

# 14.452 Economic Growth: Lecture 8, Overlapping Generations

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# Growth with Overlapping Generations

- In many situations, the assumption of a *representative household* is not appropriate because
  - ① households do not have an infinite planning horizon
  - ② new households arrive (or are born) over time.
- New economic interactions: decisions made by older “generations” will affect the prices faced by younger “generations”.
- *Overlapping generations models*
  - ① Capture potential interaction of different generations of individuals in the marketplace;
  - ② Provide tractable alternative to infinite-horizon representative agent models;
  - ③ Some key implications different from neoclassical growth model;
  - ④ Dynamics in some special cases quite similar to Solow model rather than the neoclassical model;
  - ⑤ Generate new insights about the role of national debt and Social Security in the economy.

# Problems of Infinity I

- Static economy with countably infinite number of households,  $i \in \mathbb{N}$
- Countably infinite number of commodities,  $j \in \mathbb{N}$ .
- All households behave competitively (alternatively, there are  $M$  households of each type,  $M$  is a large number).
- Household  $i$  has preferences:

$$u_i = c_i^i + c_{i+1}^i,$$

- $c_j^i$  denotes the consumption of the  $j$ th type of commodity by household  $i$ .
- Endowment vector  $\omega$  of the economy: each household has one unit endowment of the commodity with the same index as its index.
- Choose the price of the first commodity as the numeraire, i.e.,  $p_0 = 1$ .

## Problems of Infinity II

**Proposition** In the above-described economy, the price vector  $\bar{p}$  such that  $\bar{p}_j = 1$  for all  $j \in \mathbb{N}$  is a competitive equilibrium price vector and induces an equilibrium with no trade, denoted by  $\bar{x}$ .

- **Proof:**

- At  $\bar{p}$ , each household has income equal to 1.
- Therefore, the budget constraint of household  $i$  can be written as

$$c_i^i + c_{i+1}^i \leq 1.$$

- This implies that consuming own endowment is optimal for each household,
- Thus  $\bar{p}$  and no trade,  $\bar{x}$ , constitute a competitive equilibrium.

## Problems of Infinity III

- However, this competitive equilibrium is not Pareto optimal. Consider alternative allocation,  $\tilde{x}$ :
  - Household  $i = 0$  consumes its own endowment and that of household 1.
  - All other households, indexed  $i > 0$ , consume the endowment of than neighboring household,  $i + 1$ .
  - All households with  $i > 0$  are as well off as in the competitive equilibrium  $(\bar{p}, \bar{x})$ .
  - Individual  $i = 0$  is strictly better-off.

**Proposition** In the above-described economy, the competitive equilibrium at  $(\bar{p}, \bar{x})$  is not Pareto optimal.

## Problems of Infinity IV

- Source of the problem must be related to the infinite number of commodities.
- Extended version of the First Welfare Theorem covers infinite number of commodities, but only assuming  $\sum_{j=0}^{\infty} p_j^* \omega_j < \infty$  (written with the aggregate endowment  $\omega_j$ ).
- Here the only endowment is labor, and thus  $p_j^* = 1$  for all  $j \in \mathbb{N}$ , so that  $\sum_{j=0}^{\infty} p_j^* \omega_j = \infty$  (why?).
- This abstract economy is “isomorphic” to the baseline overlapping generations model.
- The Pareto suboptimality in this economy will be the source of potential inefficiencies in overlapping generations model.

## Problems of Infinity V

- Second Welfare Theorem did not assume  $\sum_{j=0}^{\infty} p_j^* \omega_j < \infty$ .
- Instead, it used convexity of preferences, consumption sets and production possibilities sets.
- This exchange economy has convex preferences and convex consumption sets:
  - Pareto optima must be decentralizable by some redistribution of endowments.

**Proposition** In the above-described economy, there exists a reallocation of the endowment vector  $\omega$  to  $\tilde{\omega}$ , and an associated competitive equilibrium  $(\bar{p}, \tilde{x})$  that is Pareto optimal where  $\tilde{x}$  is as described above, and  $\bar{p}$  is such that  $\bar{p}_j = 1$  for all  $j \in \mathbb{N}$ .

