Dynamic Moral Hazard with Persistent States*

Suehyun Kwon

July 30, 2012

Abstract

This paper studies a model of principal-agent problem in a partially persistent environment. The costly unobservable action of the agent produces a good outcome with some probability, and the probability of good outcome corresponds to the state. The states are unobservable and follow an irreducible Markov chain with positive persistence. The paper finds that an informational rent arises in this environment. The principal can, however, reduce the rent by taking an inefficient outside option in some periods. In some situations, the second best contract resembles a tenure system: the agent is paid nothing during the probationary period after which the principal implements the first best action in every period. In some circumstances, the second best contract becomes stationary after the agent is tenured. The paper provides a recursive formulation for complete characterization of the second best. For discount factors close to one, the principal can approximate his first best payoff.

Keywords: Persistence, Moral hazard.

1 Introduction

There is a large literature on repeated moral hazard with i.i.d. states or fully persistent states.1 But there are circumstances that are better described by partially persistent states.

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* Kwon: Massachusetts Institute of Technology, suehyun.kwon@gmail.com, Department of Economics, University College London, Gower Street, London, WC1E 6BT United Kingdom. I’m very grateful to Glenn Ellison, Jumso Toikka and Muhamet Yildiz for time, advice and encouragement. This paper has benefited from suggestions of Gabriel Carroll, Robert Gibbons, Bengt Holmstrom, Jean Tirole, Alexander Wolitzky and seminar participants at MIT, Pennsylvania State University, Princeton and University College London. I thank Samsung Scholarship for financial support.
Productivity of an economic sector varies from year to year, but high productivity is more likely to come after a productive year than after an unproductive year. Technological advances are also often clustered, and much advance can be made in a short amount of time. It is worthwhile to consider dynamic moral hazard with partially persistent states. In this paper, I examine such a model; I note that the persistent states create an informational rent for the agent and explore properties of the second best contracts. The main result is that in some circumstances, the second-best contract resembles a tenure contract.

This paper develops a model of principal-agent problem in which the underlying environment is partially persistent. The principal hires the agent over an infinite horizon, and each period, the agent can work or shirk. The costly unobservable action, work, produces a good outcome with some probability, and the probability of the good outcome depends on the state. The states are unobservable, and they follow an irreducible Markov chain with positive persistence. The principal observes the outcome and pays the agent. The principal can commit to a long term contract. Both parties are risk-neutral, and the agent is subject to limited liability. In the base model, the first best is to have the agent work in every period.

I start in Section 3 with an analysis of one and two period versions of the model. The two period model illustrates one of the basic insights: the deviations of the agent create information asymmetry, and the agent receives an informational rent if the principal wants him to work in both periods. When the states are persistent, the outcome has informational value in addition to its payoff consequence. If the principal believes that the agent worked this period, the principal updates his belief about the state after observing the outcome. The agent’s deviation leads to both the unproductive outcome and the information asymmetry between the principal and the agent. Since the bad outcome lowers the prior of being in the good state, the principal assigns a strictly lower probability to the good state than the agent in the period after the agent deviates. Working in the following period after shirking, the agent can ensure himself a strictly positive amount of rent by a one-shot deviation. I characterize the informational rent as a function of the discount factor, the persistence of the states and the informativeness of the outcomes, and I give conditions under which the principal wants the agent to work in both periods.

Section 4 turns to the infinite horizon case, and it describes the informational rent and derives an upper bound on the informational rent. The agent’s deviation creates information asymmetry between the principal and the agent, and in all periods following the deviation, the agent assigns weakly higher probabilities to the good state than the principal does. After any history, if the principal wants the agent to work, the principal has to provide as much rent as what the agent can get from deviating. However, I show that the principal can attain an upper bound on informational rents by offering a contract that is stationary from the second
period. When a contract is stationary, the information asymmetry doesn’t play a role in the continuation values of the agent, and the agent works in any given period if the expected payment is greater than the cost. The upper bound increases with the discount factor, the persistence of the states, and the informativeness of the outcomes. If the discount factor, the persistence of the states, or the informativeness of the outcomes is small, the principal can approximate his first best payoff with a contract that is stationary from the second period.

In Section 5, I consider a case particularly tractable in which the states correspond to the outcome that occurs if the agent exerts effort, i.e., the probability of good outcome at $t + 1$ is $M_{11}$ if good outcome is obtained at $t$ (or would have been obtained if the agent had worked) and $M_{21}$ otherwise. This case allows one a number of additional results. First, I consider contracts that induce the agent to work in every period. The main result is that the upper bound on informational rents from Section 4 turns out to be the lower bound on the rent also. Hence, the cost-minimizing contract that induces the agent to work in every period is stationary from the second period. The deterministic mapping from the states to the outcomes turns out to be a necessary condition for a history-independent contract to be optimal.

Section 6 presents the main result about tenure contracts. It considers the case studied in Section 5 and characterizes the second best contract. This contract turns out to have features of a tenure system. The agent is paid nothing during the probationary period, and once the agent is paid, the principal never takes his outside option again. By backloading the payment, the principal can provide better incentives to the agent, and he can offer a continuation contract with a higher expected outcome. The principal doesn’t benefit from backloading the payment only if he is already inducing the agent to work in every period, and therefore, once the agent is getting paid, he is tenured, and the principal never takes his outside option again. The second best contract also becomes stationary after the agent is tenured. Since the principal never takes his outside option again, he can offer the cost-minimizing contract as the continuation contract. When the state variable is whether or not working will produce a good outcome, the cost-minimizing contract is stationary from the second period, and the second best contract becomes stationary. The principal’s information changes over time, but if the principal uses his information, the agent can deviate and create information asymmetry; the principal is better off by committing not to use his information. I provide a recursive formulation to complete the characterization with the length of the probationary periods.

The paper also considers what happens in the limit as the discount factor goes to one. Here, I show that the principal can approximate his first-best payoff when the discount factor is close to one. The proof of the result is not based on computing the second best contract.
Instead, I just note that the principal could employ a review strategy with a sufficiently long block so that the probability of meeting the ergodic distribution by working in every period is close to one. Having a lump sum transfer and continuing with the contract when the agent meets the quota, the principal can ensure that the agent works in every period. When the discount factor is close to one, the principal’s payoff from each block gets arbitrarily close to his first best payoff with a sufficiently high probability, and the principal can approximate his first best payoff.

In Section 8, I consider the second best contract of an alternative specification of the model, when the first best involves taking the outside option in some periods. Under the second best contract, there is a probationary period during which the agent works for no payment. Once the principal pays the agent, he implements the first best action in every period. The results of this section highlight that probationary periods followed by the first best actions are common features of the second best contracts, whether or not the first best involves the outside option. However, the second best contract after the probationary period is not stationary if the first best involves the outside option. The stationarity of the second best contract depends on parameter values.

The informational aspect of the rent in my paper is related to the ratchet effect in dynamic adverse selection. Including Laffont and Tirole (1988), there have been numerous papers on ratchet effect. In these papers, the principal cannot commit to a long term contract, and there is much pooling in the first period. The principal can commit to a long term contract in my model, but if the principal were to use his information, the agent can deviate and create information asymmetry between the principal and the agent. The persistence of the states leads to the informational component of the agent’s action, and the rent is informational.

The second best takes a particular form in my model: after a probationary period, the agent is tenured, and the continuation contract becomes stationary or has a finite memory. The probationary period of the second best contract shares similarities with Chassang (2010). Fong and Li (2010) also find that the optimal contract has a probationary phase. In these papers, the principal cannot commit to a long term contract and the environment is i.i.d., whereas in my model, the principal can commit to a long term contract, and the states are partially persistent. Backloading of incentives is also related to Thomas and Worrall (1994, 2010), where the states are i.i.d. or observable, and the contracts must be self-enforcing. The stationarity of the continuation contract after tenure is related to the literature on sticky wages. In Townsend (1982), long-term contracts and inefficient tie-ins can be optimal under private information. The stationary payments of the second best of my model shows that the stationarity of a long-term contract is not necessarily because of enforcement costs. Fernandez and Phelan (2000) provides a recursive formulation with persistent private
information.

The first best approximation under little discounting is related to folk theorem results. Review strategies are first introduced by Radner (1981, 1985), and Fudenberg, Holmstrom and Milgrom (1990) show conditions under which the first best can be approximated with short term contracts. Other papers on the approximation of the first best include Rubinstein (1979), Rubinstein and Yaari (1983), and Dutta and Radner (1994).

Other papers with persistent states include Bhaskar (2012), where he considers moral hazard with fully persistent states. In Kwon (2012), I consider relational contracts with persistent, observable states. Another paper in which the outcome carries information about both the agent’s effort and the future profitability is DeMarzo and Sannikov (2011). Their model is in continuous time, and the firm’s fundamental evolves over time according to a Brownian motion. Garrett and Pavan (2011, 2012) study models of principal-agent problem where there is both moral hazard and exogenous, persistent private information. There is also literature on dynamic adverse selection with persistent private information. With partially persistent types, the optimal contract is often history-contingent, but the principal achieves efficiency in the limit. Papers with Markovian types include Battaglini (2005) and Athey and Bagwell (2008). Battaglini (2005) considers consumers with Markovian types, and Athey and Bagwell (2008) study collusion with persistent private shocks. Tchistyi (2006) considers security design when cash flow is correlated over time. Escobar and Toikka (2012) show the folk theorem result with Markovian types and communication.

Lastly, my paper is related to papers on innovation and venture capital. Bergemann and Hege (2005) study financing innovation, where the good project succeeds with a probability proportional to investment. Manso (2011) considers a model of two-armed bandit problem with one known arm.

The rest of the paper is organized as the following. Section 2 describes the model, and one and two period examples are described in Section 3. Section 4 discusses the informational rent. Section 5 discusses the case when the states correspond to the outcomes of working, and I characterize the second best of this case in Section 6. The first best approximation is considered in Section 7. I study the second best of an alternative specification of the model in Section 8. Section 9 concludes.

2 Model

The principal hires the agent over an infinite horizon $t = 1, 2, \cdots$. The common discount factor is $\delta < 1$, and the principal can commit to a long term contract.

Each period, the agent can work or shirk, and the outcome is either 1 or 0, which I call
the good outcome and the bad outcome. Shirking costs nothing to the agent, but it produces the bad outcome with probability 1. Work costs $c > 0$ to the agent, but it produces a good outcome with some probability. The probability of the good outcome depends on the state. The agent’s action is unobservable to the principal, and the principal only observes the outcome.

There are two states, the good state (state 1) and the bad state (state 2). Throughout the paper, the subscript 1 refers to the good state, and the subscript 2 refers to the bad state. The probability of the good outcome is $p_H$ in the good state, and it is $p_L$ in the bad state. The probability is strictly higher in the good state than in the bad state, and $0 \leq p_L < p_H \leq 1$. The state is unobservable to both parties. The states follow an irreducible Markov chain with positive persistence. Specifically, let $M$ be the Markov transition matrix for the state transition with entries

$$M_{ij} = \Pr(s_{t+1} = j \mid s_t = i),$$

where $s_t$ is the state in period $t$. The next assumption states that the states are partially persistent.

**Assumption 1 (Persistence).** The Markov matrix $M$ for the state transition satisfies

$$\text{det } M > 0, \ 0 < M_{ij} < 1, \forall i, j.$$ 

The positive persistence of the states is captured by the condition $\text{det } M > 0$. The determinant of the Markov matrix is

$$\text{det } M = M_{11}M_{22} - M_{12}M_{21} = M_{11}(1 - M_{21}) - (1 - M_{11})M_{21} = M_{11} - M_{21}.$$ 

Since $M_{11}$ and $M_{21}$ are the probabilities of the good state after the good state and the bad state, respectively, the determinant is the difference in the probabilities of being in the good state. When the states have positive persistence, the probability of being in the good state is strictly higher after being in the good state than after being in the bad state. Note also that I’m assuming that there is always a positive probability of transiting from state $i$ to state $j$, for all $i, j = 1, 2$, which implies that the Markov chain is irreducible.

The principal can take an outside option in any period, but in the first best, the principal wants the agent to work in every period. Let $\pi^t$ be the principal’s prior on the state at the
beginning of period $t$. $\pi^t$ is a vector of beliefs, and $\pi_i^t$ is the probability that the state in period $t$ will be state $i$. I assume that the initial prior $\pi^1$ satisfies $M_{21} \leq \pi^1_1 \leq M_{11}$. Then, for all $t \geq 1$, we have $M_{21} \leq \pi^t_1 \leq M_{11}$. Let $u$ be the payoff to the principal from his outside option. I assume that the payoff to the agent is zero if the principal takes his outside option. The following assumption says that it is efficient to have the agent work for any given prior on the state. I also assume that taking the outside option is better than not inducing the agent to work, which implies that if the principal doesn’t want the agent to work in a given period, the principal takes his outside option.

**Assumption 2 (Efficiency).** The parameters $M, c, p_H, p_L$ and $u$ are such that

$$M_{21}p_H + M_{22}p_L - c \geq u > 0.$$ 

Assumption 2 is relaxed in Section 8; I consider the second best contract when the first best involves taking the outside option in some periods. The rest of the paper assumes Assumption 2.

Both the principal and the agent are risk-neutral, and the agent is subject to limited liability. There are three constraints to consider, IR, IC and limited liability. IR means that in period 1, the agent receives at least the payoff from his outside option in expectation by participating in the contract. Limited liability requires that the agent receives non-negative payments. I normalize the outside option of the agent to zero, and IR is implied by the IC constraint and limited liability. The IC constraint has to be satisfied at every node at which the principal wants the agent to work. I assume that if indifferent between working and shirking, the agent chooses to work.

Throughout the paper, $h_t$ refers to the outcome in period $t$. When the principal takes the outside option, I denote it by $h_t = -1$. The history $h^t$ is the sequence of outcomes up to period $t$, and I denote it by $h^t = h_1 \cdots h_t$. The initial history is an empty set and is denoted by $h^0 = h_0 = \emptyset$. $h^t \sqcup h^k$ refers to the history that $h^t$ is followed by $h^k$.

In the first best, if the actions of the agent were observable, the principal can pay the cost if and only if the agent works. However, the states and the actions are unobservable, and the principal only observes the outcomes. A contract specifies the principal’s decisions to take his outside option and history-contingent payments $w(h^t)$ for all histories $h^t$.

### 3 One and Two Period Examples

This section describes the one and two period examples of the model. The main observation these models bring out is that an informational rent arises when the principal wants the agent
to work in both periods. The informational rent doesn’t exist in the one period example, but it arises in the two period example when the principal wants the agent to work in both periods. The rent is proportional to the discount factor and the persistence of the states. When the discount factor is low, it can be optimal to have the agent work in every period, and the optimal contract is independent of the history.

3.1 One Period

In this section, I consider the case when the principal hires the agent for one period and show that the principal can leave no rent to the agent.

Let $\pi = (\pi_1, \pi_2)$ be the common prior; $\pi_1$ is the probability of the good state. Since the principal only observes the outcome, and not the action of the agent, he offers payments $w(0)$ and $w(1)$ as a function of the outcome. The agent’s IC constraint is given by

$$-c + \pi \left( \frac{p_H}{p_L} \right) w(1) + \left( 1 - \pi \frac{p_H}{p_L} \right) w(0) \geq w(0).$$

The agent is subject to limited liability, and this imposes

$$w(0) \geq 0, \quad w(1) \geq 0.$$

The optimal contract for the principal is to provide

$$w(0) = 0, \quad w(1) = \frac{c}{\pi \frac{p_H}{p_L}}.$$

The expected rent to the agent is

$$-c + \pi \left( \frac{p_H}{p_L} \right) w(1) + \left( 1 - \pi \frac{p_H}{p_L} \right) w(0) = 0,$$

and the agent receives zero rent in the one period model.

3.2 Two Periods

In this section, I consider the two period example of the model. I note that the principal has to leave an informational rent to the agent if he wants the agent to work in both periods. The rent is proportional to the discount factor and the persistence of the states. For low discount factors, it can be optimal to leave the rent and have the agent work in both periods. The principal can provide a history-independent contract to have the agent work in both
Proposition 1. Suppose Assumptions 1-2 hold. Suppose the principal wants the agent to work in both periods. The rent to the agent is bounded from below by

\[ \delta c \det M \frac{\pi_1^1 \pi_2^1 (p_H - p_L)^2}{\pi_1^1 (1-p_H) \tilde{\pi}_2^2 (p_H)} \]

where

\[ \tilde{\pi}_2^2 = \left( \frac{\pi_1^1 (1 - p_H)}{\pi_1^1 (1-p_H)}, \frac{\pi_2^1 (1 - p_L)}{\pi_1^1 (1-p_L)} \right) M \]

is the principal’s prior in the beginning of period 2 after the bad outcome in period 1.

Proof. We have the following expressions for the priors in the beginning of period 2. After the good outcome, the prior becomes

\[ \pi_2^2 = \left( \frac{\pi_1^1 p_H}{\pi_1^1 (1-p_H)}, \frac{\pi_2^1 p_L}{\pi_1^1 (1-p_L)} \right) M. \]

After the bad outcome, when the principal believes that the agent worked in the first period, his prior in the following period is

\[ \tilde{\pi}_2^2 = \left( \frac{\pi_1^1 (1 - p_H)}{\pi_1^1 (1-p_H)}, \frac{\pi_2^1 (1 - p_L)}{\pi_1^1 (1-p_L)} \right) M. \]

There are two IC constraints to consider in period 1. Denote by \( w(0), w(1), w(00), w(01), w(10) \) and \( w(11) \) the history-contingent payments. Also denote by \( R_1 \) and \( R_0 \) the rents to the agent in period 2 after the good outcome and the bad outcome, respectively. The rents are given by the following expressions:

\[ R_1 = -c + \pi^2 \left( \frac{p_H}{p_L} \right) w(11) + (1 - \pi^2 \left( \frac{p_H}{p_L} \right)) w(10), \]

\[ R_0 = -c + \tilde{\pi}^2 \left( \frac{p_H}{p_L} \right) w(01) + (1 - \tilde{\pi}^2 \left( \frac{p_H}{p_L} \right)) w(00). \]

The one-shot deviation of the agent is to shirk in period 1 and work again in period 2. When the agent deviates and shirks in period 1, he doesn’t update his posterior after observing the outcome. The agent’s prior in the beginning of period 2 is \( \pi^1 M \). The IC constraint for the one-shot deviation is given by
\[-c + \pi^1 \left( \frac{p_H}{p_L} \right) (w(1) + \delta R_1) + (1 - \pi^1 \left( \frac{p_H}{p_L} \right))(w(0) + \delta R_0) \]
\[\geq w(0) + \delta(-c + \pi^1 M \left( \frac{p_H}{p_L} \right) w(01) + (1 - \pi^1 M \left( \frac{p_H}{p_L} \right))w(00)).\]

The second IC constraint is for which the agent deviates twice in a row and shirks in both periods. The IC constraint is given by

\[-c + \pi^1 \left( \frac{p_H}{p_L} \right) (w(1) + \delta R_1) + (1 - \pi^1 \left( \frac{p_H}{p_L} \right))(w(0) + \delta R_0) \]
\[\geq w(0) + \delta w(00).\]

On the other hand, the IC constraints for period 2 are the following: after the good outcome, the IC constraint is

\[R_1 \geq w(10),\]
and after the bad outcome, the IC constraint is

\[R_0 \geq w(00).\]

The second period IC constraint after the bad outcome is equivalent to

\[w(01) - w(00) \geq \frac{c}{\pi^2 \left( \frac{p_H}{p_L} \right)}.\]

From the positive persistence of the states, we know that

\[-c + \pi^1 M \left( \frac{p_H}{p_L} \right) w(01) + (1 - \pi^1 M \left( \frac{p_H}{p_L} \right))w(00) \geq w(00)\]

whenever \(R_0 \geq w(00)\) holds. Therefore, in period 1, it is sufficient to consider the one-shot deviation of the agent.

