalizes to an induction that shows that \(v(L_1, L_2, \ldots, L_i) > 1 - \sum_{j=1}^{i} \epsilon_j\), and the claim follows.

Note that the above proposition only relies on the second assumption about agents that are \(\epsilon\)-informed about peaches (the assumption about the value decrease), and not on the first assumption (about the probability of getting the peaches signal). Moreover, it clearly generalizes to the case of non-binary signals (any finite number of signals for each agent), as long as for each agent \(i\) the signal \(L_i\) is the worst signal for agent \(i\).

We next show that if at least one agent is \(\epsilon\)-informed about lemons, and all better informed agents are \(\epsilon\)-informed about peaches, revenue will collapse (independent of the information structure of the agents that are less informed than the agent that is \(\epsilon\)-informed about lemons).

**Proposition 17.** Consider the strong TRE that is the outcome of the procedure defined in Section 5.2 for the above connected domain.

If there exists an agent \(i\) such that \(i\) is \(\epsilon_i\)-informed about lemons and every agent \(j < i\) is \(\epsilon_j\)-informed about peaches, then the revenue of the SPA is at most

\[
2 \sum_{j=1}^{i} \epsilon_j
\]

In particular, if \(\sum_{j=1}^{i} \epsilon_j\) tends to 0 the revenue will tend to 0.

For the proof please refer to Appendix C.3.

5.3 Two Agents, Each with a Binary Signal

When more than one agent is partially informed about the state of the world and the conditions that ensure existence of a strong TRE are not satisfied, the situation becomes much more complicated. In this section we present a complete analysis for any monotonic domain with two bidders, each getting a binary signal.

Let \(\{L_1, H_1\}\) be the signals of agent 1, and \(\{L_2, H_2\}\) be the signals of agent 2. Assume that the domain is monotonic and that the order of signals is \(H_1 > L_1\) and \(H_2 > L_2\). With some abuse of notation for any agent \(i\) with will use \(H_i\) to also denote the event that the signal of agent \(i\) was realized to \(H_i\), and similarly for \(L_i\). Assume without loss of generality that \(v(H_1, H_2) = 1\) and that \(v(L_1, L_2) = 0\). In monotonic domain it holds that \(v(L_1, H_2), v(H_1, L_2) \in [0, 1]\).

A domain with two bidders, each with a binary signal is non-degenerated if \(Pr[H_1, H_2] > 0\), and for any bidder \(i \in \{1, 2\}\) it holds that \(1 > Pr[H_i] > 0\). The main result in this section is for non-degenerated monotonic domains when none of the bidders is complete informed, yet for completeness we first discuss the rather simple cases when the domain is degenerated or at least one bidder is completely informed.

If for some bidder \(i \in \{1, 2\}\) it holds that \(Pr[H_i] = 0\) or \(Pr[H_i] = 1\) then that bidder is completely uninformed, and the results of Section 5.1 apply (unless both are completely uninformed, in that case both have a dominant strategy to bid the unconditional expectation). We are left to consider domains for which for any bidder \(i \in \{1, 2\}\) it holds that \(1 > Pr[H_i] > 0\).
0. We begin by considering two special cases. The first is the case that both agents when getting the high signal learn nothing from the other agent’s signal, that is, \( \Pr[H_1, H_2] = 0 \) or \( v(H_1, H_2) = v(H_1) = v(H_2) \). The second is the case that the above holds only for one agent.

First consider the case that \( \Pr[H_1, H_2] = 0 \) or \( v(H_1, H_2) = v(H_1) = v(H_2) \).

**Observation 18.** In a monotonic domain with two agents with a binary signal each, if \( \Pr[H_1, H_2] = 0 \) or \( v(H_1, H_2) = v(H_1) = v(H_2) \) then there is a strong TRE in pure strategies:

- Every agent \( i \) bids \( v(H_i) \) when getting signal \( H_i \).
- Every agent \( i \) bids \( v(L_1, L_2) \) when getting signal \( L_i \).

Moreover, this is the unique TRE in undominated strategies.

Bidding \( v(H_i) \) is the unique dominant bid for agent \( i \) with signal \( H_i \), and given these bids, when agent \( i \) gets signal \( L_i \) any bid other larger than \( v(L_1, L_2) \) results in negative utility, while bidding below \( v(L_1, L_2) \) is dominated.

Next consider non-degenerated monotonic domains when \( v(H_1, H_2) = v(H_1) > v(H_2) \).

**Observation 19.** In a non-degenerated monotonic domain with two agents with a binary signal each, if \( v(H_1, H_2) = v(H_1) > v(H_2) \) then there is a strong TRE in pure strategies:

- Agent 1 bids \( v(H_1) \) when getting signal \( H_1 \).
- Agent 2 bids \( v(L_1, L_2) \) when getting signal \( H_2 \).
- Every agent \( i \) bids \( v(L_1, L_2) \) when getting signal \( L_i \).

Moreover, this is the unique TRE in undominated strategies.

Bidding \( v(H_1) \) is the unique dominant bid for agent 1 with signal \( H_1 \), and given this bid, when agent 2 gets signal \( s_2 \in \{L_2, H_2\} \) any bid other larger than \( v(L_1, s_2) \) results in negative utility, while bidding below \( v(L_1, s_2) \) is dominated. Finally, the same logic implies that agent 1 with signal \( L_1 \) must bid \( v(L_1, L_2) \).

After handling all the trivial cases above, we can finally focus on the non-degenerated case when \( v(H_1, H_2) > \max\{v(H_1), v(H_2)\} \). Observe that these conditions are equivalent to \( 1 > \Pr[H_1, L_2](1 - v(H_1, L_2)) > 0 \) and \( 1 > \Pr[L_1, H_2](1 - v(L_1, H_2)) > 0 \). When these two conditions hold we can assume without loss of generality that \( 0 < \Pr[H_1, L_2](1 - v(H_1, L_2)) \leq \Pr[L_1, H_2](1 - v(L_1, H_2)) < 1 \) (otherwise we exchange the agents’ names). The following theorem is proven under the assumption that no agent ever submit a dominated bid, that is, agent \( i \) with signal \( s_i \) never bids below \( v(s_i, L_j) \).

**Theorem 20.** Consider any non-degenerated monotonic domain with two bidders, each with a binary signal. Assume that \( 0 < \Pr[H_1, L_2](1 - v(H_1, L_2)) \leq \Pr[L_1, H_2](1 - v(L_1, H_2)) < 1 \).

The unique TRE of the SPA game is the profile of strategies \( \mu \) in which:

- Every bidder \( i \) bids \( v(L_1, L_2) = 0 \) when getting signal \( L_i \).
- Bidder 1 with signal \( H_1 \) always bids \( v(H_1, H_2) = 1 \).
- Bidder 2 with signal \( H_2 \)
  - bids \( v(H_1, H_2) = 1 \) with probability \( \frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} \cdot \frac{1 - v(H_1, L_2)}{1 - v(L_1, H_2)} \), and
  - bids \( v(L_1, H_2) \) with the remaining probability.
Before discussing the theorem we consider the implications of this result on the revenue of the seller. As an immediate corollary we get a prediction about the revenue in the unique TRE of this game. No revenue is generated unless both bidders receive a high signal, and even in this case the revenue is only some fraction of the value created.

**Corollary 21.** In any monotonic domain with 2 agents and binary-signal each which satisfies

\[ 0 < Pr[H_1, L_2](1 - v(H_1, L_2)) \leq Pr[L_1, H_2](1 - v(L_1, H_2)) < 1, \]

the revenue of the seller in the unique TRE of the SPA game is only

\[
Pr[H_1, H_2] \cdot \left( v(L_1, H_2) + (1 - v(L_1, H_2)) \cdot \frac{Pr[H_1, L_2]}{Pr[L_1, H_2]} \cdot \frac{1 - v(H_1, L_2)}{1 - v(L_1, H_2)} \right)
\]

Note that this can be an arbitrarily small fraction of the welfare, that is the case for example when \( v(H_1, L_2) = 0 \) and \( Pr[H_1, L_2] \) tends to 0.

We next consider the intuition behind the result. We have assumed that \( Pr[H_1, L_2](1 - v(H_1, L_2)) \leq Pr[L_1, H_2](1 - v(L_1, H_2)) \), how should one interpret this assumption? For bidder \( i \in \{1, 2\} \) the expression \( Pr[H_i, L_j](1 - v(H_i, L_j)) = Pr[H_i, L_j](v(H_i, H_2) - v(H_i, L_j)) \) is the loss that bidder \( i \) has when he gets signal \( H_i \), bids \( v(H_1, H_2) = 1 \) and pays his bid. The smaller this loss is, the "stronger" the bidder is. The bidder with the (weakly) smaller loss, bidder 1, is the (weakly) better informed bidder.

The theorem shows that in the unique TRE the weakly better informed agent (agent 1) is bidding 0 when getting the low signal, and bidding \( v(H_1, H_2) = 1 \), as if both got their high signals, when he gets his high signal. The weakly worse informed agent (agent 2) is also bidding 0 when he gets his low signal, but he does not always bid \( v(H_1, H_2) = 1 \) when he gets the high signal \( H_2 \). When he gets the high signal \( H_2 \) he bids \( v(H_1, H_2) = 1 \) with probability \( \frac{Pr[H_1, L_2]}{Pr[L_1, H_2]} \cdot \frac{1 - v(H_1, L_2)}{1 - v(L_1, H_2)} \), and he bids \( v(L_1, H_2) \) with the remaining probability. The ratio in which agent 2 is bidding \( v(H_1, H_2) = 1 \) is exactly the ratio between the strength of the two bidders. This ratio is 1 when the agents are ex ante symmetric, and becomes smaller and smaller as the asymmetry grows.

A particulary interesting special case is the ax ante symmetric case, \( v_1 = v_2 < 1 \) and \( Pr[H_1, L_2] = Pr[L_1, H_2] > 0 \). In this case both agents are ax ante symmetric and not completely informed about the value. In this case in the unique TRE both agents are always bidding 1 (a pure strategy) when getting the high signal, and the entire surplus when both get high signals is extracted as revenue by the seller (yet the revenue is 0 if exactly one agent gets the high signal, although the value might be positive, thus not all surplus is extracted). Thus when bidders are symmetric ex ante, our TRE refinement selects the symmetric equilibrium studied by Milgrom and Weber (1982a) and others. Hence, Milgrom and Weber’s (1982a) result ranking second-price auction revenue higher than first-price auction revenue applies. (Milgrom and Weber’s (1982a) result is proved for continuous signals, but the authors point out in footnote 15 that it is true more generally.) Comparing this to the result in Section 5.1 that first-price auction revenue is always higher than second-price auction revenue when only one bidder is informed illustrates that the revenue ranking depends on the level of ex ante asymmetry. While second-price auctions dominate under symmetric conditions, first-price auctions generate more revenue in sufficiently asymmetric settings.

Given ex ante asymmetry, the unique TRE identified by Theorem 20 is in mixed strategies (agent 2 is mixing between bidding 0 and bidding 1, both with positive probability) and we conclude that there is no pure TRE. Moreover, it is easy to see that the unique TRE is not a strong TRE, as one can observe that in any \((\epsilon, R)\)-tremble of the game agent 2 has negative utility by bidding 1, while bidding 0 ensures 0 utility.
We next present a sketch of the proof of Theorem 20 for the complete proof see Appendix D.2.

Sketch of the Proof of Theorem 20

We next present a sketch of the proof of Theorem 20. 

Proof sketch: Fix any standard distribution $R$ and $\epsilon > 0$ and let $\lambda(\epsilon, R)$ be the $(\epsilon, R)$-tremble of the game. In the $(\epsilon, R)$-tremble of the game the random bidder arrives to the auction with small probability $\epsilon > 0$ and is bidding according to a standard distribution $R$ (its support is $[0, 1]$).

Assume that agent $i$ with signal $H_i$ is bidding according to distribution $G_i$, let $g_i$ denote the density of $G_i$ whenever $G_i$ is differentiable (note that since $G_i$ is non-decreasing it is differentiable almost everywhere, see, for example, Theorem 31.2 in Billingsley (1995)). We note that this is an abuse of notation as $G_i$ and $g_i$ both depend on $R$ and $\epsilon$. 

To prove the theorem we show that for any standard distribution $R$ and small enough $\epsilon$ a mixed NE $\eta$ in each of the games $\lambda(\epsilon, R)$ exists (Lemma 76). We then show that the limit of any sequence of NE strategies in the games $\lambda(\epsilon, R)$ must converges to $\mu$ as $\epsilon$ goes to zero. Combined with the existence of a mixed NE in each of the games $\lambda(\epsilon, R)$ this show that $\mu$ is the limit of the some sequence of NE strategies in the games $\lambda(\epsilon, R)$, thus a TRE. As the limit of any sequence of NE strategies in the games $\lambda(\epsilon, R)$ must converges to $\mu$ as $\epsilon$ goes to zero, $\mu$ is the unique TRE.

We next presents the high level arguments that prove the uniqueness of $\mu$. Fix a standard distribution $R$ and $\epsilon > 0$ and consider the game $\lambda(\epsilon, R)$. A NE $\eta$ of the $(\epsilon, R)$-tremble of the game $\lambda$ consists of four bid distributions, one for each bidder for each signal he may receive. Thus $\eta = (G_1, G_1^L, G_2, G_2^L)$ where $G_i$ and $G_i^L$ are the bid distributions when $i \in \{1, 2\}$ gets the signals $H_i$ and $L_i$, respectively. We first observe that if bidders never submit dominated bids, bidder $i \in \{1, 2\}$ that receives signal $L_i$ must bid $v(L_1, L_2) = 0$. We next focus on the bids when bidder $i$ gets the high signal $H_i$. To simplify the notation we denote $v_1 = v(H_1, L_2)$ and $v_2 = v(L_1, H_2)$.

For a given $\eta$ we define the following notations. Let $b_i = \inf\{b : G_i(b) > 0\}$ and $\bar{b}_i = \inf\{x : G_i(x) = 1\}$ for agent $i \in \{1, 2\}$. Define $\bar{b} = \min\{b_1, b_2\}$, $b_{\min} = \max\{b_1, b_2\}$ and $b_{\max} = \max\{\bar{b}_1, \bar{b}_2\}$. Note that when agent never submit dominated bids by definition it holds that $1 \geq b_{\max} \geq b_{\min} \geq \bar{b} \geq 0$.

We start with some necessary conditions that any mix NE $\eta$ in a fixed $\lambda(\epsilon, R)$ must satisfy.

Lemma 22. At $\eta$ the following must hold.

1. For some $j \in \{1, 2\}$ it holds that $\bar{b} = b_j = v_j$ and $b_{\min} = b_i \geq v_i$ for $i \neq j$.
2. Both $G_1$ and $G_2$ are continuous and strictly increasing on $(b_{\min}, b_{\max})$. It holds that $G_1(b_{\max}) = G_2(b_{\max}) = 1$. Moreover, if $b_{\max} > b_{\min}$ then $b_{\max} = \bar{b}_1 = \bar{b}_2$.
3. For every bidder $i \in \{1, 2\}$ it holds that $G_i(b) = 0$ for every $b \in [0, \bar{b})$, and $G_i(b) = G_i(\bar{b})$ for every $b \in [\bar{b}, b_{\min})$.
4. If $b_{\min} = \bar{b}$ then $\bar{b} = \max\{v_1, v_2\}$. Additionally, if $v_1 = v_2$ then $b_{\min} = \bar{b} = v_1 = v_2$ and no bidder has any atom anywhere. If $v_1 > v_2$ then $b_{\min} = \bar{b} = v_i$ and $i$ has an atom at $b$, while $j$ has no atoms.
5. If $b_{\min} > \bar{b}$ then for one agent, say $j$, it holds that $\bar{b} = b_j = v_j$. Bidder $j$ has an atom at
Lemma 23. If \( \epsilon \) is small enough at \( \eta \) the following must hold. There must exist \( b_{min} \) and \( b_{max} \) such that \( 1 > b_{max} > b_{min} \geq 0 \) and:

- The two bidders are symmetric \( (Pr[H_1, L_2] = Pr[L_1, H_2] \) and \( v_1 = v_2 \) if and only if \( b_{min} = b = v_1 = v_2 \) and \( G_1(b_{min}) = G_2(b_{min}) = 0 \) (no atoms).

- If \( Pr[H_1, L_2](1-v_1) = Pr[L_1, H_2](1-v_2) \) but the bidders are not symmetric, and it holds that \( v_1 > v_2 \) and \( Pr[H_1, L_2] < Pr[L_1, H_2] \), then bidder 1 has an atom at \( b_{min} = b_1 \) of size \( G_1(b_{min}) > 0 \), and bidder 2 has an atom at \( v_2 = b_2 = b < b_{min} \) of size \( G_2(v_2) > 0 \).

It holds that

\[
b_{min} = b_1^*(G_2(v_2)) = \frac{Pr[H_2|H_1]G_2(v_2) + v_1Pr[L_2|H_1]}{Pr[H_2|H_1]G_2(v_2) + Pr[L_2|H_1]} > \max \{v_1, v_2\} \tag{4}
\]

\[
G_1(b_{min}) = \frac{Pr[L_1|H_2]}{Pr[H_1|H_2]} \int_{v_2}^{b_{min}} (x - v_2) \hat{r}(x) \, dx \tag{5}
\]

\[
G_2(v_2) = \frac{\hat{R}(b_{max})}{\hat{R}(b_{min})} - \left( \frac{\hat{R}(b_{max})}{\hat{R}(b_{min})} - G_1(b_{min}) \right) \cdot \frac{Pr[L_1, H_2]}{Pr[H_1, L_2]} \int_{b_{min}}^{b_{max}} \frac{x - v_1}{1-x} \hat{r}(x) \, dx \tag{6}
\]

- Assume \( Pr[H_1, L_2](1-v_1) < Pr[L_1, H_2](1-v_2) \). Then either

  - \( b_{min} = b \), bidder 1 has no atom \( (G_1(b_{min}) = 0) \) and bidder 2 has an atom at \( b = b_2 = v_2 \geq v_1 \) of size \( G_2(v_2) > 0 \) specified by Equation \( [6] \), or

  - \( b_{min} > b \), bidder 1 has an atom at \( b_{min} = b_1 \) specified by Equation \( [4] \), its size \( G_1(b_{min}) > 0 \) is specified by Equation \( [5] \), and bidder 2 has an atom at \( v_2 = b_2 = b < b_{min} \) of size \( G_2(v_2) > 0 \) specified by Equation \( [6] \).

Moreover, it always hold that

\[
G_1(b) = \begin{cases} 
0 & \text{if } 0 \leq b < b_{min}; \\
\frac{Pr[L_1|H_2]}{Pr[H_1|H_2]} \cdot \frac{\epsilon}{\hat{R}(b)} \cdot \int_{b_{min}}^{b} \frac{x - v_2}{1-x} \hat{r}(x) \, dx + G_1(b_{min}) \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b)} & \text{if } b_{min} \leq b \leq b_{max}; \\
1 & \text{if } b_{max} < b \leq 1.
\end{cases}
\]
and

\[
G_2(b) = \begin{cases} 
0 & \text{if } 0 \leq b < v_2; \\
\frac{G_2(v_2)}{\Pr[H_1|L_2]} \cdot \frac{\epsilon}{R(b)} \int_{b_{\min}}^{b} \frac{x - v_1}{1-x} \cdot r(x)dx + \frac{R(b_{\min})}{R(b)} & \text{if } v_2 \leq b < b_{\min}; \\
G_2(v_2) \cdot \hat{R}(b_{\min}) & \text{if } b_{\min} \leq b \leq b_{\max}; \\
1 & \text{if } b_{\max} < b \leq 1.
\end{cases}
\]

From this we derive that \(G_2(v_2)\) approaches \(1 - \frac{\Pr[H_1,L_2]}{\Pr[L_1,H_2]} \cdot \frac{1-v(H_1,L_2)}{1-v(L_1,H_2)}\) when \(\epsilon\) goes to 0, and that \(G_1(b_{\min})\) tends to 0. Moreover, we derive that for any large enough bid \(b \in (0,1)\) both \(G_1(b)\) and \(G_2(b) - G_2(v_2)\) are bounded from above, by some function that tends to 0 as \(\epsilon\) goes to 0. We conclude that the limit of the sequence of these mixed NE is exactly \(\mu\), as claimed in the theorem statement.

6 Discussion: Mechanism Design

The previous section shows that in the common value model the revenue of the SPA might be significantly smaller than the welfare. In the section we consider the problem of maximizing the revenue the seller receives by selling the item.

In the common value model there is a trivial mechanism that is ex-ante individually rational and maximizes the welfare as well as the revenue: we offer the first buyer a take-it-or-leave-it offer to buy the item for the price equal to the unconditional expectation of the item.

Yet this trivial mechanism does not extend to the case that there is some private component to the value of the item. For example, in the domain of online advertisement it is reasonable to assume that an informed buyer (advertiser) that has a high quality signal (cookie) on the user machine can tailor a specific advertisement to the specific user, generating some additional value over the common value created by placing a generic advertisement that is not user specific.

This motivates us to consider the following generalization of the model with a single informed bidder, in which the informed bidder is also advantaged. In this model there are \(n\) potential buyers. One random buyer \(i\) is informed about the state of the world (gets a signal \(s_i \in S_i\)), while the others are uninformed. Assume that the signals are sorted by the expected common value to an uninformed bidder. Assume that for the maximal signal \(s_{\max}\) the value for the informed bidder is larger than the common value by some \(B > 0\) (this is his Bonus). Let \(s_{\min}\) denote the lowest signal.

Let \(E\) be the unconditional expected value of the item to an uninformed bidder. Let \(p_{\max}\) be the probability of the signal to be realized to \(s_{\max}\), and let \(L\) be the expected value of the item conditional on the signal being \(s_{\min}\). The expected social welfare when the realized informed bidder always gets the item is \(E + p_{\max}B\). In this model selling the item ex-ante to a fixed agent at his expected value will generate revenue of \(E + \frac{p_{\max}B}{n}\), which can be significantly lower than the maximal social welfare.

A mechanism that gets revenue that equals the maximal welfare must allocate the item efficiently. Running the second price auction in this scenario will indeed maximize the social welfare. Yet, one can easily extend Theorem 5 to this model and see that for any realized informed bidder the unique TRE in this model is exactly the same as the one described by the theorem (with the adjustment that the informed bidder with signal \(s_{\max}\) bids his value that includes the bonus).
Yet, we can build a mechanism that is ex-ante individually rational, is socially efficient and extracts (almost) the entire welfare as revenue. All this in the unique outcome of the mechanism under our refinement as we explain below.

The mechanism has two stages. The mechanism first presents each bidder with a take-it-or-leave-it offer to buy the right to bid in a second price auction (SPA), and then runs a SPA with the bids of every agent that has bought the right to enter the SPA. Theorem 5 (and its extension to this model) predicts a unique TRE in undominated strategies. The payment in the SPA is always going to be \( L \). The take-it-or-leave-it price is set to be slightly less than the expected utility that the agent gets by participating in the SPA, assuming all agents participate in the SPA and bid according to the unique TRE in undominated strategies in that game. The entry price is set to be slightly less than \( (E + p_{\text{max}} B - L)/n \).

As TRE in undominated strategies provides a unique prediction to the outcome of the second stage, agents have a unique rational decision when facing the entry decision, and they choose to pay the entry fee. Thus, in the unique subgame-perfect-equilibrium that uses the TRE refinement, agents will all choose to enter (pay the entry fee), and allocation will be socially efficient in the SPA. The utility of each agent is essentially 0 (his gain goes to 0 as the entry price tends to \( (E + p_{\text{max}} B - L)/n \)). Although the revenue in the SPA is low, the entire utility an agent gets in this auction in expectation is essentially charged as entry fee. The revenue from entry would be \( n(E + p_{\text{max}} B - L)/n = E + p_{\text{max}} B - L \), while the revenue in the SPA would be \( L \), and the total revenue is exactly the social welfare \( E + p_{\text{max}} B \). Thus the total revenue essentially equals to the social welfare.

The above mechanism can only be used when agent can reasonably predict the outcome of the SPA that takes place at the second stage, and it can be extended to any other scenario in which a uniqueness result can be proven about the outcome of the SPA game under some solution concept.

**Interim Individually Rational Mechanism**

We note that the above mechanism is not interim individually rational. We next consider the problem of designing an interim individually rational mechanism for this setting, when the informed player has only two signals \( s_{\text{min}} \) and \( s_{\text{max}} \). The mechanism we design is dominant strategy incentive compatible. Let \( L \) be the value conditioned on \( s_{\text{min}} \) and \( P + B \) be the value of the advantaged bidder conditioned on \( s_{\text{max}} \).

While our model is not one of independent private value, it is sufficiently close that it seems useful to consider the optimal auction when each players value is sampled independently and identically from the following distribution: the value is \( L \) with probability \( 1 - 1/n \), and \( P + B \) with probability \( 1/n \). For this instance, Myerson’s optimal auction is to have some reserve price \( r \) and some floor price \( f \). If some bidders bid at least \( r \) then we run a second price auction with reserve \( r \), otherwise we randomly choose a winner among those who bid at least \( f \) and charge the winner \( f \).

In our model we can indeed set \( f = L \) and \( r = P + B - z \), where \( z = (P + B - f)/n \) is the expected utility of agent \( i \) bidding \( f \) given signal \( s_{\text{max}} \) (conditioned on every other agent \( j \) bidding \( f \)).

The revenue obtained is \( (1 - p_{\text{max}})f + p_{\text{max}} r \). Note that this is at least \( (1 - 1/n) \)-fraction of the social welfare which is \( (1 - p_{\text{max}})L + p_{\text{max}}(P + B) \).
A Detail Free Mechanism

Consider the case where the auctioneer is not aware of the values of $L$, $P$ and $B$ (yet he is aware of the rest of the properties of the environment). Then a two stage process can be implemented to get relatively high revenue without this information. In the first stage, bidders bid for the right to bid in the second stage. Specifically each bidder submits a floor value $f_i$ and a reserve value $r_i$. The auctioneer then chooses the floor looser (denoted $i$) as a random bidder among the bidder with minimal floor values and the reserve looser (denoted $j$) as a random bidder among the bidders with minimal reserve values. In the second stage the auctioneer runs the auction described above with floor price $f = f_i$ and reserve price $r = r_j$. Moreover, the auctioneer forbids the reserve looser $i$ from bidding $r$ or higher and forbids the floor looser $j$ (of the first stage) from bidding $r$ or lower. (Note that in the case that $i = j$ this implies that the looser cannot bid at all in the second stage).

In the second stage bidders have a dominant strategy to bid truthfully. Given that all bidders do that, in the first stage each bidder has a dominant bid $L$ for the floor price, and a dominant bid $P + B - z$, where $z = (P + B - f)/n$, for the reserve price. Under these strategies the revenue obtained is $(1 - \max_p (1 - 1/n)) f + \max_p (1 - 1/n) r$. Note that this is at least $(1 - 1/n)$-fraction of the revenue of the previous mechanism hence at least $(1 - 1/n)^2$ of the social welfare.

References


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### A Multiplicity of Equilibria under Perfect Equilibrium

Considering refinements for our game, one natural candidate is Selten’s (1975) Tremble Hand Perfect Equilibrium (PE). In this section we show that in our common value SPA with asymmetric information, PE does not provide the natural unique prediction one would expect in the most basic setting with two agents: one informed agent with a binary signal, and one uninformed agent. Note that in this setting there is a unique TRE in undominated strategies, and it is a strong TRE in pure strategies. In this natural equilibrium, the informed bids his posterior value while the uninformed bids to match the lowest possible bid of the informed.