## Proof of Proposition

- Consider the following reallocation of  $\omega$ : endowment of household  $i \geq 1$  is given to household  $i - 1$ .
  - At the new endowment vector  $\tilde{\omega}$ , household  $i = 0$  has one unit of good  $j = 0$  and one unit of good  $j = 1$ .
  - Other households  $i$  have one unit of good  $i + 1$ .
- At the price vector  $\bar{p}$ , household 0 has a budget set

$$c_0^0 + c_1^1 \leq 2,$$

thus chooses  $c_0^0 = c_1^1 = 1$ .

- All other households have budget sets given by

$$c_i^i + c_{i+1}^{i+1} \leq 1,$$

- Thus it is optimal for each household  $i > 0$  to consume one unit of the good  $c_{i+1}^{i+1}$
- Thus  $\tilde{x}$  is a competitive equilibrium.



# The Baseline Overlapping Generations Model

- Time is discrete and runs to infinity.
- Each individual lives for two periods.
- Individuals born at time  $t$  live for dates  $t$  and  $t + 1$ .
- Assume a general (separable) utility function for individuals born at date  $t$ ,

$$U(t) = u(c_1(t)) + \beta u(c_2(t+1)), \quad (1)$$

- $u : \mathbb{R}_+ \rightarrow \mathbb{R}$  satisfies the usual Assumptions on utility.
- $c_1(t)$ : consumption of the individual born at  $t$  when young (at date  $t$ ).
- $c_2(t+1)$ : consumption when old (at date  $t+1$ ).
- $\beta \in (0, 1)$  is the discount factor.

# Demographics, Preferences and Technology I

- Exponential population growth,

$$L(t) = (1 + n)^t L(0). \quad (2)$$

- Production side same as before: competitive firms, constant returns to scale aggregate production function, satisfying Assumptions 1 and 2:

$$Y(t) = F(K(t), L(t)).$$

- Factor markets are competitive.
- Individuals can only work in the first period and supply one unit of labor inelastically, earning  $w(t)$ .

## Demographics, Preferences and Technology II

- Assume that  $\delta = 1$ .
- $k \equiv K/L$ ,  $f(k) \equiv F(k, 1)$ , and the (gross) rate of return to saving, which equals the rental rate of capital, is

$$1 + r(t) = R(t) = f'(k(t)), \quad (3)$$

- As usual, the wage rate is

$$w(t) = f(k(t)) - k(t) f'(k(t)). \quad (4)$$

# Consumption Decisions I

- Savings by an individual of generation  $t$ ,  $s(t)$ , is determined as a solution to

$$\max_{c_1(t), c_2(t+1), s(t)} u(c_1(t)) + \beta u(c_2(t+1))$$

subject to

$$c_1(t) + s(t) \leq w(t)$$

and

$$c_2(t+1) \leq R(t+1)s(t),$$

- Old individuals rent their savings of time  $t$  as capital to firms at time  $t+1$ , and receive gross rate of return  $R(t+1) = 1 + r(t+1)$
- Second constraint incorporates notion that individuals only spend money on their own end of life consumption (no altruism or bequest motive).

## Consumption Decisions II

- No need to introduce  $s(t) \geq 0$ , since negative savings would violate second-period budget constraint (given  $c_2(t+1) \geq 0$ ).
- Since  $u(\cdot)$  is strictly increasing, both constraints will hold as equalities.
- Thus first-order condition for a maximum can be written in the familiar form of the consumption Euler equation,

$$u'(c_1(t)) = \beta R(t+1) u'(c_2(t+1)). \quad (5)$$

- Problem of each individual is strictly concave, so this Euler equation is sufficient.
- Solving for consumption and thus for savings,

$$s(t) = s(w(t), R(t+1)), \quad (6)$$

## Consumption Decisions III

- $s : \mathbb{R}_+^2 \rightarrow \mathbb{R}$  is strictly increasing in its first argument and may be increasing or decreasing in its second argument.
- Total savings in the economy will be equal to

$$S(t) = s(t) L(t),$$

- $L(t)$  denotes the size of generation  $t$ , who are saving for time  $t + 1$ .
- Since capital depreciates fully after use and all new savings are invested in capital,

$$K(t+1) = L(t) s(w(t), R(t+1)). \quad (7)$$

# Equilibrium I

**Definition** A competitive equilibrium can be represented by a sequence of aggregate capital stocks, individual consumption and factor prices,

$\{K(t), c_1(t), c_2(t), R(t), w(t)\}_{t=0}^{\infty}$ , such that the factor price sequence  $\{R(t), w(t)\}_{t=0}^{\infty}$  is given by (3) and (4), individual consumption decisions  $\{c_1(t), c_2(t)\}_{t=0}^{\infty}$  are given by (5) and (6), and the aggregate capital stock,  $\{K(t)\}_{t=0}^{\infty}$ , evolves according to (7).