The limited liability implies \(w(0) + \delta w(00) \geq 0\), and the first period IC constraint for the one-shot deviation becomes
\[-c + \pi_1 \left( \frac{p_H}{p_L} \right) (w(1) + \delta R_1) + (1 - \pi_1 \left( \frac{p_H}{p_L} \right)) (w(0) + \delta R_0) \geq w(0) + \delta (-c + \pi_1 M \left( \frac{p_H}{p_L} \right) w(01) + (1 - \pi_1 M \left( \frac{p_H}{p_L} \right)) w(00))
\]
\[= w(0) + \delta w(00) + \delta (-c + \pi_1 M \left( \frac{p_H}{p_L} \right) (w(01) - w(00))) \geq \delta c (-1 + \frac{\pi_1 M \left( \frac{p_H}{p_L} \right)}{\pi_2 \left( \frac{p_H}{p_L} \right)})
\]
\[= \delta c \det M \frac{\pi_1 \pi_2 (p_H - p_L)^2}{\pi^2 (1 - p_H) \pi^2 (p_H)}.
\]

Proposition 1 shows that the IC constraint and the limited liability imply a minimum rent to the agent when the principal wants the agent to work in both periods. Note that the bound on the rent is proportional to the discount factor and the persistence of the states. If the states were i.i.d., i.e., \( \det M = 0 \) or \( p_H = p_L \), the principal can have the agent work in both periods and leave no rent.

Note also that the rent to the agent depends on the payments in the second period after the bad outcome. The payments in the second period after the good outcome doesn’t matter for the minimum rent to the agent. When the agent deviates, the principal believes that the agent worked but the outcome is bad; the principal offers the payments in the second period as if the outcome of working was bad in the first period, and the agent’s continuation value after deviation is determined by the payments in the second period after the bad outcome.

Proposition 1 shows that there exists a lower bound on the rent the principal has to leave in order to have the agent work in both periods. In Proposition 2, I show that the lower bound is tight, and I also show a possible form of the contract the principal can provide.

**Proposition 2.** Suppose Assumptions 1-2 hold. Suppose the principal wants the agent to work in both periods. The principal can achieve the minimum rent by offering the identical contract in period 2 independent of the outcome in period 1:
\[ w(0) = w(00) = w(10) = 0, \]
\[ w(1) = \frac{c}{\pi^1(p_H)}, \]
\[ w(11) = w(01) = \frac{c}{\pi^2(p_H)}, \]

where
\[ \tilde{\pi}^2 = \left( \frac{\pi_1^1(1 - p_H)}{\pi_1^1(1 - p_L)}, \frac{\pi_2^1(1 - p_L)}{\pi_1^1(1 - p_L)} \right) M \]

is the principal’s prior in the beginning of period 2 after the bad outcome in period 1.

Throughout the paper, any proof that is not provided after the proposition is in the appendix.

The contract in Proposition 2 is one of the many contracts that the principal can provide to attain the lower bound on the rent. Since both the principal and the agent are risk-neutral, the principal can always delay the payment and pay the agent later. The above contract is nice because of the stationarity and the simplicity. The principal makes positive payments only for the good outcome, and in particular, the payments in the second period are independent of the outcome in the first period. The principal is leaving the rent to the agent because the deviation of the agent leads to information asymmetry between the principal and the agent, and one way to take care of the information asymmetry is to provide identical payments in the second period, regardless of the outcome in the first period.

Proposition 1 and Proposition 2 assume that the principal wants the agent to work in both periods. I will next show what happens when the principal takes his outside option in some periods. Since the outside option is inefficient, the principal incurs a loss in outcome by taking his outside option. However, depending on the timing of the outside option, the principal can prevent the information asymmetry from the agent’s deviation and therefore, reduce the rent to the agent.

If the principal takes his outside option in period 2 after the good outcome, the IC constraint in period 1 doesn’t change. The principal has to leave just as much rent as he would if the agent were to work in period 2 after the good outcome. Since the outside option is inefficient, the principal won’t take his outside option after the good outcome.

On the other hand, if the principal takes his outside option in period 2 after the bad outcome, the principal doesn’t have to leave the rent to the agent. The IC constraint in
period 1 becomes
\[-c + \pi^1 \left( \frac{p_H}{p_L} \right) (w(1) + \delta R_1) + (1 - \pi^1 \left( \frac{p_H}{p_L} \right)) (w(0) + \delta R_0) \geq w(0),\]
and \(R_0 = 0.\) By offering the following contract, the principal leaves no rent to the agent:
\[
\begin{align*}
w(0) &= w(10) = 0, \\
w(1) &= \frac{c}{\pi^1 \left( \frac{p_H}{p_L} \right)}, \\
w(11) &= \frac{c}{\pi^2 \left( \frac{p_H}{p_L} \right)},
\end{align*}
\]
where
\[
\pi^2 = \left( \frac{\pi^1 \left( \frac{p_H}{p_L} \right)}{\pi^1 \left( \frac{p_H}{p_L} \right)}, \frac{\pi^1 \left( \frac{p_H}{p_L} \right)}{\pi^1 \left( \frac{p_H}{p_L} \right)} \right) M
\]
is the principal’s prior in the beginning of period 2 after the good outcome in the first period.

The principal also leaves no rent to the agent if he takes his outside option in period 1. When the principal takes his outside option in period 1, period 2 is identical to the one period model with prior \(\pi^1 M,\) and the agent gets no rent.

If the principal mixes the continuation contracts, it has the same effect as taking the linear combination of the IC constraints, and it convexifies the set of payoffs.

Therefore, the agent gets a rent only if the principal wants him to work in period 1 and also in period 2 after the bad outcome in period 1 with a strictly positive probability.

However, if the discount factor is small for the given persistence of the states, it can be optimal to have the agent work in both periods and leave the rent. The amount of outcome the principal loses by taking his outside option in period 1 is
\[-c + \pi^1 \left( \frac{p_H}{p_L} \right) - u.
\]
If he takes his outside option in period 2 after the good or bad outcomes in period 1, it is
\[
\delta \pi^1 \left( \frac{p_H}{p_L} \right) (-c + \pi^2 \left( \frac{p_H}{p_L} \right) - u)
\]
and
\[
\delta \pi^1 \left( \frac{1 - p_H}{1 - p_H} \right) (-c + \pi^2 \left( \frac{p_H}{p_L} \right) - u),
\]
respectively.

If the loss in outcome is greater than the rent to the agent, the principal will choose to
have the agent work in both periods and leave the rent. This is the case if

$$\delta c \det M \frac{\pi_1 \pi_2 (p_H - p_L)^2}{\pi_1 (1-p_H) \bar{\pi}_2 (p_H)} < -c + \pi_1 \left(\frac{p_H}{p_L}\right) - u$$

and

$$\delta c \det M \frac{\pi_1 \pi_2 (p_H - p_L)^2}{\pi_1 (1-p_H) \bar{\pi}_2 (p_H)} < \delta \pi_1 \left(\frac{1-p_H}{1-p_L}\right) (-c + \bar{\pi}_2 \left(\frac{p_H}{p_L}\right) - u).$$

These calculations show that the principal leaves the rent only if he wants the agent to work in both periods. If the principal takes his outside option in the first period or in the second period after the bad outcome, the principal can leave no rent to the agent.

The above calculations also show that it can be optimal to have the agent work both periods and leave him with the rent. Since the outside option is inefficient, the principal incurs loss in outcome by taking the outside option, and if the loss in outcome is greater than the rent to the agent, the principal prefers to leave the rent and have the agent work in both periods.

**Proposition 3.** Suppose Assumptions 1-2 hold. Suppose the parameters satisfy the following two inequalities:

$$\delta \leq \frac{-c + \pi_1 \left(\frac{p_H}{p_L}\right) - u}{c \det M \frac{\pi_1 \pi_2 (p_H - p_L)^2}{\pi_1 (1-p_H) \bar{\pi}_2 (p_H)}}.$$

Then it is optimal to induce work in both periods and leave the rent to the agent. The optimal contract is the contract given in Proposition 2. Otherwise, it is optimal to take the outside option in some periods.

There are a few things to note here that will hold in the general model with the infinite horizon.

**Remark 1.** The principal has to leave the rent only if he wants the agent to work in both period 1 and period 2 after the bad outcome in period 1. Also, in the one period example, the principal doesn’t have to leave the rent. This shows that the rent is an informational rent; the principal is leaving the rent because the agent’s deviation leads to a different prior in the following period.

**Remark 2.** The minimum rent is proportional to the discount factor and the persistence of the states. When the environment is i.i.d., the principal can leave no rent and have the
agent work in both periods.

Remark 3. The composition of the continuation value in period 2 after the bad outcome matters for the IC constraint in period 1. After the good outcome in period 1, only the continuation value matters, and the composition of it in period 2 doesn’t matter.

Remark 4. If the principal wants the agent to work in both periods, he can offer a history-independent contract. The principal’s prior in the second period depends on the outcome, but the contract doesn’t depend on the principal’s information.

4 Informational Rent in the Infinite Horizon Model

This section discusses the general model with an infinite horizon. As was the case with the two-period model, the informational rent arises in this environment. In this section, I provide upper and lower bounds on the informational rent. If the discount factor, the persistence of the states or the informativeness of the outcome is small, the principal can approximate his first-best payoff.

Before discussing the IC constraints of the agent, I will first define two notations. Let $P(h^t \sqcup h^k|h^t, \pi)$ be the conditional probability of history $h^t \sqcup h^k$ given $h^t$ when the prior on the states in the period after $h^t$ is $\pi$ and the agent works in every period in which the principal doesn’t take his outside option. Note that $\pi$ is the prior on the states in the period following $h^t$. The principal updates his posterior given history $h^t$, and the posterior needs to be multiplied by the Markov matrix $M$ as it transits to the following period.

The second notation is $V(h^t, \pi)$. A contract specifies history-contingent payments, $w(h^t)$ for all histories $h^t$. $V(h^t, \pi)$ is the agent’s continuation value from working in every period when the continuation contract is conditional on the history $h^t$ and $\pi$ is the prior on the states in the period following $h^t$:

$$V(h^t, \pi) = \sum_{k=0}^{\infty} \delta^k \sum_{h^k} u(h^t \sqcup h^k),$$

where $u(h^t \sqcup h^k)$ is the expected payoff from working after $h^t \sqcup h^k$. Formally, $u(h^t \sqcup h^k)$ is given by

$$u(h^t \sqcup h^k) = q(h^t \sqcup h^k)(-P(h^t \sqcup h^k|h^t, \pi)c + P(h^t \sqcup h^k1|h^t, \pi)w(h^t \sqcup h^k1)$$

$$+ P(h^t \sqcup h^k0|h^t, \pi)w(h^t \sqcup h^k0)),$$

where the principal takes his outside option with probability $1 - q(h^t \sqcup h^k)$ after $h^t \sqcup h^k$. 15
Consider the IC constraints of the agent in period $t$ given history $h^{t-1}$. Let $\pi^t$ be the prior on the states. The agent can deviate for $T$ periods before he starts working again, and there is an infinite sequence of IC constraints. The IC constraint for deviating for $T$ periods is

$$V(h^{t-1}, \pi^t) \geq \sum_{k=1}^{T} \delta^{k-1} q(h^{t-1} \sqcup \tilde{h}^{k-1}) w(h^{t-1} \sqcup \tilde{h}^k) + \delta^T V(h^{t-1} \sqcup \tilde{h}^T, \pi^t M^T),$$

where $\tilde{h}^0 = \emptyset$ and $h^{t-1} \sqcup \tilde{h}^k, 1 \leq k \leq T$, are defined by

$$h_{t-1+k} = \begin{cases} 
0 & \text{if the agent is induced to work but shirks,} \\
-1 & \text{if the principal takes his outside option.}
\end{cases}$$

When the agent deviates, it has two effects. The first effect is the outcome consequence; by shirking in period $t$, the agent produces the bad outcome, and the continuation contract corresponds to the bad outcome in period $t$. The second effect is the information asymmetry between the principal and the agent. Since the states are unobservable, the principal and the agent have priors on the states. When the principal believes that the agent worked in a given period, the principal updates his prior after observing the outcome. However, if the agent deviates in period $t$, the agent doesn’t update his prior after observing the bad outcome. In period $t+1$, the principal and the agent have different priors on the state; when the principal believes that the agent worked in period $t$, the bad outcome lowers the prior on the good state, and the agent assigns strictly higher probability on the good state than the principal does.

In the IC constraints of the agent, the agent receives the payments for the bad outcomes while he deviates. After deviating for $T$ periods, the continuation contract is the one for the history $h^{t-1} \sqcup \tilde{h}^T$. The second effect, the information asymmetry, is captured by the term $\pi^t M^T$. The updating of the priors and the Markov transition preserve the ordering of the priors, and once the agent deviates, in all future periods, he assigns a weakly higher probability to the good state than the principal does.

After each history $h^{t-1}$, there is a sequence of IC constraints for the agent. If the principal wants the agent to work, all the IC constraints have to be satisfied, and the maximum continuation value of all possible deviations is the rent the agent gets by working in period $t$.

**Proposition 4.** Suppose Assumptions 1-2 hold. After history $h^{t-1}$, the rent for the agent is bounded from below by
\[
\max_{T \geq 1} \sum_{k=1}^{T} \delta^{k-1} q(h^{t-1} \sqcup \bar{h}^{k-1}) w(h^{t-1} \sqcup \bar{h}^{k}) + \delta^T V(h^{t-1} \sqcup \bar{h}^T, \pi^t M^T),
\]

where \( \pi^t \) is the prior given history \( h^{t-1} \), and the principal takes his outside option with probability \( 1 - q(h^t \sqcup h^k) \) after \( h^t \sqcup h^k \). \( \bar{h} = \emptyset \) and \( h^{t-1} \sqcup \bar{h}^{k}, 1 \leq k \leq T \), are defined by

\[
h_{t-1+k} = \begin{cases} 
0 & \text{if the agent is induced to work but shirks}, \\
-1 & \text{if the principal takes his outside option}. 
\end{cases}
\]

Proposition 4 shows that there is a lower bound on the rent to the agent. On the other hand, the principal can attain an upper bound on the informational rent by offering the following contract:

\[
w(h^t 1) = \frac{c}{\pi^t (p_{HL})}, w(h^t 0) = 0, \forall t \geq 0,
\]

where \( \pi^t \) is the prior on the states given history \( h^t = 0 \ldots 0 \).

When the contract is stationary, the payments don’t depend on the history of the outcomes, and the information asymmetry between the principal and the agent has no effect on the continuation contract. As long as the expected payment in the given period is above the cost, the agent is willing to work. Since \( \pi^t \) assigns the lowest probability on the good state among the priors that can arise after any history of the outcomes of \( t-1 \) periods, the above contract provides enough incentives for the agent to work in every period. The expected rent to the agent is bounded from above by

\[
\sum_{t=1}^{\infty} \delta^{t-1} (-c + \pi^1 M^{t-1} (p_{HL}) \frac{c}{\pi^t (p_{HL})}) = \sum_{t=2}^{\infty} \delta^{t-1} (-c + \pi^1 M^{t-1} (p_{HL}) \frac{c}{\pi^t (p_{HL})}) \leq \sum_{t=1}^{\infty} \delta^t (-c + \pi^1 M^t (p_{HL}) \frac{c}{M^2 (p_{HL})}) = \frac{c}{M^2 (p_{HL})} \frac{\delta}{1 - \delta} M \frac{\delta}{1 - \delta} M + \pi_1^1.
\]

Note that the upper bound is zero when the states are i.i.d.. The principal can induce working in every period and leave no rent to the agent.

When the discount factor is small for the given parameters, or when the persistence of the states or the informativeness of the outcomes is small, the principal can approximate his first best payoff by offering the stationary contract.
Proposition 5. Suppose Assumptions 1-2 hold. There exists an upper bound on the rent given by
\[
ungc \delta \det M (p_H - p_L)(\delta M_{21} + (1 - \delta) \pi_1^1).
\]

Given \( \epsilon > 0 \), there exists \( \bar{\delta} \) such that for \( \delta < \bar{\delta} \), the principal can approximate his first best payoff by \( \epsilon \) with a contract that is stationary from period 2. Conversely, for given \( \delta \), there exists \( \bar{D} \) and \( \Delta_p \) such that if \( \det M < \bar{D} \) or \( p_H / p_L < \Delta_p \), the principal can approximate his first best payoff with a contract that is stationary from period 2.

5 States as Outcomes: Stationary Contracts as Cost-Minimizing Contracts

In the results so far, I have assumed a very flexible specification for persistence - there is both an underlying Markov process and probabilistic outcomes as a function of the state. In this section, I consider a more restricted Markov model that still allows for persistence and in which I can provide more complete characterizations. The probability of a successful outcome in period \( t + 1 \) only depends on whether the project would have been successful at period \( t \) had the agent put in effort. This can be seen as a special case of the previous model in which work produces the good outcome in the good state with probability one, and work produces the bad outcome in the bad state with probability one. In this environment, I show here that the cost-minimizing contract that induces the agent to work in every period can be stationary, and the principal can offer a constant payment for the good outcome and minimize the rent to the agent. I obtain a tight lower bound on the rent to the agent when the principal wants the agent to work in every period, and the lower bound increases with the discount factor and the persistence of the states. The second best contracts are always fully history-contingent, and the stationary contracts are not optimal in general, but I will show in Section 6 that the stationary contract is part of the second best contract.

When the states correspond to the outcomes of working, the mapping from the states to the outcomes is deterministic. If the states were observable, the principal would know the outcome of working from the state. Having a constant prior means that the states are distributed i.i.d. in every period. However, when the states follow a Markov matrix, the prior after the good outcome is \( M_1 = (M_{11}, M_{12}) \) and the prior after the bad outcome is \( M_2 = (M_{21}, M_{22}) \). This implies that work produces the good outcome with probability \( M_{11} \) after the good outcome, and it produces the good outcome with probability \( M_{21} \) after the bad outcome. Having different probabilities \( M_{11} > M_{21} \) precisely captures the persistence of
the outcomes of working. An alternative interpretation would be that the good state follows a good outcome in which work is productive with probability $M_{11}$; the bad state follows a bad outcome, and work is productive with probability $M_{21} < M_{11}$.

Consider an agent who is in charge of making an innovation. The agent’s effort will be productive if an innovation is available, and it will be unproductive if an innovation is unavailable. The good state then will be the state in which an innovation is available, and the bad state is when it is not. Since the states are unobservable, the principal and the agent will have priors on the states from the previous outcome and whether the agent has worked or not, and the priors are the probabilities that they think work will produce a good outcome. Other examples can be modeled in a similar way.

**Assumption 3.** States correspond to the outcomes of working: $p_H = 1$, $p_L = 0$.

When the states correspond to the outcomes of working, the cost-minimizing contract to have the agent work in every period can be completely stationary from the second period. The principal can offer a constant payment for the good outcome and minimize the rent to the agent.