Formally, consider the setting with two agents, one informed agent with a binary signal, and one uninformed agent. Assume that the common value is 0 conditional on the informed low signal, and 1 conditional on his high signal. Each signal is realized with probability $1/2$. Each agent’s action space (bid space) is the set $[0, 1]$ (an infinite set). In the unique TRE in undominated strategies the informed bids 0 on the low signal and 1 on the high signal, while the uninformed always bids 0.

We note that PE is usually defined for *finite normal form games* while our game is a game of incomplete information with infinite strategy spaces (finite type spaces but infinite action spaces). The adaptation of the solution concept to incomplete information is relatively straightforward. The move to infinite games is more delicate and we discuss two adaptations that were suggested in Simon and Stinchcombe (1995) (extending these adaptations to the incomplete information setting) and show that neither provide a unique prediction.

We start by presenting Simon and Stinchcombe’s (1995) reformulation Selten’s (1975) Tremble Hand Perfect Equilibrium (PE) for finite (normal form) games with complete information. Let $N$ be a finite set of agents. For agent $i \in N$ let $A_i$ be a finite set of pure actions, and let $A = \times_{i \in N} A_i$. Let $\Delta_i$ (resp. $\Delta_{fs,i}$) be the set of probability distributions (resp. full support probability distributions) on $A_i$. Let $\Delta = \times_{i \in N} \Delta_i$ and $\Delta_{fs} = \times_{i \in N} \Delta_{fs,i}$. For $\mu \in \Delta$, let $Br_i(\mu_{-i})$ denote $i$’s set of mixed-strategy best-responses to the vector of strategies of the others $\mu_{-i}$.

**Definition 24.** (Selten (1975)) Consider a finite game. Fix $\epsilon > 0$. A vector $\mu^\epsilon = (\mu_i^{\epsilon})_{i \in N}$ in $\Delta_{fs}$ is an $\epsilon$-Perfect Equilibrium if for each agent $i \in N$ it holds that\(^{12}\)

$$d_i(\mu_i^{\epsilon}, Br_i(\mu_{-i}^{\epsilon})) < \epsilon$$

where $d_i(\mu_i, \nu_i) = \sum_{a_i \in A_i} |\mu_i(a_i) - \nu_i(a_i)|$.

A vector $\mu = (\mu_i)_{i \in N}$ in $\Delta$ is a Perfect Equilibrium if there exists an infinite sequence of positive numbers $\epsilon_1, \epsilon_2, \ldots$ which converges to 0 such that (1) for each $j$, $\mu^{\epsilon_j}$ is an $\epsilon_j$-Perfect Equilibrium and (2) for every $i \in N$ it holds that $\mu_i^{\epsilon_j}$ converges in distribution to $\mu_i$ when $j$ goes to infinity.

\(^{12}\)Informally, his strategy is at most $\epsilon$ away from being a best response.
Loosely speaking, for a finite (normal form) game a Perfect Equilibrium is a limit, as \( \epsilon \) goes to 0, of a sequence of full support strategy vectors, each element of such a vector is \( \epsilon \) close to being a best response to the other agent’s strategies in that element of the sequence of strategy vectors.

We next discuss two adaptations, suggested in (Simon and Stinchcombe 1995), of PE to infinite games. The first is called "limit-of-finite" which considers the limit of a sequence of strategies in a sequence of finite games, in each game only a finite subset of actions is allowed and every player’s strategy has full support. The distance from every action to the set of allowed actions goes to zero and the sequence of strategies converges to the "limit-of-finite". The second is called strong perfect equilibrium which looks directly at the infinite game and requires strictly positive mass to every nonempty open subset and the sequence of strategies converges to the strong perfect equilibrium.

Next, we adjust these concepts to games with incomplete information, finite types spaces but infinite action spaces, and show that neither predict a unique equilibrium in the simple setting discussed above.\(^{13}\)

A.1 Limit of Finite Games

We next define the notion of limit-of-finite Perfect Equilibrium for games with incomplete information, finite types spaces but infinite action spaces. The approach is to define perfect equilibrium as the limit of \( \epsilon \)-perfect equilibria for sequences of successively larger (more refined) finite games.

Let \( N \) be a finite set of agents. For agent \( i \in N \) let \( T_i \) be a finite set of types for agent \( i \). Assume that the agents have a common prior over types. Let \( A_i \) be a compact (infinite) set of actions. Let \( B_i \) be a nonempty finite subset of \( A_i \), and let \( B = \times_{i \in N} B_i \). For such a \( B_i \), let \( \Delta_i(B_i) \) (resp. \( \Delta_i^{fs}(B_i) \)) be the set of probability distributions (resp. full support probability distributions) on \( B_i \).

A \( B_i \)-supported mixed strategy \( \mu_i(B_i) \) for agent \( i \) is a mapping from his type \( t_i \) to an element of \( \Delta_i(B_i) \). For a profile of mixed strategies \( \mu(B) = (\mu_i(B_i))_{i \in N} \), agent \( i \) and type \( t_i \in T_i \), let \( B_{i}^{\mu_i}(B_{-i}, \nu_{-i}) \) denote \( i \)'s set of \( B_i \)-supported mixed-strategy best-responses to the vector of strategies of the others \( \mu_{-i}(B_{-i}) \) (with respect to the given prior and the utility functions) when his type is \( t_i \).

**Definition 25.** Consider a game with incomplete information, finite types spaces but infinite action spaces. Fix \( \epsilon > 0 \) and \( \delta > 0 \). For each agent \( i \in N \) let \( B^\delta_i \) denote a finite subset of \( A_i \) within (distance) \( \delta \) of \( A_i \). A vector \( \mu_i^{(\epsilon, \delta)} = (\mu_i^{(\epsilon, \delta)})_{i \in N} \) such that for each \( i \) and \( t_i \in T_i \) it holds that \( \mu_i^{(\epsilon, \delta)}(t_i) \in \Delta_i^{fs}(B^\delta_i) \) is an \((\epsilon, \delta)\)-Perfect Equilibrium if for each agent \( i \in N \) and type \( t_i \in T_i \) it holds that

\[
d_i^{\delta}(\mu_i^{(\epsilon, \delta)}(t_i), B_{i}^{\mu_{-i}}(B^\delta_i, \mu_{-i}^{(\epsilon, \delta)})) < \epsilon
\]

where \( d_i^{\delta}(\mu_i, \nu_i) = \sum_{a_i \in B^\delta_i} |\mu_i(a_i) - \nu_i(a_i)| \).

\(^{13}\)We note that with tremble that is independent of the signal of the informed agent, such multiplicity of equilibria result cannot be proven. Yet, the unique equilibrium that is the result of any such tremble is not the one we would expect. In the same setting of an item of a common value 0 or 1, with equal probability, and two agents, one perfectly informed and one uninformed, we observe the following. For any tremble of the informed that is independent of the informed agent’s signal, the best response of the uninformed agent is to bid the unconditional expectation (half) as this is the value of the item conditional on winning in the case the informed trembles (and if he does not, the uninformed agent just pays the exact value of the item if winning, as the price is set by the informed agent).
A vector $\mu = (\mu_i)_{i \in N}$ is a limit-of-finite Perfect Equilibrium if there exists two infinite sequences of positive numbers $\epsilon_1, \epsilon_2, \ldots$ and $\delta_1, \delta_2, \ldots$ both converging to 0 such that (1) for each $j$, $\mu^{(\epsilon_j, \delta_j)}$ is an $(\epsilon_j, \delta_j)$-Perfect Equilibrium and (2) for every $i \in N$ and $t_i \in T_i$ it holds that $\mu_i^{(\epsilon_j, \delta_j)}(t_i)$ converges in distribution to $\mu_i(t_i)$ when $j$ goes to infinity.

We next show that there are multiple strong PE in the infinite game with one informed agent with a binary signal and one uninformed agent.

**Proposition 26.** Consider the infinite game with one informed agent with a binary signal and one uninformed agent as defined above. For any $y \in (0, 1)$, the following is a (pure strategy) limit-of-finite perfect equilibrium in this infinite game: The informed bids according to his dominant strategy (his posterior: 0 on low signal, 1 on high signal), while the uninformed always bids $y$.

**Proof.** Consider the following natural way to make our game finite by discretizing the bids: fix a large natural number $m$ and only allow bids of the form $k/m$ for $k \in \{0, 1, \ldots, m\}$. Note that as $m$ grows to infinity the distance between any bid $y$ and such a set of bids decreases to zero.

Fix $\epsilon > 0$ that is small enough. Fix $m$ that is large enough and fix $k_0 \in \{1, \ldots, m-1\}$ such that $(k_0 + 1)/m$ has minimal distance to $y$ out of all bids of form $k/m$. To prove the claim we present a profile of strategies with full support over the discrete set of bids that is close to the profile in which the informed bids according to his dominant strategy while the uninformed always bids $y$. The strategies that we build have an atom of size at least $1 - \epsilon$ on the specified bids. For the informed with low signal, the probability on every bid other than 0 is proportional to $\epsilon^2$, while for the informed with high signal the probability of every bid other than 1 is proportional to $\epsilon^3$, except for $k_0/m$ for which he assigns probability of about $\epsilon$. This motivates the uninformed to bid $(k_0 + 1)/m$, right above this "gift" given by the informed bidder with high signal, and we show that such a bid is his best response. We next define the strategies formally.

The informed agent with low signal is bidding 0 with probability $1 - \epsilon^2$, and for any $k \in \{1, \ldots, m\}$ he bids $k/m$ with probability $\epsilon^2/m$. The informed agent with high signal is bidding 1 with probability $1 - \epsilon$. He bids $k_0/m$ with probability $\epsilon - \epsilon^3$, and for any $k \in \{0, \ldots, m-1\}$ such that $k \neq k_0$, he bids $k/m$ with probability $\epsilon^3/(m-1)$.

The uninformed agent is bidding $(k_0 + 1)/m$ with probability $1 - \epsilon$, and for any $k \in \{0, \ldots, m\}$ such that $k \neq k_0 + 1$ he bids $k/m$ with probability $\epsilon/m$.

The informed agent has a dominant strategy to bid his posterior value, and his strategy is clearly $\epsilon$ close to that strategy. It remains to show that the strategy of the uninformed is $\epsilon$ close to his best response (to the strategy of the informed). We claim that if $\epsilon$ is small enough the best response of the uninformed to the strategy of the informed is to bid $(k_0 + 1)/m$ with probability 1. Indeed, consider any bid $j/m$:

- If $j = k_0 + 1$ then the informed has positive utility as when the value is high he has utility of at least $1/m$ with probability at least $(\epsilon - \epsilon^3)$. When the value is low his loss is at most $(k_0 + 1)/m$ and this happens only with probability at most $\epsilon^2$. For small enough $\epsilon$ the loss will be smaller than the gain.
- If $j = 0$ then the uninformed has utility 0.
- If $0 < j < k_0$ then the uninformed wins item of value 1 with probability at most $j\epsilon^3/(2 \cdot (m-1))$ (as the quality is high with probability 1/2 and in such case he only wins if the informed is bidding below him), thus his expected value is at most $j\epsilon^3/(2 \cdot (m-1))$. 


On the other hand his expected payment is at least \( (1/4) \cdot (e^2/m) \cdot (1/m) \) (in case it is low value he pays at least \( 1/m \) with probability \( (1/2) \cdot (e^2/m) \) - the probability of the other bidding \( 1/m \) and tie is broken in favor of him). Thus his expected utility is at most \\
\[ \left( \frac{j^3}{2 \cdot (m - 1)} \right) - \frac{e^2}{4m^2} \] \\
which is negative for small enough \( \epsilon > 0 \).

- If \( j = k_0 \) then we claim that this bid is dominated by bidding \( (k_0 + 1)/m \). Due to random tie breaking the bid of \( k_0/m \) only wins half of the times when the value is high and the informed is also bidding \( k_0/m \). By increasing his bid to \( (k_0 + 1)/m \) the uninformed will always win in this case. The affect of this change is linear in \( \epsilon \). The negative effect due to winning more when the informed gets the low signal is only of the order of \( \epsilon^2 \), thus for small enough \( \epsilon \) it will be smaller.

- If \( j > k_0 + 1 \) then we claim that this bid is dominated by bidding \( (j - 1)/m \). This follow since the probability of winning high value items decreases by order of \( \epsilon^3 \), while the probability of not paying for low value items decreases by order of \( \epsilon^2 \).

\[ \square \]

Note that the proof of the proposition shows that PE does not provide a unique prediction even if we consider finite discrete action spaces. This seems to indicate that the problem with PE (with respect to our setting) is deeper than just its extension to games with infinite action spaces.

### A.2 Strong Perfect Equilibrium

We next define the notion of strong Perfect Equilibrium for games with incomplete information, finite types spaces but infinite action spaces. Let \( N \) be a finite set of agents. For agent \( i \in N \) let \( T_i \) be a finite set of types for agent \( i \). Assume that the agents have a common prior over types. Let \( A_i \) be a compact (infinite) set of actions. Let \( \Delta_i \) be the set of probability measures on \( A_i \), while \( \Delta_i^{fs} \) be the set of probability measures on \( A_i \) assigning strictly positive mass to every nonempty open subset of \( A_i \). We measure the distance between two measures \( \mu, \nu \) on an infinite actions space using the following metric:

\[ \rho(\mu, \nu) = \sup\{|\mu(B) - \nu(B)| : B \text{ measurable}\} \]

A mixed strategy \( \mu_i \) for agent \( i \) is a mapping from his type \( t_i \in T_i \) to an element of \( \Delta_i \). For a profile of mixed strategies \( \mu = (\mu_i)_{i \in N} \) agent \( i \) and type \( t_i \in T_i \), let \( Br_{t_i}^{i} (\mu_{-i}) \) denote \( i \)'s set of mixed-strategy best-responses to the vector of strategies of the others \( \mu_{-i} \) (with respect to the given prior and the utility functions) when his type is \( t_i \).

**Definition 27.** Consider a game with incomplete information, finite types spaces but infinite action spaces. Fix \( \epsilon > 0 \). A vector \( \mu^\epsilon = (\mu_i^\epsilon)_{i \in N} \) such that for each \( i \) and \( t_i \in T_i \) it holds that \( \mu_i(t_i) \in \Delta_i^{fs} \) is a strong \( \epsilon \)-Perfect Equilibrium if for each agent \( i \in N \) and type \( t_i \in T_i \) it holds that \\
\[ \rho_i(\mu_i^\epsilon(t_i), Br_{t_i}^{i} (\mu_{-i}^\epsilon)) < \epsilon \]

A vector \( \mu = (\mu_i)_{i \in N} \) is a strong Perfect Equilibrium if there exists an infinite sequence of positive numbers \( \epsilon_1, \epsilon_2, \ldots \) which converges to 0 such that (1) for each \( j \), \( \mu^\epsilon_j \) is a strong \( \epsilon_j \)-Perfect Equilibrium and (2) for every \( i \in N \) and \( t_i \in T_i \) it holds that \( \mu_i^\epsilon_j(t_i) \) converges in distribution to \( \mu_i(t_i) \) when \( j \) goes to infinity.
We next show that there are multiple strong PE in the infinite game with one informed agent with a binary signal and one uninformed agent. The construction of the strategies in the next proposition is very similar to the one in Proposition 26.

**Proposition 28.** Consider the infinite game with one informed agent with a binary signal and one uninformed agent as defined above. For any $y \in (0, 1)$, the following is a (pure strategy) strong perfect equilibrium in this infinite game: The informed bids according to his dominant strategy (his posterior: 0 on low signal, 1 on high signal), while the uninformed always bids $y$.

**Proof.** Fix some $y \in (0, 1)$. Consider the following tremble for a given $\epsilon > 0$ that is small enough.

The informed agent with low signal is bidding with CDF $F_L(x) = 1 - \epsilon^2 + x\epsilon^2$ for $x \in [0, 1]$. (He bids 0 with probability $1 - \epsilon^2$ or uniformly between 0 and 1 with probability $\epsilon^2$.)

The informed agent with high signal is bidding with CDF $F_H$: For $x \in [0, y - \epsilon]$ it holds that $F_H(x) = x\epsilon^3$. For $x \in (y, y - \epsilon)$ it holds that $F_H(x) = F_H(y - \epsilon) + (x - y + \epsilon)(1 - \epsilon^2)$. For $x \in (y, 1)$ it holds that $F_H(x) = F_H(y) + (x - y)^2\epsilon^3$, and finally, $F_H(1) = 1$. (He bids 1 with probability $1 - \epsilon + \epsilon^4$, uniformly between $y - \epsilon$ and $y$ with probability $\epsilon - \epsilon^3$, and uniformly over all other bids in $[0, 1]$ with the remaining probability $\epsilon^3(1 - \epsilon)$.)

The uninformed agent is bidding with CDF $G$: For $x \in [0, y)$ it holds that $G(x) = x\epsilon$. For $x = y$ it holds that $G(x) = G(y) = y + 1 - \epsilon$. For $x \in (y, 1]$ it holds that $G(x) = G(y) + (x - y)\epsilon$. (He bids $y$ with probability $1 - \epsilon$ or uniformly between 0 and 1 with probability $\epsilon$.)

Clearly these strategies have full support and their limit as $\epsilon$ goes to 0 is as required.

The informed agent has a dominant strategy to bid his posterior value, and his strategy is clearly $\epsilon$ close to that strategy. It remains to show that the strategy of the uninformed is $\epsilon$ close to his best response (to the strategy of the informed). We claim that if $\epsilon$ is small enough the best response of the uninformed to the strategy of the informed is to bid $y$ with probability 1. Indeed, consider any bid $z$:

- If $z = 0$ then the agent has utility 0.
- If $z = y$ then for small enough $\epsilon > 0$ the agent has positive utility. Indeed his expected gain from high value items is at least $1/2 \cdot F_H(y)(1 - y) = (\epsilon - \epsilon^3(1 - y + \epsilon))/(1 - y)/2 \geq c\epsilon$ for some constant $c > 0$ (for small enough $\epsilon > 0$), while his expected loss from low value items is at most $1/2 \cdot (1 - F_L(0))y \leq (y/2)\epsilon^2 \leq \epsilon^2$.
- If $0 < z < y$ then for small enough $\epsilon > 0$ it holds that $0 < z < y - \epsilon$. Moreover, for small enough $\epsilon > 0$ the agent has negative utility. Indeed his expected gain is at most $1/2 \cdot F_H(z) \cdot 1 \leq z\epsilon^3$, while his expected loss is at least $1/2 \cdot (F_L(z) - F_L(z/2)) \cdot z/2 \geq z^2\epsilon^2/4$.
- If $z > y$ then for small enough $\epsilon > 0$ the agent can increase his utility by bidding $y$ instead of bidding $z$. Indeed his expected loss of value by bidding $y$ instead of $z$ is at most $1/2 \cdot (F_H(z) - F_H(y)) \cdot 1 = (y - z)\epsilon^3/2$, while his expected reduction in payment is at least $1/2 \cdot (F_L(z) - F_L(y)) \cdot y \geq (z - y)\epsilon^2/2$.

\[ \square \]

## B One Informed Agent

### B.1 FPA Revenue in Example 7

Let $h = E[v(w)|s]$ be the informed bidder’s interim value given signal $s$, and $F$ be its cumulative distribution. Then by Engelbrecht-Wiggans et al.’s (1983) Theorem 4, FPA revenue is
\[ \int_0^\infty (1 - F(h))^2 dh \] First consider the peaches case. Let \( E^- = \frac{E[v(w)] - \varepsilon pw(P)}{1 - \varepsilon p} \) be the posterior given the signal \( \emptyset \). As shown in the main text, \( h \in \{ E^-, v(P) \} \) and \( \Pr(h = v(P)) = \varepsilon p \). Therefore

\[
F_{\text{peaches}}(h) = \begin{cases} 
0, & \text{if } h < E^- \\
1 - \varepsilon p, & \text{if } h \in [E^-, v(P)] \\
1, & \text{if } h \geq v(P)
\end{cases}
\]

and hence

\[
R_{FPA}^{\text{peaches}} = \left(1 - (\varepsilon p)^2\right) \frac{E[v(w)] - \varepsilon pw(P)}{1 - \varepsilon p} + (\varepsilon p)^2 v(P) = E[v(w)] - \varepsilon p (1 - p) (v(P) - v(L)).
\]

Second, consider the lemons case. Let \( E^+ = \frac{E[v(w)] - \varepsilon (1-p)v(L)}{1 - \varepsilon (1-p)} \) be the posterior given the signal \( \emptyset \). Now, \( h \in \{ v(L), E^+ \} \) and \( \Pr(h = v(L)) = \varepsilon (1 - p) \). Therefore

\[
F_{\text{lemons}}(h) = \begin{cases} 
0, & \text{if } h < v(L) \\
\varepsilon (1 - p), & \text{if } h \in [v(L), E^+] \\
1, & \text{if } h \geq E^+
\end{cases}
\]

and hence

\[
R_{FPA}^{\text{lemons}} = v(L) + (1 - \varepsilon (1 - p))^2 (E^+ - v(L)) = E[v(w)] - \varepsilon p (1 - p) (v(P) - v(L)).
\]

### B.2 Bounding the FPA Revenue from Below

In this section we prove Proposition 9.

**Proposition 29.** Consider any common value domain with items of value in \([0, 1]\) and expected value of \( E \). Assume that there are \( n \) agents, \( n-1 \) of them are uninformed. For any information structure for the informed agent the revenue of the FPA is at least \( E^2 \).

**Proof.** Define the informed bidder’s interim expected value conditional on receiving signal \( s \) as \( h(s) = E[v(\omega)|s] \). Further, let \( F \) be the cumulative distribution function of \( h \). Note that as items have value in \([0, 1]\), \( h \in [0, 1] \) and \( F(1) = 1 \). According to Theorem 4 of Engelbrecht-Wiggans et al. (1983), FPA revenue is

\[
\int_0^1 (1 - F(h))^2 dh
\]

and the informed agent expected profit is

\[
\int_0^1 F(h)(1 - F(h)) dh
\]

Note that the revenue and the informed agent’s profit sum up to \( E \), the expected value of the item (the social welfare). To bound the revenue from below we bound the informed agent’s profit from above. We use the following result due to Ahlswede and Daykin (1979).

**Lemma 30.** If, for 4 non-negative functions \( g_1, g_2, g_3, g_4 \) mapping \( \mathbb{R} \to \mathbb{R} \), the following holds:

\[
\text{for all } x, y \in \mathbb{R}, \ g_1(\max(x, y)) \cdot g_2(\min(x, y)) \geq g_3(x) \cdot g_4(y),
\]

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then it follows that
\[ \int_a^b g_1(t)dt \cdot \int_a^b g_2(t)dt \geq \int_a^b g_3(t)dt \int_a^b g_4(t)dt. \]

We apply this lemma by setting
\[ g_1(t) = F(t), \quad g_2(t) = 1 - F(t), \quad g_3(t) = F(t) \cdot (1 - F(t)), \quad g_4(t) = 1. \]

Monotonicity of \( F \) implies that the conditions of the lemma hold. Indeed, if \( x'' > x' \),
\[ F(x'') \cdot (1 - F(x')) \geq F(x'') \cdot (1 - F(x'')) \]
and
\[ F(x'') \cdot (1 - F(x')) \geq F(x') \cdot (1 - F(x')). \]

Then, it follows that
\[ E \cdot (1 - E) = \int_0^1 F(t)dt \cdot \int_0^1 (1 - F(t))dt \geq \int_0^1 F(t)(1 - F(t))dt. \]

As the revenue equals to the welfare minus the informed agent’s profit we conclude that the revenue is bounded from above by \( E^2 \):
\[ \int_0^1 (1 - F(h))^2 dh = E - \int_0^1 F(t)(1 - F(t))dt \leq E^2 \]

\[ \square \]

C Many Agents, each with Finitely Many Signals

C.1 Proof of Theorem 11

To prove Theorem 11 we need some helpful lemmas.

Lemma 31. Consider any domain with multiple bidders and multiple signals each in which the high-signal-is-never-bad-news property holds. If \( \mu_j(s_j) \) for agent \( j \) with signal \( s_j \) was set before \( \mu_k(s_k) \) for agent \( k \) and signal \( s_k \), and both were set at stage \( 2 \), then \( \mu_j(s_j) \geq \mu_k(s_k) \).

Proof. To prove the first claim it is sufficient to show that for any agent \( j \) and signal \( s_j \) that was picked at stage \( 2 \) and any agent \( k \) (possibly \( k = j \)) and signal \( s_k \) picked next at stage \( 2 \), it holds that \( \mu_j(s_j) \geq \mu_k(s_k) \).

Let \((T_j, T_{-j})\) be the sets of signals for all agents that were not determined when the bid of \( j \) with signal \( s_j \) was set.

First consider the case that \( k = j \). In this case \( \mu_j(s_j) = v(\{s_j\}, T_{-j}) \) and \( \mu_k(s_k) = v(\{s_k\}, T_{-k}) \) for \( T_{-j} = T_{-k} \) As both \( j \) and \( k \) were considered when \( j \) was picked it holds that \( v(\{s_j\}, T_{-j}) \geq v(\{s_k\}, T_{-j}) = v(\{s_k\}, T_{-k}) \) thus \( \mu_j(s_j) \geq \mu_k(s_k) \).

Next consider the case that \( k \neq j \). As the bid was set for \( s_j \) it must be the case that \((\{s_j\}, T_{-j})\) is feasible and the bid was set to \( \mu_j(s_j) = v(\{s_j\}, T_{-j}) = v(\{s_j\}, T_k, T_{-\{j,k\}}) \). As signal \( s_k \) for agent \( k \neq j \) was picked next at stage \( 2 \) it holds that \((\{s_k\}, T_j \setminus \{s_j\}, T_{-\{j,k\}})\) is feasible and \( \mu_k(s_k) = v(\{s_k\}, T_j \setminus \{s_j\}, T_{-\{j,k\}}) \). As \((\{s_k\}, T_j \setminus \{s_j\}, T_{-\{j,k\}})\) is feasible
then \(((\{s_k\}, T_j, T_{\{j,k\}}) = (\{s_k\}, T_{\{-k\}})\) is also feasible. Since \(s_j\) was picked before \(s_k\) and the procedure picks a feasible signal with maximal expected value, it holds that \(v(\{s_j\}, T_{-j}) \geq v(\{s_k\}, T_{-k})\).

When the bid \(\mu_k(s_k)\) was set, the sets of signals of \(j\) for which the bid was not determined yet was \(T_j \setminus \{s_j\}\), and the set for any agent \(i \neq j, k\) was \(T_i\). As the high-signal-is-never-bad-news property holds, if \(((\{s_k\}, \{s_j\}, T_{\{-j,k\}})\) is feasible then \(v(\{s_k\}, \{s_j\}, T_{\{-j,k\}}) \geq v(\{s_k\}, T_j \setminus \{s_j\}, T_{\{-j,k\}}) = \mu_k(s_k)\). It holds that \(\mu_k(s_k) = v(\{s_k\}, T_{-k})\) is a convex combination of \(v(\{s_k\}, \{s_j\}, T_{\{-j,k\}})\) and \(v(\{s_k\}, T_j \setminus \{s_j\}, T_{\{-j,k\}}) = \mu_j(s_j)\) and as a convex combination of two values is at least the minimum of the two, \(\mu_k(s_k) \geq \mu_j(s_j)\).

Observation 32. Consider any domain with multiple bidders and multiple signals each.

If the bid \(\mu_i(s_i)\) of agent \(i\) with signal \(s_i\) was set at stage [3] of the procedure, then the following holds. For any bid \(\mu_k(s_k) = v(\{s_k\}, T_i, T_{\{-i,k\}})\) of agent \(k \neq i\) with signal \(s_k\) that was set at stage [2] of the procedure, if \(((\{s_k\}, \{s_i\}, T_{\{-i,k\}})\) is feasible then \(\mu_k(s_k) \geq \mu_i(s_i)\).