- Steady-state equilibrium defined as usual: an equilibrium in which  $k \equiv K/L$  is constant.
- To characterize the equilibrium, divide (7) by  $L(t+1) = (1+n)L(t)$ ,

$$k(t+1) = \frac{s(w(t), R(t+1))}{1+n}.$$

## Equilibrium II

- Now substituting for  $R(t+1)$  and  $w(t)$  from (3) and (4),

$$k(t+1) = \frac{s(f(k(t)) - k(t)f'(k(t)), f'(k(t+1)))}{1+n} \quad (8)$$

- This is the fundamental law of motion of the overlapping generations economy.
- A steady state is given by a solution to this equation such that  $k(t+1) = k(t) = k^*$ , i.e.,

$$k^* = \frac{s(f(k^*) - k^*f'(k^*), f'(k^*))}{1+n} \quad (9)$$

- Since the savings function  $s(\cdot, \cdot)$  can take any form, the difference equation (8) can lead to quite complicated dynamics, and multiple steady states are possible.



# Restrictions on Utility and Production Functions I

- Suppose that the utility functions take the familiar CRRA form:

$$U(t) = \frac{c_1(t)^{1-\theta} - 1}{1-\theta} + \beta \left( \frac{c_2(t+1)^{1-\theta} - 1}{1-\theta} \right), \quad (10)$$

where  $\theta > 0$  and  $\beta \in (0, 1)$ .

- Technology is Cobb-Douglas,

$$f(k) = k^\alpha$$

- The rest of the environment is as described above.
- The CRRA utility simplifies the first-order condition for consumer optimization,

$$\frac{c_2(t+1)}{c_1(t)} = (\beta R(t+1))^{1/\theta}.$$

## Restrictions on Utility and Production Functions II

- This Euler equation can be alternatively expressed in terms of savings as

$$s(t)^{-\theta} \beta R(t+1)^{1-\theta} = (w(t) - s(t))^{-\theta}, \quad (11)$$

- Gives the following equation for the saving rate:

$$s(t) = \frac{w(t)}{\psi(t+1)}, \quad (12)$$

where

$$\psi(t+1) \equiv [1 + \beta^{-1/\theta} R(t+1)^{-(1-\theta)/\theta}] > 1,$$

- Ensures that savings are always less than earnings.

## Restrictions on Utility and Production Functions III

- The impact of factor prices on savings is summarized by the following and derivatives:

$$s_w \equiv \frac{\partial s(t)}{\partial w(t)} = \frac{1}{\psi(t+1)} \in (0, 1),$$

$$s_R \equiv \frac{\partial s(t)}{\partial R(t+1)} = \left( \frac{1-\theta}{\theta} \right) (\beta R(t+1))^{-1/\theta} \frac{s(t)}{\psi(t+1)}.$$

- Since  $\psi(t+1) > 1$ , we also have that  $0 < s_w < 1$ .
- Moreover, in this case  $s_R > 0$  if  $\theta > 1$ ,  $s_R < 0$  if  $\theta < 1$ , and  $s_R = 0$  if  $\theta = 1$ .
- Reflects counteracting influences of income and substitution effects.
- Case of  $\theta = 1$  (log preferences) is of special importance, may deserve to be called the *canonical overlapping generations model*.

# Restrictions on Utility and Production Functions IV

- Equation (8) implies

$$\begin{aligned} k(t+1) &= \frac{s(t)}{(1+n)} \\ &= \frac{w(t)}{(1+n)\psi(t+1)}, \end{aligned} \quad (13)$$

- Or more explicitly,

$$k(t+1) = \frac{f(k(t)) - k(t)f'(k(t))}{(1+n)[1 + \beta^{-1/\theta}f'(k(t+1))^{-(1-\theta)/\theta}]} \quad (14)$$

- The steady state then involves a solution to the following implicit equation:

$$k^* = \frac{f(k^*) - k^*f'(k^*)}{(1+n)[1 + \beta^{-1/\theta}f'(k^*)^{-(1-\theta)/\theta}]}.$$

## Restrictions on Utility and Production Functions V

- Now using the Cobb-Douglas formula, steady state is the solution to the equation

$$(1+n) \left[ 1 + \beta^{-1/\theta} (\alpha(k^*)^{\alpha-1})^{(\theta-1)/\theta} \right] = (1-\alpha)(k^*)^{\alpha-1}. \quad (15)$$