**Proposition 6.** Suppose Assumptions 1-3 hold. If the principal wants the agent to work in every period, a cost-minimizing contract is to provide

\[
\begin{align*}
w(1) &= \frac{c}{\pi_1^1}, \quad w(0) = 0, \\
w(h^t1) &= \frac{c}{M_{21}}, \quad w(h^t0) = 0, \quad \forall h^t, t \geq 1.
\end{align*}
\]

Before proving Proposition 6, I will first characterize the lower bound on the rent to the agent when the principal wants him to work in every period. The proof of Proposition 6 follows by showing that the principal attains the lower bound with the stationary contract.

**Proposition 7.** Suppose Assumptions 1-3 hold. If the principal wants the agent to work in every period, the average rent to the agent is bounded from below by

\[
\frac{\delta \det M}{1 - \delta \det M} c(\delta + (1 - \delta) \frac{\pi_1^1}{M_{21}}).
\]

**Proof of Proposition 7.** Consider the IC constraint for the one-shot deviation after history $h^t$. When the states correspond to the outcomes of working, the prior on the state is completely determined by the state in the previous period. In particular, the prior on the state after the good outcome is $M_1 = (M_{11}, M_{12})$, and the prior after the bad outcome believing that the agent worked in the previous period is $M_2 = (M_{21}, M_{22})$. Therefore, any subgame after the good outcome is identical, and any subgame after the bad outcome is also identical.
If the principal wants the agent to work in every period, he doesn’t take his outside option after any history $h^t$. The IC constraint for the one-shot deviation is given by

$$
-c + \pi^{t+1} \left( \frac{V_1}{V_2} \right) \geq w(h^{t}0) + \delta(-c + \pi^{t+1} M \left( \frac{V_1'}{V_2'} \right)) \\
\iff V_1 - V_2 \geq \frac{c}{\pi_1^{t+1}} + \delta \det M (V_1' - V_2'),
$$

where $\pi^{t+1}$ is the principal’s prior after $h^t$ and $V_1$ is the sum of the present compensation and the continuation value after $h^t1$. $V_2, V_1'$ and $V_2'$ are defined similarly for $h^t0, h^t01$ and $h^t00$.

Let $V^t_1, V^t_2$ be the sum of the present compensation and the continuation value after $h^t = 0 \cdots 1$ and $h^t = 0 \cdots 0$. We have the following set of IC constraints:

$$
V^t_1 - V^t_2 \geq \frac{c}{\pi^t_1} + \delta \det M (V^2_1 - V^2_2),
$$

$$
V^t_1 - V^t_2 \geq \frac{c}{M_{21}^t} + \delta \det M (V^{t+1}_1 - V^{t+1}_2), \forall t \geq 2.
$$

From Proposition 5, the IC constraint and the limited liability condition, we know that $V^t_1 - V^t_2$ is bounded for all $t$ under an optimal contract, and summing over the IC constraints, we get

$$
V^t_1 - V^t_2 \geq \frac{1}{M_{21}^t 1 - \delta \det M}, \forall t \geq 2,
$$

$$
V^1_1 - V^1_2 \geq \frac{c}{\pi^1_1} + \frac{c}{M_{21}^t 1 - \delta \det M}.
$$

Together with

$$
V^t_2 = w(0 \cdots 0) + \delta(-c + M_{21} V^{t+1}_1 + M_{22} V^{t+1}_2)
$$

$$
\geq \delta(-c + M_{21} (V^{t+1}_1 - V^{t+1}_2) + V^{t+1}_2), \forall t \geq 1
$$

we have

$$
V^t_2 \geq \delta \left( \frac{\delta \det M}{1 - \delta \det M} c + V^{t+1}_2 \right)
$$

and

$$
V^1_2 \geq \frac{\delta}{1 - \delta 1 - \delta \det M} c.
$$
The average rent to the agent is bounded from below by

\[
(1 - \delta)(-c + \pi_1^1 V_1^1 + \pi_2^1 V_2^1) \\
= (1 - \delta)(-c + \pi_1^1 (V_1^1 - V_2^1) + V_2^1) \\
\geq (1 - \delta)(-c + \pi_1^1 \left( \frac{c}{\pi_1^1} + \frac{\delta \det M}{M_{21}} \frac{1}{1 - \delta \det M} \right) + \frac{\delta}{1 - \delta} \frac{\delta \det M}{1 - \delta \det M} c) \\
= \frac{\delta \det M}{1 - \delta \det M} c (\delta + (1 - \delta) \frac{\pi_1^1}{M_{21}}).
\]

\[\square\]

From Proposition 7, we can show Proposition 6 as the following.

**Proof of Proposition 6.** When the principal offers a stationary contract, in any period following a history \(h^t\), the agent chooses to work as long as

\[-c + \pi^{t+1} \left( \frac{w(h^t1)}{w(h^t0)} \right) \geq w(h^t0).\]

The agent’s IC constraints become myopic because there is no gain from creating information asymmetry. If the principal offers a constant payment for the good outcome, the payment is unaffected by the principal’s prior, and the continuation value after a deviation is the same as the continuation value on the equilibrium path. Therefore, in deciding whether to work or shirk, the agent only cares about the payment in the current period, and as long as the expected payoff from working is greater than the payment for the bad outcome, the agent chooses to work.

The contract specified in Proposition 6 provides the following constant payments:

\[w(h^t1) = \frac{c}{M_{21}}, \quad w(h^t0) = 0, \quad \forall h^t, t \geq 1.\]

Since the agent’s prior on the state satisfies \(\pi_1^t \geq M_{21}\), the agent’s IC constraints are satisfied after any history \(h^t, t \geq 1\). In period 1, the principal pays

\[w(1) = \frac{c}{\pi_1^1}, \quad w(0) = 0,\]

and again, the agent chooses to work.

To show the optimality of the contract, we need to show that the rent to the agent is minimized with this contract. Under the specified contract, the agent gets rent if and only
if the outcome in the previous period is good, and in each of those periods, he gets

$$-c + M_{11}w(h^t1) = \frac{\det M_c}{M_{21}}.$$

The probability of a good outcome in period $t$ given the initial prior $\pi^1$ is

$$\pi^1 M^{t-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and the average rent to the agent under the contract is

$$(1 - \delta) \sum_{t=1}^{\infty} \delta^t \pi^1 M^{t-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{\det M_c}{M_{21}} = \frac{\delta \det M_c}{1 - \delta \det M} (\delta + (1 - \delta) \frac{\pi^1}{M_{21}}),$$

which is the lower bound on the rent to the agent given in Proposition 7. Therefore, the principal can attain the lower bound on the rent with a stationary contract, and the cost-minimizing contract to have the agent work in every period can be stationary.

Proposition 6 shows that the cost-minimizing contract that induces the agent to work in every period can be made to be stationary when the state variable is whether working will produce a good outcome. I will show in Section 6 that the stationary contract is a part of the second best, but the next two propositions show that stationary contracts are not optimal more generally: the assumption of a deterministic mapping from states to outcomes is needed, and even under the deterministic mapping, the principal prefers to take his outside option after an enough number of bad outcomes. First, Proposition 8 shows that in order for an optimal contract to be history-independent, it is necessary that the mapping from the states to the outcomes is deterministic. I define the history-independent contracts as contracts with payments of the following form:

$$w(h^t1) = w_t(1), \forall h^t,$$

$$w(h^t0) = w_t(0), \forall h^t.$$

**Proposition 8.** Suppose Assumptions 1-2 hold. The second best contract can be independent of the history only if the following condition holds:

$$p_H = 1, \ p_L = 0.$$

The proof of Proposition 8 goes as the following. When a contract is history-independent, the agent’s IC constraints become myopic since there is no gain from creating information
asymmetry. As long as the expected payoff from working in the given period is greater than the payment for the bad outcome, the agent chooses to work. Then the principal can adjust the payments after some histories, and by front-loading the payments for which the IC constraints don’t bind, the principal can keep the continuation value constant while reducing the deviation payoff of the agent. It allows the principal to lower the payments after some histories and reduce the rent. The only exception is when \( p_H = 1, p_L = 0 \) and the principal doesn’t benefit from front-loading the payment.

The next proposition shows that in the case with \( p_H = 1, p_L = 0 \), the principal wants to take his outside option after some histories. Together with Proposition 8, the proposition shows that the second best contracts in the general model will be fully history-contingent and the stationary contracts are not optimal.

**Proposition 9.** Suppose Assumptions 1-3 hold. There exists \( t_0 > 0 \) such that for any \( \delta > 0 \), a contract that involves taking the outside option after \( t_0 \) bad outcomes since period 1 gives a strictly higher payoff to the principal than the cost-minimizing contract that induces working in every period.

The proof of Proposition 9 consists of two steps. The first step is to show that for \( p_H = 1, p_L = 0 \), it is sufficient to consider the one-shot deviations of the agent. When the IC constraints for the one-shot deviations are satisfied, the agent doesn’t deviate more than once at a time, and all IC constraints are satisfied.

The second step is to show that by taking the outside option after some history, the principal can lower the payments to the agent leading up to the specified history. If the principal takes his outside option after \( t_0 \) bad outcomes from period 1, the principal can lower the payments \( w(0 \cdots 0 1) \) for \( 0 \leq k < t_0 \). When the reduction in the rent is greater than the loss in outcome from taking the outside option, the principal prefers to take the outside option after \( h^{t_0} = 0 \cdots 0_{t_0} \).

The number of bad outcomes before the principal takes his outside option holds uniformly for all discount factors \( \delta \). Given the Markov matrix \( M \), there exists \( t_0 \) such that for any discount factor \( \delta \), taking the outside option after \( t_0 \) bad outcomes strictly dominates inducing the agent to work in every period.

### 6 Second Best: Optimality of Tenure Contracts

This section characterizes the second best contracts in the same case of the model studied in Section 5: I assume that the state variable is whether or not the project will succeed
in period $t$. I show that in this case, the second best contracts take the form of a tenure system. The agent is paid nothing during the probationary period, and once the principal makes the positive payment, the agent is tenured, and the principal never takes his outside option again. There is no loss of generality in assuming that the principal makes positive payments only for the good outcome, and after two periods since the initial payments, the principal can offer a stationary contract. I also provide a recursive formulation to decide how long the probationary period lasts and what the initial payment is.

The first result of this section is that the second best contracts take the form of a tenure system. During the probationary period, the continuation value of the agent and the decision to take the outside option depends on the history of the outcomes, but the agent is paid nothing during this period. Once the principal makes a positive payment, the agent is tenured, and the principal never takes his outside option again.

**Proposition 10.** Suppose Assumptions 1-3 hold. Under the second best contract, once the principal makes a positive payment, the principal never takes his outside option again. For any history $h^t$ such that $w(h^t) > 0$, the principal induces work after history $h^t \sqcup h^k, \forall h^k, k \geq 0$.

The proof of Proposition 10 relies on the next proposition and the fact that the composition of the continuation value after the good outcome doesn’t matter for the agent’s IC constraints.

**Proposition 11.** Suppose Assumptions 1-3 hold. In characterizing the second best, there is no loss of generality in restricting attention to contracts under which the principal makes positive payments only for the good outcome.

The proof of Proposition 11 is in the appendix.

**Proof of Proposition 10.** Given a contract, let $R$ and $L$ be the rent and the loss in outcome under the contract. $L$ is defined to be

$$L = (1 - \delta)(Y_{FB} - Y),$$

where $Y_{FB}$ is the expected discounted sum of the outcome in the first best, and $Y$ is the expected discounted sum of the outcome under the given contract.

Consider the space of $(R, L)$ for the initial prior $\pi$. I allow the principal to randomize continuation contracts, and the set of all feasible $(R, L)$ is a convex set. In particular, there is a one to one mapping

$$f : [0, L_\pi] \rightarrow [0, \infty)$$
such that the set of feasible \((R, L)\) is given by

\[
X_\pi \equiv \{(R, L) | R \geq f(L), 0 \leq L \leq L_\pi\}
\]

and \(L_\pi\) is the minimum expected loss in outcome when the principal leaves no rent to the agent.

From Proposition 7, we know that

\[
f(0) = \frac{\delta \det Mc}{1 - \delta \det M} \left( \delta + (1 - \delta) \frac{\pi_1}{M_{21}} \right)
\]

is the minimum rent to the agent under the cost-minimizing contract. We also know from \(f(0) > 0\) that \(L_\pi > 0\). From the fact that \(X_\pi\) is convex, we know that \(f(\cdot)\) is strictly decreasing in \(L\).

Since both the principal and the agent are risk-neutral, the principal can always delay the payment. Suppose the principal makes a positive payment for \(h^t\) under the second best contract. From Proposition 11, we can assume that the principal makes the positive payment for a good outcome, and \(h_t = 1\). After the good outcome, only the sum of the present compensation and the continuation value matters for the agent’s IC constraint, and the principal can replace the continuation contract. Instead of paying \(w(h^t)\) and continuing with \(V(h^t, M_1)\), the principal can offer \(\hat{w}(h^t) = 0\) and

\[
\hat{V}(h^t, M_1) = \frac{1}{\delta} w(h^t) + V(h^t, M_1).
\]

If \(V(h^t, M_1) < f(0)\), we get

\[
f^{-1}(\hat{V}(h^t, M_1)) < f^{-1}(V(h^t, M_1)) .
\]

The principal can replace the continuation contract with a contract with a lower \(L\), and the principal’s payoff strictly increases.

Therefore, if the principal makes a positive payment under the second best contract, he doesn’t gain from delaying that payment, which means that the agent’s continuation value \(V(h^t, M_1)\) is at least as big as the minimum rent under the cost-minimizing contract; the principal doesn’t lose anything in outcome under the continuation contract. The principal never takes his outside option after he makes a positive payment.

Proposition 10 and 11, together with Proposition 6, imply that the principal can restrict attention to the following form of contracts.
Proposition 12. Suppose Assumptions 1-3 hold. There is no loss of generality in restricting attention to the following contracts: suppose the principal makes a positive payment for the first time after $h^t = h^{t-1}$. In the following period, he offers

$$w(h^t1) = \frac{c}{M_{11}}, w(h^t0) = 0,$$

and the next period and on, he offers

$$w(h^t \sqcup h^k1) = \frac{c}{M_{21}}, w(h^t \sqcup h^k0) = 0, \forall h^k, k \geq 1.$$

Note that the contract becomes completely stationary after two periods since the initial payment. In particular, we will observe that the principal makes the constant payments for the good outcomes on the equilibrium path. The states are changing, and the prior of the principal also changes over time, but it is optimal to commit not to use his information and offer a stationary contract. We know from Section 5 that the second best contracts are fully history-contingent, and it is never optimal to induce working in every period. However, the second best contracts turn out to be history-contingent only until the agent is tenured. Once the agent is tenured, it is optimal to induce working in every period, and the principal can offer a stationary contract regardless of his information.

Proposition 12 allows us to offer a stationary contract after two periods since the initial payment. Until the agent is tenured, the contract is history-contingent, and the timing of the initial payment and the outside options depends on the history of the outcomes. I will provide a recursive formulation to characterize the timing of the initial payment and the outside options, but before doing so, I will prove one more proposition and a corollary on the dynamics of the continuation values.

I will call the periods before tenure probationary. The timing of the tenure is history-contingent, and any history before the tenure is granted is probationary. During the probationary period, the agent’s continuation value strictly increases after the good outcome.

Proposition 13. Suppose Assumptions 1-3 hold. Given the history $h^t$, let $R$ and $R_1$ be the agent’s continuation values after $h^t$ and $h^t1$, respectively. For any $h^t$ such that $h^t1$ is in the probationary period, we have

$$R < R_1.$$

Proof. From Assumption 2, the principal takes his outside option when the agent is not induced to work. Therefore, any outcome is on the equilibrium path from the principal’s perspective, and the principal provides the payments specified in the contract. In particular, after each good outcome, the continuation value of the agent is exactly the amount the
principal intends to provide under the contract.

During the probationary period, the agent is not paid anything. The agent’s continuation value $R$ can be written as

$$ R = -c + \pi_1 V_1 + \pi_2 V_2 $$

$$ < -c + V_1 $$

$$ = -c + \delta R_1 $$

$$ < R_1, $$

where $\pi$ is the prior on the states after $h^t$, and the first inequality follows from $V_1 > V_2$ under the optimal contract. 

\[ \square \]

**Corollary 1.** Suppose Assumptions 1-3 hold. The second best contract never terminates after a good outcome.

**Proof.** If the contract terminates, the agent’s continuation value is zero. From $R_1 > R \geq 0$, the agent’s continuation value is strictly positive after a good outcome, and the contract doesn’t terminate. \[ \square \]

The recursive formulation consists of two steps. The first step is to characterize the pairs of continuation values $(V_1, V_2)$ with which the principal can induce working in the given period. The second step incorporates the loss in outcome under a given contract. The principal chooses a contract that minimizes the sum of the rent to the agent and the loss in outcome.

**Proposition 14.** Suppose Assumptions 1-3 hold. The second best contract can be found from the following two sets: $S$ is the largest self-generating set

$$ S = conv\{(\pi, V_1, V_2) \exists T \geq 0, (\pi', V_1', V_2') \in S \text{ such that} $$

$$(i)\pi' = M_2 M^T, $$

$$(ii)V_2 = \delta^{T+1} (-c + \pi'(V_1' V_2'), $$

$$(iii)V_1 - V_2 \geq \frac{c}{\pi_1} + \delta^{T+1} (\det M)^{T+1}(V_1' - V_2'), \} \},
and

\[ X_\pi = \text{conv}(\{(R, L) | \exists T \geq 0, (R_1, L_1) \in X_{M_1}, (R_2, L_2) \in X_{M_2} \text{ such that} \]
\[ (i) \pi' = \pi M^T, \]
\[ (ii) R = \delta^T (-c + \delta \pi'(R_1 R_2)), \]
\[ (iii) L = (1 - \delta) \sum_{k=1}^{T} \delta^{k-1}(-c + M^{k-1} (1 0)^T - u) + \delta^{T+1} \pi'(L_1 L_2), \]
\[ (iv)(\pi', \delta R_1, \delta R_2) \in S) \}

is generated from \( X_0^1 \) and \( X_0^2 \).

\( X_{M_1} \) and \( X_{M_2} \) are jointly determined as the limits of \( X_n^1, X_n^2 \):

\[ X_1^0 = \{(R, 0) | R \geq R_1^* \equiv \frac{\delta \det M c}{1 - \delta \det M} (\delta + (1 - \delta) \frac{M_{11}}{M_{21}}) \}, \]
\[ X_2^0 = \{(R, 0) | R \geq R_2^* \equiv \frac{\delta \det M c}{1 - \delta \det M} \}, \]

where \( R_1^* \) and \( R_2^* \) are the rents to the agent under the cost-minimizing contracts for priors \( M_1 \) and \( M_2 \), and

\[ X_{1 + 1} = \text{conv}(\{(R, L) | \exists T \geq 0, (R_1, L_1) \in X_1^1, (R_2, L_2) \in X_2^1 \text{ such that} \]
\[ (i) \pi' = M_1 M^T, \]
\[ (ii) R = \delta^T (-c + \delta \pi'(R_1 R_2)), \]
\[ (iii) L = (1 - \delta) \sum_{k=1}^{T} \delta^{k-1}(-c + M_1 M^{k-1} (1 0)^T - u) + \delta^{T+1} \pi'(L_1 L_2), \]
\[ (iv)(\pi', \delta R_1, \delta R_2) \in S) \}

\[ X_{2 + 1} = \text{conv}(\{(R, L) | \exists T \geq 0, (R_1, L_1) \in X_1^2, (R_2, L_2) \in X_2^2 \text{ such that} \]
\[ (i) \pi' = M_2 M^T, \]
\[ (ii) R = \delta^T (-c + \delta \pi'(R_1 R_2)), \]
\[ (iii) L = (1 - \delta) \sum_{k=1}^{T} \delta^{k-1}(-c + M_2 M^{k-1} (1 0)^T - u) + \delta^{T+1} \pi'(L_1 L_2), \]
\[ (iv)(\pi', \delta R_1, \delta R_2) \in S) \}.