Proof. The signal \(s_j\) is a member of \(T_i\) as all signals of \(i\) for which the bid is set in stage [2] are never removed from the set \(T_i\). Now, as \(((\{s_k\}, \{s_i\}, T_{\{-i,k\}})\) is feasible it holds that \((T_k, \{s_j\}, T_{\{-i,k\}})\) is also feasible. When the bid \(\mu_k(s_j) = v(\{s_j\}, T_k, T_{\{-i,k\}})\) of agent \(k\) with signal \(s_j\) was set at stage [2], \((T_k, \{s_j\}, T_{\{-i,k\}})\) was also feasible thus \(\mu_k(s_k) = v(\{s_k\}, T_i, T_{\{-i,k\}}) \geq v(T_k, \{s_j\}, T_{\{-i,k\}})\).

Since \(\mu_i(s_i)\) was set at stage [3] of the procedure it is the minimal of all feasible vectors that include \(s_i\), as \((T_k, \{s_j\}, T_{\{-i,k\}})\) is feasible it holds that \(v(T_k, \{s_i\}, T_{\{-i,k\}}) \geq \mu_i(s_i)\). We conclude that \(\mu_k(s_k) \geq \mu_i(s_i)\).

Observation 33. Consider any domain with multiple bidders and multiple signals each in which the low-signal-is-never-good-news property holds.

Assume that the bid \(\mu_i(s_i)\) of agent \(i\) with signal \(s_i\) was set at stage [3] of the procedure. If \(\mu_i(s_i)\) is a winning bid then in such case the payment is \(\mu_i(s_i)\) (there exists another agent \(k \neq i\) and signal \(s_k\) bidding \(\mu_k(s_k) = \mu_i(s_i)\)). Thus, in \(\lambda\) (and in any \((\epsilon, R)\)-tremble of the game), agent \(i\) with signal \(s_i\) has utility is 0 from the bid \(\mu_i(s_i)\).

Additionally, in \(\lambda\) (and thus in any \((\epsilon, R)\)-tremble of the game), when \(\mu_{-i}\) is played, any other bid \(b_i\) of agent \(i\) with signal \(s_i\) gives him non-positive utility.

Proof. Assume that \(\mu_i(s_i)\) is a winning bid when the vector of signals was realize to \(s\). As the bid \(\mu_i(s_i)\) of agent \(i\) with signal \(s_i\) was set at stage [3] of the procedure, it means that when stage [2] ended the vector \(s\) was not feasible. This means that for some agent \(k \neq i\) and signal \(s_k\) taken from the vector \(s\), the bid \(\mu_k(s_k)\) was already set at stage [2]. By Observation 32 it holds that \(\mu_k(s_k) \geq \mu_i(s_i)\). As \(\mu_i(s_i)\) is a winning bid it holds that \(\mu_k(s_k) \leq \mu_i(s_i)\) thus \(\mu_k(s_k) = \mu_i(s_i)\).

Next we show that any other bid \(b_i\) of agent \(i\) with signal \(s_i\) gives him non-positive utility. First assume that \(b_i < \mu_i(s_i)\). By the first claim the bid \(\mu_i(s_i)\) can be winning only if there is another agent \(k\) with signal \(s_k\) that is bidding \(\mu_k(s_k) = \mu_i(s_i)\). This means that any lower bid \(b_i < \mu_i(s_i)\) is a losing bid and the utility is still 0.

Next assume that \(b_i > \mu_i(s_i)\). For this bid to provide positive utility it must be winning when the vector of signals was realize to some vector \(s\). As the bid \(\mu_i(s_i)\) of agent \(i\) with signal \(s_i\) was set at stage [3] of the procedure, it means that when stage [2] ended the vector \(s\) was not feasible. This means that for some agent \(k \neq i\) and signal \(s_k\) taken from the vector \(s\), the bid \(\mu_k(s_k) = v(\{s_k\}, T_{-k})\) was already set at stage [2]. By Observation 32 it holds that \(\mu_k(s_k) \geq \mu_i(s_i)\), thus \(s_i \in T_{i}\) were \(T_{i}\) was the set of signal of agent \(i\) that were not
set yet when \( s_k \) was picked. Now the low-signal-is-never-good-news property implies that if \((\{s_i\}, \{s_k\}, T_{-(i,k)})\) is feasible then \(v(\{s_i\}, \{s_k\}, T_{-(i,k)}) \leq v(\{s_i\}, T_k, T_{-(i,k)}) = \mu_k(s_k)\), that is, the value is at most \(\mu_k(s_k)\). As the payment is at least \(\mu_k(s_k)\), the utility in \(\lambda\) is non-positive.

**Lemma 34.** Consider any domain with multiple bidders and multiple signals each in which the low-signal-is-never-good-news property and the high-signal-is-never-bad-news property hold. For such a domain the procedure computes a pure NE of the SPA game \(\lambda\).

**Proof.** Consider agent \(k\) with signal \(s_k\), bidding \(\mu_k(s_k)\) as defined by the procedure. To prove that the strategies computed by the procedure are a pure Nash equilibrium of the game it is sufficient to show that agent \(k\) with signal \(s_k\) cannot increase his utility by bidding \(b_k \neq \mu_k(s_k)\).

By Observation 33 if the bid \(\mu_k(s_k)\) was set in stage \(2\) of the procedure then it ensures 0 utility while any other bid \(b_k\) results in non-positive utility, thus is not a beneficial deviation.

Below we consider a bid \(\mu_k(s_k)\) that was set in stage \(2\) of the procedure. First consider a bid \(b_k > \mu_k(s_k)\). If the utility of agent \(k\) increases by bidding \(b_k\) instead of \(\mu_k(s_k)\), it must be the case that agent \(k\) wins some additional goods that give him positive utility. For any such good there exists some agent \(j\) with signal \(s_j\) that is bidding \(\mu_j(s_j) \geq \mu_k(s_k)\), and is setting the payment for \(k\). There are two cases.

- **\(\mu_j(s_j)\) was set in stage \(3\) of the procedure.** By Observation 32 \(\mu_k(s_k) \geq \mu_j(s_j)\), thus \(\mu_k(s_k) = \mu_j(s_j)\). As \(\mu_k(s_k)\) that was set in stage \(2\) it holds that \(\mu_k(s_k) = v(\{s_k\}, T_{-k})\) where \(T_i\) is the set that corresponds to agent \(i \neq k\) when \(s_k\) was set at stage \(2\). As \(\mu_j(s_j)\) was set in stage \(3\) it holds that \(s_j \in T_j\). Now the low-signal-is-never-good-news property implies that if \((\{s_j\}, \{s_k\}, T_{-(j,k)})\) is feasible then the value is \(v(\{s_j\}, \{s_k\}, T_{-(j,k)}) \leq v(\{s_k\}, T_{-k}) = \mu_k(s_k) = \mu_j(s_j)\), and as the payment is \(\mu_j(s_j)\) the gain for any such good is non-positive.

- **\(\mu_j(s_j)\) was set in stage \(2\).** By Lemma 31 the sequence of bids set at stage \(2\) is non-increasing, thus either the bid \(\mu_j(s_j)\) was set before \(\mu_k(s_k)\) in stage \(2\), and it holds that \(b_k \geq \mu_j(s_j) \geq \mu_k(s_k)\), or it was set after \(\mu_k(s_k)\) and \(b_k > \mu_k(s_k) = \mu_j(s_j)\).

  - In the first case \(\mu_j(s_j)\) was set before \(\mu_k(s_k)\) in stage \(2\), and it holds that \(b_k \geq \mu_j(s_j) \geq \mu_k(s_k)\). As \(\mu_j(s_j)\) that was set in stage \(2\) it holds that \(\mu_j(s_j) = v(\{s_j\}, T_{-j})\) where \(T_i\) is the set that corresponds to agent \(i \neq j\) when \(s_j\) was set at stage \(2\). For any such \(j\) and \(s_j\), agent \(k\) now wins additional items of expected value \(v(\{s_j\}, \{s_k\}, T_{-(j,k)})\). For these goods agent \(k\) pays \(\mu_j(s_j) = v(\{s_j\}, T_k, T_{-(j,k)})\). By the low-signal-is-never-good-news property if \((\{s_j\}, \{s_k\}, T_{-(j,k)})\) is feasible then the value of the good agent \(k\) wins is \(v(\{s_j\}, \{s_k\}, T_{-(j,k)}) \leq v(\{s_j\}, T_k, T_{-(j,k)}) = \mu_j(s_j)\), and as the payment is \(\mu_j(s_j)\) the gain for any such good is non-positive.

  - In the second case \(\mu_j(s_j)\) was set after \(\mu_k(s_k)\) and \(b_k > \mu_k(s_k) = \mu_j(s_j)\). As \(\mu_k(s_k)\) that was set in stage \(2\) it holds that \(\mu_k(s_k) = v(\{s_k\}, T_{-k})\) where \(T_i\) is the set that corresponds to agent \(i \neq k\) when \(s_k\) was set at stage \(2\). By the low-signal-is-never-good-news property (with \(j\) being the low signal) if \((\{s_j\}, \{s_k\}, T_{-(j,k)})\) is feasible then the value is \(v(\{s_j\}, \{s_k\}, T_{-(j,k)}) \leq v(\{s_k\}, T_k, T_{-(j,k)}) = \mu_k(s_k) = \mu_j(s_j)\), and as the payment is \(\mu_j(s_j)\) the gain for any such good is non-positive.

Next we consider a bid \(b_k < \mu_k(s_k)\). If the utility of agent \(k\) with \(s_k\) increases by bidding \(b_k\) instead of \(\mu_k(s_k)\), it must be the case that agent \(k\) no longer wins some goods that give him negative utility. Any such good is a good for which agent \(k\) with bid \(\mu_k(s_k)\) was paying \(\mu_j(s_j) \leq \mu_k(s_k)\) \((\mu_j(s_j) \geq b_k)\) for some agent \(j\) and signal \(s_j\). There are two cases.
• $\mu_j(s_j)$ was set in stage (3) of the procedure. As $\mu_k(s_k)$ that was set in stage (2) it holds that $\mu_k(s_k) = v(\{s_k\}, T_{-k})$ where $T_i$ is the set that corresponds to agent $i \neq k$ when $s_k$ was set at stage (2). As $\mu_j(s_j)$ was set in stage (3) it holds that $s_j \in T_j$. If $(\{s_j\}, \{s_k\}, T_{-(j,k)})$ is feasible then the value of the good agent $k$ wins is $v(\{s_j\}, \{s_k\}, T_{-(j,k)}) \geq \mu_j(s_j)$ as $\mu_j(s_j)$ was set to be the minimal value for any feasible vector that contains $s_j$. As the payment is $\mu_j(s_j)$ the gain for any such good is non-negative.

• $\mu_j(s_j)$ was set in stage (2) of the procedure. By Lemma 31 the sequence of bids set at stage (2) is non-increasing, thus either the bid $\mu_j(s_j)$ was set after $\mu_k(s_k)$ in stage (2), and it holds that $b_k \leq \mu_j(s_j) \leq \mu_k(s_k)$, or it was set before $\mu_k(s_k)$ and $b_k < \mu_k(s_k) = \mu_j(s_j)$.

– In the first case $\mu_j(s_j)$ was set set after $\mu_k(s_k)$ in stage (2), and it holds that $b_k \leq \mu_j(s_j) \leq \mu_k(s_k)$. As $\mu_j(s_j)$ that was set in stage (2) it holds that $\mu_j(s_j) = v(\{s_j\}, T_{-j})$ where $T_i$ is the set that corresponds to agent $i \neq j$ when $s_j$ was set at stage (2). For any such $j$ and $s_j$, agent $k$ now loses items of expected value $v(\{s_j\}, \{s_k\}, T_{-(j,k)})$. For these goods agent $k$ pays $\mu_j(s_j) = v(\{s_j\}, T_k, T_{-(j,k)})$. By the high-signal-is-never-bad-news property if $(\{s_j\}, \{s_k\}, T_{-(j,k)})$ is feasible then the value is $v(\{s_j\}, \{s_k\}, T_{-(j,k)}) \geq v(\{s_j\}, T_k, T_{-(j,k)}) = \mu_j(s_j)$, and as the payment is $\mu_j(s_j)$ the gain for any such good is non-negative.

– In the second case $\mu_j(s_j)$ was set before $\mu_k(s_k)$ and $b_k < \mu_k(s_k) = \mu_j(s_j)$. Assume that $\mu_k(s_k) = v(\{s_k\}, T_{-k})$ where $T_i$ is the set that corresponds to agent $i \neq k$ when $s_k$ was set at stage (2). By the high-signal-is-never-bad-news property (with $j$ being the high signal) if $(\{s_j\}, \{s_k\}, T_{-(j,k)})$ is feasible then the value is $v(\{s_j\}, \{s_k\}, T_{-(j,k)}) \geq v(\{s_j\}, T_j, T_{-(j,k)}) = \mu_j(s_j) = \mu_k(s_k)$, and as the payment is $\mu_j(s_j)$ the gain for any such good is non-negative.

\[ \square \]

Let $v_i(b_i, s_i, \eta_{-i})$ be the expected value of the good that agent $i$ with signal $s_i$ and bid $b_i$ receives, when all other agents are playing according to the strategies $\eta_{-i}$.

**Lemma 35.** Fix any game $\lambda$ and an $(\epsilon, R)$-tremble of the game $\lambda(\epsilon, R)$. Let $\eta$ be a profile of pure strategies. For every agent $i$ and signal $s_i$, if $\eta_i(s_i)$ is an optimal bid for agent $i$ with signal $s_i$, given $\eta_{-i}$ in $\lambda(\epsilon, R)$, and agent $i$ has positive expected utility from such a bid, then $\eta_i(s_i)$ must be equal to agent $i$’s expected value conditional on winning, given his signal $s_i$ and the strategies $\eta$, formally, $\eta_i(s_i) = v_i(\eta_i(s_i), s_i, \eta_{-i})$.

**Proof.** As agents are playing pure strategies and the set of signals is finite, there exists a $\delta > 0$ such that no agent ever bids in the open interval $(\eta_i(s_i) - \delta, \eta_i(s_i) + \delta)$, except possibly at $\eta_i(s_i)$.

Assume in contradiction that $\eta_i(s_i) \neq v_i(\eta_i(s_i), s_i, \eta_{-i})$.

Let $\gamma \geq 0$ be the probability that agent $i$ with signal $s_i$ wins the item conditional on being tied with some agent $j$ and signal $s_j$ ($\eta_i(s_i) = \eta_j(s_j)$). If $\gamma > 0$, let $T$ be the expected value that agent $i$ with signal $s_i$ gets when bidding $\eta_i(s_i)$, conditional on being tied with some agent $j$ and signal $s_j$ ($\eta_i(s_i) = \eta_j(s_j)$, there is another bidding with an atom at the same bid).

If $\gamma > 0$ and $T > \eta_i(s_i)$ then for small enough $\beta \in (0, \delta)$ bidding $\eta_i(s_i) + \beta$ is a beneficial deviation as we prove next. Because of the tie breaking rule, a bid of $\eta_i(s_i) + \beta$ has probability at least $2\gamma$ to win items for which some other agent is bidding $\eta_i(s_i)$. Thus the bid change will cause the utility to increase by $\gamma(T - \eta_i(s_i)) - P(\beta)$, where $\gamma(T - \eta_i(s_i)) > 0$ and $P(\beta)$ is the utility loss caused by winning items and paying between $\eta_i(s_i)$ and $\eta_i(s_i) + \beta$ due to a bid...
by the random bidder. As \( P(\beta) \) tends to 0 when \( \beta \) goes to zero, the gain by such deviation is positive if \( \beta \) is small enough.

Similar arguments show that if \( \gamma > 0 \) and \( T < \eta_i(s_i) \) then for small enough \( \beta \in (0, \delta) \) bidding \( \eta_i(s_i) - \beta \) is a beneficial deviation (as the expected utility from winning conditional on being tied with some agent is negative, decreasing the winning probability from \( \gamma > 0 \) to 0 is beneficial).

Next consider the case that \( \gamma = 0 \), or \( \gamma > 0 \) and \( T = \eta_i(s_i) \). Let \( \zeta \) be the probability that agent \( i \) with signal \( s_i \) wins with the bid \( \eta_i(s_i) \), conditional on not being tied with another bidder. As agent \( i \) with signal \( s_i \) has positive expected utility from bidding \( \eta_i(s_i) \), it must be the case that \( \zeta > 0 \).

In the case that \( \eta_i(s_i) > v_i(\eta_i(s_i), s_i, \eta_{-i}) \) for any \( \beta \in (0, \min\{\delta, \eta_i(s_i) - v_i(\eta_i(s_i), s_i, \eta_{-i})\}) \) bidding \( \eta_i(s_i) - \beta \) is a beneficial deviation. This is so as for any item won for which the random bidder is bidding in the interval \( (\eta_i(s_i) - \beta, \eta_i(s_i)) \), agent \( i \) receives an item of expected value \( v_i(\eta_i(s_i), s_i, \eta_{-i}) \) (the expected value does not change on the interval \((\eta_i(s_i) - \delta, \eta_i(s_i))\), no other bidder ever bids in this interval) and he pays at least \( \eta_i(s_i) - \beta > v_i(\eta_i(s_i), s_i, \eta_{-i}) \), ending up with negative utility. As \( \zeta > 0 \) and the random bidder is bidding in the interval with positive probability, such a utility gain happens with positive probability.

In the case that \( \eta_i(s_i) < v_i(\eta_i(s_i), s_i, \eta_{-i}) \) for any \( \beta \in (0, \min\{\delta, \eta_i(s_i) - v_i(\eta_i(s_i), s_i, \eta_{-i})\}) \) bidding \( \eta_i(s_i) + \beta \) is a beneficial deviation. The argument is very similar to the argument presented in the former case and is omitted.

The following Lemma is an immediate corollary.

**Lemma 36.** Fix any game \( \lambda \) and an \((\epsilon, R)\)-tremble of the game \( \lambda(\epsilon, R) \). If \( \eta \) is a Nash Equilibrium in pure strategies of \( \lambda(\epsilon, R) \) then the following must hold. For every agent \( i \) and signal \( s_i \), if \( \eta_i(s_i) \) is agent \( i \)'s bid when we gets signal \( s_i \), and agent \( i \) has positive expected utility from such a bid, then \( \eta_i(s_i) \) must equal to agent \( i \)'s expected value conditional on winning, given his signal \( s_i \) and the strategies \( \eta \), formally, \( \eta_i(s_i) = v_i(\eta_i(s_i), s_i, \eta_{-i}) \).

Note that the lemma implies the same claim in the special case that \( \mu = \eta \) is a strong Tremble Robust Equilibrium in pure strategies.

We next show that if there are multiple bids for which the bid equals to the expected value conditional on winning then the highest is optimal.

**Lemma 37.** Consider any monotonic domain with multiple bidders and multiple signals each. Fix any game \( \lambda \) and an \((\epsilon, R)\)-tremble of the game \( \lambda(\epsilon, R) \). Let \( \mu \) be a profile of pure strategies. For \( \epsilon > 0 \) that is small enough the following holds:

For every agent \( i \) and signal \( s_i \), if \( \mu_i(s_i) \) is an optimal bid for agent \( i \) with signal \( s_i \) given \( \mu_{-i} \) in \( \lambda(\epsilon, R) \), and agent \( i \) has positive expected utility from such a bid, then \( \mu_i(s_i) = \sup\{b_i | b_i = v_i(b_i, s_i, \mu_{-i})\} \).

**Proof.** As \( \mu \) is a pure strategy profile there are only finitely many bids \( b_i \) such that \( b_i = v_i(b_i, s_i, \mu_{-i}) \). Thus the supremum is obtained. Assume in contradiction that \( \mu_i(s_i) < \max\{b_i | b_i = v_i(b_i, s_i, \mu_{-i})\} \). Thus for some bid \( b_i^+ > \mu_i(s_i) \) it holds that \( v_i(\mu_i(s_i), s_i, \mu_{-i}) = \mu_i(s_i) < b_i^+ = v_i(b_i^+, s_i, \mu_{-i}) \). We show that \( b_i^+ \) gives agent \( i \) with signal \( s_i \) higher utility than \( \mu_i(s_i) \), in contradiction to \( \mu_i(s_i) \) being an optimal bid.

The expected value of agent \( i \) with signal \( s_i \) and bid \( b_i^+ \) conditional on winning can be decompose to two: The first is the expected value conditional on the maximum of the others' agents bid (including the random bidder) being at most \( \mu_i(s_i) \). As the random bidder is bidding independent of the value, this equals to \( v_i(\mu_i(s_i), s_i, \mu_{-i}) \). This event happens with positive
probability as \( \mu_i(s_i) \) gives positive utility. The second is the expected value conditional on the maximum of the others’ bid (including the random bidder) being more than \( \mu_i(s_i) \) but at most \( b_i^+ \), denote that value as \( V \). As \( v_i(b_i^+, s_i, \mu_{-i}) > v_i(\mu_i(s_i), s_i, \mu_{-i}) \) the value \( v_i(b_i^+, s_i, \mu_{-i}) \) is a non-trivial convex combination of \( v_i(\mu_i(s_i), s_i, \mu_{-i}) \) and \( V \), and thus it must hold that \( v_i(b_i^+, s_i, \mu_{-i}) < V \). Note that \( v_i(b_i^+, s_i, \mu_{-i}) > v_i(\mu_i(s_i), s_i, \mu_{-i}) \) means that this event has positive probability of happening, and that probability that does not go to 0 as \( \epsilon \) goes to 0.

Now we can decompose the utility that \( i \) gets by bidding \( b_i^+ \) to three, conditional on three disjoint events in which \( i \) wins, agent \( i \)'s utility when winning is the sum of the three. First, the utility when no other agent, including the random bidder, is bidding more than \( \mu_i(s_i) \). The utility conditional on winning in this case is exactly the same as the utility when bidding \( \mu_i(s_i) \), and as \( b_i^+ > \mu_i(s_i) \) agent \( i \) wins with bid \( b_i^+ \) every time he was winning with \( \mu_i(s_i) \). Second, the utility when no other agent, excluding the random bidder, is bidding more than \( \mu_i(s_i) \) (and the random bidder is bidding in \( (\mu_i(s_i), b_i^+] \)). The probability of this event goes to 0 as \( \epsilon \) goes to zero, and so is its (negative) contribution to the utility of \( i \) conditional on winning with bid \( b_i^+ \). Finally, the utility when some other agent (not the random bidder) is bidding more than \( \mu_i(s_i) \) but at most \( b_i^+ \) (and the random bidder is bidding at most \( b_i^+ \)). In this case the utility is positive as items won have expected value \( V \) but the payment is at most \( v_i(b_i^+, s_i, \mu_{-i}) < V \). Moreover, this event happens with positive probability that does not diminish when \( \epsilon \) goes to 0. We conclude that the utility from bidding \( b_i^+ \) is the utility from bidding \( \mu_i(s_i) \) (the first case) plus two terms, one is negative but tends to 0 as \( \epsilon \) goes to 0, the other is positive and does not tend 0 as \( \epsilon \) goes to 0. We conclude that for small enough \( \epsilon \) the utility from bidding \( b_i^+ \) is larger than the utility from bidding \( \mu_i(s_i) \), a contradiction. \( \square \)

**Lemma 38.** Consider any monotonic domain with multiple bidders and multiple signals each. For such a domain the following holds. If a profile \( \mu \) satisfies the following conditions:

- it is a pure Nash Equilibrium in \( \lambda \), and
- for every agent \( i \) and signal \( s_i \) it holds that \( \mu_i(s_i) = v_i(\mu_i(s_i), s_i, \mu_{-i}) \) (bid equals the expected value conditional on winning),

then \( \mu \) is a strong Tremble Robust Equilibrium in pure strategies.

**Proof.** To prove that \( \mu \) is a strong Tremble Robust Equilibrium we need to show that for every \( \epsilon > 0 \) and a standard distribution \( R \), it is a Nash equilibrium in the \((\epsilon, R)\)-tremble of the game.

Fix a standard distribution \( R \) and \( \epsilon > 0 \) and let \( \lambda(\epsilon, R) \) be the \((\epsilon, R)\)-tremble of the game. Assuming that \( \mu \) is NE in \( \lambda \) we show that \( \mu \) is a Nash equilibrium in \( \lambda(\epsilon, R) \). To show this it is sufficient to show that for any agent \( i \) and signal \( s_i \), bidding \( \mu_i(s_i) \) is a best response to \( \mu_{-i} \) in \( \lambda(\epsilon, R) \).

Let \( U_i(c_i, s_i, \mu_{-i}) \) and \( Q_i(c_i, s_i, \mu_{-i}) \) be the expected utilities of agent \( i \) with signal \( s_i \) that is bidding \( c_i \), when the other agents are playing according to \( \mu_{-i} \), in \( \lambda \) and in \( \lambda(\epsilon, R) \), respectively.

Consider bidding \( b_i \neq \mu_i(s_i) \) and assume that bidding \( b_i \) is a beneficial deviation in \( \lambda(\epsilon, R) \), that is \( Q_i(b_i, s_i, \mu_{-i}) > Q_i(\mu_i(s_i), s_i, \mu_{-i}) \).

First assume that \( b_i > \mu_i(s_i) \). Any change to the utility of \( i \) must come from winning goods for which the bid of \( \mu_i(s_i) \) was losing yet the bid of \( b_i \) is winning. There must exist at least one agent \( j \) and signal \( s_j \) such that \( \mu_i(s_i) \leq \mu_j(s_j) \leq b_i \). If that was not the case, since \( \mu_i(s_i) = v_i(\mu_i(s_i), s_i, \mu_{-i}) \) and the random bidder’s bid is independent of the value, all additional items won would give negative utility (the expected value is \( v_i(\mu_i(s_i), s_i, \mu_{-i}) \) but payment is at least that).

If such a bid of \( b_i \) increases agent \( i \)'s utility compared to the bid of \( \mu_i(s_i) \) in \( \lambda(\epsilon, R) \), it means that the expected utility from winning goods conditional on some other agent \( j \) bidding
optimal bid but does not satisfy \( \mu_i(s_i) \) for \( \mu_i(s_i) \leq \mu_j(s_j) \leq b_i \) is positive, and the probability of such an event is positive. As the gain was positive even with the presence of the random bidder (that can only cause payment increase when winning, and winning such goods less often), removing the random bidder can only increase the expected utility of \( i \) conditional on winning such goods. Thus \( U_i(b_i, s_i, \mu_{-i}) - U_i(\mu_i(s_i), s_i, \mu_{-i}) \geq Q_i(b_i, s_i, \mu_{-i}) - Q_i(\mu_i(s_i), s_i, \mu_{-i}) > 0 \). We conclude that it must be the case that \( U_i(b_i, s_i, \mu_{-i}) > U_i(\mu_i(s_i), s_i, \mu_{-i}) \) a contradiction to \( \mu \) being a Nash equilibrium in \( \lambda \).

As we ruled out \( b_i > \mu_i(s_i) \) it must be the case that \( b_i < \mu_i(s_i) \). Assume that \( b_i \) is the optimal deviation of agent \( i \) with signal \( s_i \). As the bid \( \mu_i(s_i) \) must give non-negative utility to \( i \) (\( \mu \) is a NE and bidding 0 ensures non-negative utility), and \( b_i \) is a beneficial deviation, agent \( i \) receives positive utility by bid \( b_i \). Now, by Lemma \[35\] it holds that \( b_i = v_i(b_i, s_i, \mu_{-i}) \). Thus \( v_i(\mu_i(s_i), s_i, \mu_{-i}) = \mu_i(s_i) > b_i = v_i(b_i, s_i, \mu_{-i}) \). But this contradict Lemma \[37\] as \( b_i \) is an optimal bid but does not satisfy \( b_i = \sup\{c_i|c_i = v_i(c_i, s_i, \mu_{-i})\} \).