- For simplicity, define  $R^* \equiv \alpha(k^*)^{\alpha-1}$  as the marginal product of capital in steady-state, in which case, (15) can be rewritten as

$$(1+n) \left[ 1 + \beta^{-1/\theta} (R^*)^{(\theta-1)/\theta} \right] = \frac{1-\alpha}{\alpha} R^*. \quad (16)$$

- Steady-state value of  $R^*$ , and thus  $k^*$ , can now be determined from equation (16), which always has a unique solution.
- To investigate the stability, substitute for the Cobb-Douglas production function in (14)

$$k(t+1) = \frac{(1-\alpha)k(t)^\alpha}{(1+n) \left[ 1 + \beta^{-1/\theta} (\alpha k(t+1)^{\alpha-1})^{-(1-\theta)/\theta} \right]}. \quad (17)$$

# Restrictions on Utility and Production Functions VI

**Proposition** In the overlapping-generations model with two-period lived households, Cobb-Douglas technology and CRRA preferences, there exists a unique steady-state equilibrium with the capital-labor ratio  $k^*$  given by (15), this steady-state equilibrium is globally stable for all  $k(0) > 0$ .

- In this particular (well-behaved) case, equilibrium dynamics are very similar to the basic Solow model
- Figure shows that convergence to the unique steady-state capital-labor ratio,  $k^*$ , is monotonic.

# Canonical Model I

- Even the model with CRRA utility and Cobb-Douglas production function is relatively messy.
- Many of the applications use log preferences ( $\theta = 1$ ).
- Income and substitution effects exactly cancel each other: changes in the interest rate (and thus in the capital-labor ratio of the economy) have no effect on the saving rate.
- Structure of the equilibrium is essentially identical to the basic Solow model.
- Utility of the household and generation  $t$  is,

$$U(t) = \log c_1(t) + \beta \log c_2(t+1), \quad (18)$$

- $\beta \in (0, 1)$  (even though  $\beta \geq 1$  could be allowed).
- Again  $f(k) = k^\alpha$ .

## Canonical Model II

- Consumption Euler equation:

$$\frac{c_2(t+1)}{c_1(t)} = \beta R(t+1)$$

- Savings should satisfy the equation

$$s(t) = \frac{\beta}{1+\beta} w(t), \quad (19)$$

- Constant saving rate, equal to  $\beta / (1 + \beta)$ , out of labor income for each individual.



## Canonical Model III

- Combining this with the capital accumulation equation (8),

$$\begin{aligned}
 k(t+1) &= \frac{s(t)}{(1+n)} \\
 &= \frac{\beta w(t)}{(1+n)(1+\beta)} \\
 &= \frac{\beta(1-\alpha)[k(t)]^\alpha}{(1+n)(1+\beta)},
 \end{aligned}$$

- Second line uses (19) and last uses that, given competitive factor markets,  $w(t) = (1-\alpha)[k(t)]^\alpha$ .
- There exists a unique steady state with

$$k^* = \left[ \frac{\beta(1-\alpha)}{(1+n)(1+\beta)} \right]^{\frac{1}{1-\alpha}}. \quad (20)$$

- Equilibrium dynamics are identical to those of the basic Solow model and monotonically converge to  $k^*$ .

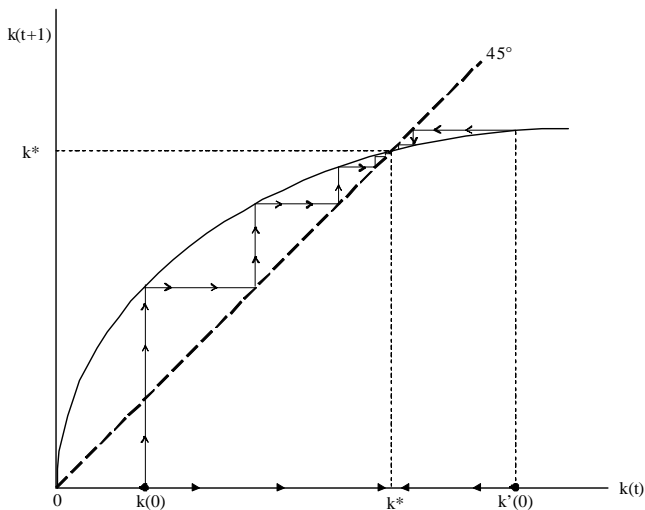


Figure: Equilibrium dynamics in the canonical overlapping generations model.