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The second best contract given the initial prior $\pi$ is the one that minimizes $R + L$ such that $(R, L) \in X_\pi$.

Once we find $(R, L) \in X_\pi$ that minimizes $R + L$, the contract can be constructed as the following. Given $(R, L) \in X_\pi$, there exist $T \geq 0, (R_1, L_1) \in X_{M_1}, (R_2, L_2) \in X_{M_2}$ supporting $(R, L)$. The principal takes the outside option for $T$ periods, and after the first period the agent is induced to work, the continuation contract is determined by $(R_1, L_1)$ and $(R_2, L_2)$. If the outcome is good and $R_1 < R_1^*$, the contract continues with $(R_1, L_1)$. If the outcome is good and $R_1 \geq R_1^*$, the agent is tenured and is paid $\delta(R_1 - R_1^*)$ this period. The contract continues with $(R_1^*, 0)$, the contract specified in Proposition 12. If the outcome is bad and $0 < R_2 < R_2^*$, the contract continues with $(R_2, L_2)$. If the outcome is bad, but $R_2 = R_2^*$, the agent is tenured, and the principal pays

$$w(h^1) = \frac{c}{M_{21}}, w(h^0) = 0$$

from the following period. If the outcome is bad and $R_2 = 0$, the contract terminates.

### 7 First Best Approximation

In this section, I return to the more general formulation with $p_H$ and $p_L$ and discuss the first best approximation. As the discount factor approaches one, the principal can get arbitrarily close to his first best payoff.

**Proposition 15.** Suppose Assumptions 1-2 hold. Given $\epsilon > 0$, there exists $\tilde{\delta}$ such that for any $\delta > \tilde{\delta}$, the principal’s average per period payoff in the second best is within $\epsilon$ of his first best payoff.

Consider the following review contract. The contract specifies a review block of $T$ periods, a quota and a lump sum transfer. A quota is on the number of successful outcomes from the block. If the agent meets the quota, the principal pays the agent the discounted sum of the outcome subtracted by the lump sum transfer at the end of the review block, and the contract continues. If the agent fails to meet the quota, the principal pays the agent the discounted sum of the outcome, and the contract terminates. If the outcome is bad and $R_2 = 0$, the contract terminates.

First, consider the principal’s payoff. Let $s$ be the agent’s strategy in the equilibrium, $p(s)$ be the minimum probability he meets the quota and $X$ be the lump sum transfer. In general, the probability the agent meets the quota with a strategy depends on the prior on the state at the beginning of the review block, but we can take the minimum of the
probabilities over the priors. Then the principal’s average per period payoff is at least

\[
(1 - \delta)\delta^{T-1}p(s)X (1 + \delta^T p(s) + \delta^{2T} p(s)^2 + \cdots)
= \frac{\delta^{T-1}p(s)(1 - \delta)X}{1 - \delta^T p(s)}
= \frac{\delta^T(1 - \delta^T)p(s) \ 1 - \delta \ X}{1 - \delta^T \ p(s) \ 1 - \delta^T \frac{1}{\delta}}.
\]

Since the states exhibit positive persistence, the expected discounted sum of the outcome in the first best is the maximum when the pair starts with \(\pi^1 = (1, 0)\). Let \(\bar{y}\) be the average expected discounted sum of the outcome over the infinite horizon in the first best when \(\pi^1 = (1, 0)\). When the following two inequalities hold,

\[
\frac{\delta^T(1 - \delta^T)p(s)}{1 - \delta^T p(s)} \geq 1 - \frac{\epsilon}{2},
\frac{1 - \delta \ X}{1 - \delta^T \frac{1}{\delta}} \geq (1 - \frac{\epsilon}{2})\bar{y} - c,
\]

the principal’s payoff is at least \((1 - \epsilon)\bar{y} - c \geq \bar{y} - c - \epsilon\).

Roughly speaking, the first inequality says that the agent meets the quota with a high enough probability. The review block is sufficiently long to have \(p(s)\) close to one. Then, there is also a lower bound on the discount factor so that \(\delta^T\) is close to one. If the review block is too long for the given discount factor, the lump sum transfer the principal gets at the end of the review block is discounted too much for the principal’s payoff to be close to his first best payoff. Therefore, the review block is sufficiently long to have a high probability of meeting the quota and yet not too long for the given discount factor so that the principal’s payoff is not discounted too much.

The second inequality says that the lump sum transfer the principal gets on meeting the quota is close to his first best payoff. The expected outcome in the first best in any given period increases with the prior the pair puts on the good state, and together with the positive persistence, the expected discounted sum of the outcome is maximum when they start believing they are in the good state. If the lump sum transfer is above \((1 - \frac{\epsilon}{2})\bar{y} - c\), it is above \(1 - \frac{\epsilon}{2}\) times the first best outcome for any initial prior, subtracted by the cost.

It remains to verify the agent’s incentives that the agent will pass the quota with \(p(s)\) close to one. Let \(V(\pi)\) be the agent’s continuation value when the review block starts with prior \(\pi\). Since the agent can always choose to work in every period, letting \(s\) be the strategy of working in every period, we have

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\[ V(\pi) \geq (1 - \delta)(Y(\pi) - \frac{1 - \delta^T}{1 - \delta} c) + \delta^T p(s)(\mathbb{E}[V(\tilde{\pi})] - \frac{1 - \delta}{\delta} X), \]

where \( Y(\pi) \) is the expected discounted sum of the outcome from working in every period from a block with the initial prior \( \pi \), and \( \tilde{\pi} \) is the prior in the beginning of the new block.

Let \( V \) be the minimum of \( V(\pi) \) over all priors \( \pi \). Together with the fact that \( Y(\pi) \) increases with \( \pi_1 \), we get the following inequality:

\[ V \geq \frac{(1 - \delta)(Y((0, 1)) - \frac{1 - \delta^T}{1 - \delta} c) - \delta^T p(s) \frac{1 - \delta}{\delta} X}{1 - \delta^T p(s)}. \]

If \( V \geq \frac{1 - \delta}{\delta} X \), the agent always prefers to increase the probability of meeting the quota. Since the principal pays the agent the discounted sum of the outcome, subtracted by \( X \) on meeting the quota, the agent works in every period on the equilibrium path.

When the lump sum transfer is specified to

\[ X \leq \delta(Y((0, 1)) - \frac{1 - \delta^T}{1 - \delta} c), \]

the inequality \( V \geq \frac{1 - \delta}{\delta} X \) is always satisfied. The last condition is to ensure that the discounted sum of the outcome on meeting the quota is weakly greater than \( X \) so that the principal can actually take away the lump sum transfer. A slightly stronger condition is

\[ Q \geq X, \]

where \( Q \) is the number of good outcomes for the quota.

Therefore, when a review contract satisfies

\[ \frac{\delta^T (1 - \delta^T)p(s)}{1 - \delta^T p(s)} \geq 1 - \epsilon \frac{2}{\delta}, \]

\[ \frac{1 - \delta}{1 - \delta^T} X \geq (1 - \epsilon \frac{2}{\delta})\bar{y} - c, \]

\[ X \leq \delta(Y((0, 1)) - \frac{1 - \delta^T}{1 - \delta} c), \]

\[ Q \geq X, \]

the agent chooses to work in every period, and the principal’s payoff is within \( \epsilon \) of his first best payoff.

By the uniform weak law of large numbers, the principal can find \( \tilde{\delta} \) such that for any \( \delta \geq \tilde{\delta} \), there exist \( Q \) and \( T \) that satisfy the above conditions.
The above contract sets the quota on the number of good outcomes. Alternatively, we can set the quota directly on the discounted sum of the outcomes, which allows the principal to approximate his first best payoff in a more general environment. Generally speaking, we need the uniform weak law of large numbers for the discounted sum rather than the time average.

**Proposition 16.** Let \( x_0 \in \hat{X} \) be an initial condition and let \( \{X_n\}_{n \geq 1} \) be an \( \mathbb{R}_+ \)-valued stochastic process satisfying the following condition: there exists \( \mu \) such that for given \( \epsilon > 0 \), there exists \( \delta_0, T_0 \) such that for every \( k, (x_0, x_1, \cdots, x_k), \delta \geq \delta_0, T \geq T_0, \)

\[
\left| \frac{1 - \delta}{1 - \delta^T} \left( \sum_{t=1}^{T} \delta^{t-1} \mathbb{E}[X_{k+t}] \right) - \mu \right| < \epsilon
\]

and

\[
\Pr(\left| \frac{1 - \delta}{1 - \delta^T} \left( \sum_{t=1}^{T} \delta^{t-1} X_{k+t} \right) - \mu \right| > \epsilon) < \epsilon.
\]

Suppose the outcome of working follows a stochastic process \( \{X_n\} \) and \( \mathbb{E}[X_n] \geq \bar{u} \) for each \( n \). For given \( \epsilon > 0 \), there exists \( \bar{\delta} \) such that for \( \delta \geq \bar{\delta} \), the principal’s average per period payoff in the second best is within \( \epsilon \) of his first best payoff.

The conditions in Proposition 16 say that the expected discounted sum of the outcomes converges uniformly and the uniform weak law of large numbers holds. A sufficient condition is that there are a finite number of states with an ergodic distribution and the outcome of working is bounded. First-order Markov chains that are irreducible and aperiodic have ergodic distributions, and we have the following proposition.

**Proposition 17.** Suppose there are a finite number of states following an irreducible, aperiodic first-order Markov chain. In every state, the outcome of working is bounded, and after each state, the expected outcome of working is greater than the outside option. Given \( \epsilon > 0 \), there exists \( \bar{\delta} \) such that for \( \delta \geq \bar{\delta} \), the principal’s average per period payoff in the second best is within \( \epsilon \) of his first best payoff.

### 8 Alternative Specification: Inefficient to Work in Every Period

This section considers an alternative specification of the model: the principal takes his outside option in some periods in the first best. In the first best, the principal wants the agent to work after the good outcome, but he wants to take his outside option for some periods after
the bad outcome. Under the second best contract, the agent works for no payment until they reach some history, after which the principal implements the first best action in every period. The second best contract to implement the first best contract is not stationary; the principal takes his outside option in some periods, and the payment for the good outcome is also not stationary.

Results of this section highlight the general properties of the second best contracts. Compared to the results on the second best from Section 6, the second best contracts in both environments have probationary periods during which the agent gets no payment, regardless of the outcome. In both environments, once the probationary period is over, the principal implements the first best action in every period. However, the cost-minimizing contract to implement the first best action in every period depends on the specification. If the principal wants the agent to work in every period in the first best, the cost-minimizing contract is stationary; if in the first best, the principal doesn’t want the agent to work in some periods, the cost-minimizing contract is not stationary.

The following assumption is the efficiency assumption I make throughout this section. In the static setting, it is efficient to work only with the prior after the good outcome. After the bad outcome, it takes \( k \) periods until the expected outcome from working in the given state becomes higher than the outside option.

**Assumption 4.** The parameters \( M, c \) and \( u \) are such that

\[
\begin{align*}
    M_{21} - c &< u < M_{11} - c, \\
    M_2 M^{k-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - c &< u \leq M_2 M^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} - c
\end{align*}
\]

for some \( k > 0 \).

### 8.1 First Best

I will first describe the first best as a benchmark. In the first best, the principal observes the agent’s action, but neither the principal or the agent observes the state. The principal wants the agent to work after the good outcome, but after the bad outcome, the principal takes his outside option for some periods.

**Proposition 18.** Suppose Assumptions 1, 3 and 4 hold. In the first best, the principal wants the agent to work after the good outcome. The principal wants to take the outside option for \( l \) periods after the bad outcome. \( l \) is weakly less than \( k \), and whether \( l < k \) depends on parameter values.
When the parties don’t observe the state, the outcome has an informational aspect in addition to the payoff consequence. If the informational value of having a more precise prior in the following period outweighs the expected loss in outcome, the principal will want to take his outside option after \( l < k \) periods.

### 8.2 Second Best

This section characterizes the second best contract given Assumptions 1, 3 and 4 hold. The second best contract is characterized by two phases; the agent works for no payment during the probationary periods, and once the agent is paid, the principal implements the first best action in every period.

**Proposition 19.** Suppose Assumptions 1, 3 and 4 hold. Under the second best contract, once the principal makes a positive payment, the principal implements the first best action in every period.

The proof runs parallel to the proof of Proposition 10. I note that Proposition 11 holds under Assumption 4 and the principal can make positive payments only for the good outcomes. Define \( L \) to be the loss in outcome under a given contract. \( L \) is the difference in expected outcome under the given contract and in the first best. If the principal makes a positive payment in some period and \( L \) is strictly positive under the continuation contract, the principal can delay the payment and replace the continuation contract with a contract with a lower \( L \). Backloading the payment strictly improves the principal’s payoff as long as \( L \) is strictly positive under the continuation contract. Therefore, if the principal makes the positive payment under the second best contract, the principal implements the first best action in every period under the continuation contract.

The next proposition shows that the cost-minimizing contract to implement the first best action in every period is no longer stationary. When \( l > 0 \) in the first best, the principal wants to take his outside option in some periods, even after the principal starts making positive payments, and the payments for the good outcome are not stationary. However, the continuation contract after tenure has a finite memory.

**Proposition 20.** Suppose Assumptions 1, 3 and 4 hold. The cost-minimizing contract to implement the first best actions is stationary if and only if \( l = 0 \) in the first best. For \( l > 0 \), the principal never takes his outside option after the good outcome, but he takes his outside option for \( l \) periods after the bad outcome. The payments for the good outcome can be made in two levels, one for prior \( M_1 \) and the other for prior \( M_2 M^l \).
Proposition 20 shows that for \( l > 0 \), the second best contract is not stationary. However, the second best contract has a finite memory once the principal makes a positive payment. The payment for the good outcome only depends on the prior on the state, and the only relevant information is whether the previous outcome was good and how many periods to take the outside option before inducing the agent to work again.

9 Conclusion

I study a model of principal-agent problem in a persistent environment in this paper and show that an informational rent arises when the states are partially persistent. When the states are partially persistent, the agent’s effort has both a payoff consequence and informational value. If the principal believes that the agent worked this period, the principal infers about the state by observing the outcome, and the agent’s deviation leads to a lower outcome and information asymmetry between the principal and the agent. Following a deviation, the agent assigns weakly higher probabilities to the good state than the principal does in all future periods, and the principal has to provide the maximum of all deviation payoffs the agent can get.

When the states correspond to the outcomes of working, the second best contract resembles a tenure system. In this environment, the principal makes positive payments only for the good outcome, and after the good outcome, only the sum of the present compensation and the continuation value matters for the agent’s IC constraint. If the principal makes a positive payment after some history and takes his outside option with a positive probability under the continuation contract, the principal can backload the payment and replace the continuation contract with a contract with a higher expected payoff to the principal. The principal doesn’t benefit from backloading the payment only if he is already inducing the agent to work in every period under the continuation contract. Therefore, if the agent is paid, the principal never takes his outside option again, and the agent is tenured.

After the agent is tenured, the principal offers the cost-minimizing contract, which is stationary from the second period. When the states correspond to the outcomes of working, we can express the deviation payoffs of the agent by his continuation values on the equilibrium path, without having to keep track of all future payments. If the principal uses his information to reduce the rent, the agent can deviate and get a positive rent from the one-shot deviation. One way to prevent the deviation is to always provide the payments as if the previous state was bad, and the stationary contract minimizes the rent to the agent.

The same principles hold for the second best contract when the first best involves taking the outside option in some periods. The second best contract has a probationary period
during which the agent works for no payment, and after the agent is tenured, the principal implements the first best action in every period. The intuition is the same; if the principal makes a positive payment but doesn’t achieve the first best outcome under the continuation contract, he can delay the payment and strictly improve his payoff. The continuation contract after tenure is not stationary, but it can be done with two levels of payments for the good outcome. The second best contract has a finite memory after the agent is tenured.

For discount factors close to one, the principal can approximate his first best payoff. The contract combines the review contracts and the residual claimant argument. At the end of each review block, the agent is paid the discounted sum of the outcome from the block, subtracted by a lump sum transfer if he meets the quota. The principal and the agent continue with the contract only if the agent meets the quota, and the lump sum transfer is chosen so that the agent chooses to work in every period. The principal can use the law of large numbers to ensure that the agent meets the quota with a high probability by working in every period. The law of large numbers also allows the lump sum transfer to be close to the principal’s payoff in the first best, and the principal’s payoff gets arbitrarily close to his first best payoff as the discount factor goes to one.

In this paper, I assumed that the principal can commit to a long term contract. When the states are persistent, the expected outcome in the future varies with the state in the given period. If the principal cannot commit to a long term contract, the persistence of the states puts further restrictions on the payments the principal can make, and the effect of the lack of commitment power will be magnified. It will be interesting to consider relational contracts in this environment and see what forms of commitment power or contracts can mitigate the effect of persistence.

A Proofs

Proof of Proposition 2. There are three IC constraints to consider: there are two IC constraints in the second period after the good outcome or the bad outcome in the first period, and there is the IC constraint for the one-shot deviation in the first period. We know from the proof of Proposition 1 that the positive persistence implies that the IC constraint for the one-shot deviation is sufficient for the IC constraint for the double deviation.
The IC constraints are

\[ R_1 = -c + \pi^2 \left( \frac{p_H}{p_L} \right) w(11) + (1 - \pi^2 \left( \frac{p_H}{p_L} \right)) w(10) \geq w(10), \]

\[ R_0 = -c + \tilde{\pi}^2 \left( \frac{p_H}{p_L} \right) w(01) + (1 - \tilde{\pi}^2 \left( \frac{p_H}{p_L} \right)) w(00) \geq w(00), \]

\[ -c + \pi^1 \left( \frac{p_H}{p_L} \right) (w(1) + \delta R_1) + (1 - \pi^1 \left( \frac{p_H}{p_L} \right)) (w(0) + \delta R_0) \geq w(0) + \delta (-c + \pi^1 M \left( \frac{p_H}{p_L} \right) w(01) + (1 - \pi^1 M \left( \frac{p_H}{p_L} \right)) w(00)), \]

where \( \pi^2 \) and \( \tilde{\pi}^2 \) are the priors in the second period after the good and the bad outcomes in the first period.

One can verify that the IC constraints are satisfied under the given contract, and the given contract yields the rent specified in Proposition 1.

Proof of Proposition 3. From Proposition 1, if the principal wants the agent to work in both periods, the minimum rent is

\[ \delta c \det M \frac{\pi^1 \pi^1 \left( \frac{p_H}{p_L} - p_L \right)^2}{\pi^1 \left( 1 - p_H \right) \tilde{\pi}^2 \left( \frac{p_H}{p_L} \right)}. \]

If the principal takes his outside option after the good outcome in the first period but wants the agent to work both in the first period and the second period after the bad outcome, he has to leave the same amount of rent as inducing the agent to work in both periods. Since the outside option is inefficient, the principal never wants to take his outside option only after the good outcome in the first period.