We are now ready to present the proof of Theorem \[11\]

Proof. (of Theorem \[11\]) By Lemma \[34\] the profile \( \mu \) defined by the procedure is indeed a pure Nash equilibrium of the SPA game. To show that \( \mu \) is a strong Tremble Robust Equilibrium it is sufficient to show that the rest of the conditions required by Lemma \[38\] are indeed satisfied. Indeed, for every agent \( i \) and signal \( s_i \) for which the bid \( \mu_i(s_i) \) was set at stage \( (2) \) it holds that \( \mu_i(s_i) = v_i(\mu_i(s_i), s_i, \mu_{-i}) \), by the way the procedure is defined. Additionally, by Observation \[33\] if the bid \( \mu_i(s_i) \) was set at stage \( (3) \) it holds that \( \mu_i(s_i) = v_i(\mu_i(s_i), s_i, \mu_{-i}) \) (as the utility if winning is 0).

C.2 Relation to the work of Einy et al. (2002)

Einy et al. (2002) study common value second price auction in domains that are connected. For connected domains Einy et al. consider the concept of sophisticated equilibrium, which makes successive rounds of dominated strategy eliminations. This process might result in multiple equilibria and that paper points out a single sophisticated equilibrium that Pareto-dominates the rest in terms of bidders resulting utilities, and it is also the only sophisticated equilibrium that guarantees every bidder non-negative utility. Moreover, this is the only sophisticated equilibrium that survives the elimination process if an uninformed bidder is added to the domain.

In this section we observe that Theorem \[11\] can be applied to any connected domain, as any such domain satisfies the low-signal-is-never-good-news and the high-signal-is-never-bad-news properties. Moreover, we observe that for connected domains the set of strategies picked by the procedure is exactly the one pointed out by Einy et al. (2002). Finally, we show that our procedure also works in some domains that are not connected. Some obvious such domains are monotonic domains in which the mapping from the state of the world to signals is not deterministic (yet they still satisfy the low-signal-is-never-good-news property), but we also present examples of domains in which the mapping is deterministic yet they are no connected and for which Theorem \[11\] applies.

Before formally presenting connected domains we present an example due to Einy et al. (2002) and the equilibrium that our procedure (as well as Einy et al.) pick for that domain.
Example 39. Assume that there are two buyers and four states of the world $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, with $v(\omega_i) = i$ and states all are equally probable ( $H(\omega_i) = 1/4$ for all $i \in \{1, 2, 3, 4\}$ ). If the state is $\omega_1$ then agent 1 gets the signal $L_1$, otherwise he gets $H_1$. If the state is $\omega_4$ then agent 2 gets the signal $H_2$, otherwise he gets $L_2$. The procedure would first set $\mu_2(H_2) = v(\{H_2\}, \{H_1, L_1\}) = 4$. Then it will set $\mu_1(H_1) = v(\{H_1\}, \{L_2\}) = 2.5$ and finally it will set $\mu_1(L_1) = \mu_2(L_2) = v(\{L_1\}, \{L_2\}) = 1$.

We next define connected domains.

Definition 40. A domain is a connected domain if the following hold. Each agent $i$ has a partition $\Pi_i$ of the state of nature and his signal is the element of the partition that include the realized state. The information partition $\Pi_i$ of bidder $i$ is connected (with respect to the common value $v$) if every $\pi_i \in \Pi_i$ has the property that, when $\omega_1, \omega_2 \in \pi_i$ and $v(\omega_1) \leq v(\omega_2)$ then every $\omega \in \Omega$ with $v(\omega_1) \leq v(\omega) \leq v(\omega_2)$ is necessarily in $\pi_i$. A common-value domain is connected (with respect to the common value) if for every agent $i$ his information partition $\Pi_i$ is connected.

Proposition 41. For every connected domain the outcome of our procedure is exactly the same as the unique sophisticated equilibrium picked by Einy et al. (2002) (the sophisticated equilibrium that survives the elimination process if an uninformed bidder is added to the domain).

Proof. Einy et al. show that unique sophisticated equilibrium that they pick can be computed as follows. One can look at $\Pi^*$, the coarsest partition of $\Omega$ that refines the partition $\Pi_j$ for every agent $j$. Let $\sigma$ denote an element of $\Pi^*$. Let $v(\sigma)$ denote the expected value of the item conditional on $\sigma$. An order over elements $\sigma_1, \sigma_2 \in \Pi^*$ is naturally defined by the order on the corresponding values $v(\sigma_1)$ and $v(\sigma_2)$. For agent $j$ with signal $\pi_j \in \Pi_j$ the bid is defined to be $\min_{\sigma \in \pi_j} v(\sigma)$.

We next show that for connected domain our procedure also sets the bid of $j$ with signal $\pi_j \in \Pi_j$ to be $\min_{\sigma \in \pi_j} v(\sigma)$. Let $S$ be the set of bids determined by the unique sophisticated equilibrium picked by Einy et al., that is $S = \{\min_{\sigma \in \pi_j} v(\sigma)\}_{j \in N, \pi_j \in \Pi_j}$. We prove by induction that if ties are consistently broken the same way in sorting $S$ and in picking signals in stage (2) of our procedure, then the $k$ highest bids in both are exactly the same (and correspond to the same signal). For signals for which the bid is set in stage (3) of our procedure the equivalence of the bids is trivial by the definition of our procedure.

For signal $\pi_j$ of agent $j$ denote the bid set by our procedure by $\mu_j(\pi_j)$. Map every signal $\pi_j$ of agent $j$ to the element $\sigma_{\pi_j} \in \Pi^*$ that determines its bid in $S$. Consider the multiset of such elements of $\Pi^*$ and sort them in the order of $v(\sigma_{\pi_j})$, breaking ties the same they are broken when sorting $S$. The claim that we prove by induction is that after $k$ steps, if $\sigma_{\pi_j}$ is the $t$-th highest elements of this multiset for $t \leq k$, then the $t$-th highest bid set by the procedure is $\mu_j(\pi_j) = v(\pi_j, T_{-j})$ and it is equal to $v(\sigma_{\pi_j})$. Moreover, the set of elements of $\Pi^*$ that are feasibly contained in the top $k$ signals for which the bid was already set, is exactly the set that contains the $k$ highest elements in $S$. We need to show that if the above holds after $k$ steps then it holds after $k + 1$ steps.

Recall that when the procedure is about to fix the $k + 1$ highest bid it has a set $T_i$ for each agent $i$, that set is the set of all signals of agent $i$ for which the bid was not set yet. Assume that when setting the $k + 1$ highest bid the procedure picks agent $i$ and signal $\pi_i$ such that $v(\pi_i, T_{-i})$ is maximal, when ties are broken in the same way as in sorting $S$. Assume that $\sigma_{\pi_j}$ is the $k + 1$ highest element of $S$. We need to show that $\pi_i = \pi_j$ and that $v(\pi_i, T_{-i}) = v(\sigma_{\pi_j})$. 

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We first observe that as the set of $k$ highest signals are the same for both cases, the maximization in both cases is over the same set of signals. In particular, $\pi_j$ is one of the signals our procedure considers in the maximization and clearly it holds that $v(\pi_j, T_{-j}) \geq v(\sigma_{\pi_j})$. This implies that to prove the claim it is sufficient to show that there does not exist an agent $l$ with signal $\pi_l \in T_j$ such that $v(\pi_l, T_{-l}) > v(\sigma_{\pi_l})$. Assume in contradiction that such a signal exists. It cannot be the case that $v(\pi_l, T_{-l}) = v(\sigma_{\pi_l})$ as $\sigma_{\pi_l}$ is not one of the $k$ highest elements in $S$ and it belongs to $S$ (but not maximal).

By the induction hypothesis the set of elements of $\Pi^*$ that are feasibly contained in the top $k$ signals for which the bid was already set, is exactly the set that contains the $k$ highest elements in $S$. This means that $v(\pi_l, T_{-l})$ is a convex combination of the values of $\sigma_{\pi_l}$ and at least one more consecutive and larger element of $\Pi^*$, call it $\sigma$. As $\Pi^*$ is the coarsest partition of $\Omega$ that refines the partition $\Pi_r$ for every agent $r$, there must exist an agent $r$ and signal $\pi_r$ such that $\sigma$ is the minimal element in $\pi_r$. But signal $\pi_r$ must have been of the $k$ highest signals (as $\sigma > \sigma_{\pi_l}$ and all elements larger than $\sigma_{\pi_l}$ where already removed from $S$), which contradict the claim that $(\pi_l, T_{-l})$ contains $\sigma$ as a feasible element. Finally, we observe that what we have proved implies that after the $k + 1$ step, the set of elements of $\Pi^*$ that are feasibly contained in the top $k + 1$ signals for which the bid was already set, is exactly the set that contains the $k + 1$ highest elements in $S$.

\[\Box\]

**Lemma 42.** For every connected domain the low-signal-is-never-good-news and the high-signal-is-never-bad-news properties hold.

**Proof.** By Proposition 41 and the result of Einy et al., for connected domains it holds that for agent $j$ with signal $\pi_j \in \Pi_j$ the bid is defined to $\mu_j(\pi_j) = \min_{\sigma \in \pi_j} v(\sigma)$. This means that no signal can convey any bad news, in particular, the high-signal-is-never-bad-news property trivially hold.

Assume that $\mu_j(\pi_j) = v(\pi_j, T_{-j})$ where the set $T_i$ is the set of signals for which the procedure did not set the bid of agent $i \neq j$ when the bid $\mu_j(\pi_j)$ was set. For any signal $\pi_k$ of agent $k$ such that $\mu_k(\pi_k) = \min_{\sigma \in \pi_k} v(\sigma)$ and $\mu_k(\pi_k) \leq \mu_j(\pi_j)$ (only such a signal $s_k$ can be a "low" signal), the induction in the proof of Proposition 41 shows that $v(\pi_j, T_{-j}) = v(\pi_j, \pi_k, T_{-(j,k)})$ (as all elements of $\Pi^*$ that are larger than $\mu_j(\pi_j)$ are covered by signals larger than $s_j$), thus the low-signal-is-never-good-news property hold.

\[\Box\]

We next show that our procedure is strictly more general than the one of Einy et al. (2002) by presenting two examples for domains that satisfy the required properties. For such domains Theorem 11 applies although they are not connected. We start with a simple example with only one informed bidder.

**Example 43.** Consider a domain with two buyers and three states of the world $\Omega = \{\omega_1, \omega_2, \omega_3\}$, with $v(\omega_1) = 0$, $v(\omega_2) = 4$, $v(\omega_3) = 10$ and all states are equally probable ($H(\omega_i) = 1/3$ for all $i \in \{1, 2, 3\}$). If the state is $\omega_1$ or $\omega_3$ then agent 1 gets the signal $H_1$, otherwise he gets $L_1$. Agent 2 is not informed at all. This example is covered by Theorem 11 and moreover it is covered by Theorem 3. Yet, this domain is not connected, as signal $H_1$ of agent 1 indicates that the state is $\omega_1$ or $\omega_3$ and does not include $\omega_2$. We also present an example with more than one informed bidder, in this example there are 2 agents and each has a binary signal.

**Example 44.** Assume that there are two buyers and four states of the world $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ with $v(\omega_1) = 0$, $v(\omega_2) = 4$, $v(\omega_3) = 6$, $v(\omega_4) = 10$, and all states are equally probable
(\(H(\omega_i) = 1/4\) for all \(i \in \{1, 2, 3, 4\}\). If the state is \(\omega_4\) then agent 1 gets the signal \(H_1\), otherwise he gets \(L_1\). If the state is \(\omega_1\) or \(\omega_3\) then agent 2 gets the signal \(L_2\), otherwise he gets \(H_2\). (note that this is not connected as \(\omega_2\) does not belong to \(L_2\)). The procedure would first set \(\mu_1(H_1) = v(\{H_1\}, \{H_2, L_2\}) = 10\). Then it will set \(\mu_2(H_2) = v(\{L_1\}, \{H_2\}) = 4\) and finally it will set \(\mu_1(L_1) = \mu_2(L_2) = v(\{L_1\}, \{L_2\}) = 3\).

Note that if we change the above example by replacing the value of \(v(\omega_2)\) from 4 to 2, then the bid for signal \(L_2\) would be set by our procedure before the bid for \(H_2\).

While Example 43 presents a very simple domain that is not connected, it is clear that there exists a different representation of the states of the world for which a domain with exactly the same signal structure and posteriors, is indeed connected. In this new representation each state corresponds to one of the informed agent’s signals and the value corresponds to the posterior value given that signal. That is, we can define \(\Omega' = \{\omega'_1, \omega'_2\}\), with \(v(\omega'_1) = 5\), \(v(\omega'_2) = 4\), and the probabilities are \(H(\omega'_1) = 2/3\) and \(H(\omega'_2) = 1/3\). If the state is \(\omega'_1\) then agent 1 gets the signal \(H_1\), otherwise he gets \(L_1\). Agent 2 is not informed at all. Clearly under the new representation the domain is connected, and the domain is equivalent to the original domain.

One might wonder if any domain that satisfies the low-signal-is-never-good-news property can be transformed to an equivalent connect domain. We next show that this is not the case, presenting a domain that satisfies the property and cannot be represented by a connect domain. This shows that Theorem 11 applies to domains that do not have a representation as connected domains.

The domain we consider is the domain presented in Example 44, with \(v(\omega_2)\) assigned a value of 2 instead of 4. Clearly in a connected domain that is equivalent to that domain it must be the case that signals \(H_1\) and \(H_2\) are both received for some subset of states of the world such that for each such state the value is at least as high as the value if signal \(H_1\) is not received. Now connectivity for \(H_2\) implies that \(v(L_1) \geq v(L_2)\) which does not hold for the domain we are considering.

C.3 Generalizing "Lemons and Peaches" to \(n\) agents

We next restate Proposition 17 and prove it.

**Proposition 45.** Consider the strong TRE that is the outcome of the procedure defined in Section 5.2 for the above connected domain.

If there exists an agent \(i\) such that \(i\) is \(\epsilon_i\)-informed about lemons and every agent \(j < i\) is \(\epsilon_j\)-informed about peaches, then the revenue of the SPA is at most

\[
2 \sum_{j=1}^{i} \epsilon_j
\]

In particular, if \(\sum_{j=1}^{i} \epsilon_j\) tends to 0 the revenue will tend to 0.

**Proof.** A simple induction shows that since each \(j < i\) is \(\epsilon_j\)-informed about peaches it holds that

\[
Pr[L_1, L_2, \ldots, L_{i-1}] > 1 - \sum_{j=1}^{i-1} \epsilon_j
\]
Now, since \( i \) is \( \epsilon_i \)-informed about lemons it holds that \( \Pr[L_1, L_2, \ldots, L_{i-1}, L_i] < \epsilon_i \), and thus

\[
\Pr[L_1, L_2, \ldots, L_{i-1}, H_i] = \Pr[L_1, L_2, \ldots, L_{i-1}] - \Pr[L_1, L_2, \ldots, L_{i-1}, L_i] > 1 - \sum_{j=1}^{i} \epsilon_j
\]

The revenue obtained when the signals of agents 1, 2, \ldots, \( i \) are not realized to \( L_1, L_2, \ldots, L_{i-1}, H_i \) is at most 1, and that happens with probability at most \( \sum_{j=1}^{i} \epsilon_j \).

The revenue obtained when the signals of agents 1, 2, \ldots, \( i \) are realized to \( L_1, L_2, \ldots, L_{i-1}, H_i \) is bounded as follows. If \( i = n \) then the payment is set by a bid of an agent \( j < i \) for some signal \( L_j \), such a bid must be equal to \( v(L_1, \ldots, L_n) \). This value is bounded from above by \( \sup\{v(\omega)|\omega \text{ implies } L_1, L_2, \ldots, L_{i-1}, L_i\} \), and as agent \( i \) is \( \epsilon_i \)-informed about lemons this is at most \( \epsilon_i \).

If on the other hand \( i < n \), the payment of \( i \) is bounded by maximum of any bid of any of the agents \( i+1, i+2, \ldots, n \), for any of their signals. These bids are all bounded from above by \( \sup\{v(\omega)|\omega \text{ implies } L_1, L_2, \ldots, L_{i-1}, L_i\} \). As agent \( i \) is \( \epsilon_i \)-informed about lemons this is at most \( \epsilon_i \). We conclude that the revenue is bounded from above by

\[
\sum_{j=1}^{i} \epsilon_j + \epsilon_i (1 - \sum_{j=1}^{i} \epsilon_j) < \sum_{j=1}^{i} \epsilon_j + \epsilon_i < 2 \sum_{j=1}^{i} \epsilon_j
\]

\[\square\]

### D Two Agents, Each with a Binary Signal

#### D.1 Proof of Lemma [22]

Let \( G \) be a distribution function. We say that \( G \) has an \textit{atom} at \( b \) if \( b = 0 \) and \( G(0) > 0 \), or if \( b > 0 \) and \( G \) is discontinuous at \( b \). We define \( G^-(b) = \sup_{x \leq b} G(x) \). We say that a bid \( b \) of bidder \( j \) is \textit{optimal} (or is in the \textit{support}) if the utility from that bid (given the other agent’s strategy and the random bidder) is at least as high as with any other bid.

In this section we use \( i \) to denote a bidder, either bidder 1 or 2. When we want to refer to the other bidder we use \( j \) to denote that bidder, and assume that \( j \neq i \).

Let \( R \) be a standard distribution and fix some \( \epsilon > 0 \). A NE \( \eta \) of the \((\epsilon, R)\)-tremble of the game \( \lambda \) consists of four bid distributions, one for each bidder for each signal he may receive. Thus \( \eta = (G_1, G^L_1, G_2, G^L_2) \) where \( G_1 \) and \( G^L_1 \) are the bid distributions when \( i \in \{1, 2\} \) gets the signals \( H_i \) and \( L_i \), respectively. In \( \eta \), for bidder \( i \) with signal \( H_i \): let \( \Pi_i(b_i) \) denote the utility (profit) of bidder \( i \) when he bids \( b_i \), and let \( v^{\text{win}}(b_i) \) denote the expected value of the items \( i \) gets, conditional on winning, when he bids \( b_i \).

To simplify the notation we denote \( v_1 = v(H_1, L_2) \) and \( v_2 = v(L_1, H_2) \). We assume that

\[0 < \Pr[H_1, L_2](1 - v_1) \leq \Pr[L_1, H_2](1 - v_2) < 1, \text{ and that in case of equality } v_1 \geq v_2.\]

Note that this implies that \( \min\{\Pr[H_1, L_2], \Pr[L_1, H_2]\} > 0 \).

We first show that if bidders never submit dominated bids bidder \( i \in \{1, 2\} \) that receives signal \( L_i \) must bid \( v(L_1, L_2) = 0 \).

**Lemma 46.** At \( \eta \) the following must hold. For each bidder \( i \in \{1, 2\} \) it holds that \( G^L_i(0) = 1 \). That is, bidder \( i \) with signal \( L_i \) always bids \( v(L_1, L_2) = 0 \). Also, it holds that \( G^-(v_1) = \sup_{b < v_1} G_i(b) = 0 \). That is, bidder \( i \) with signal \( H_i \) always bids at least \( v_1 \).
**Proof.** By assumption, bidders do not make weakly dominated bids. Therefore, bidder $i$ bids at least $0$ given signal $L_i$ and at least $v(H_i, L_j)$ given signal $H_i$. Similarly, bidder $i$ bids no more than $v(L_i, H_j)$ given signal $L_i$ and no more than $1$ given signal $H_i$. Bidder 1 with signal $L_1$ cannot bid $b \in (0, v_2)$ because she would only win when bidder 2 has a low signal and the value is zero but she would pay a positive amount due to the random bidder. Increasing the bid to $v_2$ incurs the same losses conditional on $L_2$ as bidding just below $v_2$ and earns zero conditional on $H_2$ because any wins are priced at their value $v_2$. Therefore bidder 1 must bid $0$ given a low signal, and the same is true for bidder 2 by similar logic. □

Given this lemma we focus in the rest of the proof at the bidding of each bidder $i$ given his high signal $H_i$. (e.g. if we say that some bid "is optimal for $i$" we mean to say that this bid "is optimal for $i$ with signal $H_i".

**Lemma 47.** At $\eta$ the following must hold. Assume that $G_j$ is discontinuous at $b < 1$ (has an atom at $b$), then $\exists \delta > 0$ such that bidding in the interval $(b - \delta, b]$ is not optimal for $i$ as it is dominated by bidding $b + \delta$.

**Proof.** Let $\Delta$ be the discrete increase in $G_j$ at $b$. For $\delta > 0$ small enough bidding $b + \delta$ is strictly better than bidding in $(b - \delta, b]$ as the probability of winning increases by at least $\Delta/2$ (moving from $b$ to $b + \delta$ means always winning against the atom instead of tie-breaking), while the increase in payment when winning low value items tends to $0$ as $\delta$ go to zero (as the random bidder is bidding continuously). □

Let $b^-$ and $b^+$ be two bids such that $0 \leq b^- < b^+ \leq 1$.

For $b > 0$ define $G_j^-(b)$ as the left hand limit of $G_j$ evaluated at $b$:

$$G_j^-(b) = \sup_{x < b} G_j(x).$$

**Lemma 48.** At $\eta$ the following must hold. For every bidder $j$ the expected value of the items he gets, conditional on winning, is monotonic in his bid. That is, if $v_j^{\text{win}}(b)$ is the expected value of the items $j$ gets, conditional on winning, with bid $b$, then $v_j^{\text{win}}(b)$ is non-decreasing in $b$.

**Proof.** If $i$ is bidding an atom of size $\Delta_i(b)$ at $b$ it holds that $\Delta_i(b) = G_i(b) - G_i^-(b)$. If bidder $j$ is bidding $b > 0$, then $j$’s expected value conditional on winning $v_j^{\text{win}}(b)$ can be computed by separating the case that $i$ bids below $b$, and the case that $i$ is bidding at $b$:

$$v_j^{\text{win}}(b) = \frac{\Pr[H_i \mid H_j] (G_i^-(b) + \frac{1}{2} \Delta_i(b)) + \Pr[L_i \mid H_j] v_j}{\Pr[H_i \mid H_j] (G_i^-(b) + \frac{1}{2} \Delta_i(b)) + \Pr[L_i \mid H_j]},$$

where the factor half comes from tie breaking in case both are bidding at $b$. Note that the first term is non-decreasing in $b$ because it is increasing in $G_i^-(b)$ and $G_i^-(b)$ is non-decreasing in $b$. Moreover, any increase in the bid will make sure the bidder always wins against the atom at $b$, instead of only half of the time. □

The next lemma shows than an optimal bid $b$ for bidder $j$ must be at least the expected value of the item $j$ wins, conditional on winning.

**Lemma 49.** At $\eta$ the following must hold. If $b \in [0, 1)$ is an optimal bid of bidder $j$ then $b \geq v_j^{\text{win}}(b)$. 48
Proof. If \( v^\text{win}_j(b) = v_j \) the claim follows from \( b \geq v_j \) (Lemma 46).

Now assume in contradiction that \( b < v^\text{win}_j(b) \leq 1 \) and that \( v^\text{win}_j(b) > v_j \). It must hold that \( G_i(b) > 0 \), since \( G_i(b) = 0 \) implies \( v^\text{win}_j(b) = v_j \). If \( i \) has an atom at \( b < 1 \) then \( b \) is not optimal for \( j \) by Lemma 47, contradicting our assumption that \( b \) is optimal for \( j \). Thus, bidder \( i \) does not have an atom at \( b \).

We show that for \( \delta > 0 \) that is small enough \( \delta < v^\text{win}_j(b) - b \), bidding \( b + \delta \) gives higher utility. To show that the bid \( b + \delta \) gives higher utility than \( b \), we consider the difference in utility due to such an increase in the bid. There are two cases: first, if bidder \( j \) wins with \( b + \delta \) but would have lost with \( b \) due to a bid of \( i \) in \((b, b+\delta)\) then \( j \) wins an item of value 1 and pays at most \( b + \delta < 1 \), having positive utility. Second, if \( i \) was not bidding in \((b, b+\delta)\) but the random bidders does, by bidding \( b + \delta \) bidder \( j \) is now winning items with expected value at least \( v^\text{win}_j(b) \) (by Lemma 48) is non-decreasing) and paying at most \( b + \delta \). As \( \delta < v^\text{win}_j(b) - b \) the expected value from such a win is positive. Moreover, this second event happens with strictly positive probability because the random bidder is bidding continuously over \([0,1]\) and given no atoms of bidder \( i \) at \( b > 0 \) bidder \( i \) bids less than \( b \) with probability \( \Pr[L_i|H_i] + \Pr[H_i|H_i]G_i(b) > 0 \). We conclude that such an increase in bid strictly increases the utility.

\[ \text{Lemma 50. At } \eta \text{ the following must hold. For } 1 \geq b^+ > b^- \geq 0 \text{ suppose that } G_j(b^-) = G_j^-(b^+) \text{ (} j \text{ does not bid on } (b^-, b^+)) \text{. Let } \Gamma = G_j(b^-). \text{ Let} \]

\[ b^*_i(\Gamma) = \frac{\Pr [H_j|H_i] \Gamma + \Pr [L_j|H_i] v_i}{\Pr [H_j|H_i] \Gamma + \Pr [L_j|H_i]} \]

If \( b^*_i(\Gamma) \in (b^-, b^+] \) then \( b^*_i(\Gamma) \) strictly dominates any other bid by \( i \) in \((b^-, b^+]. \) If \( b^*_i(\Gamma) > b^+ \), then \( i \)'s payoff is strictly increasing in \( b \) over \((b^-, b^+]. \) If \( b^*_i(\Gamma) \leq b^- \), then \( i \)'s payoff is strictly decreasing in \( b \) over \((b^-, b^+]. \)

Proof. \( G_j(b_j) \) is constant over \((b^-, b^+] \) and thus \( g_j(b_j) = 0 \) over \((b^-, b^+] \). Therefore \( \Pi_i(b_i) \) is continuous and differentiable on \((b^-, b^+] \). Moreover, since \( g_j(b_j) \) is zero, the derivative is

\[ \frac{d\Pi_i(b_i)}{db_i} = \hat{r}(b_i)(\Pr [H_j|H_i] \Gamma (1 - b_i) + \Pr [L_j|H_i] (v_i - b_i)) \]

As we assume that \( 1 > \Pr [L_j|H_i] > 0 \) and it holds that \( \hat{r}(b_i) > 0 \), \( \Pr [H_j|H_i] \geq 0 \), this function of \( b_i \) is not identically 0. The function has a unique 0 at \( b^* (\Gamma) \), it is positive for \( b_i > b^* (\Gamma) \), and it is negative for \( b_i > b^* (\Gamma) \). Thus if \( i \) has an optimal bid in the interval \((b^-, b^+] \) it can only be at \( b^* (\Gamma) \). To extend the result to the interval \((b^-, b^+] \) consider four cases:

(1) \( b^* (\Gamma) \in (b^-, b^+] \): In this case, \( b^* (\Gamma) \) is the unique best bid within the interval \((b^-, b^+] \). At either endpoint \( b^- \) or \( b^+ \), either \( j \) bids an atom and the corresponding endpoint cannot be an optimal bid (Lemma 47) or \( j \) does not bid an atom and \( \Pi_i(b_i) \) is continuous at that point, meaning it is dominated by the interior bid \( b^*_i(\Gamma) \). In either case, \( b^*_i(\Gamma) \) is the only possible optimal bid within \((b^-, b^+] \).