# Canonical Model IV

**Proposition** In the canonical overlapping generations model with log preferences and Cobb-Douglas technology, there exists a unique steady state, with capital-labor ratio  $k^*$  given by (20). Starting with any  $k(0) \in (0, k^*)$ , equilibrium dynamics are such that  $k(t) \uparrow k^*$ , and starting with any  $k'(0) > k^*$ , equilibrium dynamics involve  $k(t) \downarrow k^*$ .

# Overaccumulation I

- Compare the overlapping-generations equilibrium to the choice of a social planner wishing to maximize a weighted average of all generations' utilities.
- Suppose that the social planner maximizes

$$\sum_{t=0}^{\infty} \beta_S^t U(t)$$

- $\beta_S$  is the discount factor of the social planner, which reflects how she values the utilities of different generations.

## Overaccumulation II

- Substituting from (1), this implies:

$$\sum_{t=0}^{\infty} \beta^t (u(c_1(t)) + \beta u(c_2(t+1)))$$

subject to the resource constraint

$$F(K(t), L(t)) = K(t+1) + L(t)c_1(t) + L(t-1)c_2(t).$$

- Dividing this by  $L(t)$  and using (2),

$$f(k(t)) = (1+n)k(t+1) + c_1(t) + \frac{c_2(t)}{1+n}.$$

## Overaccumulation III

- Social planner's maximization problem then implies the following first-order necessary condition:

$$u'(c_1(t)) = \beta f'(k(t+1)) u'(c_2(t+1)).$$

- Since  $R(t+1) = f'(k(t+1))$ , this is identical to (5).
- Not surprising: allocate consumption of a given individual in exactly the same way as the individual himself would do.
- No “market failures” in the over-time allocation of consumption at given prices.
- However, the allocations across generations may differ from the competitive equilibrium: planner is giving different weights to different generations
- In particular, competitive equilibrium is **Pareto suboptimal** when  $k^* > k_{gold}$ ,

## Overaccumulation IV

- When  $k^* > k_{gold}$ , reducing savings can increase consumption for every generation.
- More specifically, note that in steady state

$$\begin{aligned} f(k^*) - (1+n)k^* &= c_1^* + (1+n)^{-1}c_2^* \\ &\equiv c^*, \end{aligned}$$

- First line follows by national income accounting, and second defines  $c^*$ .
- Therefore

$$\frac{\partial c^*}{\partial k^*} = f'(k^*) - (1+n)$$

- $k_{gold}$  is defined as

$$f'(k_{gold}) = 1 + n.$$

## Overaccumulation V

- Now if  $k^* > k_{gold}$ , then  $\partial c^* / \partial k^* < 0$ : reducing savings can increase (total) consumption for everybody.
- If this is the case, the economy is referred to as *dynamically inefficient*—it involves overaccumulation.
- Another way of expressing dynamic inefficiency is that

$$r^* < n,$$

- Recall in infinite-horizon Ramsey economy, transversality condition required that  $r > g + n$ .
- Dynamic inefficiency arises because of the heterogeneity inherent in the overlapping generations model, which removes the transversality condition.
- Suppose we start from steady state at time  $T$  with  $k^* > k_{gold}$ .



## Overaccumulation VI

- Consider the following variation: change next period's capital stock by  $-\Delta k$ , where  $\Delta k > 0$ , and from then on, we immediately move to a new steady state (clearly feasible).
- This implies the following changes in consumption levels:

$$\Delta c(T) = (1+n)\Delta k > 0$$

$$\Delta c(t) = -\left(f'(k^* - \Delta k) - (1+n)\right)\Delta k \text{ for all } t > T$$

- The first expression reflects the direct increase in consumption due to the decrease in savings.
- In addition, since  $k^* > k_{gold}$ , for small enough  $\Delta k$ ,  $f'(k^* - \Delta k) - (1+n) < 0$ , thus  $\Delta c(t) > 0$  for all  $t \geq T$ .
- The increase in consumption for each generation can be allocated equally during the two periods of their lives, thus necessarily increasing the utility of all generations.