If the principal takes his outside option after the bad outcome in the first period, the principal doesn’t have to leave any rent to the agent. The IC constraint in the first period becomes

\[ -c + \pi^1 \left( \frac{p_H}{p_L} \right) (w(1) + \delta R_1) + (1 - \pi^1 \left( \frac{p_H}{p_L} \right)) w(0) \geq w(0), \]

and the principal can offer
\[ w(0) = w(10) = 0, \]
\[ w(1) = \frac{c}{\pi^1(p_H, p_L)}, \]
\[ w(11) = \frac{c}{\pi^2(p_H, p_L)}, \]

where \( \pi^2 \) is the principal’s prior in the second period after the good outcome in the first period. Since the principal is already leaving no rent, the principal prefers to have the agent work in the first period and the second period after the good outcome. The loss in outcome in this case is

\[ \delta \pi^1 \left( 1 - p_H \right) \left( -c + \tilde{\pi}^2 \left( \frac{p_H}{p_L} \right) - u \right). \]

If the principal takes his outside option in the first period, the second period problem becomes the same as the one period model, and the principal can induce working in the second period without leaving any rent. The loss in outcome is

\[ -c + \pi^1 \left( \frac{p_H}{p_L} \right) - u. \]

If the principal mixes the continuation contract, it has the same effect as taking the linear combination of the IC constraints, and it convexifies the set of payoffs.

Therefore, the principal’s problem is to choose the contract that minimizes the sum of the rent and the loss in outcome, and he prefers to leave the rent to the agent if the loss from taking the outside option is greater than the rent. This happens when

\[ \delta c \det M \frac{\pi^1 \pi^2 (p_H - p_L)^2}{\pi^1 \left( 1 - p_H \right) \pi^2 \left( \frac{p_H}{p_L} \right)} \leq -c + \pi^1 \left( \frac{p_H}{p_L} \right) - u, \]

\[ \delta c \det M \frac{\pi^1 \pi^2 (p_H - p_L)^2}{\pi^1 \left( 1 - p_H \right) \pi^2 \left( \frac{p_H}{p_L} \right)} \leq \delta \pi^1 \left( 1 - p_H \right) \left( -c + \tilde{\pi}^2 \left( \frac{p_H}{p_L} \right) - u \right). \]

Rearranging the inequalities, we get the conditions given in the proposition.

If one of the inequalities doesn’t hold, the expected loss in outcome from taking the outside option in some period is smaller than the rent to the agent, and the principal prefers to take his outside option in that period. \( \square \)
Proof of Proposition 4. After history $h^{t-1}$, the IC constraint for deviating for $T$ periods is

$$V(h^{t-1}, \pi^t) \geq \sum_{k=1}^{T} \delta^{k-1}q(h^{t-1} \sqcup \tilde{h}^{k-1})w(h^{t-1} \sqcup \tilde{h}^{k}) + \delta^T V(h^{t-1} \sqcup \tilde{h}^T, \pi^t M^T),$$

where the principal takes his outside option with probability $1 - q(h^t \sqcup h^k)$ after $h^t \sqcup h^k$. $\tilde{h}^0 = \emptyset$ and $h^{t-1} \sqcup \tilde{h}^k, 1 \leq k \leq T$, are defined by

$$h_{t-1+k} = \begin{cases} 
0 & \text{if the agent is induced to work but shirks}, \\
-1 & \text{if the principal takes his outside option}.
\end{cases}$$

There is a sequence of IC constraints for $T \geq 1$, and the maximum of the arguments on the right hand side of the IC constraints is the minimum rent to the agent. Therefore, the rent to the agent is bounded from below by

$$\max_{T \geq 1} \left[ \sum_{k=1}^{T} \delta^{k-1}q(h^{t-1} \sqcup \tilde{h}^{k-1})w(h^{t-1} \sqcup \tilde{h}^{k}) + \delta^T V(h^{t-1} \sqcup \tilde{h}^T, \pi^t M^T) \right].$$

Proof of Proposition 5. Let $\pi^t$ be the principal’s prior given history $h^{t-1} = 0 \cdots 0$. If the principal offers

$$w(h^t1) = \frac{c}{\pi^{t+1}(p_H p_L)}, w(h^t0) = 0, \forall t \geq 0,$$

the agent is induced to work in every period. Since the continuation value of the agent doesn’t depend on the history, the agent’s IC constraint becomes myopic, and the probability the agent assigns on the good state is the lowest when all outcomes have been bad. Therefore, the agent chooses to work in every period with the above contract. The rent to the agent is given by

$$\sum_{t=1}^{\infty} \delta^{t-1}(-c + \pi^1 M^{t-1} \left( \frac{p_H}{p_L} \right) \frac{c}{\pi^t(p_H p_L)}),$$

and there’s no loss in outcome.

The above contract gives an upper bound on the difference in the principal’s payoff between the first best and the second best. Using

$$\pi^t(p_H p_L) \geq M_2 \left( \frac{p_H}{p_L} \right),$$

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we have

\[
\sum_{t=1}^{\infty} \delta^{t-1}(-c + \pi^1 M^{t-1} \left( \frac{p_H}{p_L} \right) \frac{c}{\pi^1 \left( \frac{p_H}{p_L} \right)}) \\
= \sum_{t=2}^{\infty} \delta^{t-1}(-c + \pi^1 M^{t-1} \left( \frac{p_H}{p_L} \right) \frac{c}{\pi^1 \left( \frac{p_H}{p_L} \right)}) \\
\leq \sum_{t=1}^{\infty} \delta^t(-c + \pi^1 M^t \left( \frac{p_H}{p_L} \right) \frac{c}{M_2 \left( \frac{p_H}{p_L} \right)}) \\
= \frac{c}{M_2 \left( \frac{p_H}{p_L} \right)} \frac{\delta \det M}{1 - \delta \det M} (p_H - p_L)( \frac{\delta}{1 - \delta} M_{21} + \pi^1_1).}

The difference in the average per period payoff of the principal between the first best and the second best is at most

\[
(1 - \delta) \frac{c}{M_2 \left( \frac{p_H}{p_L} \right)} \frac{\delta \det M}{1 - \delta \det M} (p_H - p_L)( \frac{\delta}{1 - \delta} M_{21} + \pi^1_1) \\
= \frac{c}{M_2 \left( \frac{p_H}{p_L} \right)} \frac{\delta \det M}{1 - \delta \det M} (p_H - p_L)( \delta M_{21} + (1 - \delta) \pi^1_1).}

Therefore, given \( \epsilon > 0 \), there exists \( \bar{\delta} \) such that for \( \delta < \bar{\delta} \), the principal can approximate his first best payoff by \( \epsilon \). Conversely, for given \( \delta \), there exists \( \bar{D} \) and \( \Delta_p \) such that if \( \det M < \bar{D} \) or \( p_H/p_L < \Delta_p \), the principal can approximate his first best payoff by \( \epsilon \).

In addition, the principal can offer

\[
w(1) = \frac{c}{\pi^1 \left( \frac{p_H}{p_L} \right)}, \quad w(0) = 0, \\
w(h^t 1) = \frac{c}{M_2 \left( \frac{p_H}{p_L} \right)}, \quad w(h^t 0) = 0, \quad \forall t \geq 1,
\]

and he can approximate his first best payoff with a contract that is stationary from the second period. \( \Box \)

**Proof of Proposition 8.** Consider the IC constraints of the agent. By deviating in period \( t \) given history \( h^{t-1} \), the agent effectively replaces the continuation contract for \( h^{t-1} 1 \) with the continuation contract for \( h^{t-1} 0 \). When the payments are independent of the history, the continuation contract from period \( t + 1 \) is identical whether \( h_t = 0 \) or \( h_t = 1 \), and the IC constraint becomes

\[
-c + \pi^t \left( \frac{p_H}{p_L} \right) w(h^{t-1} 1) + (1 - \pi^t \left( \frac{p_H}{p_L} \right)) w(h^{t-1} 0) \geq w(h^{t-1} 0).
\]
In particular, the principal doesn’t take his outside option in any period, and the agent is induced to work in every period.

Since the payments \( w_{t-1}(1) \) and \( w_{t-1}(0) \) satisfy the agent’s IC constraint at all information sets in period \( t \), it is necessary that

\[
w_{t-1}(1) \geq w_{t-1}(0) + \frac{c}{\pi_t^{(PH)}},
\]

for all \( \pi^t \) given \( t \). In particular, let \( \tilde{\pi}^t \) be the prior when all outcomes have been bad since period 1, and we get

\[
w_{t-1}(1) \geq w_{t-1}(0) + \frac{c}{\pi_t^{(PH)}},
\]

\[
w_{t-1}(1) > w_{t-1}(0) + \frac{c}{\pi_t^{(PH)}},
\]

for all \( \pi^t \neq \tilde{\pi}^t \) that can arise as a prior after some history. Since in each period \( t \), there are only two levels of payments, the principal wants to provide the positive payment only after the good outcome, and we have \( w_t(0) = 0 \) for all \( t \).

On the other hand, consider the IC constraints in period 1. From Section 4, the IC constraints in period 1 are given by

\[
V(0, \pi^1) \geq \sum_{k=1}^{T} \delta^{k-1} q(0 \cdots 0) w(0 \cdots 0) + \delta^{T} V(0 \cdots 0, \pi^1 M^T) \nonumber \\
= \delta^{T} V(0 \cdots 0, \pi^1 M^T), T \geq 1.
\]

We know from Proposition 5 that there exists a uniform upper bound on \( V(0 \cdots 0, \pi) \) under an optimal contract. \( w(1) > 0 \) implies that there exists \( T > 0 \) such that the IC constraint for deviating \( T \) times in a row binds. Lowering \( \delta^{T} V(0 \cdots 0, \pi^1 M^T) \), the deviation payoff under the wrong continuation contract, relaxes the IC constraint, and it will allow the principal to lower \( w(1) \), increasing his payoff.

Consider updating priors \( \pi \) and \( \tilde{\pi} \) after the good outcome. Without loss of generality, suppose \( \pi_1 > \tilde{\pi}_1 \). After the good outcome, the priors become

\[
\pi' = \left( \frac{\pi_1^{PH}}{\pi^{(PH)}}, \frac{\pi_2^{PL}}{\pi^{(PL)}} \right).
\]
and
\[ 
\hat{\pi}' = \left( \frac{\hat{\pi}_1 p_H}{\hat{\pi}}, \frac{\hat{\pi}_2 p_L}{\hat{\pi}} \right),
\]
\[ 
\pi'_1 \geq \hat{\pi}'_1, \text{ and the equality holds if and only if } p_L = 0.
\]

Consider history \( h^t = 0 \cdots 0 1 \). In evaluating \( V(0 \cdots 0, \pi^1 M^T) \), the agent assigns \( \pi^1 M^T(p_{PH} p_{PL}) \) as the probability on \( h^t \) and updates his prior after observing \( h_t = 1 \). On the equilibrium path, he has the same prior in the period after \( h^t \) if and only if \( p_L = 0 \). Suppose \( p_L > 0 \) and consider adjusting \( w(h^t) \) and \( w(h^t 1) \).

Let \( \pi \) and \( \hat{\pi} \) be the priors of the agent in the period following \( h^t \) when he has deviated in the first \( T \) periods and on the equilibrium path, respectively. If the principal lowers \( w(h^t 1) \) by \( \Delta \) and raises \( w(h^t) \) by \( \delta \hat{\pi}(p_{PH}) \Delta \), the agent’s continuation value on the equilibrium path doesn’t change. However, the deviation payoff, \( V(0 \cdots 0, \pi^1 M^T) \), changes by
\[
\delta P(0 \cdots 0 1|0 \cdots 0, \pi^1 M^T)(\hat{\pi} \left( p_{PH} \right) \Delta - \pi \left( p_{PH} \right) \Delta) = \delta P(0 \cdots 0 1|0 \cdots 0, \pi^1 M^T)(\hat{\pi} - \pi) \left( p_{PH} \right) \Delta.
\]

From \( p_L > 0 \), we know that \( \pi_1 - \hat{\pi}_1 < 0 \), and the change in \( V(0 \cdots 0, \pi^1 M^T) \) is strictly negative. Since \( h_t = 1 \), the IC constraint doesn’t bind at \( h^t \), and the principal can make the adjustment for \( \Delta \) sufficiently small. Therefore, the principal can lower the rent to the agent by raising \( w(h^t) \) and lowering \( w(h^t 1) \); if it is optimal to provide a history-independent contract, \( p_L \) has to be zero.

Similarly, we can consider raising \( w(10 \cdots 0) \) and lowering \( w(10 \cdots 0 1) \). This will lower \( V(10 \cdots 0, \pi^2 M^{T-1}) \), where \( \pi^2 \) is the agent’s prior after the good outcome in period 1, and the principal can lower the payment \( w(11) \).

When the agent updates priors \( \pi \) and \( \hat{\pi} \) such that \( \pi_1 > \hat{\pi}_1 \), after the bad outcome, the priors become
\[
\hat{\pi}'' = \left( \frac{\hat{\pi}_1 (1 - p_H)}{\hat{\pi}(1 - p_H)}, \frac{\hat{\pi}_2 (1 - p_L)}{\hat{\pi}(1 - p_L)} \right),
\]
and
\[
\hat{\pi}'' = \left( \frac{\hat{\pi}_1 (1 - p_H)}{\hat{\pi}(1 - p_H)}, \frac{\hat{\pi}_2 (1 - p_L)}{\hat{\pi}(1 - p_L)} \right).
\]
\[ 
\pi''_1 \geq \hat{\pi}''_1, \text{ and the equality holds if and only if } p_H = 1.
\]
Let \( \pi \) and \( \hat{\pi} \) be the priors of the agent in the period following \( h^t = 10 \cdots 0 \) when he has deviated for \( T \) periods from period 2 and on the equilibrium path, respectively. If the principal lowers \( w(10 \cdots 0) \) by \( \Delta \) and raises \( w(10 \cdots 0) \) by \( \delta \hat{\pi}(\frac{p_H}{p_L}) \Delta \), the agent’s continuation payoff on the equilibrium path doesn’t change. On the other hand, the deviation payoff, \( V(10 \cdots 0, \pi^2M^{T-1}) \), changes by

\[
\delta P(10 \cdots 0|10 \cdots 0, \pi^2M^{T-1})(\hat{\pi}(\frac{p_H}{p_L}) - \pi(\frac{p_H}{p_L})) \Delta.
\]

Unless \( p_H = 1, \hat{\pi}_1 - \pi_1 < 0 \), and the principal can make the adjustment since the IC constraint doesn’t bind. Therefore, if the optimal contract is history-independent, \( p_H \) must equal one.

For an optimal contract to be history-independent, it is necessary that \( p_H = 1, p_L = 0 \).

**Lemma 1.** Suppose Assumptions 1-3 hold. The IC constraints for the one-shot deviations are sufficient conditions for all IC constraints.

**Proof of Lemma 1.** Randomizing the continuation contracts is the same as taking the linear combination of the IC constraints, and it is sufficient to prove the lemma for pure strategies. Consider the agent who deviated in every period he is induced to work from period \( t + 1 \) to period \( t + T \). Denote this history by \( h^{t+T} \). Without loss of generality, we only need to consider the case the principal doesn’t take his outside option in period \( t + T + 1 \) after the given history. Suppose the principal takes his outside option for \( k \geq 0 \) times after history \( h^{t+T} \). The IC constraint for the one-shot deviation after \( h^{t+T} \) is given by

\[
-c + \pi \left( \begin{array}{c}
V_1 \\
V_2
\end{array} \right) \geq w(h^{t+T}0) + \delta^{k+1}(-c + \pi M^{k+1} \left( \begin{array}{c}
V'_1 \\
V'_2
\end{array} \right)),
\]

where \( \pi \) is the principal’s prior after \( h^{t+T} \) when he believes that the agent worked in every period, and \( V_1 \) is the sum of the present compensation and the continuation value after \( h^{t+T}1 \). \( V_2, V'_1, \) and \( V'_2 \) are defined analogously for \( h^{t+T}0, h^{t+T}0 - 1 \cdots 11 \) and \( h^{t+T}0 - 1 \cdots 10 \).

Remember \(-1\) refers to a period in which the principal takes his outside option.

By subtracting \( V_2 = w(h^{t+T}0) + \delta^{k+1}(-c + M^{k}(\frac{V'_1}{V'_2})) \) from both sides, we know that
the IC constraint is equivalent to
\[-c + \pi_1(V_1 - V_2) \geq \delta^{k+1}(\pi M - M_2)M^k\begin{pmatrix} V_1' \\ V_2' \end{pmatrix},\]
which is again equivalent to
\[V_1 - V_2 \geq \frac{c}{\pi_1} + (\delta \det M)^{k+1}(V_1' - V_2').\]

When the agent has deviated from period \(t + 1\) to \(t + T\), his prior at the beginning of period \(t + T + 1\) is given by \(\pi^{t+1}M^T\). From the positive persistence, the agent assigns a strictly higher probability on the good state than the principal does, and we have
\[\pi^{t+1}M^T\begin{pmatrix} 1 \\ 0 \end{pmatrix} > \pi_1.\]

After having deviated from period \(t + 1\) to \(t + T\), the agent’s IC constraint for working in period \(t + T + 1\) is given by
\[-c + \pi^{t+1}M^T\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \geq w(h^{t+T}0) + \delta^{k+1}(-c + \pi^{t+1}M^{T+k+1}\begin{pmatrix} V_1' \\ V_2' \end{pmatrix})\]
⇔ \[V_1 - V_2 \geq \frac{c}{\pi^{t+1}M^T(1 \ 0)} + (\delta \det M)^{k+1}(V_1' - V_2').\]

From
\[\pi^{t+1}M^T\begin{pmatrix} 1 \\ 0 \end{pmatrix} > \pi_1,\]
the agent prefers to work in period \(t + T + 1\) even if he has deviated from period \(t + 1\) to \(t + T\), as long as the IC constraint for the one-shot deviation after \(h^{t+T}\) is satisfied. Therefore, after each history \(h^t\), it is sufficient to consider the IC constraint for the one-shot deviation. \(\square\)

**Proof of Proposition 9.** Suppose the principal wants the agent to work in every period. Let \(V_1', V_2'\) be the sum of the present compensation and the continuation value after history \(0 \cdots 0\) and \(0 \cdots 0\). In period 1, the IC constraint for the one-shot deviation is given by
\[-c + \pi_1 V_1 + \pi_2 V_2 \geq w(0) + \delta(-c + \pi_1 M\begin{pmatrix} V_1^2 \\ V_2^2 \end{pmatrix})\]
⇔ \[V_1^1 - V_2^1 \geq \frac{c}{\pi_1} + \delta M(V_1^2 - V_2^2),\]
and for $t \geq 2$, the IC constraint for the one-shot deviation is given by

$$-c + M_{21} V_1^t + M_{22} V_2^t \geq w(t) + \delta(-c + M_2 M \left( \frac{V_{1}^{t+1}}{V_{2}^{t+1}} \right))$$

$$\iff V_1^t - V_2^t \geq \frac{c}{M_{21}} + \delta \det M (V_1^{t+1} - V_2^{t+1}).$$

We know from Proposition 6 that the principal can offer

$$w(1) = \frac{c}{\pi_1}, w(0) = 0,$$

$$w(h^t 1) = \frac{c}{M_{21}}, w(h^t 0) = 0, \forall h^t, t \geq 1.$$ to have the agent work in every period. Under the optimal contract, the IC constraints are strictly binding after $0 \cdots 0, \forall t \geq 0$.