(2) \( b^* (\Gamma) \leq b^- \): In this case, \( d\Pi_i(b_i)/db_i < 0 \) for \((b^-, b^+] \) and there is no optimal bid in \((b^-, b^+] \). If \( j \) bids an atom at \( b^+ \) then \( b^+ \) is not an optimal bid for \( i \) by Lemma 47. If \( j \) does not bid an atom at \( b^+ \), then \( \Pi_i(b_i) \) is continuous from the left at \( b^+ \). Therefore \( \Pi_i(b_i) \) is strictly lower at \( b^+ \) than at any other \( b_i \in (b^-, b^+] \). In either case there is no optimal bid within \((b^-, b^+] \), a contradiction.

(3) \( b^* (\Gamma) > b^+ \): Lemmas 48 and 49 imply that any optimal bid \( b_i > b^- \) must be at least \( b^*_i(\Gamma) \) because \( v^\text{win}_i(b_i) = b^*_i(\Gamma) \) for all \( b_i \in (b^-, b^+] \). Therefore there is no optimal bid at or below \( b^+ \), a contradiction.

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(4) \( b^*(\Gamma) = b^+ \). In this case, \( d\Pi_i(b_i)/db_i > 0 \) for \((b^-, b^+)\). Therefore \( \Pi_i(b_i) \) is strictly higher at \( b^+ \) than at any \( b_i \in (b^-, b^+) \) because \( \Pi_i(b_i) \) is either continuous at \( b^+ \) or increases discretely at \( b^+ \) (depending on whether or not \( j \) has an atom at \( b^+ \)). Therefore \( b^*(\Gamma) = b^+ \) is the only possible optimal bid for \( i \) in the interval \((b^-, b^+)\).

\[ \square \]

**Corollary 51.** At \( \eta \) the following must hold. If bidder \( i \in \{1, 2\} \) bids an atom at \( b \in [0, 1] \), then \( b = b_i^*(G_j(b)) \).

**Proof.** Lemma 47 implies that \( j \) does not bid in the interval \((b - \delta, b)\) for some \( \delta > 0 \). Therefore Lemma 50 implies the result.

Define \( \bar{b}_i \) to be the infimum bid by \( i \in \{1, 2\} \), \( \bar{b}_i = \inf\{b : G_i(b) > 0\} \). Let \( \bar{b} = \min\{\bar{b}_1, \bar{b}_2\} \) be the infimum of all bids of any bidder with a high signal. Let \( b_{\min} = \max\{\bar{b}_1, \bar{b}_2\} \).

**Corollary 52.** At \( \eta \) the following must hold. Suppose that \( j \in \{1, 2\} \) has an optimal bid \( b \) at or below \( \bar{b}_j \). Then \( b = v_j \).

**Proof.** Note that \( G_j(b) = 0 \) because Lemma 47 implies that \( i \) does not have an atom at \( \bar{b}_i \) if \( b = \bar{b}_i \). Thus, as \( i \) does not bid strictly below \( \bar{b}_i \) but \( j \) has an optimal bid \( b \) weakly below \( \bar{b}_i \), Lemma 50 implies \( b = b_j^*(0) = v_j \).

**Lemma 53.** At \( \eta \) the following must hold. Assume that both bids \( b^- \geq 0 \) and \( b^+ > b^- \) are optimal bids for bidder \( i \in \{1, 2\} \). Then for bidder \( j \neq i \) it holds that \( G_j(b^+) > G_j(b^-) \).

**Proof.** Assume in contradiction that both bids \( b^+ \) and \( b^- < b^+ \) are optimal bids for bidder \( i \), while \( G_j(b^+) = G_j(b^-) = \Gamma \) (note that \( G_j \) is non decreasing thus \( G_j(b^+) \geq G_j(b^-) \)). Because \( G_j(b) \) is constant over \([b^-, b^+]\), \( \Pi_i(b_i) \) is continuous and differentiable on \((b^-, b^+)\) and is continuous from the left at \( b^+ \). By Lemma 47 the fact that \( b^- \) is an optimal bid for \( i \) implies that \( j \) does not bid an atom at \( b^- \). Therefore there is no tie-breaking at \( b^- \) that would be resolved by bidding slightly more than \( b^- \) and \( \Pi_i(b_i) \) is also continuous from the right at \( b^- \). Following the argument in the proof of Lemma 50, within the interval \((b^-, b^+)\), \( d\Pi_i(b_i)/db_i \) is zero at \( b^*(\Gamma) \), strictly positive for \( b_i > b^*(\Gamma) \), and strictly negative for \( b_i > b^*(\Gamma) \). There are three cases: (1) \( b^*(\Gamma) \leq b^- \). Then \( d\Pi_i(b_i)/db_i < 0 \) for \((b^-, b^+)\) and, by right-continuity at \( b^- \) and left-continuity at \( b^+ \), \( \Pi_i(b_i) < \Pi_i(b^-) \) contradicting optimality of \( b^+ \). (2) \( b^*(\Gamma) \geq b^+ \). Then by a symmetric argument \( b^- \) cannot be optimal. (3) \( b^*(\Gamma) \in (b^-, b^+) \). Then by similar argument both \( b^- \) and \( b^+ \) are strictly dominated by \( b^*(\Gamma) \).

\[ \square \]

**Lemma 54.** At \( \eta \) the following must hold.

1. Suppose both bidders have the same infimum bid: \( \bar{b}_i = \bar{b}_j = \bar{b} = b_{\min} \). Then \( \bar{b} = \max\{v_i, v_j\} \). If \( v_i = v_j \), then neither bidder bids an atom at \( \bar{b} \) (that is, \( G_j(\bar{b}) = G_i(\bar{b}) = 0 \)). However, if \( v_i < v_j \) then \( j \) bids an atom at \( \bar{b} = v_j \) and \( i \) does not bid at \( \bar{b} \).

2. Suppose bidder \( i \) has a strictly higher infimum bid: \( \bar{b}_i > \bar{b}_j \). Then \( \bar{b} = \bar{b}_j = v_j \) and \( j \) bids an atom with some positive weight \( \Gamma > 0 \) at \( v_j \) but nowhere else at or below \( \bar{b}_i \):

\[
G_j(b) = \begin{cases} 
0 & b < v_j \\
\Gamma & b \in [v_j, \bar{b}_i] 
\end{cases}
\]

Moreover, \( b_{\min} = \bar{b}_i > v_i \).
Proof. It cannot be the case that both bidders have an atom at \( b \). Suppose \( i \) does not have an atom at \( b \). Then \( \Pi_j(b) \) is continuous at \( b \) and therefore \( b \) is an optimal bid for \( j \). (\( b_j = b \) implies that \( j \) bids with positive probability at \( b \) or in every neighborhood above \( b \).) Because \( j \) has an optimal bid at \( b \), Corollary \( \ref{corollary:52} \) implies that \( b_i = v_j \). Moreover, \( b_i \geq v_i \) by Lemma \( \ref{lemma:46} \). Therefore \( v_i \leq v_j \) and \( b = \max\{v_i, v_j\} \).

Suppose that \( v_i < v_j \) and \( j \) does not bid an atom at \( b \). Then \( \Pi_i(b) \) is continuous at \( b \) and hence \( b \) is an optimal bid for \( i \) and Corollary \( \ref{corollary:52} \) implies \( b = v_i \), which is a contradiction. Thus \( v_i < v_j \) implies \( j \) has an atom at \( b \). (Hence by Lemma \( \ref{lemma:47} \) \( i \) does not bid at \( b \).)

Suppose that \( v_i = v_j \) and \( j \) has an atom of weight \( \Gamma > 0 \) at \( b \). Then by Lemma \( \ref{lemma:49} \) bidder \( i \) must bid at least \( v_i^{\text{win}}(b) > v_i \), which is a contradiction. Thus \( v_i = v_j \) implies neither bidder has an atom at \( b \).

2) The assumption \( b_j < b_i \) implies that \( j \) bids with some positive probability \( \Gamma > 0 \) below \( b_j \). By Corollary \( \ref{corollary:52} \) \( j \) can only bid below \( b_j \) at \( v_j \). Therefore \( j \) bids with atom \( \Gamma \) at \( b_j = v_j \) and nowhere else below \( b_j \). Moreover, Lemma \( \ref{lemma:49} \) implies that for all bids \( b \geq b_j \), bidder \( i \) must bid at least \( v_i^{\text{win}}(b_j) = b_i^*(\Gamma) > v_i \).

Lemma \( \ref{lemma:55} \). If for all \( \delta > 0 \), bidder \( i \) has an optimal bid in the interval \( (b - \delta, b] \) then \( b \) is an optimal bid for \( i \).

Proof. By Lemma \( \ref{lemma:47} \) \( j \) does not have an atom at \( b \) and hence \( \Pi_i(b_i) \) is continuous from the left at \( b_i = b \). Since \( i \) has an optimal bid at \( b \) or arbitrarily close to \( b \), continuity implies that \( b \) must be an optimal bid.

Suppose that bidder \( j \) has an atom at \( b > 0 \). By Lemma \( \ref{lemma:47} \) bidder \( i \) does not bid in \( (b - \delta, b] \) for some \( \delta > 0 \). Define \( x_i(b) \) to be the supremum point below \( b \) at which bidder \( i \) does place a bid

\[
x_i(b) = \sup \{ x : G_i(x) < G_i(b) \} = \inf \{ x : G_i(x) = G_i(b) \}.
\]

Lemma \( \ref{lemma:47} \) implies \( x_i(b) < b \). Similarly, let

\[
x_j(b) = \inf \{ x : G_j(x) = G_j^{-1}(b) \}.
\]

Note that if \( i \) does not bid below \( b \) (\( b_i \geq o \) then \( x_i(b) = -\infty \).

Our goal is to prove that if \( j \) has an atom at \( b \) then \( b \) is \( j \)'s infimum bid. We first prove some helpful claims.

Lemma \( \ref{lemma:56} \). If \( j \) has an atom at \( b > 0 \) and \( b \) is not \( j \)'s infimum bid \((0 \leq b_j < b) \) then:

1. It holds that \( v_j \leq x_j(b) < x_i(b) < b \).
2. In the interval \((x_j(b), b], i \) bids an atom at \( x_i(b) = b_i^*(G_j(x_i(b))) \) but nowhere else.
3. \( j \) bids with an atom at \( x_j(b) = b_j^*(G_i(x_j(b))) \).
4. \( b = b_j^*(G_i(b)) \).

Proof. We prove the claims:

1. We prove that \( v_j \leq x_j(b) < x_i(b) < b \):
   
   \( x_j(b) \geq v_j \): By assumption \( (b_j < b) \) bidder \( j \) bids with positive probability below \( b \). Such bids must be at least \( v_j \).
• $x_i(b) < b$: follows from Lemma 47

• $x_j(b) < x_i(b)$: suppose not and $x_i(b) \leq x_j(b) < b$. There are two cases. (i) First, if $x_j(b) > x_i(b)$, then there exists some bid $b^- \in [x_i(b), b)$ where $j$ bids. Then by Lemma 53 $G_j(b^-) < G_i(b)$ which contradicts $G_j(x_i(b)) = G_i(b)$ and $x_i(b) < b^- < b$. (ii) Second, if $x_j(b) = x_i(b)$ then by Lemma 55, $x_i(b)$ is an optimal bid for $j$. Then by Lemma 53 $G_j(x_i(b)) < G_i(b)$ which contradicts $G_j(x_i(b)) = G_i(b)$.

2. By part (1) and definition of $x_i(b)$, $j$ does not bid with positive probability in the interval $(x_j(b), b)$ but $i$ does. As a result, Lemma 50 implies part (2).

3. There are two cases, either $b_j = x_i(b)$ or $b_j < x_i(b)$. (i) By part (1), $j$ bids with positive probability below $x_i(b)$. Therefore, if bidder $i$’s infimum bid is at $b_j = x_i(b)$, Lemma 54 implies that $j$ bids with an atom at $x_j(b) = v_j$. (ii) bidder $i$ bids with positive probability below $x_i(b)$ and $b_j < x_i(b)$. Parts (1) and (2) of the Lemma can be applied to the atom at $x_i(b)$ and these imply that $j$ bids with an atom at $x_j(b) = b_j^*(G_i(x_j(b))) > v_i$.

4. Since $i$ does not have an atom at $b$ (Lemma 47), $v_j^{\mathrm{min}}(b) = b_j^*(G_i(b))$. Therefore Lemma 49 implies that $b \geq b_j^*(G_i(b))$. Thus it is sufficient to show $b \leq b_j^*(G_i(b))$. Suppose not and $b > b_j^*(G_i(b))$. Within the interval $(x_j(b), b)$, the proof of Lemma 50 implies $\Pi_j$ is strictly increasing as the bid is moved towards $b_j^*(G_i(b))$ from above or below. Since $i$ does not have an atom at $b$, $\Pi_j$ is left-continuous at $b$. Thus if $b > b_j^*(G_i(b))$, $b$ could not be optimal for $j$ since it would be dominated by bidding $b - \delta$ for some $\delta > 0$.

Lemma 57. If $j \in \{1,2\}$ has an atom at $b$ then $b$ is $j$’s infimum bid: $b = b_j$.

Proof. Suppose not and $j$ bids with positive probability in a neighborhood of $b^- < b$. Then by Lemma 56 $j$ bids with an atom at $x_j(b) = b_j^*(G_i(x_j(b)))$, $i$ bids with an atom at $x_i(b) \in (x_j(b), b)$, $b = b_j^*(G_i(b))$, and there are no other bids in the interval $(x_j(b), b)$. We will show a contradiction by showing that $\Pi_j(b) > \Pi_j(x_j(b))$. Let $\Gamma_1 = G_i(x_j(b))$ and $\Gamma_2 = G_i(x_i(b)) = G_i(b)$.

Let $\Pi_j^-$ and $\Pi_j^+$ be the left and right hand limits of $\Pi_j$ respectively. I will write down the difference in profit between bidding at $x_j(b)$ and $b$ for bidder $j$ in three parts corresponding to $\Pi_j^-(x_j(b)) - \Pi_j(x_j(b))$, $\Pi_j^+(x_j(b)) - \Pi_j^-(x_j(b))$, and $\Pi_j(b) - \Pi_j^+(x_j(b))$:

$$\Pi_j(b) - \Pi_j(x_j(b)) = (G_i(x_j(b)) \Pr[H_i|H_j] + Pr[L_i|H_j]) \int_{x_j(b)}^{x_i(b)} (b_j^*(\Gamma_1) - t) \hat{r}(t) dt + \Pr[H_i|H_j] (G_i(x_i(b)) - G_i(x_j(b))) \hat{R}(x_i(b)) (1 - x_i(b)) + (G_i(x_i(b)) \Pr[H_i|H_j] + Pr[L_i|H_j]) \int_{x_i(b)}^{b} (b_j^*(\Gamma_2) - t) \hat{r}(t) dt$$

The third term $\Pi_j(b) - \Pi_j(x_j(b))$ is positive since $b = b_j^*(\Gamma_2)$ implies the following integral is positive:

$$\int_{x_i(b)}^{b} (b_j^*(\Gamma_1) - t) \hat{r}(t) dt = \int_{x_i(b)}^{b} (b - t) \hat{r}(t) dt > 0. \quad (7)$$

The fact that $b_j^*(\Gamma_1) = x_j(b)$ provides a lower bound to the integral in the first term:

$$\int_{x_j(b)}^{x_i(b)} (b_j^*(\Gamma_1) - t) \hat{r}(t) dt \geq - \left(\hat{R}(x_i(b)) - \hat{R}(x_j(b))\right) (x_i(b) - x_j(b)) \geq -\hat{R}(x_i(b)) (x_i(b) - x_j(b)). \quad (8)$$

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The inequalities in equations (7) and (8) imply that
\[
\Pi_j(b) - \Pi_j(x_j(b)) > -(G_i(x_j(b)) \Pr [H_i|H_j] + \Pr [L_i|H_j]) \hat{R}(x_j(b)) (x_i(b) - x_j(b)) \tag{9}
\]
\[
+ \Pr [H_i|H_j] (G_i(x_i(b)) - G_i(x_j(b))) \hat{R}(x_j(b)) (1 - x_i(b)) \tag{10}
\]
Substituting \(x_j(b) = b^*_j(G_i(x_j(b))) = \frac{G_i(x_j(b)) \Pr [H_i|H_j] + \Pr [L_i|H_j] v_j}{G_i(x_j(b)) \Pr [H_i|H_j] + \Pr [L_i|H_j]} \) into the right-hand side of equation (10) and canceling and regrouping terms gives
\[
\hat{R}(x_i(b)) (G_i(x_i(b)) \Pr [H_i|H_j] + \Pr [L_i|H_j]) \left( G_i(x_i(b)) \Pr [H_i|H_j] + \Pr [L_i|H_j] v_j \right) - x_i(b)
\]
Finally, since \(G_i(x_i(b)) = G_i(b)\) and \(b = b^*_j(G_i(b))\) we can substitute in \(b = \frac{G_i(x_i(b)) \Pr [H_i|H_j] + \Pr [L_i|H_j] v_j}{G_i(x_i(b)) \Pr [H_i|H_j] + \Pr [L_i|H_j]}\)
yielding
\[
\hat{R}(x_i(b)) (G_i(x_i(b)) \Pr [H_i|H_j] + \Pr [L_i|H_j]) (b - x_i(b)),
\]
which is positive since \(b > x_i(b)\). Thus \(\Pi_j(b) - \Pi_j(x_j(b)) > 0\).

Recall the definition \(b_{min} = \max\{b_1, b_2\}\). In addition, define \(\bar{b}_i = \inf\{x : G_i(x) = 1\}\) and \(\bar{b}_j = \inf\{x : G_j(x) = 1\}\). Finally, define \(b_{max} = \max\{\bar{b}_1, \bar{b}_2\}\). Notice that \(b_{max} \geq b_{min}\).

**Lemma 58.** At \(\eta\) the following must hold.

1. If \(b_{max} > b_{min}\) then both bidders have the same supremum bid: \(\bar{b}_i = \bar{b}_j = b_{max}\).
2. Both \(G_1\) and \(G_2\) are continuous for all \(b > b_{min}\). Moreover, both \(G_1\) and \(G_2\) are strictly increasing over the interval \((b_{min}, b_{max})\).
3. Suppose that \(\bar{b}_j > \bar{b}_i\) so that \(b_{min} = \bar{b}_i > b = \bar{b}_j\). Then \(j\) bids an atom at \(b = \bar{b}_j = v_j\) and \(i\) bids an atom at \(b_{min} = \bar{b}_i = b^*_i(G_j(v_j))\) with weight \(\Gamma_i\). Moreover the size of \(i\)'s atom at \(b_{min}\) is 1 if \(b_{max} = b_{min}\) and otherwise is:

\[
\Gamma_i = \frac{\Pr [L_i|H_j] J_{b_{min}}^{v_{min}} (x - v_j) \hat{r}(x) \, dx}{\Pr [H_i|H_j] \hat{R}(b_{min}) (1 - b_{min})}
\]

**Proof.** (1) Suppose not and \(b_{max} = \bar{b}_i > \bar{b}_j\). Then \(j\) does not bid over \((\bar{b}_i, \bar{b}_j]\) but \(i\) bids with positive probability in \((\bar{b}_i, \bar{b}_j]\). By Lemma 50 and the definition of \(\bar{b}_i\), this positive probability must be concentrated at a single atom at \(\bar{b}_i\). By Lemma 54, \(\bar{b}_i\) is \(i\)'s infimum bid, that is \(\bar{b}_i = \bar{b}_j\), thus \(\bar{b}_i = \bar{b}_j \leq b_{min} \leq b_{max} = \bar{b}_i\), so \(b_{min} = b_{max}\), a contradiction.

(2) By Lemma 57, \(G_i\) and \(G_j\) are continuous for all \(b > b_{min}\). To show that they must also be strictly increasing over \((b_{max}, b_{min})\) we consider and rule out two types of flat spots. Throughout, we assume \(b_{max} > b_{min}\) (the claim is trivially satisfied for \(b_{max} = b_{min}\)).

First, suppose that at least one bidder, say \(i\), does not bid in an interval \((b_{min}, b^+\) so that \(G_i(b_{min}) = G_i(b^+) = \Gamma\) for some \(b^+ > b_{min}\). Note that \(b_{max} > b_{min}\) and part (1) imply \(\Gamma < 1\) and no atoms above \(b_{min}\) implies \(G_i(b^+) = G_i(b^+)^\). Moreover, let \(b^+\) be the upper bound of the flat spot: \(b^+ = \sup\{b : G_i(b) = G_i(b_{min})\}\). By Lemma 50, \(j\) can place at most one bid over \((b_{min}, b^+)\). By definition, \(b_{min}\) must be the infimum bid of one or both bidders. As neither bidder bids in \((b_{min}, b^+)\), this implies one (but not both by Lemma 50) bidders has an atom at \(b_{min}\). By the definition of \(b^+\) and the fact that \(j\) does not bid an atom at \(b^+, b^+\) must be an optimal bid for \(i\).

Suppose (i) \(i\) has the atom at \(b_{min}\). Then \(i\) has optimal bids at \(b_{min}\) and \(b^+\) but \(G_j(b_{min}) = G_j(b^+)\), contradicting Lemma 53.
Suppose instead (ii) that \( \hat{b} \) has the atom at \( b_{\min} \). By Lemma 53, \( b^+ \) is not an optimal bid for \( j \) because \( b_{\min} \) is optimal but \( G_i(b_{\min}) = G_i(b^+) \). Because \( i \) does not bid an atom at \( b^+ \), \( \Pi_j(b) \) is continuous at \( b^+ \) and \( j \) does not have an optimal bid in a neighborhood \( (b^+ - \beta, b^+ + \beta) \) for \( \beta > 0 \) sufficiently small. However \( i \) must bid with positive probability in this interval by definition of \( b^+ \), by Lemma 50 it must be concentrated at an atom, and this contradicts no atoms above \( b_{\min} \).

Second, suppose that at least one bidder, say \( i \), does not bid in an interval \( (b^-, b^+) \) such that \( G_i(b^-) = G_i(b^+) = \Gamma \) where

\[
 b_{\min} < b^- = \inf\{b : G_i(b) = \Gamma \} < b^+ = \sup\{b : G_i(b) = \Gamma \} < b_{\max}.
\]

Note that \( b^- > b_{\min} \) implies \( \Gamma > 0 \) and \( b_{\max} > b^+ \) implies \( \Gamma < 1 \). Because there are no atoms above \( b_{\min} \), both agent’s utility functions are continuous at \( b^- \) and \( b^+ \). Thus the definitions of \( b^- \) and \( b^+ \) (and \( \Gamma \in (0, 1) \)) therefore imply that \( b^- \) and \( b^+ \) are both optimal bids for \( i \). By Lemma 53, \( j \) can place at most one bid over \( (b^- , b^+) \), and because \( j \) cannot have an atom, this implies \( G_j(b^-) = G_j(b^+) \). By Lemma 53, this contradicts optimality of \( b^- \) and \( b^+ \) for \( i \).

Lemma 54 and \( b_j < b \) imply that \( j \) bids an atom at \( b = v_j \) but nowhere else below \( b_{\min} \). The final step in the proof is to show that \( i \) bids an atom at \( b_{\min} \). Then Corollary 51 implies \( b_{\min} = b_i^*(G_j(b_{\min})) \). Finally \( G_j(b_{\min}) = G_j(v_j) \) because \( j \) does not bid in \( (v_j, b_{\min}) \) (Lemmas 47 and 54).

To show that \( i \) bids an atom at \( b_{\min} \), there are two cases. (1) \( b_{\max} = b_{\min} \): This implies that \( i \)’s atom at \( v_j \) has mass 1 and that \( i \) bids \( b_{\min} \) with probability 1. (2) \( b_{\max} > b_{\min} \): Then by part (1) of this Lemma, for any \( \delta > 0 \) bidder \( j \) has optimal bid within the interval \( (b_{\min}, b_{\min} + \delta) \). This means that bidder \( i \) must have an atom at \( b_{\min} = b_i \). Suppose not and \( G_i(b_{\min}) = G_i(0) \). Then \( b_{\min} \) will be an optimal bid for \( j \) by continuity but \( \bar{b} \) is also an optimal bid for \( j \). This contradicts Lemma 53 given \( G_i(b_{\min}) = G_i(0) \).

To compute \( \Gamma_i \) we observe that the utility of \( j \) is the same across all bids in the support, in particular at his atom at \( v_j \). And at any optimal bid \( b_j > b_{\min} \) that is arbitrarily close to \( b_{\min} \) (such bid exists for any \( \delta > 0 \) in the interval \( (b_{\min}, b_{\min} + \delta) \) since \( b_{\max} > b_{\min} \)). Thus the change in utility from increasing the bid from \( v_j \) to such \( b_j \) is zero. The next equation presents this utility change in the limit when \( b_j \) tends to \( b_{\min} \) from above.

\[
\hat{R}(b_{\min}) \Pr[H_i|H_j] \Gamma_i (1 - b_{\min}) - \Pr[L_i|H_j] \int_{v_j}^{b_{\min}} (x - v_j) \hat{r}(x) \, dx = 0
\]

Or equivalently,

\[
\Gamma_i = \frac{\Pr[L_i|H_j] \int_{v_j}^{b_{\min}} (x - v_j) \hat{r}(x) \, dx}{\Pr[H_i|H_j] \hat{R}(b_{\min}) (1 - b_{\min})}
\]

Recall that \( \bar{b} = \inf\{b : G_i(b) > 0\} \) and \( \bar{b} = \inf\{x : G_i(x) = 1\} \) for agent \( i \in \{1, 2\} \). Note that when agent never submit dominated bids by definition it holds that \( 1 \geq b_{\max} = \max\{b_i, b_j\} \geq b_{\min} = \max\{\bar{b}_1, \bar{b}_2\} \geq b = \min\{\bar{b}_1, \bar{b}_2\} \geq 0 \).

We are now ready to restate Lemma 22 and prove it.

**Lemma 59.** At \( \eta \) the following must hold.

1. For some \( j \in \{1, 2\} \) it holds that \( \bar{b} = \bar{b}_j = v_j \) and \( b_{\min} = \bar{b}_i \geq v_i \) for \( i \neq j \).

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2. Both $G_1$ and $G_2$ are continuous and strictly increasing on $(b_{\min}, b_{\max})$. It holds that $G_1(b_{\max}) = G_2(b_{\max}) = 1$. Moreover, if $b_{\max} > b_{\min}$ then $b_{\max} = b_1 = b_2$.

3. For every bidder $i \in \{1, 2\}$ it holds that $G_i(b) = 0$ for every $b \in [0, b_{\min}]$, and $G_i(b) = G_i(b)$ for every $b \in [b_{\min}, b_{\max})$.

4. If $b_{\min} = b$ then $b = \max\{v_1, v_2\}$. Additionally, if $v_1 = v_2$ then $b_{\min} = b = v_1 = v_2$ and no bidder has any atom anywhere. If $v_i > v_j$ then $b_{\min} = b = v_i$ and $i$ has an atom at $b$, while $j$ has no atoms.

5. If $b_{\min} > b$ then for one agent, say $j$, it holds that $b = b_j = v_j$. Bidder $j$ has an atom at $v_j$ and bidder $i \neq j$ has an atom at

$$b_{\min} = b_j^*(G_j(v_j)) = \frac{\Pr[H_i|H_j]G_j(v_j) + v_i \Pr[L_j|H_i]}{\Pr[H_i|H_j]G_j(v_j) + \Pr[L_j|H_i]} > \max\{v_i, v_j\} \tag{11}$$

and $b_{\min}$ satisfies $b_{\min} \leq v(H_i)$, and $b_{\min} = v(H_i)$ if and only if $G_j(v_j) = 1$.