# Pareto Optimality and Suboptimality in the OLG Model

**Proposition** In the baseline overlapping-generations economy, the competitive equilibrium is not necessarily Pareto optimal. More specifically, whenever  $r^* < n$  and the economy is dynamically inefficient, it is possible to reduce the capital stock starting from the competitive steady state and increase the consumption level of all generations.

- Pareto inefficiency of the competitive equilibrium is intimately linked with *dynamic inefficiency*.

# Interpretation

- Intuition for dynamic inefficiency:
  - Individuals who live at time  $t$  face prices determined by the capital stock with which they are working.
  - Capital stock is the outcome of actions taken by previous generations.
  - Pecuniary externality from the actions of previous generations affecting welfare of current generation.
  - Pecuniary externalities typically second-order and do not matter for welfare.
  - But not when an infinite stream of newborn agents joining the economy are affected.
  - It is possible to rearrange in a way that these pecuniary externalities can be exploited.

## Further Intuition

- Complementary intuition:
  - Dynamic inefficiency arises from overaccumulation.
  - Results from current young generation needs to save for old age.
  - However, the more they save, the lower is the rate of return and may encourage to save even more.
  - Effect on future rate of return to capital is a pecuniary externality on next generation
  - If alternative ways of providing consumption to individuals in old age were introduced, overaccumulation could be ameliorated.

# Role of Social Security in Capital Accumulation

- Social Security as a way of dealing with overaccumulation
- Fully-funded system: young make contributions to the Social Security system and their contributions are paid back to them in their old age.
- Unfunded system or a *pay-as-you-go*: transfers from the young directly go to the current old.
- Pay-as-you-go (unfunded) Social Security discourages aggregate savings.
- With dynamic inefficiency, discouraging savings may lead to a Pareto improvement.

# Fully Funded Social Security I

- Government at date  $t$  raises some amount  $d(t)$  from the young, funds are invested in capital stock, and pays workers when old  $R(t+1)d(t)$ .
- Thus individual maximization problem is,

$$\max_{c_1(t), c_2(t+1), s(t)} u(c_1(t)) + \beta u(c_2(t+1))$$

subject to

$$c_1(t) + s(t) + d(t) \leq w(t)$$

and

$$c_2(t+1) \leq R(t+1)(s(t) + d(t)),$$

for a given choice of  $d(t)$  by the government.

- Notice that now the total amount invested in capital accumulation is  $s(t) + d(t) = (1+n)k(t+1)$ .

# Fully Funded Social Security II

- No longer the case that individuals will always choose  $s(t) > 0$ .
- As long as  $s(t)$  is free, whatever  $\{d(t)\}_{t=0}^{\infty}$ , the competitive equilibrium applies.
- When  $s(t) \geq 0$  is imposed as a constraint, competitive equilibrium applies if given  $\{d(t)\}_{t=0}^{\infty}$ , privately-optimal  $\{s(t)\}_{t=0}^{\infty}$  is such that  $s(t) > 0$  for all  $t$ .

## Fully Funded Social Security III

**Proposition** Consider a fully funded Social Security system in the above-described environment whereby the government collects  $d(t)$  from young individuals at date  $t$ .

- ① Suppose that  $s(t) \geq 0$  for all  $t$ . If given the feasible sequence  $\{d(t)\}_{t=0}^{\infty}$  of Social Security payments, the utility-maximizing sequence of savings  $\{s(t)\}_{t=0}^{\infty}$  is such that  $s(t) > 0$  for all  $t$ , then the set of competitive equilibria without Social Security are the set of competitive equilibria with Social Security.
  - ② Without the constraint  $s(t) \geq 0$ , given any feasible sequence  $\{d(t)\}_{t=0}^{\infty}$  of Social Security payments, the set of competitive equilibria without Social Security are the set of competitive equilibria with Social Security.
- Moreover, even when there is the restriction that  $s(t) \geq 0$ , a funded Social Security program cannot lead to the Pareto improvement.



# Unfunded Social Security I

- Government collects  $d(t)$  from the young at time  $t$  and distributes to the current old with per capita transfer  $b(t) = (1+n)d(t)$
- Individual maximization problem becomes

$$\max_{c_1(t), c_2(t+1), s(t)} u(c_1(t)) + \beta u(c_2(t+1))$$

subject to

$$c_1(t) + s(t) + d(t) \leq w(t)$$

and

$$c_2(t+1) \leq R(t+1)s(t) + (1+n)d(t+1),$$

for a given feasible sequence of Social Security payment levels  $\{d(t)\}_{t=0}^{\infty}$ .