I will now show that by taking his outside option after $h^t = 0 \cdots 0$, the principal can lower the payments $w(0 \cdots 0), t < t_0$, and therefore, the rent to the agent is reduced. If the reduction in rent is greater than the loss in outcome by taking the outside option, the principal will prefer to take his outside option after $h^t = 0 \cdots 0$.

From Lemma 1, it is sufficient to consider the IC constraints for the one-shot deviations. Suppose the principal takes his outside option once after $h^t = 0 \cdots 0$. After taking the outside option, the principal continues to pay $w(h^t 1) = \frac{c}{M_{21}}, w(h^t 0) = 0$ for all histories $h^t = h^t \sqcup h^k, \forall k \geq 1, h^k$. Consider the IC constraint after $h^{t_0 - 1} = 0 \cdots 0$. Since the continuation games after the good outcomes are identical and the continuation games after the bad outcomes are identical, we have

$$V_1^{t_0} - V_2^{t_0} \geq \frac{c}{M_{21}} + (\delta \det M)^2 (\hat{V}_1^{t_0+2} - \hat{V}_2^{t_0+2}),$$

where $\hat{V}_1^{t_0+2}$ is the sum of the present compensation and the continuation value after $h^{t_0} - 11$. $\hat{V}_2^{t_0+2}$ is defined for $h^{t_0} - 10$. We also have

$$\hat{V}_1^{t_0+2} - \hat{V}_2^{t_0+2} = \hat{V}_1^{t_0+1} - \hat{V}_2^{t_0+1} = \frac{c}{M_{21}} \frac{1}{1 - \delta \det M},$$

where $\hat{V}_1^{t_0+1}$ is the sum of the present compensation and the continuation value after the good outcome, from the contract in Proposition 6. $\hat{V}_2^{t_0+1}$ is defined for the bad outcome.
Note that the IC constraint after $h^{t_0-1}$ in the contract from Proposition 6 is

$$V_1^{t_0} - V_2^{t_0} \geq \frac{c}{M_{21}} + \delta \det M (\tilde{V}_1^{t_0+1} - \tilde{V}_2^{t_0+1}).$$

Therefore, by taking his outside option after $h^{t_0} = 0 \cdots 0$, the IC constraint after $h^{t_0-1} = 0 \cdots 0$ is relaxed by

$$\delta \det M (1 - \delta \det M) \frac{c}{M_{21}} 1 - \delta \det M = \delta \det M \frac{c}{M_{21}}.$$

The principal can lower the payment $w(h^{t_0-1})$ by $\delta \det M \frac{c}{M_{21}}$. By an inductive argument, we have that $V_1^k - V_2^k$ for $1 \leq k \leq t_0$ can be reduced by

$$(\delta \det M)^{t_0-k+1} \frac{c}{M_{21}}.$$

$V_1^k$ and $V_2^k$ for $1 \leq k \leq t_0$ are each reduced by

$$\delta^{t_0-k+1} \det M \left( \frac{1 - (\det M)^{t_0-k}}{1 - \det M} + \frac{1}{M_{21}} (\det M)^{t_0-k} c \right)$$

and

$$\delta^{t_0-k+1} \det M \frac{1 - (\det M)^{t_0-k}}{1 - \det M} c.$$
From period 1, the rent to the agent is reduced by

\[ \pi_1^1 \Delta V_1^1 + \pi_2^1 \Delta V_2^1 \]
\[ = \pi_1^1 \delta^t_0 \det M \left( \frac{1 - (\det M)^{t_0-1}}{1 - \det M} + \frac{1}{M_{21}} (\det M)^{t_0-1} \right) c \]
\[ + \pi_2^1 \delta^t_0 \det M \frac{1 - (\det M)^{t_0-1}}{1 - \det M} c \]
\[ = \delta^t_0 \det M \left( \frac{1 - (\det M)^{t_0-1}}{1 - \det M} + \frac{\pi_1^1}{M_{21}} (\det M)^{t_0-1} \right) c \]
\[ \geq \delta^t_0 \det M \frac{1 - (\det M)^{t_0}}{1 - \det M} c. \]

On the other hand, the loss in outcome from taking the outside option is

\[ \delta^t_0 \pi_2^1 (M_{22})^{t_0-1} (-c + M_{21} - \mu). \]

Both the loss in outcome and the reduction in rent are discounted by \( \delta^t_0 \). Apart from the discounting, the loss in outcome converges to zero as \( t_0 \) goes to infinity, while the reduction in the rent is bounded away from zero. Therefore, there exists \( t_0 \) such that

\[ \frac{1 - (\det M)^{t_0}}{1 - \det M} \det M c > \pi_2^1 (M_{21})^{t_0-1} (-c + M_{21} - \mu), \]
and for any discount factor \( \delta > 0 \), the principal strictly prefers to take his outside option after \( h^{t_0} = 0 \cdots 0 \) than to have the agent work in every period.

Proof of Proposition 11. Suppose the principal makes a positive payment for history \( h^t \) with \( h_t = 0 \). Let \( k \) be the maximum \( k < t \) such that \( h_k = 1 \) in the history \( h^t \). The principal can frontload the payment so that \( \hat{w}(h^{k-1}) = w(h^{k-1}) + \delta^{t-k} M_{12} M_{22}^{t-k-1} w(h^t) \) and \( \hat{w}(h^t) = 0 \). If \( k = 0 \), lower the payment for \( h^t \) to \( \hat{w}(h^t) = 0 \). Since the composition of the continuation value after the good outcome doesn’t matter for the agent’s IC constraint, the IC constraints leading up to history \( h^k \) are not affected by the adjustment. On the other hand, the IC constraints after \( h^k \sqcup h^l \) in the history \( h^t \) are relaxed under the new contract.

Under the new contract, the agent is induced to work after exactly the same set of histories as under the previous contract, and the agent’s IC constraints are satisfied after every history after which the principal wants the agent to work. The rent to the agent is weakly lower under the new contract. Therefore, the principal can frontload the payment whenever he makes a positive payment for a bad outcome, and there is no loss of generality in assuming that the principal makes positive payments only for the good outcomes.
Proof of Proposition 12. The proof follows directly from Proposition 6, 10 and 11.

Proof of Proposition 14. The first step is to find the set of pairs of \((V_1, V_2)\) with which the agent is induced to work in a given period. Define \(S\) to be the largest self-generating set of the form

\[
S = \text{conv}(\{ (\pi, V_1, V_2) | \exists T \geq 0, (\pi', V'_1, V'_2) \in S \text{ such that } \\
(i) \pi' = M_2 M^T, \\
(ii) V_2 = \delta^{T+1}(-c + \pi' \left( \frac{V'_1}{V'_2} \right)), \\
(iii) V_1 - V_2 \geq \frac{c}{\pi_1} + \delta^{T+1}(\det M)T+1(V'_1 - V'_2) \}).
\]

Since mixing the continuation contracts is the same as taking the linear combination of the IC constraints, we can focus on the pure strategies and take the convex hull. From Lemma 1, it is sufficient to consider the one-shot deviations. Using Proposition 11, I will consider contracts under which the principal makes positive payments only for the good outcomes.

Suppose the principal wants the agent to work after history \(h^t\) and he takes his outside option for \(T \geq 0\) periods after history \(h'^0\). Let \(V_1\) be the sum of the present compensation and the continuation value after \(h'^1\). Define \(V_2, V'_1\) and \(V'_2\) similarly for \(h'^0, h'^0 - 1 \cdots - 11\) and \(h'^0 - 1 \cdots - 10\). Let \(\pi\) be the prior on the state after history \(h^t\). The IC constraint after history \(h^t\) is

\[-c + \pi \left( \frac{V_1}{V_2} \right) \geq \delta^{T+1}(-c + \pi M^{T+1} \left( \frac{V'_1}{V'_2} \right)).\]

Subtracting

\[V_2 = \delta^{T+1}(-c + \pi' \left( \frac{V'_1}{V'_2} \right))\]

from both sides, we get

\[V_1 - V_2 \geq \frac{c}{\pi_1} + \delta^{T+1}(\det M)T+1(V'_1 - V'_2).\]

Conversely, if there exists \(T \geq 0, V'_1, V'_2\) such that

\[V_2 = \delta^{T+1}(-c + \pi' \left( \frac{V'_1}{V'_2} \right)),\]

\[V_1 - V_2 \geq \frac{c}{\pi_1} + \delta^{T+1}(\det M)T+1(V'_1 - V'_2)\]

hold, and the agent is induced to work given prior \(\pi' = M_2 M^T\) and \(V'_1, V'_2\), then given prior
\( \pi \) and \( V_1, V_2 \), the agent is induced to work.

Therefore, the set of feasible continuation values to induce work is given by the largest self-generating set

\[
S = \text{conv}(\{(\pi, V_1, V_2) | \exists T \geq 0, (\pi', V_1', V_2') \in S \text{ such that } \quad \\
(i) \pi' = M_2M^T, \\
(ii) V_2 = \delta^{T+1}(-c + \pi'(V_1')), \\
(iii) V_1 - V_2 \geq \frac{c}{\pi_1} + \delta^{T+1}(\det M)^{T+1}(V_1' - V_2')\}).
\]

The next step is to characterize the space of \((R, L)\) for all incentive compatible contracts. Let \( X_\pi \) be the space of \((R, L)\) for all incentive compatible contracts with the initial prior \( \pi \), where \( R \) is the rent to the agent and \( L \) is the loss in outcome under the contract. \( L \) is defined to be

\[
L = (1 - \delta)(Y_{FB} - Y),
\]

where \( Y_{FB} \) is the expected discounted sum of the outcome in the first best, and \( Y \) is the expected discounted sum of the outcome under the given contract. Given prior \( \pi \), there exists a contract with the rent \( R \) and the loss \( L \) if the following is satisfied: the principal takes his outside option for \( T \geq 0 \) periods, and in the first period the agent is induced to work, the continuation contracts after the good outcome and the bad outcome have \((R_1, L_1)\) and \((R_2, L_2)\), respectively. Specifically, \( X_\pi \) is given by

\[
X_\pi = \text{conv}(\{(R, L) | \exists T \geq 0, (R_1, L_1) \in X_{M_1}, (R_2, L_2) \in X_{M_2} \text{ such that } \quad \\
(i) \pi' = \pi M^T, \\
(ii) R = \delta^T(-c + \pi'(R_1)), \\
(iii) L = (1 - \delta) \sum_{k=1}^{T} \delta^{k-1}(-c + \pi M^{k-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - u) + \delta^{T+1}\pi'(L_1), \\
(iv) (\pi', \delta R_1, \delta R_2) \in S\}),
\]

where Condition (ii) and (iv) use the fact that there is no loss of generality in delaying the payments.

I will show that \( X_{M_1} \) and \( X_{M_2} \) can be found as limits of two sequences of sets. Once we find \( X_{M_1} \) and \( X_{M_2} \), \( X_\pi \) is generated from \( X_{M_1} \) and \( X_{M_2} \). Consider the sequences of sets,
\{X_{n}^{1}\} \text{ and } \{X_{n}^{2}\}:

\begin{align*}
X_{1}^{0} &= \{ (R, 0) | R \geq R_{1}^{*} \equiv \frac{\delta \det M c}{1 - \delta \det M} (\delta + (1 - \delta) \frac{M_{11}}{M_{21}}) \}, \\
X_{2}^{0} &= \{ (R, 0) | R \geq R_{2}^{*} \equiv \frac{\delta \det M c}{1 - \delta \det M} \},
\end{align*}

where \( R_{1}^{*} \) and \( R_{2}^{*} \) are the rents to the agent under the cost-minimizing contracts for initial priors \( M_{1} \) and \( M_{2} \), and

\begin{align*}
X_{1}^{n+1} &= \text{conv}(\{(R, L) | \exists T \geq 0, (R_{1}, L_{1}) \in X_{1}^{n}, (R_{2}, L_{2}) \in X_{2}^{n} \text{ such that} \}) \\
& \quad (i) \pi' = M_{1} M^{T}, \\
& \quad (ii) R = \delta^{T} (-c + \delta \pi' \begin{pmatrix} R_{1} \\ R_{2} \end{pmatrix}), \\
& \quad (iii) L = (1 - \delta) \sum_{k=1}^{T} \delta^{k-1} (-c + M_{1} M^{k-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \mu) + \delta^{T+1} \pi' \begin{pmatrix} L_{1} \\ L_{2} \end{pmatrix}, \\
& \quad (iv) (\pi', \delta R_{1}, \delta R_{2}) \in S}, \\
X_{2}^{n+1} &= \text{conv}(\{(R, L) | \exists T \geq 0, (R_{1}, L_{1}) \in X_{1}^{n}, (R_{2}, L_{2}) \in X_{2}^{n} \text{ such that} \}) \\
& \quad (i) \pi' = M_{2} M^{T}, \\
& \quad (ii) R = \delta^{T} (-c + \delta \pi' \begin{pmatrix} R_{1} \\ R_{2} \end{pmatrix}), \\
& \quad (iii) L = (1 - \delta) \sum_{k=1}^{T} \delta^{k-1} (-c + M_{2} M^{k-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \mu) + \delta^{T+1} \pi' \begin{pmatrix} L_{1} \\ L_{2} \end{pmatrix}, \\
& \quad (iv) (\pi', \delta R_{1}, \delta R_{2}) \in S}.
\end{align*}

Define

\begin{align*}
X_{1}^{\infty} &= \lim_{n \to \infty} X_{1}^{n}, \\
X_{2}^{\infty} &= \lim_{n \to \infty} X_{2}^{n}.
\end{align*}

\( X_{1}^{\infty} \) and \( X_{2}^{\infty} \) are the sets of \((R, L)\) we are looking for. Before proving \( X_{M_{1}} = X_{1}^{\infty} \) and \( X_{M_{2}} = X_{2}^{\infty} \), I will first show that \( X_{1}^{\infty} \) and \( X_{2}^{\infty} \) are well-defined.
For all $n \geq 1$, we have
\[
L \leq (1 - \delta) \sum_{k=1}^{\infty} \delta^{k-1} (-c + M_1 M^{k-1}(1) - \delta) \equiv L_1^*, \forall (R, L) \in X_1^n,
\]
\[
L \leq (1 - \delta) \sum_{k=1}^{\infty} \delta^{k-1} (-c + M_2 M^{k-1}(1) - \delta) \equiv L_2^*, \forall (R, L) \in X_2^n.
\]

Each $X_i^n$ is of the form
\[
X_i^n = \{(R, L)|R \geq f_i^n(L), 0 \leq L \leq L_i^*\}
\]
for some function
\[
f_i^n : [0, L_i^*] \rightarrow [0, R_i^*].
\]

For each $i$ and $n$, $f_i^n(\cdot)$ is strictly decreasing in $L$, and from $X_i^n \subset X_i^{n+1}, \forall n \geq 0, i = 1, 2$, we know that
\[
f_i^{n+1}(L) \leq f_i^n(L), \forall 0 \leq L \leq L_i^*, \forall n \geq 1, i = 1, 2.
\]

Together with $R \geq 0$, the monotone convergence theorem gives that the limits
\[
\bar{X}_i^\infty \equiv \{(R, L)|R = \lim_{n \to \infty} f_i^n(L), 0 \leq L \leq L_i^*\}, i = 1, 2,
\]
are well-defined and downward-sloping. Therefore,
\[
X_i^\infty = \{(R, L)|R \geq \lim_{n \to \infty} f_i^n(L), 0 \leq L \leq L_i^*\}, i = 1, 2
\]
are well-defined.

That any $(R, L) \in X_1^\infty$ and $X_2^\infty$ are feasible given priors $M_1$ and $M_2$ can be shown as the following. Let $Y_i$ be the set generated by $X_1^\infty$ and $X_2^\infty$ for the initial prior $M_i$. Given $\epsilon > 0$ and $(R, L) \in Y_i$, there exists $n$ such that $(R, L + \epsilon) \in X_i^n$. Therefore, $Y_i$ lies in the limit of $X_i^n$, and we have
\[
Y_i \subset X_i^\infty, i = 1, 2.
\]

On the other hand, we know that each $Y_i$ is closed, and
\[
Y_i \supset X_i^\infty \setminus \bar{X}_i^\infty.
\]
Together, $Y_i = X_i^\infty$ for $i = 1, 2$ and $X_1^\infty$ and $X_2^\infty$ are jointly self-generating.

Conversely, if $(R, L)$ is feasible given $M_1$ or $M_2$, we can show that it’s in $X_1^\infty$ or $X_2^\infty$, respectively. Note that if the principal takes his outside option for $T$ blocks under the
given contract, \((R, L) \in X_i^T \subset X_i^\infty\). For contracts under which the principal takes his outside option for an infinite number of times, we can construct a truncated contract as the following. Given \(T\), for each history \(h^T\), pay the sum of the present compensation and the continuation value after history \(h^T\). From period \(T + 1\) and on, take the outside option forever. This replacement contract weakly relaxes the IC constraints of the agent, and it provides exactly the same amount of rent to the agent. The loss in outcome under the contract differs from the original contract by at most \(\delta^T L_i^*\), depending on whether \(h_T = 1\) or 0. For given \(\epsilon > 0\), the principal can choose \(T\) sufficiently large so that the replacement contract lies within \(\epsilon\) from the original contract in the space of \((R, L)\). This implies that any feasible \((R, L)\) lies in the limits \(X_1^\infty\) and \(X_2^\infty\).

Together, we get \(X_{M_1} = X_1^\infty\) and \(X_{M_2} = X_2^\infty\). Once we have \(X_{M_1} = X_1^\infty\) and \(X_{M_2} = X_2^\infty\), for any prior \(\pi\) can be constructed from \(X_{M_1}\) and \(X_{M_2}\). The second best contract given the prior \(\pi\) is the contract that minimizes \(R + L\) in \(X_\pi\), and it can be constructed as the following. Given \((R, L) \in X_\pi\), there exist \(T \geq 0, (R_1, L_1) \in X_{M_1}, (R_2, L_2) \in X_{M_2}\) supporting \((R, L)\). The principal takes the outside option for \(T\) periods, and after the first period the agent is induced to work, the continuation contract is determined by \((R_1, L_1)\) and \((R_2, L_2)\). The contract continues in a probationary period if the outcome is good and \(0 < R_1 < R_1^*\) or if the outcome is bad and \(0 < R_2 < R_2^*\). The continuation contract is \((R_1, L_1)\) if the outcome is good, and it’s \((R_2, L_2)\) if the outcome is bad. If the outcome is bad and \(R_2 = 0\), the contract terminates, and the principal takes his outside option forever. If the outcome is good and \(R_1 \geq R_1^*\), the agent is tenured, and \(\delta(R_1 - R_1^*)\) is provided as the initial payment. From the following period, the contract continues with \((R_1^*, 0)\), and the payments are given by the contract in Proposition 12. If the outcome is bad but \(R_2 = R_2^*\) given \(h^0\), again, the agent is tenured, and the principal provides

\[
w(h^0 \sqcup h^k) = \frac{c}{M_{21}}, \quad w(h^0 \sqcup h^k) = 0, \forall h^k, k \geq 0,
\]

from the following period.