It also holds that either

- $b_{\max} = b_{\min}$, in this case $G_i(b_{\min}) = 1$, $G_j(v_j) = 1$ ($j$ always bids $v_j$, $i$ always bids $b_{\min}$). Or
- $b_{\max} > b_{\min}$, $G_i(b_{\min}) > 0$ and

$$G_i(b_{\min}) = \frac{\Pr[L_i|H_j] \int_{v_j}^{b_{\min}} (x - v_j) \hat{r}(x) dx}{\Pr[H_i|H_j] R(b_{\min})(1 - b_{\min})} \tag{12}$$

Proof. (1) Follows from Lemma \ref{lemma:continuity}. (2) Follows from the definition of $b_{\max}$ and Lemma \ref{lemma:definition} parts \ref{lemma:definition:part1} and \ref{lemma:definition:part2} (3) $G_i(b) = 0$ for $b < b$ follows from the definition of $b$. $G_i(b) = G_i(b)$ for $b \in [b_{\min}, b_{\max})$ follows from Lemma \ref{lemma:continuity} part \ref{lemma:definition:part2} (4) Follows from Lemma \ref{lemma:continuity} part \ref{lemma:definition:part1} (5) Follows almost entirely from Lemma \ref{lemma:definition} part \ref{lemma:definition:part3} The fact that $\max\{v_i, v_j\} < b_{\min} \leq v(H_i)$ and $b_{\min} = v(H_i)$ if and only if $G_j(v_j) = 1$ follows from the definition of $b_{\min}$, inspection of equation (11), and the fact that $v(H_i) = \Pr[H_j|H_i] + v_i \Pr[L_j|H_i]$.

\begin{lemma}
At $\eta$ the following must hold. If $\epsilon > 0$ is small enough then $b_{\max} > b_{\min}$.
\end{lemma}

Proof. Assume that $b_{\max} = b_{\min}$. Clearly it cannot be the case that $b_{\min} = b$ as it means that both agents are bidding an atom (of size 1) at $b$. If $b_{\min} < 1$, this contradicts Lemma \ref{lemma:atom}. If $b_{\min} \geq 1$, bidder $i \in \{1, 2\}$ could earn strictly more by deviating to bid $v_i$. Reducing the bid to $v_i$ means that bidder $i$ loses every time bidder $j \neq i$ has a high signal. In these cases the value is 1, but the payment would have been 1, so bidder $i$ is indifferent to losing rather than tying. In addition, reducing the bid to $v_i$ means that bidder $i$ now loses every time that bidder $j$ has a low signal and the random bidder bids between $v_i$ and $i$. Thus the bid reduction avoids overpayment with positive probability. This contradicts optimality of bidder $i$ bidding 1. We conclude that $b_{\min} > b$.

Given $b_{\max} = b_{\min} > b$, Lemma \ref{lemma:atom} implies that one agent, say $j$, is bidding an atom of size 1 at $v_j$, while the other agent $i$ is bidding an atom of size 1 at $b_{\min} = b_j^*(1)$. We note that Equation (11) shows that for $v_i < 1$ there exists $\zeta < 1$ which is independent of $\epsilon$ such that $b_{\min} < \zeta$. When $\epsilon$ is small enough agent $j$ can deviate and get strictly higher utility by bidding $b^+ \in (b_{\min}, 1)$. This deviation has two effects. First it means that $j$ has additional wins when $i$ has a low signal and the random bidder bids between $v_j$ and $b^+$ causing $j$ to pay more than the value $v_j$. This costs bidder $j$

$$\epsilon \Pr[L_i|H_j] \int_{v_j}^{b^+} (x - v_j) r(x) dx < \epsilon$$

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which is proportional to $\epsilon$. In addition, the deviation means that $j$ has additional wins when $i$ has a high signal and the random bidder bids below $b^+$. All of these incremental wins are valued at 1 but cost no more than $b^+$ so increase $j$’s payoff. Considering just those incremental wins for which the random bidder bids below $b_{\text{min}}$, this benefit is bounded below by $\Pr[H_i|H_j](1-b_{\text{min}}) > \Pr[H_i|H_j](1-\zeta)$. Thus $\epsilon < \Pr[H_i|H_j](1-\zeta)$ is a sufficient condition for the deviation to be strictly profitable. This contradiction shows $b_{\text{max}} > b_{\text{min}}$. 

\section*{D.2 Proofs of Lemma 23 and of Theorem 20}

We next restate Theorem 20 and prove it.

**Theorem 61.** Consider any non-degenerated monotonic domain with 2 bidders, each with a binary signal. Assume that $0 < \Pr[H_1,L_2](1-v(H_1,L_2)) \leq \Pr[L_1,H_2](1-v(L_1,H_2)) < 1$.

The unique TRE of the SPA game is the profile of strategies $\mu$ in which:

- Every bidder $i$ bids $v(L_1,L_2) = 0$ when getting signal $L_i$.
- Bidder 1 with signal $H_1$ always bids $v(H_1,H_2) = 1$.
- Bidder 2 with signal $H_2$
  - bids $v(H_1,H_2) = 1$ with probability $\frac{\Pr[H_1,L_2]}{\Pr[L_1,H_2]} \frac{1-v(H_1,L_2)}{1-v(L_1,H_2)}$, and
  - bids $v(L_1,H_2)$ with the remaining probability.

Recall that to simplify the notation we denote $v_1 = v(H_1,L_2)$ and $v_2 = v(L_1,H_2)$. As the domain is non-degenerated, $\Pr[H_1,H_2] > 0$ and for any bidder $i \in \{1,2\}$ it holds that $1 > \Pr[H_i] > 0$. The assumption that $0 < \Pr[H_1,L_2](1-v_1) \leq \Pr[L_1,H_2](1-v_2) < 1$ implies $\max\{v_1,v_2\} < 1$ and that $\min\{\Pr[H_1,L_2],\Pr[L_1,H_2]\} > 0$ and combining with the above implies that $\min\{\Pr[L_2|H_1],\Pr[L_2|H_2]\} > 0$. Additionally, as the domain is non-degenerated, $\min\{\Pr[H_1|H_2],\Pr[H_2|H_1]\} > 0$.

Consider the game with the random bidder that is bidding according to a standard distribution (its support is $[0,1]$). The random bidder arrives to the auction with small probability $\epsilon > 0$.

Assume that agent $i$ with signal $H_i$ is bidding according to distribution $G_i$, let $g_i$ denote the density of $G_i$ whenever $G_i$ is differentiable (note that since $G_i$ is non-decreasing it is differentiable almost everywhere, see, for example, Theorem 31.2 in (Billingsley 1995)). We note that this is an abuse of notation as $G_i$ and $g_i$ both depend on $R$ and $\epsilon$.

To prove the theorem we show that for any standard distribution $R$ and small enough $\epsilon$ a mixed NE in each of the games $\lambda(\epsilon,R)$ exists (Lemma 76). We then show that the limit of any sequence of NE strategies in the games $\lambda(\epsilon,R)$ must converges to $\mu$ as $\epsilon$ goes to zero. Combined with the existence of a mixed NE in each of the games $\lambda(\epsilon,R)$ this show that $\mu$ is the limit of the some sequence of NE strategies in the games $\lambda(\epsilon,R)$, thus a TRE. As the limit of any sequence of NE strategies in the games $\lambda(\epsilon,R)$ must converges to $\mu$ as $\epsilon$ goes to zero, $\mu$ is the unique TRE.

Fix a standard distribution $R$ and $\epsilon > 0$ and consider the game $\lambda(\epsilon,R)$. Let $\eta = (G_1,G_1^\epsilon,G_2,G_2^\epsilon)$ be a NE of $\lambda(\epsilon,R)$. For agent $i$ let $b_i = \inf\{b:G_i(b) > 0\}$. Define $\bar{b} = \min\{b_1,b_2\}$ and $b_{\text{min}} = \max\{b_1,b_2\}$.

Lemma 22 characterizes candidates for NE in $\lambda(\epsilon,R)$. We next take it as given and defer the proof of it to Section D.1.

The following two well known theorems (see for example (Billingsley 1995)) will be useful for proving our lemmas.
Theorem 62 (Theorem 31.2 in (Billingsley 1995)). A non-decreasing function $G$ is differentiable almost everywhere, the derivative $g$ is non-negative, and $G(b) - G(a) \geq \int_a^b g(x) dx$ for all $a$ and $b$.

Theorem 63 (Theorem 31.3 in (Billingsley 1995)). If $g$ is non-negative and integrable, and if $G(b) = \int_{-\infty}^b g(x) dx$, then $\frac{\partial G(b)}{\partial b} = g(b)$ except on a set of Lebesgue measure 0.

In addition, the following well known differential-equation result is also useful for proving our lemmas.

Theorem 64. Assume that $q(x) = u'(x) + p(x) \cdot u(x)$ holds for every $x \in (b_{\min}, b)$ but a set of measure zero, and $p(x)$ and $q(x)$ are continuous on the interval. Define $z(x) = e^{\int_{b_{\min}}^b p(y) dy}$. Then every function $u(x)$ that satisfies the assumption is of the form

$$u(b) - \frac{u(b_{\min})}{z(b)} = \frac{1}{z(b)} \int_{b_{\min}}^b z(x)q(x)dx + C$$

for some $C$.

D.2.1 Characterizations of the CDFs of $G_1$ and $G_2$

Lemma 65. At $\eta$ the following must hold. For every bid $b \geq b_{\min}$ in the support of agent 2’s distribution $G_2$, it must holds that

$$Pr[L_1|H_2] \cdot \epsilon \int_{b_{\min}}^b (x - v_2) \cdot r(x)dx = Pr[H_1|H_2] \left( s_1^{(\min)}(b) + s_1^{(+)}(b) \right)$$

where

$$s_1^{(\min)}(b) = \epsilon \cdot G_1(b_{\min}) \int_{b_{\min}}^b (1 - y) \cdot r(y)dy$$

and

$$s_1^{(+)}(b) = \int_{b_{\min}}^b g_1(x) \left( \hat{R}(b)(1 - x) - \epsilon \cdot \int_x^b (y - x)r(y)dy \right) dx$$

Proof. By Lemma [22] for any $\delta > 0$ agent 2 has an optimal bid in $[b_{\min}, b_{\min} + \delta)$. Agent 2 must be indifferent between all his bids, in particular, between bidding $b$ and bidding arbitrarily close to $b_{\min}$. The left hand side is the decrease in the expected utility of agent 2 when the agent 1 receives signal $L_1$ (happens with probability $Pr[L_1|H_2]$). As agent 1 with signal $L_1$ bids 0 (Lemma [46]), agent 2 with signal $H_2$ bidding a positive value always beats agent 1. Any time agent 2’s wins he gets a value of $v_2$. A bid of $b$ wins while a bid arbitrary close to $b_{\min}$ does not, only when the random bidder arrives (happens with probability $\epsilon$). In this case the extra utility gain is $\int_{b_{\min}}^b (x - v_2) \cdot r(x)dx$.

The right side handles the net gain when agent 1 receives signal $H_1$ (happens with probability $Pr[H_1|H_2]$). Agent 1 is bidding at most $b_{\min}$, which happen with probability $G_1(b_{\min})$. By bidding $b$ and not arbitrarily close to $b_{\min}$ the presence of the random bidder creates an additional utility of $s_1^{(\min)}(b) = \epsilon \cdot G_1(b_{\min}) \int_{b_{\min}}^b (1 - y) \cdot r(y)dy$ to agent 2.

Next consider the case that agent 1 is bidding more than $b_{\min}$. With probability $1 - \epsilon$ the random bidder is bidding 0. By Lemma [22] $G_1$ is continuous for every $b \in (b_{\min}, 1)$ (has no atom at bid in $(b_{\min}, 1)$). As $G_1$ is continuous for every $b \in (b_{\min}, 1)$, the expected utility is given by $\int_{b_{\min}}^b g_1(x) (1 - x) dx$, as the value from winning is 1, the payment if set by agent 1 is
x. (note that by Theorem 63 $G_1$ if differentiable at all points but a set of measure 0, and there are no atoms at these points, thus this set cannot change the integral). Thus conditional on $H_1$, the contribution of this case to the utility of agent 2 is

$$(1 - \epsilon) \int_{b_{\min}}^{b} g_1(x) (1 - x) \, dx \tag{17}$$

Additionally, with probability $\epsilon$ the random bidder is bidding according to $R$. In this case the utility difference is

$$\epsilon \left( \int_{b_{\min}}^{b} g_1(x) \left( \int_{b_{\min}}^{x} (1 - x) r(y) \, dy + \int_{x}^{b} (1 - y) r(y) \, dy \right) \, dx \right) = r(b) \epsilon \tag{18}$$

$$\epsilon \left( \int_{b_{\min}}^{b} g_1(x) \left( R(x)(1 - x) + \int_{x}^{b} (1 - x + y) r(y) \, dy \right) \, dx \right) = r(b) \epsilon \tag{19}$$

$$\epsilon \left( \int_{b_{\min}}^{b} g_1(x) \left( R(b)(1 - x) - \int_{x}^{b} (y - x) r(y) \, dy \right) \, dx \right) = r(b) \epsilon \tag{20}$$

Now Equation 14 is derived by noting that $\hat{R}(x) = 1 - \epsilon + \epsilon \cdot R(x)$. 

\[\square\]

**Lemma 66.** At $\eta$ the following must hold. For every bid $b \geq b_{\min}$ in the support of agent 2’s distribution $G_2$, if $\frac{\partial s_1^{(+)}(b)}{\partial b}$ exists at $b$ then it holds that

$$\frac{Pr[L_1 | H_2]}{Pr[H_1 | H_2]} \cdot \frac{b - v_2}{1 - b} \cdot \frac{\hat{r}(b)}{\hat{R}(b)} = g_1(b) + \frac{\hat{r}(b)}{\hat{R}(b)} \cdot G_1(b) \tag{21}$$

**Proof.** Under the condition of the lemma, by Lemma 65 we know that Equation 14 holds at $b$. We show that if $\frac{\partial s_1^{(+)}(b)}{\partial b}$ exists we can take the derivative of Equation 14 with respect to $b$ and that this yield the desired equality.

First we look at the LHS of Equation 14. As $R$ is standard $r$ is continuous and the derivative with respect to $b$ exists. It holds that the derivative of the LHS of Equation 14 is

$$\frac{\partial (Pr[L_1 | H_2] \cdot \epsilon \int_{b_{\min}}^{b} (x - v_2) \cdot r(x) \, dx)}{\partial b} = Pr[L_1 | H_2] \cdot \epsilon \cdot (b - v_2) \cdot r(b) \tag{22}$$

The derivative of the RHS of Equation 14 exists as $\frac{\partial s_1^{(+)}(b)}{\partial b}$ exists by assumption and $\frac{\partial s_1^{(min)}(b)}{\partial b}$ exists as $r$ is continuous. The derivative is

$$Pr[H_1 | H_2] \cdot \frac{\partial \left( s_1^{(min)}(b) + s_1^{(+)}(b) \right)}{\partial b} = Pr[H_1 | H_2] \cdot \left( \frac{\partial s_1^{(min)}(b)}{\partial b} + \frac{\partial s_1^{(+)}(b)}{\partial b} \right) \tag{23}$$

Now

$$\frac{\partial s_1^{(min)}(b)}{\partial b} = \epsilon \cdot G_1(b_{\min}) \cdot (1 - b) \cdot r(b) \tag{24}$$

and

$$\frac{\partial s_1^{(+)}(b)}{\partial b} = \frac{\partial \left( \int_{b_{\min}}^{b} g_1(x) \left( \hat{R}(b)(1 - x) - \epsilon \cdot \int_{x}^{b} (y - x) r(y) \, dy \right) \, dx \right)}{\partial b} = \tag{25}$$

58
\[
\frac{\partial}{\partial b} \left( \hat{R}(b) \int_{b_{\text{min}}}^{b} g_1(x) (1 - x) \, dx \right) = \epsilon \frac{\partial}{\partial b} \left( \int_{b_{\text{min}}}^{b} g_1(x) (1 - x) \, dx \right) + \hat{R}(b) \cdot g_1(b) (1 - b)
\]

We handle each of the two terms:

\[
\frac{\partial}{\partial b} \left( \hat{R}(b) \int_{b_{\text{min}}}^{b} g_1(x) (1 - x) \, dx \right) = \epsilon \cdot r(b) \left( \int_{b_{\text{min}}}^{b} g_1(x) (1 - x) \, dx \right) + \hat{R}(b) \cdot g_1(b) (1 - b)
\]

Now define \( h(b, x) = \int_{x}^{b} (y - x)r(y) \, dy \) and note that \( h(b, b) = 0 \) and \( \frac{\partial h(b, x)}{\partial b} = (b - x)r(b) \)

\[
\frac{\partial}{\partial b} \int_{b_{\text{min}}}^{b} g_1(x) h(b, x) \, dx = \int_{b_{\text{min}}}^{b} g_1(x) \frac{\partial h(b, x)}{\partial b} \, dx = \epsilon \cdot r(b) \int_{b_{\text{min}}}^{b} g_1(x) (b - x) \, dx =
\]

We conclude that

\[
\frac{\partial s_1^{(+)}(b)}{\partial b} = \epsilon \cdot r(b) \int_{b_{\text{min}}}^{b} g_1(x) (1 - x) \, dx + \hat{R}(b) \cdot g_1(b) (1 - b) - \epsilon \cdot r(b) \int_{b_{\text{min}}}^{b} g_1(x) (b - x) \, dx =
\]

\[
\epsilon \cdot r(b) (G_1(b) - G_1(b_{\text{min}})) (1 - b) + \hat{R}(b) \cdot g_1(b) (1 - b)
\]

Summarizing:

\[
\frac{\partial s_1^{(\text{min})}(b)}{\partial b} + \frac{\partial s_1^{(+)}(b)}{\partial b} = \epsilon \cdot r(b) G_1(b) (1 - b) + \hat{R}(b) \cdot g_1(b) (1 - b)
\]

Combining all the above with the observation that \( \hat{r}(b) = \epsilon \cdot r(b) \) we conclude that

\[
\frac{Pr[L_1 | H_2]}{Pr[H_1 | H_2]} \cdot (b - v_2) \cdot \hat{r}(b) = \hat{r}(b) \cdot G_1(b) (1 - b) + \hat{R}(b) \cdot g_1(b) (1 - b)
\]

Equivalently, by reorganizing, this yields equation (21).

**Lemma 67.** At \( \eta \) the following must hold. For every bid \( b \geq b_{\text{min}} \) in the support of agent 2’s distribution \( G_2 \), it must hold that

\[
G_1(b) = \frac{Pr[L_1 | H_2]}{Pr[H_1 | H_2]} \cdot \frac{\epsilon}{R(b)} \cdot \int_{b_{\text{min}}}^{b} x - v_2 \frac{r(x)}{1 - x} \, dx + \frac{\hat{R}(b_{\text{min}})}{R(b)}
\]

**Proof.** By Theorem 62 \( G_1 \) is differentiable almost everywhere. At any point \( x \) for which \( G_1 \) is differentiable (\( g_1(x) \) exists) it holds that \( \frac{\partial s_1^{(\text{min})}(b)}{\partial b} \) exists. By Lemma 66 if \( \frac{\partial s_1^{(\text{min})}(b)}{\partial b} \) exists for an optimal \( b \in (b_{\text{min}}, 1) \), then for \( b \) it holds that

\[
\frac{Pr[L_1 | H_2]}{Pr[H_1 | H_2]} \cdot \frac{b - v_2}{1 - b} \cdot \hat{r}(b) = \frac{\hat{r}(b)}{R(b)} = g_1(b) + \frac{\hat{R}(b_{\text{min}})}{R(b)}
\]

This is a First-Order Ordinary Differential Equation. We apply Theorem 64 with \( u(b) = G_1(b) \), \( u'(b) = g_1(b) \), \( q(b) = \frac{Pr[L_1 | H_2]}{Pr[H_1 | H_2]} \cdot \frac{b - v_2}{1 - b} \cdot \hat{r}(b) \) and \( p(b) = \frac{\hat{r}(b)}{R(b)} \).
We observe that \( z(x) = \int_{b_{min}}^{x} p(y)dy = \int_{b_{min}}^{x} \frac{r(y)}{R(y)}dy = \log(\hat{R}(x)) - \log(b_{min}), \) thus \( z(x) = e^{\int_{b_{min}}^{x} p(y)dy} = \frac{\hat{R}(x)}{\hat{R}(b_{min})}. \) Thus

\[
G_1(b) - G_1(b_{min}) \frac{\hat{R}(b_{min})}{\hat{R}(b)} = \frac{\hat{R}(b_{min})}{\hat{R}(b)} \left( \int_{b_{min}}^{b} \frac{\hat{R}(x)}{\hat{R}(b_{min})} \frac{Pr[L_1|H_2]}{Pr[H_1|H_2]} \cdot \frac{x-v_2}{1-x} \cdot \frac{\hat{r}(x)}{\hat{R}(x)}dx + C \right) = \\
\frac{1}{\hat{R}(b)} \left( \int_{b_{min}}^{b} \frac{Pr[L_1|H_2]}{Pr[H_1|H_2]} \cdot \frac{x-v_2}{1-x} \cdot \frac{\hat{r}(x)}{\hat{R}(x)}dx + C \right)
\]

(36)

Thus \( C \) is determined by evaluating the above at \( b = b_{min}. \) It must hold that \( C = 0. \) We conclude that

\[
G_1(b) = \frac{Pr[L_1|H_2]}{Pr[H_1|H_2]} \cdot \frac{\epsilon}{\hat{R}(b)} \int_{b_{min}}^{b} \frac{x-v_2}{1-x} \cdot \frac{\hat{r}(x)}{\hat{R}(x)}dx + G_1(b_{min}) \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b)}
\]

(37)

(38)

By replacing each player by the other and repeating the proof above similarly we get:

**Lemma 68.** At \( \eta \) the following must hold. For every bid \( b \geq b_{min} \) in the support of agent 1’s distribution \( G_1, \) it must holds that

\[
G_2(b) = \frac{Pr[L_2|H_1]}{Pr[H_2|H_1]} \cdot \frac{\epsilon}{\hat{R}(b)} \int_{b_{min}}^{b} \frac{x-v_1}{1-x} \cdot \frac{\hat{r}(x)}{\hat{R}(x)}dx + G_2(b_{min}) \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b)}
\]

(39)

**D.2.2 Proofs of Lemma 23**

We first show that \( b_{max} \) tends to 1 as \( \epsilon \) goes to 0.

**Lemma 69.** Fix a small \( \delta > 0. \) At \( \eta \) the following must hold. If \( \epsilon > 0 \) is small enough then it holds that \( 1 > b_{max} > 1 - \delta \) (\( b_{max} \) tends to 1 as \( \epsilon \) goes to 0).

**Proof.** By Lemma 67 and Lemma 68 for each bidder \( i \in \{1, 2\} \) and \( j \neq i, \) \( b_{max} \) must satisfy:

\[
1 = \frac{Pr[L_i|H_j]}{Pr[H_j|H_i]} \cdot \frac{\epsilon}{\hat{R}(b_{max})} \int_{b_{min}}^{b_{max}} \frac{x-v_i}{1-x} \cdot \frac{\hat{r}(x)}{\hat{R}(x)}dx + G_i(b_{min}) \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b_{max})}
\]

(40)

As \( \epsilon \) approaches zero, Lemma 60, Lemma 22, and Equation (12) imply that for some bidder \( i \) either \( G_i(b_{min}) = 0 \) or \( G_i(b_{min}) \) approaches zero. For the first term to approach 1 as \( \epsilon \) approaches zero requires the integral to approach infinity. As \( r \) is continuous on a compact set its infimum is obtained. Since \( r \) is positive for every \( x, \) \( \exists \bar{x} > 0 \) such that \( r(x) \geq \bar{x} \) for every \( x. \) Hence it is clear that for the integral to approach infinity \( b_{max} \) must approach 1. Finally, for any fixed \( \epsilon > 0 \) it must hold that \( 1 > b_{max} \) as the integral to approach infinity as \( b_{max} \) tends to 1.

\[\Box\]

The following notations will be useful. Let \( \alpha_1 = \frac{Pr[L_1|H_2]}{Pr[H_1|H_2]} \) and \( \alpha_2 = \frac{Pr[L_2|H_1]}{Pr[H_2|H_1]} \).

We assume that \( 0 < Pr[H_1, L_2](1-v_1) \leq Pr[L_1, H_2](1-v_2) \) \[14\] Additionally, assume that if \( Pr[H_1, L_2](1-v_1) = Pr[L_1, H_2](1-v_2) \) then \( v_1 \geq v_2. \) The next lemma (Lemma 23) presents additional properties that candidates for NE must satisfy when \( \epsilon \) is small enough.

\[\text{If } \min\{Pr[H_1, L_2](1-v_1), Pr[L_1, H_2](1-v_2)\} > 0 \text{ this is without loss of generality, by renaming the bidders if necessary.} \]
Lemma 70. If $\epsilon$ is small enough at $\eta$ the following must hold. There must exist $b_{\min}$ and $b_{\max}$ such that $1 > b_{\max} > b_{\min} \geq 0$ and:

- The two bidders are symmetric ($Pr[H_1, L_2] = Pr[L_1, H_2]$ and $v_1 = v_2$) if and only if $b_{\min} = b = v_1 = v_2$ and $G_1(b_{\min}) = G_2(b_{\min}) = 0$ (no atoms).

- If $Pr[H_1, L_2](1 - v_1) = Pr[L_1, H_2](1 - v_2)$ but the bidders are not symmetric, and it holds that $v_1 > v_2$ and $Pr[H_1, L_2] < Pr[L_1, H_2]$, then bidder 1 has an atom at $b_{\min} = b_1$ of size $G_1(b_{\min}) > 0$, and bidder 2 has an atom at $v_2 = b_2 = b < b_{\min}$ of size $G_2(v_2) > 0$. It holds that

$$b_{\min} = b_1^*(G_2(v_2)) = \frac{Pr[H_2|H_1]G_2(v_2) + v_1Pr[L_2|H_1]}{Pr[H_2|H_1]G_2(v_2) + Pr[L_2|H_1]} > \max\{v_1, v_2\} \tag{41}$$

$$G_1(b_{\min}) = \frac{Pr[L_1|H_2]}{Pr[L_1|H_2]} \int_{b_{\min}}^{b_{\max}} (x - v_2) \hat{r}(x) dx \tag{42}$$

$$G_2(v_2) = \frac{\hat{R}(b_{\max})}{\hat{R}(b_{\min})} - \left( \frac{\hat{R}(b_{\max})}{\hat{R}(b_{\min})} - G_1(b_{\min}) \right) \cdot \frac{Pr[H_1, L_2]}{Pr[L_1, H_2]} \int_{b_{\min}}^{b_{\max}} \frac{x - v_1}{1 - x} r(x) dx \tag{43}$$

- Assume $Pr[H_1, L_2](1 - v_1) < Pr[L_1, H_2](1 - v_2)$. Then either

- $b_{\min} = b$, bidder 1 has no atom ($G_1(b_{\min}) = 0$) and bidder 2 has an atom at $b_2 = v_2 > v_1$ of size $G_2(v_2) > 0$ specified by Equation (43), or

- $b_{\min} > b$, bidder 1 has an atom at $b_{\min} = b_1$ specified by Equation (41), its size $G_1(b_{\min}) > 0$ is specified by Equation (42), and bidder 2 has an atom at $v_2 = b_2 = b < b_{\min}$ of size $G_2(v_2) > 0$ specified by Equation (43).