- Rate of return on Social Security payments is  $n$  rather than  $r(t+1) = R(t+1) - 1$ , because unfunded Social Security is a pure transfer system.

# Unfunded Social Security II

- Only  $s(t)$ —rather than  $s(t)$  plus  $d(t)$  as in the funded scheme—goes into capital accumulation.
- It is possible that  $s(t)$  will change in order to compensate, but such an offsetting change does not typically take place.
- Thus unfunded Social Security reduces capital accumulation.
- Discouraging capital accumulation can have negative consequences for growth and welfare.
- In fact, empirical evidence suggests that there are many societies in which the level of capital accumulation is suboptimally low.
- But here reducing aggregate savings may be good when the economy exhibits dynamic inefficiency.

## Unfunded Social Security III

**Proposition** Consider the above-described overlapping generations economy and suppose that the decentralized competitive equilibrium is dynamically inefficient. Then there exists a feasible sequence of unfunded Social Security payments  $\{d(t)\}_{t=0}^{\infty}$  which will lead to a competitive equilibrium starting from any date  $t$  that Pareto dominates the competitive equilibrium without Social Security.

- Similar to way in which the Pareto optimal allocation was decentralized in the example economy above.
- Social Security is transferring resources from future generations to initial old generation.
- But with *no* dynamic inefficiency, any transfer of resources (and any unfunded Social Security program) would make some future generation worse-off.

# Overlapping Generations with Impure Altruism I

- Exact form of altruism within a family matters for whether the representative household would provide a good approximation.
- Parents care about certain dimensions of the consumption vector of their offspring instead of their total utility or “impure altruism.”
- A particular type, “warm glow preferences”: parents derive utility from their bequest.
- Production side given by the standard neoclassical production function, satisfying Assumptions 1 and 2,  $f(k)$ .
- Economy populated by a continuum of individuals of measure 1.
- Each individual lives for two periods, childhood and adulthood.
- In second period of his life, each individual begets an offspring, works and then his life comes to an end.
- No consumption in childhood (or incorporated in the parent’s consumption).

## Overlapping Generations with Impure Altruism II

- No new households, so population is constant at 1.
- Each individual supplies 1 unit of labor inelastically during his adulthood.
- Preferences of individual  $(i, t)$ , who reaches adulthood at time  $t$ , are

$$\log(c_i(t)) + \beta \log(b_i(t)), \quad (21)$$

where  $c_i(t)$  denotes the consumption of this individual and  $b_i(t)$  is bequest to his offspring.

- Offspring starts the following period with the bequest, rents this out as capital to firms, supplies labor, begets his own offspring, and makes consumption and bequests decisions.
- Capital fully depreciates after use.

# Overlapping Generations with Impure Altruism III

- Maximization problem of a typical individual can be written as

$$\max_{c_i(t), b_i(t)} \log(c_i(t)) + \beta \log(b_i(t)), \quad (22)$$

subject to

$$c_i(t) + b_i(t) \leq y_i(t) \equiv w(t) + R(t)b_i(t-1), \quad (23)$$

where  $y_i(t)$  denotes the income of this individual.

- Equilibrium wage rate and rate of return on capital

$$w(t) = f(k(t)) - k(t)f'(k(t)) \quad (24)$$

$$R(t) = f'(k(t)) \quad (25)$$

- Capital-labor ratio at time  $t+1$  is:

$$k(t+1) = \int_0^1 b_i(t) di, \quad (26)$$

# Overlapping Generations with Impure Altruism IV

- Measure of workers is 1, so that the capital stock and capital-labor ratio are identical.
- Denote the distribution of consumption and bequests across households at time  $t$  by  $[c_i(t)]_{i \in [0,1]}$  and  $[b_i(t)]_{i \in [0,1]}$ .
- Assume the economy starts with the distribution of wealth (bequests) at time  $t$  given by  $[b_i(0)]_{i \in [0,1]}$ , which satisfies  $\int_0^1 b_i(0) di > 0$ .

**Definition** An equilibrium in this overlapping generations economy with warm glow preferences is a sequence of consumption and bequest levels for each household,  $\left\{ [c_i(t)]_{i \in [0,1]}, [b_i(t)]_{i \in [0,1]} \right\}_{t=0}^{\infty}$ , that solve (22) subject to (23), a sequence of capital-labor ratios,  $\{k(t)\}_{t=0}^{\infty}$ , given by (26) with some initial distribution of bequests  $[b_i(0)]_{i \in [0,1]}$ , and sequences of factor prices,  $\{w(t), R(t)\}_{t=0}^{\infty}$ , that satisfy (24) and (25).