Proof of Proposition 15. I will show the proposition by constructing a review contract that allows the principal to approximate his first best payoff. Consider the following review contract: the contract specifies a review block of \(T\) periods, a quota, \(Q\), and a lump sum transfer, \(X\). A quota is on the number of successful outcomes from the block. If the agent meets the quota, the principal pays the agent the discounted sum of the outcome subtracted by the lump sum transfer at the end of the review block, and the contract continues. If the agent fails to meet the quota, the principal pays the agent the discounted sum of the outcome, and the contract terminates.
First, consider the principal’s payoff. Let $s$ be the agent’s strategy in the equilibrium and $p(s)$ be the minimum probability he meets the quota. In general, the probability the agent meets the quota with a strategy depends on the prior on the state at the beginning of the review block, but we can take the minimum of the probabilities over the priors. Then the principal’s average per period payoff is at least
\[
(1 - \delta)\delta^{T-1}p(s)X(1 + \delta^Tp(s) + \delta^{2T}p(s)^2 + \cdots)
= \frac{\delta^{T-1}p(s)(1 - \delta)X}{1 - \delta^Tp(s)}
= \frac{\delta^T(1 - \delta^T)p(s)}{1 - \delta^Tp(s)} \frac{1 - \delta}{1 - \delta^T} X.
\]

Since the states exhibit positive persistence, the expected discounted sum of the outcome in the first best is the maximum when the pair starts with $\pi^1 = (1, 0)$. Let $\bar{y}$ be the average expected discounted sum of the outcome over the infinite horizon in the first best when $\pi^1 = (1, 0)$. When the following two inequalities hold,
\[
\frac{\delta^T(1 - \delta^T)p(s)}{1 - \delta^Tp(s)} \geq 1 - \frac{\epsilon}{2},
\]
\[
\frac{1 - \delta}{1 - \delta^T} X \geq (1 - \frac{\epsilon}{2})\bar{y} - c,
\]
the principal’s payoff is at least $(1 - \epsilon)\bar{y} - c \geq \bar{y} - c - \epsilon$. Note that the first inequality implies that both $\delta^T$ and $p(s)$ are greater than $1 - \epsilon/2$.

The second step is to verify the agent’s incentives that the agent will pass the quota with $p(s)$ close to one. Let $V(\pi)$ be the agent’s continuation value when the review block starts with the prior $\pi$. Since the agent can always choose to work in every period, letting $s$ be the strategy of working in every period, we have
\[
V(\pi) \geq (1 - \delta)(Y(\pi) - \frac{1 - \delta^T}{1 - \delta}c) + \delta^T p(s)(\mathbb{E}[V(\hat{\pi})] - \frac{1 - \delta}{\delta}X),
\]
where $Y(\pi)$ is the expected discounted sum of the outcome from working in every period from a block with the initial prior $\pi$ and $\hat{\pi}$ is the prior in the beginning of the next block.

Let $V$ be the minimum of $V(\pi)$ over all priors $\pi$. Together with the fact that $Y(\pi)$ increases with $\pi_1$, we get the following inequality:
\[
V \geq \frac{(1 - \delta)(Y((0, 1)) - \frac{1 - \delta^T}{1 - \delta}c) - \delta^T p(s)\frac{1 - \delta}{\delta} X}{1 - \delta^Tp(s)}.
\]
If $V \geq \frac{1-\delta}{\delta} X$, the agent always prefers to increase the probability of meeting the quota. Since the principal pays the agent the discounted sum of the outcome, subtracted by $X$ on meeting the quota, the agent works in every period on the equilibrium path.

When the lump sum transfer is specified to

$$ X \leq \delta(Y((0,1)) - \frac{1 - \delta^T}{1 - \delta} c), $$

the inequality $V \geq \frac{1-\delta}{\delta} X$ is always satisfied. The last condition is to ensure that the discounted sum of the outcome on meeting the quota is weakly greater than $X$ so that the principal can actually take away the lump sum transfer. A slightly stronger condition is

$$ Q \geq X. $$

Therefore, when a review contract satisfies

$$ \frac{\delta^T (1 - \delta^T) p(s)}{1 - \delta^T p(s)} \geq 1 - \frac{\epsilon}{2}, \quad (1) $$

$$ \frac{1 - \delta}{1 - \delta^T} X \geq (1 - \frac{\epsilon}{2}) \bar{y} - c, \quad (2) $$

$$ X \leq \delta(Y((0,1)) - \frac{1 - \delta^T}{1 - \delta} c), $$

the agent chooses to work in every period, and the principal’s payoff is within $\epsilon$ of his first best payoff.

Since the Markov chain is irreducible and there are two states, there exist $\delta_0$ and $T_0$ such that for all $\delta \geq \delta_0, T \geq T_0$ and the initial prior $\pi$, we have

$$ |\frac{1 - \delta}{1 - \delta^T} Y(\pi) - \bar{p}| < \frac{\epsilon}{4} \bar{p}, $$

where

$$ \bar{p} = \left( \frac{M_{21}}{M_{12} + M_{21}}, \frac{M_{12}}{M_{12} + M_{21}} \right) \left( \frac{p_H}{p_L} \right), $$

is the probability of the good outcome from the ergodic distribution of the Markov chain.

Let

$$ X = \frac{\delta(1 - \delta^T)}{1 - \delta} ((1 - \frac{\epsilon}{4}) \bar{p} - c). $$

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For $\delta \geq \delta_0, T \geq T_0$, Inequality (2) is satisfied as

$$\frac{1 - \delta}{1 - \delta T} \frac{X}{\delta} = (1 - \frac{\epsilon}{4})\bar{p} - c$$

$$> (1 - \frac{\epsilon}{2})(1 + \frac{\epsilon}{4})\bar{p} - c$$

$$\geq (1 - \frac{\epsilon}{2})\bar{y} - c.$$  

Lastly, Inequality (1) can be rearranged as a quadratic equation of $\delta T$. We get

$$1 - \frac{\epsilon}{4} - \sqrt{(1 - \frac{\epsilon}{4})^2 - \frac{1 - \frac{\epsilon}{2}}{p(s)}} \leq \delta T \leq 1 - \frac{\epsilon}{4} + \sqrt{(1 - \frac{\epsilon}{4})^2 - \frac{1 - \frac{\epsilon}{2}}{p(s)}}$$

There exists $p < 1$ such that for $p(s) \geq p$, we have

$$\sqrt{(1 - \frac{\epsilon}{4})^2 - \frac{1 - \frac{\epsilon}{2}}{p(s)}} \geq \frac{\epsilon}{8}.$$  

Let

$$Q = \lceil X \rceil,$$

where $\lceil X \rceil$ is the smallest integer greater than or equal to $X$. I will now show that we can find $T_1$ such that for $T \geq T_1$, the agent meets the quota with a probability higher than $p$ by working in every period.

From $Q = \lceil X \rceil$, the quota is satisfied whenever

$$\frac{\hat{Q}}{T} \geq \frac{X + 1}{T},$$

where $\hat{Q}$ is the number of good outcomes from the block. By the strong law of large numbers, $\hat{Q}/T$ converges to $\bar{p}$ for all initial priors $\pi$. Since the right hand side of the inequality is bounded from above by

$$((1 - \frac{\epsilon}{4})\bar{p} - c) + \frac{1}{T},$$

we can pick $T_1 \geq \frac{1}{\epsilon}$, and the right hand side is strictly bounded away from $\bar{p}$ for all $\delta, T \geq T_1$. Therefore, there exists $T_1$ such that for $T \geq T_1$, we have

$$\Pr(\frac{\hat{Q}}{T} \geq \frac{X + 1}{T}) \geq p.$$  

We can find $T_1$ that holds uniformly for all initial priors $\pi$, since $\hat{Q}/T$ for the given prior $\pi$
can be written as
\[ \frac{\hat{Q}}{T} = 1_{\{Z=1\}}X_1 + 1_{\{Z=2\}}X_2. \]

\(X_1\) is \(\hat{Q}/T\) for the prior \((1, 0)\), and \(X_2\) is \(\hat{Q}/T\) for the prior \((0, 1)\). \(Z\) is a random variable with \(\Pr(Z = 1) = \pi_1\) and \(\Pr(Z = 2) = \pi_2\).

Let \(\bar{T} = \max\{T_0, T_1\}\) and define \(\bar{\delta}\) to be
\[ \bar{\delta} = \max\{\delta_0, \frac{1 - \frac{3}{8}\epsilon}{1 - \frac{\epsilon}{8}}, \sqrt[4]{1 - \frac{3}{8}\epsilon}\}. \]

Then for any \(\delta \geq \bar{\delta}\), there exist \(T, Q, \) and \(X\) for the review contract that allows the principal to approximate his first best payoff.

\(\Box\)

**Proof of Proposition 16.** Consider the following review contract. Each review block lasts \(T\) periods, and there exist a quota, \(Q\), and a lump sum transfer, \(X\). The quota is on the discounted sum of the outcome, and if the agent meets the quota, the principal pays him the discounted sum of the outcome from the review block, subtracted by the lump sum transfer, and the contract continues. If the agent fails to meet the quota, the principal pays the discounted sum of the outcome from the review block to the agent, and the contract terminates.

Given \(\epsilon > 0\), the expected discounted sum of the outcome converges to \(\mu\), and there exists \(\delta_0, T_0\) such that for any \(k, (x_0, x_1, \cdots, x_k)\), \(\delta \geq \delta_0, T \geq T_0\), we have
\[ |1 - \frac{\delta}{1 - \delta} \left( \sum_{t=1}^{T} \delta^{t-1} \mathbb{E}[X_{k+t}] \right) - \mu| < \frac{\epsilon}{4}\mu. \]

Let
\[ Q \equiv X \equiv \frac{\delta(1 - \delta^T)}{1 - \delta} ((1 - \frac{\epsilon}{4})\mu - c). \]

Denote by \(V((x_0, x_1, \cdots, x_k))\) the agent’s continuation value given history \((x_0, x_1, \cdots, x_k)\). We have the following expression for \(\bar{V} = \min_{\mathcal{H}} V((x_0, x_1, \cdots, x_k))\), where \(\mathcal{H}\) is the set of all histories:
\[ \bar{V} \geq \frac{(1 - \delta)(\min_{\mathcal{H}}(\sum_{t=1}^{T} \delta^{t-1} \mathbb{E}[X_{k+t}]) - \frac{1 - \delta^T}{1 - \delta} c) - \delta^T p(s) \frac{1 - \delta}{\delta} X}{1 - \delta^T p(s)}. \]

For \(\delta \geq \delta_0, T \geq T_0\), we have
\[ \bar{V} \geq \frac{1 - \delta}{\delta} X, \]
and the agent always prefers to increase the probability of meeting the quota. Since the agent is paid the discounted sum of the outcome, the agent is induced to work in every
period under the contract.

It remains to show that the principal’s payoff under the contract is close to his first best payoff. Let \( p(s) \) be the infimum of the probability of meeting the quota by working in every period, where the infimum is taken over the set of histories \( \mathcal{H} \). The principal’s payoff is at least

\[
(1 - \delta)\delta^{T-1}p(s)X(1 + \delta^T p(s) + \delta^{2T} p(s)^2 + \cdots)
\]

\[
= \frac{\delta^{T-1}p(s)(1 - \delta)X}{1 - \delta^T p(s)}
\]

\[
= \frac{\delta^T(1 - \delta^T) p(s)}{1 - \delta^T p(s)} \frac{1 - \delta}{1 - \delta^T} X.
\]

Similarly as in the proof of Proposition 15, the principal’s payoff is within \( \epsilon \) of his first best payoff if the following two inequalities hold:

\[
\frac{\delta^T(1 - \delta^T) p(s)}{1 - \delta^T p(s)} \geq 1 - \frac{\epsilon}{2},
\]

\[
\frac{1 - \delta}{1 - \delta^T} X \geq (1 - \frac{\epsilon}{2}) \bar{y} - c,
\]

where \( \bar{y} \) is the supremum of the expected discounted sum of outcome over the infinite horizon in the first best, and the supremum is taken over all initial conditions \( x_0 \). The second inequalities is satisfied for any \( \delta \geq \delta_0, T \geq T_0 \), and we need to show that the first inequality also holds. By the uniform weak law of large numbers, there exist \( \delta_1, T_1 \) such that for \( \delta \geq \delta_1, T \geq T_1 \), we have

\[
\Pr(|\frac{1 - \delta}{1 - \delta^T} \sum_{t=1}^{T} \delta^{t-1} X_{k+t} - \mu| > \epsilon') < \epsilon'.
\]

Choose \( \epsilon' \) such that

\[
\epsilon' < \frac{\epsilon}{4} \mu,
\]

and that for \( p(s) \geq 1 - \epsilon' \),

\[
\sqrt{(1 - \epsilon')^2 - \frac{1 - \epsilon'}{p(s)}} \geq \frac{\epsilon}{8}.
\]

Let \( \bar{T} = \max\{T_0, T_1\} \) and define \( \bar{\delta} \) to be

\[
\bar{\delta} = \max\{\delta_0, \frac{1 - \frac{3}{8} \epsilon}{1 - \frac{1}{8} \epsilon}, \sqrt{1 - \frac{3}{8} \epsilon}\}.
\]

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Then for any $\delta \geq \bar{\delta}$, there exist $T, Q,$ and $X$ for the review contract that allows the principal to approximate his first best payoff.

Proof of Proposition 17. Suppose there are $n$ states and $M$ is the Markov transition matrix. When there are a finite number of states following an irreducible Markov chain, the prior on the state is a sufficient static for the distribution of future states, and it is sufficient to show the following: there exists $\mu > 0$ such that (i) for given $\epsilon > 0$, there exist $\delta_0, T_0$ such that for any prior $\pi$, $\delta \geq \delta_0, T \geq T_0$,

$$\left| \frac{1 - \delta}{1 - \delta^T} \left( \sum_{t=1}^{T} \delta^{t-1} \mathbb{E}[X_t(\pi)] \right) - \mu \right| < \epsilon,$$

and (ii) for given $\epsilon, \epsilon' > 0$, there exists $T_1$ such that for any prior $\pi, \delta, T \geq T_1$ with $\delta^T \geq 1 - \frac{\epsilon}{4}$,

$$\Pr\left( \frac{1 - \delta}{1 - \delta^T} \left( \sum_{t=1}^{T} \delta^{t-1} X_t(\pi) \right) < (1 - \frac{\epsilon}{4}) \mu - c \right) < \epsilon',$$

where $X_t(\pi)$ is the stochastic process for the outcome of working in period $t$ given the initial prior $\pi$. Without loss of generality, I assume $X_t(\pi)$ is non-negative for all $t, \pi$.

I will first show why Inequalities (3) and (4) are sufficient conditions for the principal to be able to approximate his first best payoff. Consider the following review contract: each review block lasts $T$ periods, and there exist a quota, $Q$, and a lump sum transfer, $X$. The quota is on the discounted sum of the outcome, and if the agent meets the quota, the principal pays him the discounted sum of the outcome from the review block, subtracted by the lump sum transfer, and the contract continues. If the agent fails to meet the quota, the principal pays the discounted sum of the outcome from the review block to the agent, and the contract terminates.

Suppose Inequalities (3) and (4) hold. Given $\epsilon > 0$, the expected discounted sum of the outcome converges to $\mu$, and there exists $\delta_0, T_0$ such that for any prior $\pi, \delta \geq \delta_0$, and $T \geq T_0$, we have

$$\left| \frac{1 - \delta}{1 - \delta^T} \left( \sum_{t=1}^{T} \delta^{t-1} \mathbb{E}[X_t(\pi)] \right) - \mu \right| < \frac{\epsilon}{4} \mu.$$

Let

$$Q \equiv X \equiv \frac{\delta(1 - \delta^T)}{1 - \delta} \left( (1 - \frac{\epsilon}{4}) \mu - c \right).$$

Denote by $V(\pi)$ the agent’s continuation value given the initial prior $\pi$. We have the following
expression for $V = \min_\pi V(\pi)$:

$$V \geq \frac{(1 - \delta)(\min_\pi(\sum_{t=1}^{T} \delta^{t-1}E[X_t(\pi)]) - \frac{1-\delta^T}{1 - \delta}c) - \delta^T p(s) \frac{1-\delta}{\delta}}{1 - \delta^T p(s)}.$$

For $\delta \geq \delta_0, T \geq T_0$, we have

$$V \geq \frac{1 - \delta}{\delta} X,$$

and the agent always prefers to increase the probability of meeting the quota. Since the agent is paid the discounted sum of the outcome, the agent is induced to work in every period under the contract.

We can also show that the principal’s payoff under the contract is close to his first best payoff. Let $p(s)$ be the minimum of the probability of meeting the quota by working in every period, where the minimum is taken over the initial priors. The principal’s payoff is at least

$$(1 - \delta)\delta^{T-1} p(s) X (1 + \delta^T p(s) + \delta^{2T} p(s)^2 + \cdots)$$

$$= \frac{\delta^{T-1} p(s) (1 - \delta) X}{1 - \delta^T p(s)}$$

$$= \frac{\delta^T (1 - \delta^T) p(s)}{1 - \delta^T p(s)} \frac{1 - \delta}{\delta} X.$$

Similarly as in the proof of Proposition 15, the principal’s payoff is within $\epsilon$ of his first best payoff if the following two inequalities hold:

$$\frac{\delta^T (1 - \delta^T) p(s)}{1 - \delta^T p(s)} \geq 1 - \frac{\epsilon}{2},$$

$$\frac{1 - \delta}{\delta} X \geq (1 - \frac{\epsilon}{2})\tilde{y} - c,$$

where $\tilde{y}$ is the maximum expected discounted sum of outcome over the infinite horizon in the first best. The second inequalities is satisfied for any $\delta \geq \delta_0, T \geq T_0$, and we need to show that the first inequality also holds.

Let $p$ be

$$p = \frac{1 - \frac{\epsilon}{2}}{1 - \frac{\epsilon}{2} + \frac{3}{64}\epsilon^2}.$$

For any $p(s) \geq p$, we have

$$\sqrt{(1 - \frac{\epsilon}{4})^2 - \frac{1 - \frac{\epsilon}{2}}{p(s)}} \geq \frac{\epsilon}{8}.$$
By Inequality (4), there exist $T_1$ such that for $T \geq T_1$ with $\delta^T \geq 1 - \frac{\epsilon}{2}$, we have

$$\Pr\left( \frac{1 - \delta}{1 - \delta^T} \left( \sum_{t=1}^{T} \delta^{t-1} X_t(\pi) \right) < (1 - \frac{\epsilon}{4}) (\mu - c) < 1 - p. \right)$$

Then, the agent meets the quota with $p(s) \geq p$ by working in every period.

We can rearrange

$$\delta^T (1 - \delta^T) p(s) \geq 1 - \frac{\epsilon}{2}$$

as

$$1 - \frac{\epsilon}{4} - \sqrt{(1 - \frac{\epsilon}{4})^2 - \frac{1 - \frac{\epsilon}{2}}{p(s)}} \leq \delta^T \leq 1 - \frac{\epsilon}{4} + \sqrt{(1 - \frac{\epsilon}{4})^2 - \frac{1 - \frac{\epsilon}{2}}{p(s)}}.$$ 

Let $\bar{T} = \max\{T_0, T_1\}$ and define $\bar{\delta}$ to be

$$\bar{\delta} = \max\{\delta_0, \frac{1 - \frac{3}{8} \epsilon}{1 - \frac{1}{8} \epsilon}, \sqrt{1 - \frac{3}{8} \epsilon}\}.$$ 

Then for any $\delta \geq \bar{\delta}$, there exist $T, Q$, and $X$ for the review contract that allows the principal to approximate his first best payoff.