Moreover, it always hold that

$$G_1(b) = \begin{cases} 0 & \text{if } 0 \leq b < b_{\min}; \\ Pr[L_1|H_2] \cdot \frac{\epsilon}{\hat{R}(b)} \int_{b_{\min}}^{b} \frac{x - v_2}{1 - x} r(x) dx + G_1(b_{\min}) \cdot \frac{\hat{R}(b_{\min})}{\hat{R}(b)} & \text{if } b_{\min} \leq b \leq b_{\max}; \\ 1 & \text{if } b_{\max} < b \leq 1. \end{cases} \tag{44}$$

and

$$G_2(b) = \begin{cases} 0 & \text{if } 0 \leq b < v_2; \\ Pr[L_2|H_1] \cdot \frac{\epsilon}{\hat{R}(b)} \int_{b_{\min}}^{b} \frac{x - v_1}{1 - x} r(x) dx + G_2(v_2) \cdot \frac{\hat{R}(b_{\min})}{\hat{R}(b)} & \text{if } v_2 \leq b \leq b_{\min}; \\ 1 & \text{if } b_{\min} \leq b \leq b_{\max}; \\ Pr[L_2|H_1] \cdot \frac{\epsilon}{\hat{R}(b)} \int_{b_{\min}}^{b} \frac{x - v_1}{1 - x} r(x) dx & \text{if } b_{\max} < b \leq 1. \end{cases} \tag{45}$$

Proof. By Lemma 60 for small enough $\epsilon$ it holds that $b_{\max} > b_{\min}$, so we assume that in the rest of the proof.

Observe that

$$\frac{a_2}{a_1} = \frac{Pr[L_2|H_1]}{Pr[L_1|H_2]} \cdot \frac{Pr[H_1|H_2]}{Pr[H_2|H_1]} = \frac{Pr[L_2|H_1]}{Pr[L_1|H_2]} \cdot \frac{Pr[H_1, H_2]}{Pr[H_2|H_1]} \cdot \frac{Pr[H_1]}{Pr[H_1]} \cdot \frac{Pr[H_1, L_2]}{Pr[L_1, H_2]} \tag{46}$$

Let $T(b, b_{\min}) = \int_{b_{\min}}^{b} \frac{1 - x}{x - r(x)} dx$ and let $\beta(b) = \frac{\hat{R}(b_{\min})}{\hat{R}(b)}$. 


Claim 1. Assume $b_{\text{max}} > b_{\text{min}}$. It holds that

$$
\frac{1 - G_2(b_{\text{min}}) \cdot \beta(b_{\text{max}})}{1 - G_1(b_{\text{min}}) \cdot \beta(b_{\text{max}})} = \frac{\alpha_2}{\alpha_1} \cdot \frac{1 - v_1 T(b_{\text{max}}, b_{\text{min}})}{1 - v_2 T(b_{\text{max}}, b_{\text{min}})}
$$

(47)

Proof. Recall that by Lemma 22 for $b_{\text{max}}$ it holds that $G_1(b_{\text{max}}) = G_2(b_{\text{max}}) = 1$.

By Lemma 68 for every bid $b \geq b_{\text{min}}$ in the support of agent 1’s distribution $G_1$, that is, for every $b \in [b_{\text{min}}, b_{\text{max}})$, Equation (39) hold. As the equation is continuous at $b_{\text{max}}$ we conclude that

$$
1 - G_2(b_{\text{min}}) \cdot \beta(b_{\text{max}}) = \alpha_2 \cdot \frac{\epsilon}{R(b_{\text{max}})} \int_{b_{\text{min}}}^{b_{\text{max}}} \frac{x - v_1}{1 - x} \cdot r(x) dx
$$

(48)

Additionally, by Lemma 67 for every bid $b \geq b_{\text{min}}$ in the support of agent 2’s distribution $G_2$, that is, for every $b \in [b_{\text{min}}, b_{\text{max}})$, Equation (34) hold. As the equation is continuous at $b_{\text{max}}$ we conclude that

$$
1 - G_1(b_{\text{min}}) \cdot \beta(b_{\text{max}}) = \alpha_1 \cdot \frac{\epsilon}{R(b_{\text{max}})} \int_{b_{\text{min}}}^{b_{\text{max}}} \frac{x - v_2}{1 - x} \cdot r(x) dx
$$

(49)

The claim follows from dividing the two equations (since for $b_{\text{max}} > b_{\text{min}}$ both sides of the two equations are not 0, thus such a division is well defined).

Claim 2. Assume $b_{\text{max}} > b_{\text{min}}$. There are no atoms ($G_1(b_{\text{min}}) = G_2(b_{\text{min}}) = 0$) if and only if both bidders are symmetric: $v_1 = v_2$ and $Pr[H_1, L_2] = Pr[L_1, H_2]$.

Proof. By Lemma 22 if $G_1(b_{\text{min}}) = G_2(b_{\text{min}}) = 0$ then $b = b_{\text{min}} = v_1 = v_2$. In such a case

Equation (47) reduces to $\alpha_2 = \alpha_1$. Now, recall that $\frac{\alpha_2}{\alpha_1} = \frac{Pr[H_1, L_2]}{Pr[L_1, H_2]}$, thus if there are no atoms in both $G_1$ and $G_2$ then $b = b_{\text{min}} = v_1 = v_2$ and $Pr[H_1, L_2] = Pr[L_1, H_2]$, that is, the two agents are completely symmetric.

Now, assume that both bidders are symmetric, that is, $v_1 = v_2$ and $Pr[H_1, L_2] = Pr[L_1, H_2]$, we want to show that no bidder has an atom. We next show that it cannot be the case that $b_{\text{min}} > b$. This is sufficient as, by Lemma 22 $b_{\text{min}} = b$ and $v_1 = v_2$ imply that no bidder has an atom, that is $G_2(b_{\text{min}}) = G_1(b_{\text{min}}) = 0$.

We next show that symmetry and $b_{\text{min}} > b$ implies a contradiction. For symmetric bidders Equation (47) implies that $G_1(b_{\text{min}}) = G_2(b_{\text{min}})$. Using Lemma 22 we observe the following. One bidder, w.l.o.g. bidder 2, bids an atom at $b = v_1 = v_2 = v$ and the other bidder (bidder 1) bids an atom at $b_{\text{min}} > b = v$. Denote $\Gamma = G_1(b_{\text{min}}) = G_2(b)$. By Equation (41),

$$
b_{\text{min}} = b^*_2(\Gamma) = \frac{Pr[H_2|H_1] \cdot \Gamma + v_1 \cdot Pr[L_2|H_1]}{Pr[H_2|H_1] \cdot \Gamma + Pr[L_2|H_1]},
$$

or equivalently,

$$
\Gamma = \frac{Pr[L_2|H_1]}{Pr[H_2|H_1]} \cdot \frac{b_{\text{min}} - v_1}{1 - b_{\text{min}}}.
$$

By Equation (42),

$$
\Gamma = \frac{Pr[L_1|H_2]}{Pr[H_1|H_2]} \cdot \frac{\int_{v_2}^{b_{\text{min}}} (x - v_2) \hat{r}(x) dx}{\hat{R}(b_{\text{min}}) (1 - b_{\text{min}})}.
$$

Thus,

$$
\frac{Pr[L_1|H_2]}{Pr[H_1|H_2]} \cdot \frac{\int_{v_2}^{b_{\text{min}}} (x - v_2) \hat{r}(x) dx}{\hat{R}(b_{\text{min}}) (1 - b_{\text{min}})} = \frac{Pr[L_2|H_1]}{Pr[H_2|H_1]} \cdot \frac{b_{\text{min}} - v_1}{1 - b_{\text{min}}},
$$

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or due to symmetry in conditional probabilities ($\alpha_1 = \alpha_2$) and values ($v_1 = v_2 = v$),

$$\int_v^{b_{\min}} (x - v) \hat{r}(x) dx = \hat{R}(b_{\min}) (b_{\min} - v).$$

Integration by parts implies that

$$\int_v^{b_{\min}} (x - v) \hat{r}(x) dx = (b_{\min} - v) \hat{R}(b_{\min}) - \int_v^{b_{\min}} \hat{R}(x) dx,$$

and this can only equal $\hat{R}(b_{\min}) (b_{\min} - v)$ when $b_{\min} = v$, a contradiction. \qed

We next consider the case that $Pr[H_1, L_2](1 - v_1) = Pr[L_1, H_2](1 - v_2)$ but the bidders are not symmetric ($v_1 > v_2$ and $Pr[H_1, L_2] < Pr[L_1, H_2]$).

**Claim 3.** Assume $b_{\max} > b_{\min}$ and that $\epsilon$ is small enough. Assume that $Pr[H_1, L_2](1 - v_1) = Pr[L_1, H_2](1 - v_2)$ but the bidders are not symmetric, and it holds that $v_1 > v_2$ and $Pr[H_1, L_2] < Pr[L_1, H_2]$. Then bidder 1 has an atom at $b_{\min} = b_1 > v_1$ and bidder 2 has an atom at $v_2 = b_2 = \hat{b} < b_{\min}$.

**Proof.** By Claim 2, as bidders are not symmetric it cannot be the case that both bidders have no atom.

We next show that it cannot be the case that only one bidder has an atom. By Lemma 22, if only one bidder has an atom and $v_1 > v_2$ it must be the case that $b = b_{\min} = v_1 > v_2$ and bidder 1 has the atom at $v_1$. But in this case, as $G_2(b_{\min}) = 0$, the LHS of Equation (47) equals to $\frac{1}{1 - G_1(b_{\min}) - \beta(b_{\max})} > 1$ (as $0 < \beta(b_{\max}) \leq 1$ and $G_1(b_{\min}) > 0$), while the RHS of Equation (47) is at most 1 since by Lemma 71 it is monotonically increasing to its limit 1, a contradiction.

We conclude that both bidders have an atom, each at his infimum bid. We next figure out which bidder has an atom at $b$ and which has an atom at $b_{\min}$. We first show that it must be the case that both $G_1(b_{\min})$ and $G_2(b_{\min})$ tend to 0 as $\epsilon$ goes to 0. By Equation (12), for one bidder $i$ it holds that $G_i(b_{\min})$ must tend to 0 as $\epsilon$ goes to 0 (as $b_{\min}$ does not tend to 1 the denominator does not tend to 0, while the numerator tends to 0). Now, as the RHS of Equation (47) tends to 1 as $\epsilon$ goes to 0, $G_1(b_{\min}) - G_2(b_{\min})$ must tend to 0. Now, as both $G_1(b_{\min})$ and $G_2(b_{\min})$ tend to 0 as $\epsilon$ goes to 0, by Equation (11), the bid of bidder $i$ that is bidding at $b_i = b_{\min}$ must tend to $v_i$, that is $b_{\min} - v_i$ tends to 0. Now recall that in that case it holds that $b_{\min} > b_j = v_j$. Thus, if $v_i < v_j$ we get a contradiction as $b_{\min} - v_i > v_j - v_i$ and $v_j - v_i$ is some positive constant (bounded away from 0). We conclude that $b_{\min} = b_1 > b = b_2 = v_2$, that is, bidder 1 has an atom at $b_{\min} = b_1 > v_1$ and bidder 2 has an atom at $v_2 = b_2 = \hat{b} < b_{\min}$, as we need to show. \qed

**Claim 4.** Assume $b_{\max} > b_{\min}$ and that $\epsilon$ is small enough. Assume that $Pr[H_1, L_2](1 - v_1) = Pr[L_1, H_2](1 - v_2)$ and $\epsilon$ is small enough. Then either bidder 1 has no atom and bidder 2 has an atom at $v_2 = b_2 = \hat{b} = b_{\min}$. Or, bidder 1 has an atom at $b_{\min} = b_1 > v_1$ and bidder 2 has an atom at $v_2 = b_2 = \hat{b} = b < b_{\min}$.

**Proof.** By Claim 2, as bidders are not symmetric it cannot be the case that both bidders have no atom. We next consider the case that at least one bidder has an atom. By Lemma 69, $b_{\max}$ tends to 1 as $\epsilon$ goes to 0. Additionally, $T(b, b_{\min})$ tends to 1 as $b$ tends to 1 (by Lemma 71). Thus, the RHS of Equation (47) tends to $\chi = \frac{Pr[H_1, L_2](1 - v_1)}{Pr[L_1, H_2](1 - v_2)} < 1$ as $\epsilon$ goes to 0. Equation (47) combined with $\chi < 1$ implies that $G_1(b_{\min}) < G_2(b_{\min})$. 

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Now, if only one bidder has an atom it must be bidder 2, since $G_2(b_{min}) = 0$ implies $G_1(b_{min}) < 0$, a contradiction. If on the other hand both bidders have an atom, we claim that bidder 1 has an atom at $b_{min} = b_1$ and bidder 2 has an atom at $v_2 = b_2 = b < b_{min}$. Observe also that $\beta(b_{\text{max}}) = \frac{R(b_{\text{min}})}{R(b_{\text{max}})}$ tends to 1 as $b$ goes to 0. Now, if bidder 2 is the bidder with the atom at $b_{min}$, by Equation (42), $G_2(b_{\text{min}})$ must tend to 0 as $b$ goes to 0 (as $b_{\text{min}}$ does not tend to 1 the denominator does not tend to 0, while the numerator tends to 0). Combining with $G_1(b_{\text{min}}) < G_2(b_{\text{min}})$ this will imply that $G_1(b_{\text{min}})$ must also tend to 0 as $b$ goes to zero. But then the LHS of Equation (47) tends to 1 while the RHS tends to $\chi < 1$, a contradiction. We conclude that bidder 1 has an atom at $b_{\text{min}} = b_1$ and bidder 2 has an atom at $v_2 = b_2 = b < b_{\text{min}}$.

By Equation (47), $G_2(b_{\text{min}})$ must satisfy

$$G_2(b_{\text{min}}) = \frac{1}{\beta(b_{\text{max}})} - \left(\frac{1}{\beta(b_{\text{max}})} - G_1(b_{\text{min}})\right) \cdot \frac{Pr[H_1, L_2]}{Pr[L_1, H_2]} \cdot \frac{1 - v_1 T(b_{\text{max}}, b_{\text{min}})}{1 - v_2 T(b_{\text{max}}, b_{\text{min}})}$$

(50)

Now Equation (43) follows from the definition of $\beta(b_{\text{max}})$ and $T(b_{\text{max}}, b_{\text{min}})$. The other claims in the lemma for the case that bidder 1 has an atom at $b_{\text{min}} = b_1$ and bidder 2 has an atom at $v_2 = b_2 = b < b_{\text{min}}$ directly follow from Lemma 22, Lemma 67 and Lemma 68.

Note that by Lemma 71 and the above observations, as $\epsilon$ goes to 0, the size of the atom $G_2(v_2)$ tends to $1 - \frac{Pr[H_1, L_2]}{Pr[L_1, H_2]}(1 - v_1)$.

**Lemma 71.** Fix any $0 \leq b_{\text{min}} < 1$ and a standard distribution $R$ with density $r$. The function

$$T(b) = \frac{\int_{b_{\text{min}}}^{b} \frac{1}{1 - x} \cdot r(x)dx}{\int_{b_{\text{min}}}^{b} \frac{1}{1 - x} \cdot r(x)dx}$$

(51)

monotonically decreases to 1 as $b$ increases from $b_{\text{min}}$ to 1. Additionally,

$$\frac{\int_{b_{\text{min}}}^{b} \frac{x - v_1}{1 - x} r(x)dx}{\int_{b_{\text{min}}}^{b} \frac{x - v_2}{1 - x} r(x)dx} = \frac{1 - v_1 T(b)}{1 - v_2 T(b)}$$

(52)

tends to $\frac{1 - v_1}{1 - v_2}$ as $b$ tends to 1. If $v_1 > v_2$ it is monotonically increasing to its limit, and if $v_1 < v_2$ it is monotonically decreasing to its limit.

**Proof.** Let $c \geq b_{\text{min}}$ be some number such that $0 < c < 1$ (say, $c = b_{\text{min}}$ unless $b_{\text{min}} = 0$, in this case $c = 1/2$). Assume $b \geq c$. Since $r$ is continuous on a compact set its infimum is obtained. Since $r$ is positive for every $x$, $\exists r > 0$ such that $r(x) \geq r$ for every $x$. Then

$$\int_{b_{\text{min}}}^{b} \frac{1}{1 - x} r(x)dx \geq \int_{b_{\text{min}}}^{b} \frac{x}{1 - x} r(x)dx \geq r \cdot \int_{c}^{b} \frac{x}{1 - x} dx \geq c \cdot r \cdot \int_{c}^{b} \frac{1}{1 - x} dx$$

Now we observe that both the numerator and the denominator of $T(b)$ tend to infinity when $b$ tends to 1 as

$$\lim_{b \to 1} \int_{c}^{b} \frac{1}{1 - x} dx = \lim_{b \to 1} \left(\ln (1 - c) - \ln (1 - b)\right) = \infty$$

Thus by L'Hôpital’s rule,

$$\lim_{b \to 1} \frac{\int_{b_{\text{min}}}^{b} \frac{1}{1 - x} r(x)dx}{\int_{b_{\text{min}}}^{b} \frac{x}{1 - x} r(x)dx} = \lim_{b \to 1} \frac{\frac{d}{db} \int_{b_{\text{min}}}^{b} \frac{1}{1 - x} r(x)dx}{\frac{d}{db} \int_{b_{\text{min}}}^{b} \frac{x}{1 - x} r(x)dx} = \lim_{b \to 1} \frac{\frac{1}{1 - b} r'(b)}{\frac{b}{1 - b} r'(b)} = \lim_{b \to 1} \frac{1}{b} = 1.$$
Next we show that \( T(b) \) monotonically decreases to 1 as \( b \) increases to 1. For any \( b < 1 \) all terms are finite, so we can compute the derivative:

\[
\frac{d}{db} \int_{b_{\text{min}}}^{b} \frac{1}{1-x} r(x)dx = \frac{1}{1-b} r(b) \left( \int_{b_{\text{min}}}^{b} \frac{x-b}{1-x} r(x)dx \right)^2 < 0.
\]

For \( b < 1, \frac{1}{b} > 0 \). For \( 0 \leq b_{\text{min}} < b < 1 \) and \( x \in [b_{\text{min}}, b], \frac{x-b}{1-x} < 1 \). Therefore \( T(b) \) is monotonically decreasing to 1 as \( b \) increases to 1.

Observe that

\[
\int_{b_{\text{min}}}^{b} \frac{x-v_1}{1-x} r(x)dx = 1 - v_1 \cdot T(b) - \frac{v_1 - v_2}{1/T(b) - v_2}.
\] (53)

When \( v_1 > v_2 \) it is monotonically increasing to \( \frac{1-v_1}{1-v_2} \) as \( b \) increases to 1, since \( T(b) \) decreases to 1 and \( v_1 - v_2 > 0 \). Similar argument shows that when \( v_1 < v_2 \) it is monotonically decreasing to its limit.

### D.2.3 Convergence to the TRE

For a standard distribution \( R \) it holds that its density function \( r \) is bounded as \( r \) is a continuous function on a compact set. Thus there exists some bound \( r_{\text{max}} < \infty \) such that \( r_{\text{max}} \geq r(x) \) for all \( x \).

**Lemma 72.** If \( \epsilon \) is small enough then the following holds. For every \( b \in (b_{\text{min}}, b_{\text{max}}) \) in the support of \( G_1 \) it must holds that:

\[
\frac{\Pr[L_1|H_2]}{\Pr[H_1|H_2]} \cdot \frac{\epsilon}{1-\epsilon} \cdot r_{\text{max}} \cdot (-b - \log(1-b)) + G_1(b_{\text{min}}) \geq G_1(b) \tag{54}
\]

where for \( r_{\text{max}} \) it holds that \( r_{\text{max}} \geq r(x) \) for all \( x \).

**Proof.** By Lemma [61]

\[
G_1(b) = \frac{\Pr[L_1|H_2]}{\Pr[H_1|H_2]} \cdot \frac{\epsilon}{R(b)} \cdot \int_{b_{\text{min}}}^{b} \frac{1}{1-x} r(x)dx + G_1(b_{\text{min}}) \cdot \frac{\hat{R}(b_{\text{min}})}{\hat{R}(b)} \tag{55}
\]

As \( v_2 \geq 0 \) and \( r(b) \leq r_{\text{max}} \) for all \( b \),

\[
\int_{b_{\text{min}}}^{b} \frac{x-v_2}{1-x} r(x)dx \leq \int_{b_{\text{min}}}^{b} \frac{x}{1-x} r(x)dx \leq r_{\text{max}} \int_{0}^{b} \frac{x}{1-x} dx = r_{\text{max}} (-b - \log(1-b)) \tag{56}
\]

As \( \hat{R}(b) \geq \hat{R}(b_{\text{min}}) \geq 1 - \epsilon \) we conclude that

\[
G_1(b) \leq \frac{\Pr[L_1|H_2]}{\Pr[H_1|H_2]} \cdot \frac{\epsilon}{1-\epsilon} \cdot r_{\text{max}} (-b - \log(1-b)) + G_1(b_{\text{min}}) \tag{57}
\]

**Lemma 73.** If \( \epsilon \) is small enough then the following holds. For every \( b \in (b_{\text{min}}, b_{\text{max}}) \) in the support of \( G_2 \) it must holds that:

\[
\frac{\Pr[L_2|H_1]}{\Pr[H_2|H_1]} \cdot \frac{\epsilon}{1-\epsilon} \cdot r_{\text{max}} (-b - \log(1-b)) \geq G_2(b) - G_2(v_2) \tag{58}
\]

where for \( r_{\text{max}} \) it holds that \( r_{\text{max}} \geq r(x) \) for all \( x \).
Proof. The proof is the same as the proof of Lemma 72 when using the following equation proved in Lemma 68 and the fact that \( G_2(b_{\text{min}}) = G_2(v_2) \):

\[
G_2(b) = \frac{\Pr[L_2|H_1]}{\Pr[H_2|H_1]} \cdot \frac{\epsilon}{\hat{R}(b)} \int_{b_{\text{min}}}^{b} \frac{x - v_1}{1 - x} \cdot r(x) dx + G_2(b_{\text{min}}) \cdot \frac{\hat{R}(b_{\text{min}})}{\hat{R}(b)} \tag{59}
\]

Lemma 23, Lemma 72, and Lemma 73 enable us to prove that any TRE must be \( \mu \).

**Corollary 74.** Fix a standard distribution \( R \). For every \( b \in [0, \min\{v_2, v_1\}] \) it holds that \( G_1(b) = G_2(b) = 0 \). For every \( b \in [\min\{v_2, v_1\}, 1] \) the limits of \( G_1(b) \) and of \( G_2(b) = G_2(v_2) \), as \( \epsilon \) goes to zero, are both zero. Additionally, \( G_2(v_2) \) tends to \( 1 - \frac{\Pr[H_1, L_2](1-v_1)}{\Pr[L_1, H_2](1-v_2)} \) as \( \epsilon \) goes to zero.

**Proof.** Fix some \( b < 1 \). For a small \( \delta > 0 \) such that \( \delta < 1 - b \), by Lemma 60 and Lemma 69 it holds that for a small enough \( \epsilon, b_{\text{max}} > 1 - \delta \) and \( b_{\text{max}} > b_{\text{min}} \). Now it holds that \( b < b_{\text{max}} \).

Since \( b_1 \geq v_1 \) and \( b_2 \geq v_2 \) it holds that \( b = \min\{b_1, b_2\} \geq \min\{v_1, v_2\} \), thus by Lemma 23

\[
G_1(b) = G_2(b) = 0 \quad \text{for every} \quad b \in [0, \min\{v_1, v_2\}].
\]

The same lemma also implies that \( G_1(b) = G_2(b) = 0 \) for every \( b \in [\min\{v_1, v_2\}, b_{\text{min}}] \).

Next we consider \( b \in [b_{\text{min}}, 1] \). We first observe that for any fixed \( b \in (b_{\text{min}}, 1) \), by Lemma 71 \( G_1(b) \) tends to \( G_1(b_{\text{min}}) \) as \( \epsilon \) goes to 0 (and clearly \( G_1(b) = G_1(b_{\text{min}}) \) for \( b = b_{\text{min}} \)).

The claim that \( G_1(b) \) tends to 0 follows from Lemma 23 which shows that \( G_1(b_{\text{min}}) \) is either 0 or tends to 0 when \( \epsilon \) goes to 0. Additionally, by Lemma 73 for any fixed \( b \in (b_{\text{min}}, 1) \) it holds that \( G_2(b) \) tends to \( G_2(v_2) \) as \( \epsilon \) goes to 0. As \( G_2 \) is continuous at \( b_{\text{min}} \) the claim also hold at that point.

Finally, Lemma 23 combined with Lemma 71 show that \( G_2(v_2) \) is 0 if and and only if the bidders are symmetric and \( \frac{\Pr[H_1, L_2](1-v_1)}{\Pr[L_1, H_2](1-v_2)} = 1 \), and otherwise \( G_2(v_2) \) tends to \( 1 - \frac{\Pr[H_1, L_2](1-v_1)}{\Pr[L_1, H_2](1-v_2)} \) as \( \epsilon \) goes to zero.

**D.2.4 Existence of NE in \( \lambda(\epsilon, R) \)**

Observe that \( R(b) = b \) is a standard distribution, so standard distributions exist. We next show that for any standard distribution \( R \), if \( \epsilon \) is small enough then there exists a mixed NE in the game \( \lambda(\epsilon, R) \).

We prove existence of one of three types of equilibria depending on parameter values. For symmetric bidders, we show the existence of an equilibrium with no atoms (case 1). For asymmetric bidders we show the existence of either a one-atom (case 2) or a two-atom (case 3) equilibrium depending on whether or not equation (63) in the proof is satisfied. The following observation indicates why equation (63) determines whether asymmetric equilibria involve one or two atoms.

**Observation 75.** If \( \epsilon \) is small enough and \( G_1(b_{\text{min}}) > 0 \) (bidder 1 has an atom, which implies that bidder 2 also has an atom) then it must hold that

\[
\alpha_2 \cdot \frac{v_2 - v_1}{1 - v_2} \leq 1 - \frac{\alpha_2}{\alpha_1} \cdot \frac{1 - v_1}{1 - v_2} \tag{60}
\]

**Proof.** If \( G_1(b_{\text{min}}) > 0 \) then Equation (41) holds. In particular it must holds that

\[
\frac{G_2(v_2) + v_1 \alpha_2}{G_2(v_2) + \alpha_2} = 1 - \frac{\alpha_2(1 - v_1)}{G_2(v_2) + \alpha_2} > v_2 \tag{61}
\]
By Corollary 74, $G_2(v_2)$ tends to $1 - \frac{Pr[H_1,L_2](1-v_1)}{Pr[L_1,H_2](1-v_2)} = 1 - \frac{\alpha_2(1-v_1)}{\alpha_1(1-v_2)}$ as $\epsilon$ goes to zero. Thus it must holds that
\[1 - \frac{\alpha_2(1-v_1)}{1 - \frac{\alpha_2(1-v_1)}{\alpha_1(1-v_2)}} + \alpha_2 \geq v_2 \quad (62)\]
and the claim follows from reorganizing the last equation.

**Lemma 76.** Fix any standard distribution $R$. For every small enough $\epsilon > 0$ there exists a mixed NE $\eta$ in the game $\lambda(\epsilon, R)$.

**Proof.** Let $\bar{v} = \max\{v_1, v_2\}$. Throughout the proof we index bidders 1 and 2 such that either 1) $\alpha_1(1-v_2) = \alpha_2(1-v_1)$ and $v_1 > v_2$, or 2) $\alpha_1(1-v_2) > \alpha_2(1-v_1)$. Moreover, we often distinguish between three cases:

1. **No atom case.** Bidders are symmetric: $v = v_1 = v_2$ and $Pr[H_1, L_2] = Pr[L_1, H_2]$. In this case we show there exists an equilibrium in which $b_{\min} = v$ and neither bidder has an atom: $G_1(b_{\min}) = G_2(v_2) = 0$.