# Overlapping Generations with Impure Altruism V

- Solution of (22) subject to (23) is straightforward because of the log preferences,

$$\begin{aligned} b_i(t) &= \frac{\beta}{1+\beta} y_i(t) \\ &= \frac{\beta}{1+\beta} [w(t) + R(t) b_i(t-1)], \end{aligned} \quad (27)$$

for all  $i$  and  $t$ .

- Bequest levels will follow non-trivial dynamics.
- $b_i(t)$  can alternatively be interpreted as “wealth” level: distribution of wealth that will evolve endogenously.
- This evolution will depend on factor prices.
- To obtain factor prices, aggregate bequests to obtain the capital-labor ratio of the economy via equation (26).



# Overlapping Generations with Impure Altruism VI

- Integrating (27) across all individuals,

$$\begin{aligned}
 k(t+1) &= \int_0^1 b_i(t) di \\
 &= \frac{\beta}{1+\beta} \int_0^1 [w(t) + R(t) b_i(t-1)] di \\
 &= \frac{\beta}{1+\beta} f(k(t)).
 \end{aligned} \tag{28}$$

- The last equality follows from the fact that  $\int_0^1 b_i(t-1) di = k(t)$  and because by Euler's Theorem,  $w(t) + R(t) k(t) = f(k(t))$ .
- Thus dynamics are straightforward and again closely resemble Solow growth model.
- Moreover dynamics do *not* depend on the distribution of bequests or income across households.

## Overlapping Generations with Impure Altruism VII

- Solving for the steady-state equilibrium capital-labor ratio from (28),

$$k^* = \frac{\beta}{1 + \beta} f(k^*), \quad (29)$$

- Uniquely defined and strictly positive in view of Assumptions 1 and 2.
- Moreover, equilibrium dynamics again involve monotonic convergence to this unique steady state.
- We know that  $k(t) \rightarrow k^*$ , so the ultimate bequest dynamics are given by steady-state factor prices.
- Let these be denoted by  $w^* = f(k^*) - k^* f'(k^*)$  and  $R^* = f'(k^*)$ .
- Once the economy is in the neighborhood of the steady-state capital-labor ratio,  $k^*$ ,

$$b_i(t) = \frac{\beta}{1 + \beta} [w^* + R^* b_i(t-1)].$$

# Overlapping Generations with Impure Altruism VIII

- When  $R^* < (1 + \beta) / \beta$ , starting from any level  $b_i(t)$  will converge to a unique bequest (wealth) level

$$b^* = \frac{\beta w^*}{1 + \beta(1 - R^*)}. \quad (30)$$

- Moreover, it can be verified that  $R^* < (1 + \beta) / \beta$ ,

$$\begin{aligned} R^* &= f'(k^*) \\ &< \frac{f(k^*)}{k^*} \\ &= \frac{1 + \beta}{\beta}, \end{aligned}$$

- Second line exploits the strict concavity of  $f(\cdot)$  and the last line uses the definition of  $k^*$  from (29).

# Overlapping Generations with Impure Altruism IX

**Proposition** Consider the overlapping generations economy with warm glow preferences described above. In this economy, there exists a unique competitive equilibrium. In this equilibrium the aggregate capital-labor ratio is given by (28) and monotonically converges to the unique steady-state capital-labor ratio  $k^*$  given by (29). The distribution of bequests and wealth ultimately converges towards full equality, with each individual having a bequest (wealth) level of  $b^*$  given by (30) with  $w^* = f(k^*) - k^* f'(k^*)$  and  $R^* = f'(k^*)$ .

# Conclusions

- Overlapping generations are more realistic than infinity-lived representative agents.
- Models with overlapping generations fall outside the scope of the First Welfare Theorem:
  - they were partly motivated by the possibility of Pareto suboptimal allocations.
- Equilibria may be “dynamically inefficient” and feature overaccumulation: unfunded Social Security can ameliorate the problem.
- Declining path of labor income important for overaccumulation, and what matters is not finite horizons but arrival of new individuals.
- Overaccumulation and Pareto suboptimality: pecuniary externalities created on individuals that are not yet in the marketplace.
- Not overemphasize dynamic inefficiency: major question of economic growth is why so many countries have so little capital.