I will next show that Inequalities (3) and (4) are satisfied. Define $X^i$ to be the stochastic process for the outcome of working in the state $i$, and we have

$$\mathbb{E}[X_t(\pi)] = \pi M^{t-1} \cdot (\mathbb{E}[X^1], \ldots, \mathbb{E}[X^n])$$

for all $t \geq 1$.

Let $\pi_0$ be the invariant distribution of the Markov chain, and define $\mu = \mathbb{E}[X_1(\pi_0)]$. Since the Markov chain is irreducible and aperiodic, $\pi_0$ and $\mu$ are well-defined. We also know that for any prior $\pi$, $\mathbb{E}[X_t(\pi)]$ converges to $\mu$ as $t$ goes to infinity.

Given any prior $\pi$ and $T$, we can rewrite $\sum_{t=1}^{T} \delta^{t-1} X_t(\pi)$ as

$$\sum_{t=1}^{T} \delta^{t-1} X_t(\pi) = \sum_{i=1}^{n} 1_{\{Z = i\}} \left( \sum_{t=1}^{T} \delta^{t-1} X_t(e_i) \right),$$

where $e_i$ is the indicator vector for the $i$-th coordinate and $Z$ is a random variable with $\Pr(Z = i) = \pi_i$. Since there are a finite number of states, it is sufficient to show that Inequality (3) is satisfied for each $e_i, 1 \leq i \leq n$. By symmetry, we only need to prove the statement for $\pi = e_1.$
For given $\epsilon > 0$, we have
\[ \mathbb{E}[X_t(e_1)] \rightarrow \mu \text{ as } t \rightarrow \infty, \]
and there exists $N$ such that
\[ |\mathbb{E}[X_t(e_1)] - \mu| < \frac{\epsilon}{2}, \forall t \geq N. \]

From the fact that
\[ \frac{1 - \delta}{1 - \delta^t} \]
is decreasing in both $\delta$ and $T$, there exist $\hat{\delta}_1, \hat{T}_1$ such that for $\delta \geq \hat{\delta}_1, T \geq \hat{T}_1$,
\[ \frac{1 - \delta}{1 - \delta^t} \sum_{t=1}^{N} \delta^{t-1} |\mathbb{E}[X_t(e_1)] - \mu| < \frac{\epsilon}{2}. \]

Therefore, for given prior $e_1$ and $\epsilon > 0$, we can always find $\hat{\delta}_1, \hat{T}_1$ such that for $\delta \geq \hat{\delta}_1, T \geq \hat{T}_1$,
\[ \frac{1 - \delta}{1 - \delta^t} \sum_{t=1}^{T} \delta^{t-1} |\mathbb{E}[X_t(e_1)] - \mu| < \epsilon. \]

Similarly, we can find $\hat{\delta}_i, \hat{T}_i$ for $2 \leq i \leq n$ such that for $\delta \geq \hat{\delta}_i, T \geq \hat{T}_i$, we have
\[ \frac{1 - \delta}{1 - \delta^t} \sum_{t=1}^{T} \delta^{t-1} |\mathbb{E}[X_t(e_i)] - \mu| < \epsilon. \]

Let $\hat{\delta} = \max_i \hat{\delta}_i, \hat{T} = \max_i \hat{T}_i$, and we get
\[ \frac{1 - \delta}{1 - \delta^t} \sum_{t=1}^{T} \delta^{t-1} |\mathbb{E}[X_t(\pi)] - \mu| < \epsilon. \] (5)
for all $\pi, \delta \geq \hat{\delta}, T \geq \hat{T}$.

On the other hand, from
\[ \sum_{t=1}^{T} \delta^{t-1} X_t(\pi) = \sum_{i=1}^{n} 1_{\{Z=i\}}(\sum_{t=1}^{T} \delta^{t-1} X_t(e_i)), \]
it is sufficient to show Inequality (4) for $\epsilon, \epsilon'$ and each $e_i, 1 \leq i \leq n$. By symmetry, we can show the inequality for $e_1$. Without loss of generality, assume $\epsilon < \frac{\epsilon}{2\mu}$. Since the Markov chain has an ergodic distribution and the outcome of working is bounded, the strong law of
large numbers holds, and there exists $\tilde{T}_1$ such that for $T \geq \tilde{T}_1$, we have
\[
\Pr\left(\frac{1}{T} \sum_{t=1}^{T} X_t(e_1) \geq \mu(1-\epsilon) \right) \geq 1 - \epsilon'.
\] (6)

For $\delta$ and $T$ such that $\delta^T \geq 1 - \frac{\epsilon}{2}$, we have
\[
\frac{(1-\delta)\delta^{T-2}}{1-\delta^T} \geq (1 - \frac{\epsilon}{2}) \frac{1}{T}.
\]

Therefore, for $T \geq \tilde{T}_1$ and $\delta$ such that $\delta^T \geq 1 - \frac{\epsilon}{2}$, we know that
\[
\Pr\left(\sum_{t=1}^{T} \delta^{t-1} X_t(e_1) \geq Q \right) \geq \Pr(\frac{(1-\delta)\delta^{T-2}}{1-\delta^T} \sum_{t=1}^{T} X_t(e_1) \geq (1 - \frac{\epsilon}{4})\mu - c)
\]
\[
\geq \Pr\left((1 - \frac{\epsilon}{2}) \frac{1}{T} \sum_{t=1}^{T} X_t(e_1) \geq (1 - \frac{\epsilon}{4})\mu - c)\right)
\]
\[
\geq 1 - \epsilon',
\]
where the last inequality follows from Inequality (6). Rearranging the inequality, we get
\[
\Pr(\frac{1-\delta}{1-\delta^T} (\sum_{t=1}^{T} \delta^{t-1} X_t(e_1)) < (1 - \frac{\epsilon}{4})\mu - c) < \epsilon'.
\]

Similarly, we can find $\tilde{T}_i$ for $2 \leq i \leq n$ such that for $T \geq \tilde{T}_i, \delta^T \geq 1 - \frac{\epsilon}{2}$, we have
\[
\Pr\left(\frac{1-\delta}{1-\delta^T} (\sum_{t=1}^{T} \delta^{t-1} X_t(e_i)) < (1 - \frac{\epsilon}{4})\mu - c) < \epsilon'.
\]

Take $\tilde{T} = \max_i \tilde{T}_i$, and we have Inequality (4).

Therefore, when there are a finite number of states following an irreducible, aperiodic first-order Markov chain, and the outcome of working is bounded, the principal can approximate his first best payoff. Given $\epsilon > 0$, there exists $\tilde{\delta}$ such that for $\delta \geq \tilde{\delta}$, the principal’s average per period payoff in the second best is within $\epsilon$ of his first best payoff.

Proof of Proposition 18. In the first best, the principal observes the agent’s actions. Since the principal and the agent don’t observe the state, their continuation values can be written as functions of the prior, and the agent’s continuation value will be zero for all priors. Let $W(\pi)$ be the continuation value of the principal given prior $\pi$.

When the agent works this period, the principal has better information in the following
period. The principal can always mimic the behavior with less information, and in particular, the principal wants the agent to work whenever

\[-c + \pi_1 \geq \tilde{u}.\]

Therefore, the principal always wants the agent to work after the good outcome.

Suppose the principal takes his outside option for \(l\) periods after the bad outcome. \(W(\pi)\) satisfies the following condition:

\[W(M_1) = -c + M_1 \left(1 + \delta W(M_1) \right) \frac{1}{\delta W(M_2 M^l)} , \]
\[W(M_2 M^l) = -c + M_2 M^l \left(1 + \delta W(M_1) \right) \frac{1}{\delta W(M_2 M^l)} .\]

\(l\) maximizes the continuation value \(W(\pi)\). From Assumption 4, we have \(-c + M_2 M^k \left( \frac{l}{0} \right) \geq \tilde{u}\).

The principal takes his outside option for at most \(k\) periods, and \(l \leq k\).

If \(l < k\), the principal takes his outside option before the expected payoff from working becomes greater than the outside option. This is because of the informational value of working, having a more precise prior in the following period. Whether \(l < k\) depends on the parameter values.

\[ \square \]

**Proposition 21 (Proposition 11’).** Suppose Assumptions 1, 3 and 4 hold. In characterizing the second best, there is no loss of generality in restricting attention to contracts under which the principal makes positive payments only for the good outcome.

**Proof.** The proof of Proposition 11 doesn’t depend on the parameter assumption, and it holds under Assumption 4 as well.

\[ \square \]

**Proof of Proposition 19.** Given a contract, let \(R\) and \(L\) be the rent and the loss in outcome under the contract. \(L\) is defined to be

\[ L = (1 - \delta)(Y_{FB} - Y), \]

where \(Y_{FB}\) is the expected discounted sum of the outcome in the first best, and \(Y\) is the expected discounted sum of the outcome under the given contract.

Consider the space of \((R, L)\) for the initial prior \(\pi\). I allow the principal to randomize continuation contracts, and the set of all feasible \((R, L)\) is a convex set. In particular, there is a one to one mapping

\[ f : [0, L_{\pi}] \to [0, \infty) \]
such that the set of feasible \((R, L)\) is given by

\[
X_\pi \equiv \{(R, L)|R \geq f(L), 0 \leq L \leq L_\pi\},
\]

and \(L_\pi\) is the minimum expected loss in outcome with leaving no rent to the agent. Since the principal can take the outside option forever, \(L_\pi\) is bounded from above. The agent can guarantee positive rent by one-shot deviations, and together with \(l \leq k\), we have \(f(0) > 0\). \(f(0) > 0\) also implies that \(L_\pi > 0\). From the fact that \(X_\pi\) is convex, we also know that \(f(\cdot)\) is strictly decreasing in \(L\).

Since both the principal and the agent are risk-neutral, the principal can always delay the payment. Suppose the principal makes a positive payment for \(h^t\) under the second best contract. From Proposition 21, we can assume that the principal makes the positive payment for a good outcome, and \(h_1 = 1\). After the good outcome, only the sum of the present compensation and the continuation value matters for the agent’s IC constraint, and the principal can replace the continuation contract. Instead of paying \(w(h^t)\) and continuing with \(V(h^t, M_1)\), the principal can offer \(\hat{w}(h^t) = 0\) and

\[
\hat{V}(h^t, M_1) = \frac{1}{\delta} w(h^t) + V(h^t, M_1).
\]

If \(V(h^t, M_1) < f(0)\), we get

\[
f^{-1}(\hat{V}(h^t, M_1)) < f^{-1}(V(h^t, M_1)).
\]

The principal can replace the continuation contract with a contract with a lower \(L\), and the principal’s payoff strictly increases.

Therefore, if the principal makes a positive payment under the second best contract, he doesn’t gain from delaying that payment, which means that the agent’s continuation value \(V(h^t, M_1)\) is at least as big as the minimum rent under the cost-minimizing contract; the principal doesn’t lose anything in outcome under the continuation contract. The principal implements the first best action in every period after he makes a positive payment.

**Proof of Proposition 20.** I will show the proposition in three steps. I first derive the lower bound on the agent’s continuation values. Then, I show that the lower bound can be achieved with two levels of payments. The last step is to show that the two levels are not equal for \(l > 0\).

First, consider the agent’s IC constraints when the principal implements the first best action in every period. Specifically, the principal never takes his outside option after the good outcome, but he takes the outside option for \(l\) periods after the bad outcome. By the
same argument as in the base model, IC constraints for one-shot deviations are sufficient conditions for all IC constraints. The IC constraint for the one-shot deviation is given by

\[-c + \pi \left( \frac{V_1}{V_2} \right) \geq w(h^t0) + \delta^{l+1}(-c + \pi M^{l+1} \left( \frac{V_1'}{V_2'} \right)),\]

where \(V_1, V_2, V_1', \) and \(V_2'\) are the sum of the present compensation and the continuation value after histories \(h^{t1}, h^t0, h^t0 \cdots 0_1\) and \(h^{t2}0 \cdots 0\). We know from Proposition 21 that the principal can make positive payment only for the good outcome, and \(V_2\) is given by

\[V_2 = \delta^{l+1}(-c + M_2 M^l \left( \frac{V_1'}{V_2'} \right)).\]

Subtracting \(V_2\) from both sides of the IC constraint and rearranging the inequality, we get

\[V_1 - V_2 \geq \frac{c}{\pi_1} + (\delta \det M)^{l+1} (V_1' - V_2'),\]

\[V_1 - V_2 \geq \frac{c}{\pi_1} + \frac{(\delta \det M)^{l+1}}{1 - (\delta \det M)^{l+1}} \frac{c}{M_2 M^l(0)},\]

\[V_2 \geq \delta^{l+1} \frac{(\delta \det M)^{l+1}}{1 - \delta^{l+1}} \frac{c}{1 - (\delta \det M)^{l+1}},\]

\[V_1 \geq \frac{c}{\pi_1} + \frac{(\delta \det M)^{l+1}}{1 - (\delta \det M)^{l+1}} \left( \frac{1}{M_2 M^l(0)} + \frac{\delta^{l+1}}{1 - \delta^{l+1}} \right) c.\]

The next step is to show that two levels of payments for the good outcome, one for prior \(M_1\) and the other for \(M_2 M^l\), implement the lower bound. Define \(P_1\) and \(P_2\) be the discounted sum of the probabilities of the good outcome when the agent works every period, for prior \(M_1\) and \(M_2 M^l\):

\[P_1 = \sum_{j=0}^{\infty} \delta^j M_1 M^j \left( \frac{1}{0} \right),\]

\[P_2 = \sum_{j=0}^{\infty} \delta^j M_2 M^{l+j} \left( \frac{1}{0} \right).\]
From

\[ P_1 = M_{11}(1 + \delta P_1) + M_{12}\delta^{i+1}P_2, \]
\[ P_2 = M_2 M^l \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 + \delta P_1) + M_2 M^l \begin{pmatrix} 0 \\ 1 \end{pmatrix} \delta^{i+1}P_2, \]

we get

\[ P_1 = \frac{M_{11} - \delta^{i+1}(M_{11}M_2 M^l \begin{pmatrix} 0 \\ 1 \end{pmatrix} - M_{12} M_2 M^l \begin{pmatrix} 1 \\ 0 \end{pmatrix})}{(1 - \delta M_{11})(1 - \delta^{i+1}M_2 M^l \begin{pmatrix} 0 \\ 1 \end{pmatrix}) - \delta^{i+2}M_{12} M_2 M^l \begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \]
\[ P_2 = \frac{M_2 M^l \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{(1 - \delta M_{11})(1 - \delta^{i+1}M_2 M^l \begin{pmatrix} 0 \\ 1 \end{pmatrix}) - \delta^{i+2}M_{12} M_2 M^l \begin{pmatrix} 1 \\ 0 \end{pmatrix}}. \]

Let the payment for the good outcome given \( M_2 M^l \) be \( \frac{c}{M_2 M^l \begin{pmatrix} 1 \\ 0 \end{pmatrix}} \), and let the payment for the good outcome given \( M_1 \) be such that

\[ P_2(M_{11} w(1) - c)\delta^{i+2} = \frac{\delta^{i+1}}{1 - \delta^{i+1}} \frac{(\delta \text{ det } M)^{i+1}}{1 - (\delta \text{ det } M)^{i+1}} c. \tag{7} \]

Given these payments, the agent gets a positive rent if and only if the previous outcome was good, and the agent’s continuation value given prior \( M_2 M^l \) is exactly the lower bound.

Now, I will show that the agent’s IC constraint is satisfied for both priors \( M_1 \) and \( M_2 M^l \). By construction, the IC constraint for \( M_2 M^l \) holds with equality. Consider the IC constraint for \( M_1 \). The IC constraint is equivalent to

\[ V_1 - V_2 \geq \frac{c}{M_{11}} + \frac{(\delta \text{ det } M)^{i+1}}{1 - (\delta \text{ det } M)^{i+1}} \frac{c}{M_2 M^l \begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \]

and when the IC constraint for \( M_2 M^l \) holds with equality, the IC constraint is equivalent to

\[ V_1 \geq \frac{c}{M_{11}} + \frac{(\delta \text{ det } M)^{i+1}}{1 - (\delta \text{ det } M)^{i+1}} \left( \frac{1}{M_2 M^l \begin{pmatrix} 1 \\ 0 \end{pmatrix}} + \frac{\delta^{i+1}}{1 - \delta^{i+1}} \right) c. \]

When the payment \( w(1) \) satisfies (7), the agent’s continuation value is given by

\[ V_1 = w(1) + (M_{11} w(1) - c)\delta(1 + \delta P_1). \]
It can easily be verified that

\[(M_{11}w(1) - c)\delta(1 + \delta P_1) = \frac{(\delta \det M)^{l+1}}{1 - (\delta \det M)^{l+1}} \left( \frac{1}{M_{2}M^l(1)} + \frac{\delta^{l+1}}{1 - \delta^{l+1}} \right) c,\]

and from \(w(1) > c/M_{11}\), the IC constraint is satisfied. Therefore, the agent’s IC constraints are satisfied with these two levels of payments.

The last step is to show that \(w(1) < \frac{c}{M_{2}M^l(1)}\) unless \(l = 0\), and the cost-minimizing is not stationary for any \(l > 0\). \(w(1) < \frac{c}{M_{2}M^l(1)}\) is equivalent to the following:

\[\delta(M_{11}w(1) - c) < \delta(M_{11} - \frac{c}{M_{2}M^l(1)}) - c\]

\[\Leftrightarrow \frac{1}{P_2} \frac{1}{1 - \delta^{l+1}} \frac{(\delta \det M)^{l+1}}{1 - (\delta \det M)^{l+1}} c < \delta \frac{M_{11} - M_{2}M^l(1)}{M_{2}M^l(1)} c,\]

which is again equivalent to

\[((1 - \delta M_{11})(1 - \delta^{l+1}M_{2}M^l(1)) - \delta^{l+2}M_{12}M_{2}M^l(1))(\delta \det M)^{l+1} \]

\[< \delta(1 - \delta^{l+1})(1 - (\delta \det M)^{l+1})(M_{11} - M_{2}M^l(1)),\]

\[\Leftrightarrow (\delta \det M)^{l+1}(1 - \delta^{l+1}) - \delta(1 - \delta^{l+1})M_{2}M^l(1)\]

\[< \delta(1 - \delta^{l+1})(M_{11} - M_{2}M^l(1)).\]

The last inequality can be shown by

\[(\delta \det M)^{l+1}(1 - \delta^{l+1} - \delta(1 - \delta^{l+1})M_{2}M^l(1))\]

\[\leq (\delta \det M)^{l+1}(1 - \delta^{l+1})\]

\[\leq \delta(1, -1)M^{l+1}(1 - \delta^{l+1})\]

\[= \delta(M_1M^l(1) - M_2M^l(1))(1 - \delta^{l+1})\]

\[\leq \delta(1 - \delta^{l+1})(M_{11} - M_{2}M^l(1)),\]

and the equalities hold if and only if \(l = 0\).
Therefore, the cost-minimizing contract to implement the first best action in every period is stationary if and only if \( l = 0 \).

References