2. **One atom case.** Bidders are asymmetric ($v_1 \neq v_2$ or $Pr[H_1, L_2] \neq Pr[L_1, H_2]$) and equation (63) holds:
\[\alpha_2 \cdot \frac{v_2 - v_1}{1 - v_2} \geq 1 - \frac{\alpha_2}{\alpha_1} \cdot \frac{1 - v_1}{1 - v_2}. \quad (63)\]
Note that asymmetry and equation (63) imply that $\alpha_1(1-v_2) > \alpha_2(1-v_1)$ and $v_2 > v_1$. This is so as by assumption the RHS of equation (63) is non-negative, this implies that $v_2 \geq v_1$. If $v_2 = v_1$ Then the equation implies that $\alpha_1 = \alpha_2$ which means the bidders are symmetric, a contradiction. Therefore $v_2 > v_1$ and thus $\alpha_1(1-v_2) > \alpha_2(1-v_1)$ (since in the case that $\alpha_1(1-v_2) = \alpha_2(1-v_1)$ we assume that $v_1 > v_2$).

In this case we show that there exists an equilibrium in which $b_{\min} = v_2$ and only bidder 2 has an atom: $G_2(v_2) > 0$ and $G_1(b_{\min}) = 0$.

3. **Two atom case.** Bidders are asymmetric ($v_1 \neq v_2$ or $Pr[H_1, L_2] \neq Pr[L_1, H_2]$) and equation (63) is violated. Note that either 1) $\alpha_1(1-v_2) = \alpha_2(1-v_1)$ and $v_1 > v_2$, or 2) $\alpha_1(1-v_2) > \alpha_2(1-v_1)$ are both feasible. In this case we show that there exists an equilibrium in which $b_{\min} > \max\{v_1, v_2\}$ and both bidders have atoms: $G_2(v_2) > 0$ and $G_1(b_{\min}) > 0$.

In all cases, bidder $i$ with signal $L_i$ is bidding $v(L_1, L_2) = 0$. We construct distributions $G_1$ and $G_2$ using the necessary conditions in Lemma 23 and show that they form a NE. Equations (44) and (45) define $G_1$ and $G_2$ as a function of the four parameters $b_{\min}$, $b_{\max}$, $G_1(b_{\min})$, and $G_2(v_2)$. There are three main steps to the proof. First we show existence of parameters $b_{\min}$, $b_{\max}$, $G_1(b_{\min})$, and $G_2(v_2)$ that satisfy the necessary conditions in Lemma 23. Second, we show that, for the chosen parameters, $G_1$ and $G_2$ are well defined distributions (non-decreasing, and satisfying $G_1(0) = G_2(0) = 0$ and $G_1(1) = G_2(1) = 1$). Third we show that the constructed bid distributions are best responses. By construction, bidder $i \in \{1,2\}$ is indifferent to all bids in the support of his bid distribution and we show that every bid outside the support gives weakly lower utility.

**Step 1.** Existence of parameters $b_{\min}$, $b_{\max}$, $G_1(b_{\min})$, and $G_2(v_2)$:

**Case 1 (no atoms):** First consider the case that the bidders are symmetric. We define $b_{\min} = v$ and $G_1(b_{\min}) = G_2(v_2) = 0$. By the necessary conditions at $b_{\max}$ it must hold that
\[1 = G_1(b_{\max}) = \frac{Pr[L_1|H_2]}{Pr[H_1|H_2]} \cdot \frac{\epsilon}{R(b_{\max})} \cdot \int_v^{b_{\max}} \frac{x - v}{1 - x} r(x) dx \quad (64)\]
The RHS increases from zero to infinity as \( b_{\text{max}} \) increases from \( v \) to 1 (Claim 6), so there exists a unique value of \( b_{\text{max}} \in (v, 1) \) that solves this equation. It is clear that \( b_{\text{max}} \) must tend to 1 as \( \epsilon \) goes to 0. Note that all the necessary conditions presented in Lemma 23 for the symmetric case are now satisfied.

**Case 2 (one atom):** Next consider the case that bidders are asymmetric and equation (63) holds (implying \( \alpha_2 \cdot (1 - v_1) < \alpha_1 \cdot (1 - v_2) \) and \( v_1 < v_2 \)). We define \( b_{\text{min}} = v_2 \) and \( G_1(b_{\text{min}}) = 0 \). As \( G_1(b_{\text{min}}) = 0 \), \( b_{\text{max}} \in (v, 1) \) can be determined exactly as in the symmetric case. Finally, we set \( G_2(v_2) \) using Equation (43). Observe that \( G_2(v_2) \) as defined tends to 1 as \( \epsilon \) tends to 0, thus for sufficiently small \( \epsilon \) it is positive.

**Case 3 (two atoms):** Finally, consider the case that bidders are asymmetric and equation (63) is violated. We define \( G_1(b_{\text{min}}) \) as a function of \( b_{\text{min}} \) and \( b_{\text{max}} \) by equation (42). We define \( G_2(v_2) \) as a function of \( b_{\text{min}} \) by equation (41), or equivalently by:

\[
G_2(v_2) = \frac{\Pr[L_2|H_1]}{\Pr[H_2|H_1]} \frac{b_{\text{min}} - v_1}{1 - b_{\text{min}}}.
\] (65)

The arguments below show that \( b_{\text{min}} \) is larger than 1, while for \( b_{\text{min}} \) increases from zero to infinity as \( \epsilon \) tends to 0, thus for sufficiently small \( \epsilon \) it is positive.

We first show that when \( \epsilon \) is small enough, for any \( b_{\text{min}} \in [\bar{v}, v(H_1)] \) we can find a unique \( b_{\text{max}} \in (b_{\text{min}}, 1) \) that solves equation (66). We denote such a solution by \( b_{\text{max}}(b_{\text{min}}) \). When \( b_{\text{max}} = b_{\text{min}} \), the RHS of equation (66) equals \( \epsilon \cdot h(b_{\text{min}}) \) for \( h(b_{\text{min}}) = \frac{\alpha_1}{R(b_{\text{min}})} \int_{v_{\min}}^{b_{\text{min}}} \frac{x - v_1}{1 - b_{\text{min}}} \cdot \hat{r}(x)dx \). As \( h \) is a continuous function on a compact set it is bounded, thus \( \epsilon \cdot h(b_{\text{min}}) < 1 \) for any \( b_{\text{min}} \in [\bar{v}, v(H_1)] \) as long as \( \epsilon \) is small enough. Now, for every fixed \( b_{\text{min}} \in [\bar{v}, v(H_1)] \), the RHS of equation (66) is continuously increasing in \( b_{\text{max}} \) (by Claim 6 below) and goes to infinity when \( b_{\text{max}} \) tends to 1. Therefore there exists a unique \( b_{\text{max}} \in (b_{\text{min}}, 1) \) that solves the equation. Note that \( b_{\text{max}}(b_{\text{min}}) \) is a continuous function of \( b_{\text{min}} \) and, for any fixed \( b_{\text{min}} \), \( b_{\text{max}}(b_{\text{min}}) \) tends to 1 as \( \epsilon \) tends to 0.

Now we substitute \( b_{\text{max}}(b_{\text{min}}) \) into equation (67) and get the following equation in \( b_{\text{min}} \):

\[
1 = \alpha_2 \cdot \frac{1}{R(b_{\text{max}}(b_{\text{min}}))} \int_{b_{\text{min}}}^{b_{\text{max}}(b_{\text{min}})} \frac{x - v_1}{1 - x} \cdot \hat{r}(x)dx + \frac{b_{\text{min}} - v_1}{1 - b_{\text{min}}} \cdot \alpha_2 \cdot \frac{\hat{R}(b_{\text{min}})}{R(b_{\text{max}}(b_{\text{min}}))}.
\] (68)

To complete the proof we need to show that there exists \( b_{\text{min}} \in [\bar{v}, v(H_1)] \) that satisfies equation (68). The RHS of this equation is a continuous function of \( b_{\text{min}} \) on the compact set \([\bar{v}, v(H_1)]\). It will therefore be sufficient to show that for \( b_{\text{min}} = v(H_1) \) the RHS is strictly larger than 1, while for \( b_{\text{min}} = \bar{v} \) the RHS is strictly smaller than 1. Once this is shown (below) we conclude that there exists \( b_{\text{min}} > \bar{v} \) such that the RHS is exactly 1. This \( b_{\text{min}} \) together with \( b_{\text{max}} = b_{\text{max}}(b_{\text{min}}) \) solve both equations (66) and (67) and satisfy \( 1 > b_{\text{max}} > b_{\text{min}} > \bar{v} \).
To prove the remaining two inequalities, define:

\[
z(b_{\text{min}}) = \alpha_1 \cdot \frac{1}{R(b_{\text{max}}(b_{\text{min}}))} \int_{b_{\text{min}}}^{b_{\text{max}}(b_{\text{min}})} \frac{x - v_2}{1 - x} \cdot \hat{r}(x) \, dx.
\]

Now, the RHS of equation (68) can be written as

\[
z(b_{\text{min}}) = \frac{\alpha_2 \cdot \int_{b_{\text{min}}}^{b_{\text{max}}(b_{\text{min}})} \frac{x - v_1}{1 - x} \cdot \hat{r}(x) \, dx}{\alpha_1 \cdot \int_{b_{\text{min}}}^{b_{\text{max}}(b_{\text{min}})} \frac{x - v_2}{1 - x} \cdot \hat{r}(x) \, dx} + \frac{b_{\text{min}} - v_1}{1 - b_{\text{min}}} \cdot \alpha_2 \cdot \frac{\hat{R}(b_{\text{max}}(b_{\text{min}}))}{\hat{R}(b_{\text{max}}(b_{\text{min}}))} \tag{69}
\]

Fix \(b_{\text{min}}\). Note that equation (66) implies that \(z(b_{\text{min}}) \leq 1\) and \(z(b_{\text{min}})\) tends to 1 as \(\epsilon\) goes to 0, as the second term of the RHS of equation (66) is positive and tends to 0. By Lemma 71, \(\frac{\alpha_2 \cdot \int_{b_{\text{min}}}^{b_{\text{max}}(b_{\text{min}})} \frac{x - v_1}{1 - x} \cdot \hat{r}(x) \, dx}{\alpha_1 \cdot \int_{b_{\text{min}}}^{b_{\text{max}}(b_{\text{min}})} \frac{x - v_2}{1 - x} \cdot \hat{r}(x) \, dx}\) tends to \(\frac{\alpha_2(1 - v_1)}{\alpha_1(1 - v_2)}\), and thus, as \(\epsilon\) tends to 0, the RHS of equation (68) tends to

\[
\frac{\alpha_2(1 - v_1)}{\alpha_1(1 - v_2)} + \frac{b_{\text{min}} - v_1}{1 - b_{\text{min}}} \cdot \alpha_2 \cdot \frac{\hat{R}(b_{\text{max}}(b_{\text{min}}))}{\hat{R}(b_{\text{max}}(b_{\text{min}}))} \tag{70}
\]

For \(b_{\text{min}} = v(H_1)\), equation (70) strictly exceeds 1 since by equation (41) it holds that \(b_{\text{min}} = v(H_1)\) if and only if \(G_2(v_2) = \frac{b_{\text{min}} - v_1}{b_{\text{min}}} \cdot \alpha_2 = 1\), and the first term is strictly positive by assumption. Thus, for sufficiently small \(\epsilon\), the RHS of equation (68) also strictly exceeds 1 for \(b_{\text{min}} = v(H_1)\).

If \(b_{\text{min}} = \bar{v}\) we show that the RHS of equation (68) is strictly less than 1 for sufficiently small \(\epsilon\). We consider two cases separately. First, if \(b_{\text{min}} = \bar{v} = v_2 \geq v_1\), equation (70) is strictly less than 1 as equation (63) is violated. Thus, for sufficiently small \(\epsilon\), the RHS of equation (68) is also strictly less than 1. Second, if \(b_{\text{min}} = \bar{v} = v_1 > v_2\), equation (70) is weakly (but not necessarily strictly) less than 1. However, we show that equation (69) (and hence the RHS of equation (68)) is strictly less than equation (70) for all \(\epsilon > 0\). This follows because \(b_{\text{min}} > v_2\) implies that the second term on the RHS of equation (68) is strictly positive so that \(z(b_{\text{min}}) < 1\) and \(v_1 > v_2\) implies (by Lemma 71) that \(\frac{\alpha_2 \cdot \int_{b_{\text{min}}}^{b_{\text{max}}(b_{\text{min}})} \frac{x - v_1}{1 - x} \cdot \hat{r}(x) \, dx}{\alpha_1 \cdot \int_{b_{\text{min}}}^{b_{\text{max}}(b_{\text{min}})} \frac{x - v_2}{1 - x} \cdot \hat{r}(x) \, dx}\) is increasing to its limit (which is at most 1).

**Step 2.** \(G_1\) and \(G_2\) are well defined: We next argue that \(G_1\) and \(G_2\), as defined above by Step 1 and equations (44) and (45), are well defined distributions. The way we have chosen the parameters in Step 1 ensures that \(\max\{v_1, v_2\} \leq b_{\text{min}} < b_{\text{max}} \leq 1\), \(G_1(b_{\text{min}}), G_2(v_2) \geq 0\), and \(G_1(b_{\text{max}}) = G_2(b_{\text{max}}) = 1\). The two distributions are continuous from the right at \(b_{\text{min}}\), and by Claim 6 and Claim 5 are strictly increasing on \((b_{\text{min}}, b_{\text{max}}]\). Thus both are monotonically non-decreasing on \([0, \infty)\) with \(G_1(0) = G_2(0) = 0\) and \(G_1(b_{\text{max}}) = G_2(b_{\text{max}}) = 1\).

**Step 3.** Constructed bid distributions are best responses: To see that \(\eta\) is indeed a mixed NE we show that each bidder is best responding to the other. Observe that, by construction, \(G_1\) and \(G_2\) ensure that each bidder is indifferent between all the bids in the support her bid distribution. It only remains to show that all other bids earn weakly lower payoffs.

First consider bids above \(b_{\text{max}}\). As \(0 < Pr[H_1, L_2](1 - v(H_1, L_2)) \leq Pr[L_1, H_2](1 - v(L_1, H_2))\) it holds that \(\max\{v(H_1), v(H_2)\} < 1\). Therefore, as \(b_{\text{max}}\) tends to 1 when \(\epsilon\) tends to 0, for small enough \(\epsilon\) it holds that \(b_{\text{max}} > \max\{v(H_1), v(H_2)\}\). Therefore, for small enough \(\epsilon\), Lemma 50 implies that for both bidders \(b_{\text{max}}\) strictly dominates any higher bid \(b > b_{\text{max}}\).

Second note that Lemma 50 also implies that for bidder \(i\), bidding \(v_i\) strictly dominates any lower bid \(b < v_i\).
Third, we consider bids \( b \in [v_i, b_{\text{min}}] \) by bidder \( i \in \{1, 2\} \) outside the support of bidder \( i \)'s bid distribution for each of the three cases.

Consider case 1 (no atoms) in which \( b_{\text{min}} = v_1 = v_2 = v \). In this case, the utility from bidding \( b_{\text{min}} = v \) equals the utility of any bid in \( [v, b_{\text{max}}] \) by continuity.

Consider case 2 (one atom) in which \( b_{\text{min}} = v_2, \alpha_2 \cdot (1 - v_1) < \alpha_1 \cdot (1 - v_2), \) and \( v_2 > v_1 \). Bidder 2 bids an atom at \( v_2 \) so there are no other bids to check. For bidder 1, Lemma 47 implies that any bid in \( (v_2, b_{\text{max}}) \) strictly dominates bidding \( v_2 \). By Lemma 50, the bid with the highest payoff strictly below \( v_2 \) is \( v_1 \). By bidding \( v_1 \), bidder 1 never wins when bidder 2 gets the high signal \( H_2 \). Since \( 1 - \alpha_2 \cdot \frac{1}{1 - v_2} > 0 \) the size of the atom of bidder 2 does not tend to 0 as \( \epsilon \) tends to 0, and clearly the gain by bidding above the atom of bidder 2 at \( v_2 \) instead of bidding \( v_1 \) is positive if \( \epsilon \) is small enough.

Consider case 3 (two atoms) in which \( b_{\text{min}} > \max\{v_1, v_2\} \). Bidder 2 bids an atom at \( v_2 \), which by Lemma 50 dominates any bid \( b < b_{\text{min}} \). Moreover, for bidder 2, bidding \( b_{\text{min}} \) is dominated by bids in the support by Lemma 47. Now turn to bidder 1. Lemmas 50 and 47 imply that \( i \)'s atom at \( b_{\text{min}} \) dominates any bid in \( [v_2, b_{\text{min}}] \) because \( b_{\text{min}} \) is defined by equation (41). For \( v_1 \geq v_2 \), \( [v_2, b_{\text{min}}] \) includes all bids \( [v_1, b_{\text{min}}) \) and we are done. For \( v_1 < v_2 \), we must also consider bids \( [v_1, v_2) \), of which \( v_1 \) gives the highest payoff by Lemma 50. As \( v_1 < v_2 \) implies \( \alpha_2 \cdot (1 - v_1) < \alpha_1 \cdot (1 - v_2) \), \( b_{\text{min}} \) must dominate \( v_1 \) for sufficiently small \( \epsilon \) by the same argument applied above in the one-atom case.

**Claim 5.** In all three cases (no atoms, one atom, two atoms) \( G_2(b) \) as defined above is increasing in \( b \) for every \( b \in (b_{\text{min}}, b_{\text{max}}) \).

**Proof.** We need to show that in all three cases \( G_2(b) \) is increasing in \( b \) for every \( b \in (b_{\text{min}}, b_{\text{max}}) \). For any such \( b \), \( G_2(b) \) satisfies Equation (39), and its derivative with respect to \( b \) is

\[
g_2(b) = \frac{\hat{r}(b)}{\hat{R}(b)} \left( \alpha_2 \cdot \frac{b - v_1}{1 - b} - G_2(b) \right).
\]

To prove the claim it is sufficient to show that for every \( b \in (b_{\text{min}}, b_{\text{max}}) \):

\[
g_2(b) \cdot \frac{\hat{R}(b)}{\hat{r}(b)} = \alpha_2 \cdot \frac{b - v_1}{1 - b} - G_2(b) > 0.
\]

If \( G_2(b) \leq 0 \) the claim follows from \( 1 \geq b_{\text{max}} > b > b_{\text{min}} \geq \max\{v_1, v_2\} \). Next assume that \( G_2(b) \geq 0 \). We observe that for small enough \( \epsilon \) this is an increasing function in \( b \) for \( b \in (b_{\text{min}}, b_{\text{max}}) \):

\[
\frac{d}{db} \left( \frac{\hat{R}(b)}{\hat{r}(b)} g_2(b) \right) = \alpha_2 \frac{1 - v_1}{(1 - b)^2} - g_2(b) = \alpha_2 \frac{1 - v_1}{(1 - b)^2} - \frac{\hat{r}(b)}{\hat{R}(b)} \left( \alpha_2 \frac{b - v_1}{1 - b} - G_2(b) \right)
\]
\[
\geq \alpha_2 \frac{1}{(1 - b)^2} \left( 1 - v_1 - \frac{\hat{r}(b)}{\hat{R}(b)} (b - v_1) (1 - b) \right)
\]
\[
\geq \alpha_2 \frac{1}{(1 - b_{\text{min}})^2} \left( 1 - v_1 - \frac{r(b)}{1 - \epsilon} \right).
\]

As \( 1 > v_1 \) and \( r(b) \) is bounded from above (\( r \) is continuous on a compact interval), for small enough \( \epsilon \) this is positive.
Thus, as the function \( \frac{R(b)}{R(v_2)} \), \( g_2(b) \) is increasing, to prove that it is positive for any \( b > b_{\text{min}} \) it would be sufficient to show that it is at least 0 at \( b_{\text{min}} \), or equivalently, that the following holds:

\[
\alpha_2 \cdot \frac{b_{\text{min}} - v_1}{1 - b_{\text{min}}} \geq G_2(b_{\text{min}}).
\] (72)

We show that equation (72) is satisfied for each of the three cases.

In the first case (no atoms), \( G_2(v_2) = 0 \), and equation (72) clearly holds because \( b_{\text{min}} \geq v_1 \). In the third case (two atoms), \( G_2(v_2) \) satisfies equation (41), which is exactly equivalent to equation (72) holding with equality.

Finally we consider the second case (one atom) in which \( \alpha_2 \cdot (1 - v_1) < \alpha_1 \cdot (1 - v_2) \), equation (63) holds and \( G_2(b_{\text{min}}) = G_2(v_2) > 0 \) satisfies equation (43) with \( G_1(b_{\text{min}}) = 0 \), and additionally, \( b_{\text{min}} = v_2 > v_1 \) (this corresponds to the case that only bidder 2 has an atom). These conditions imply that

\[
G_2(v_2) = \frac{\hat{R}(b_{\text{max}})}{\hat{R}(v_2)} \left(1 - \frac{\alpha_2 \int_{v_2}^{b_{\text{max}}} \frac{x - v_1}{1 - x} r(x) dx}{\alpha_1 \int_{v_2}^{b_{\text{max}}} \frac{x - v_2}{1 - x} r(x) dx}\right).
\]

Which means that we need to show that

\[
\alpha_2 \frac{v_2 - v_1}{1 - v_2} = \frac{\hat{R}(b_{\text{max}})}{\hat{R}(v_2)} \left(1 - \frac{\alpha_2 \int_{v_2}^{b_{\text{max}}} \frac{x - v_1}{1 - x} r(x) dx}{\alpha_1 \int_{v_2}^{b_{\text{max}}} \frac{x - v_2}{1 - x} r(x) dx}\right) = G_2(v_2)
\]

Equation (64) determines \( b_{\text{max}} \) and implies that \( \frac{\hat{R}(b_{\text{max}})}{\hat{R}(v_2)} = \alpha_1 \int_{v_2}^{b_{\text{max}}} \frac{x - v_2}{1 - x} \hat{r}(x) dx \), thus:

\[
\frac{\hat{R}(b_{\text{max}})}{\hat{R}(v_2)} = \frac{\hat{R}(b_{\text{max}})}{\hat{R}(b_{\text{max}}) - \int_{v_2}^{b_{\text{max}}} \hat{r}(x) dx} = \frac{\alpha_1 \int_{v_2}^{b_{\text{max}}} \frac{x - v_2}{1 - x} r(x) dx}{\int_{v_2}^{b_{\text{max}}} \left(\frac{\alpha_1}{1 - x} - 1\right) r(x) dx}
\]

We can now express \( G_2(v_2) \) as a function of \( b_{\text{max}} \) as follows:

\[
G_2(v_2) = \frac{\alpha_1 \int_{v_2}^{b_{\text{max}}} \frac{x - v_2}{1 - x} r(x) dx}{\int_{v_2}^{b_{\text{max}}} \left(\frac{\alpha_1}{1 - x} - 1\right) r(x) dx} \left(1 - \frac{\alpha_2 \int_{v_2}^{b_{\text{max}}} \frac{x - v_1}{1 - x} r(x) dx}{\alpha_1 \int_{v_2}^{b_{\text{max}}} \frac{x - v_2}{1 - x} r(x) dx}\right)
\]

\[
= \frac{\int_{v_2}^{b_{\text{max}}} \left(\alpha_1 (x - v_2) - \alpha_2 (x - v_1)\right) \frac{r(x) dx}{1 - x}}{\int_{v_2}^{b_{\text{max}}} \left(\alpha_1 \frac{x - v_2}{1 - x} - 1\right) r(x) dx}
\]

\( b_{\text{max}} \) tends to 1 as \( \epsilon \) goes to 0 (Lemma 69) and \( G_2(v_2) \) tends to \( 1 - \frac{\alpha_2}{\alpha_1} \cdot \frac{1 - v_1}{1 - v_2} \) as \( b_{\text{max}} \) tends to 1 (Corollary 74). By Equation (63) it is thus sufficient to prove that \( G_2(v_2) \) is nondecreasing in \( b_{\text{max}} \): \( \frac{d}{db_{\text{max}}} G_2(v_2) \geq 0 \).
\[
\frac{dG_2(v_2)}{db_{\max}} = \frac{1}{\left(\int_{v_2}^{b_{\max}} \left(\alpha_1 \frac{x-v_2}{1-x} - 1 \right) r(x) \, dx \right)^2} \cdot \frac{r(b_{\max})}{1-b_{\max}} \cdot \left( (\alpha_1 (b_{\max} - v_2) - \alpha_2 (b_{\max} - v_1)) \int_{v_2}^{b_{\max}} (\alpha_1 (x - v_2) - (1-x)) \frac{r(x)}{1-x} \, dx \\
- (\alpha_1 (b_{\max} - v_2) - (1-b_{\max})) \int_{v_2}^{b_{\max}} (\alpha_1 (x - v_2) - \alpha_2 (x - v_1)) \frac{r(x)}{1-x} \, dx \right) \\
= \frac{1}{\left(\int_{v_2}^{b_{\max}} \left(\alpha_1 \frac{x-v_2}{1-x} - 1 \right) r(x) \, dx \right)^2} \cdot \frac{r(b_{\max})}{1-b_{\max}} \cdot \int_{v_2}^{b_{\max}} \frac{b_{\max} - x}{1-x} (\alpha_1 \alpha_2 (v_2 - v_1) - \alpha_1 (1-v_2) + \alpha_2 (1-v_2)) r(x) \, dx \\
= \alpha_1 (1-v_2) \left( \frac{v_2 - v_1}{1-v_2} - \left( 1 - \frac{\alpha_2 (1-v_2)}{\alpha_1 (1-v_2)} \right) \right) \cdot \frac{r(b_{\max})}{1-b_{\max}} \cdot \int_{v_2}^{b_{\max}} \frac{b_{\max} - x}{1-x} r(x) \, dx \\
\cdot \left( \int_{v_2}^{b_{\max}} \left(\alpha_1 \frac{x-v_2}{1-x} - 1 \right) r(x) \, dx \right)^2
\]

By Equation (63), \( \alpha_2 \frac{v_2 - v_1}{1-v_2} \geq \left( 1 - \frac{\alpha_2 (1-v_2)}{\alpha_1 (1-v_2)} \right) \), thus \( \frac{dG_2(v_2)}{db_{\max}} \geq 0 \) holds. (Moreover, when \( \frac{v_2 - v_1}{1-v_2} = \left( 1 - \frac{\alpha_2 (1-v_2)}{\alpha_1 (1-v_2)} \right) \), \( \frac{dG_2(v_2)}{db_{\max}} = 0 \) and \( G_2(v_2) \) attains its limit for any \( b_{\max} < 1 \).

Claim 6. In all three cases (no atoms, one atom, two atoms) \( G_1(b) \) as defined above is increasing in \( b \) for every \( b \in (b_{\min}, b_{\max}) \).

Proof. The same arguments as the ones presented in the proof of Claim 5 show that it is sufficient to prove that

\[ \alpha_1 \cdot \frac{b_{\min} - v_2}{1-b_{\min}} \geq G_1(b_{\min}). \]  

(73)

When bidder 1 does not have an atom (when no bidder has an atom, or only bidder 2 has an atom), this trivially holds since \( b_{\min} \geq v_2 \). We are left to prove the claim when both bidders have an atom and \( G_1(b_{\min}) > 0 \) satisfies Equation (42). We need to show that

\[ \alpha_1 \cdot \frac{b_{\min} - v_2}{1-b_{\min}} \geq \alpha_1 \cdot \frac{\int_{v_2}^{b_{\min}} (x - v_2) \hat{r}(x) \, dx}{\hat{R}(b_{\min}) (1-b_{\min})} , \]

(74)

which trivially holds since \( \hat{R}(b_{\min}) \geq \int_{v_2}^{b_{\min}} \hat{r}(x) \, dx = \hat{R}(b_{\min}) - \hat{R}(v_2) \).