News and Liquidity in Markets with Asymmetrically-Informed Traders*

Brendan Daley
Duke University
The Fuqua School of Business
bd28@duke.edu
http://faculty.fuqua.duke.edu/~bd28

Brett Green
UC Berkeley
Haas School of Business
bgreen@haas.berkeley.edu
http://faculty.haas.berkeley.edu/bgreen/

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Abstract

This paper explores the role of news in financial markets with asymmetrically-informed traders. We consider a dynamic economy in which private information about a seller’s asset is revealed stochastically over time to a market of traders. Traders’ time preferences for money are subject to random liquidity shocks generating future incentive to trade. The equilibrium involves periods of no trade in which liquidity dries up: assets remain in the hands of liquidity-constrained traders despite efficient gains from trade. Equilibrium prices are determined not only by traders’ beliefs about the fundamental value of the asset, but also by expectations of future liquidity in the market. The no-trade periods lead to endogenous liquidation costs. Buyers correctly anticipate such costs, driving prices below fundamentals. Our results have implications for asset pricing as well as welfare and efficiency.

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1 Introduction

Financial markets are susceptible to periods of market breakdown. Such periods often follow revelation of negative information about fundamentals, and involve a large decrease in trade volume and market prices that diverge from fundamentals. In this paper, we propose a mechanism to help understand such phenomena and derive implications for asset prices and market efficiency.

The model is set in a dynamic economy with rational, risk-neutral agents who share a common prior. There is an indivisible asset in the economy that delivers cash flows to its owner. The cash flows depend on the asset’s type, which is privately known by the current owner. As time passes: (i) potential buyers arrive and make offers to the asset owner, (ii) the asset owner is subject to an observable liquidity shock, and (iii) stochastic information (or news) about the asset’s type is gradually revealed to the market by a Brownian diffusion process.

Liquidity shocks arrive randomly according to a Poisson process and increase the owner’s holding costs, generating gains from trade. An owner of the asset is not forced to sell upon the arrival of a liquidity shock, but she is more eager to do so. In addition, traders’ beliefs about the asset’s type evolve over time as news is revealed, e.g., noisy signals of cash flows, economic indicators, analyst forecasts. In this setting, a trader’s value for the asset depends not only on her beliefs about the asset type, but also on her expectations about future liquidity in the market. Thus, a buyer’s value (and hence the market price) arises endogenously.

We construct an equilibrium with the following features. When the owner is liquidity constrained, trading behavior is characterized by three distinct regions: (1) the market is liquid when beliefs about the asset’s type are favorable: trade occurs immediately at a “fair” price; (2) a sell-off region when the market is pessimistic about the asset: the owner is forced to either sell at rock-bottom prices or hold out; (3) a no-trade region where both sides of the market wait until either sufficient good news restores confidence to (1) or enough bad news forces (2). Finally, an owner who is not liquidity constrained never sells, because it is common knowledge that there are no gains from trade and the owner has payoff-relevant private information.

In the absence of asymmetric information, the asset would trade efficiently at its fundamental value. In the presence of asymmetric information, the no-trade region leads to an endogenous liquidation cost because liquidating a position is often associated with inefficient delay. Buyers correctly anticipate such costs, driving prices below fundamentals, and creating an endogenous (information-driven) illiquidity discount relative to the symmetric
information benchmark. Asymmetric information increases expected returns and bid-ask spreads, while decreasing trade volume. More frequent liquidity shocks reduce prices and increase liquidity premia. Over time, news eliminates the information asymmetry. However, in the short-term, higher quality news reduces liquidity and increases volatility, but may increase or decrease excess returns and liquidity premia.

Within our framework, bid-ask spreads, excess returns, volatility, and trade volume are all time-varying and stochastic. Thus, our model provides a micro-foundation for empirically observed phenomena, such as stochastic volatility and time variation in returns. Our model also predicts that excess returns (attributable to illiquidity) correlate positively with volatility, and move inversely to trade volume and liquidity. This is consistent with studies by Amihud and Mendelson (1986); Brennan and Subrahmanyam (1996); Amihud (2002).

Our results also have implications for market efficiency. Prices and efficiency decrease with the arrival rate of shocks because traders liquidate (and incur the cost of doing so) more frequently. This occurs despite the fact that the asset’s fundamental value is unaffected. As the arrival rate of liquidity shocks goes to zero, asset prices converge to fundamental values. Surprisingly, market efficiency is non-monotonic in the quality of the news. Higher quality news has two opposing effects. First, it provides more incentive for sellers of high-type assets to hold out and wait for a better price, thereby increasing the size of the no-trade region. Second, equilibrium beliefs evolve more rapidly through the no-trade region, and thus the amount of time spent in the inefficient region decreases. Which effect dominates depends both on the current state and the magnitude of the increase in news quality.

We show that the efficiency of the market can improve with the severity of the liquidity shocks (i.e., holding costs), even for relatively small increases. More severe shocks impose larger inefficiencies during periods of no-trade, but also decrease the motivation for agents to endure such periods and hence the size of the region itself. One implication is that government programs aimed at injecting liquidity into the system and easing the credit constraints of distressed financial institutions have an adverse effect that can actually reduce market efficiency.

Though parsimonious, we believe our model, and the key forces behind our results, can help to explain several phenomena commonly observed in financial markets. For example, the model predicts that a sell-off of assets at low prices can help stabilize a shaky market. Wall Street traders and analysts refer to this as “market capitulation” (Zweig, 2008; Cox 2008). In addition, a small amount of bad news can lead to a drastic decrease in volume, which explains another phenomenon that traders refer to as “liquidity drying up” (Smith 2008; Reuters 2008).

The recent collapse of the private-label mortgage-backed securities (MBS) market is per-
haps illustrative. Prior to the collapse, trade and issuance of mortgage-backed securities occurred in a liquid and well-functioning market, despite the fact that banks issuing these securities had significant information about the underlying collateral that was inaccessible to investors. In mid-2007, economic indicators of a decline in the housing market increased uncertainty regarding the value of the collateral, and led to a catastrophic drop in both liquidity and prices. Investors were unwilling to buy these securities or lend against them (even at a substantial discount/haircut) for fear of being stuck with the most “toxic” assets. Perhaps rightly so. As a result, these MBS remained on the balance sheet of numerous large banks despite their need for capital.

Figure 1 illustrates the dramatic decrease in liquidity that private-label MBS experienced during 2008. There was a significant tightening of credit starting in 2008, which certainly played a role in the decline. However, several facts suggest that asset-specific factors (e.g., information frictions) were also responsible. First, private-label MBS issuance as a percentage of the total US bond market experienced a similarly severe decline. Second, issuance of agency-backed MBS, which are less information sensitive than private-label MBS, remained roughly constant over the same time period.

![Graph showing issuance of private-label mortgage backed securities](image)

**Figure 1:** Private-label MBS issuance and percentage of private label MBS contributing to the US bond market fell drastically in 2008 (source SIFMA).

The remainder of the paper is organized as follows. Section 1.1 discusses our work within the context of the theoretical literature. Section 2 presents the model. Section 3 contains equilibrium analysis. Section 4 discusses implications for asset pricing and trading patterns, and relates our findings to the empirical literature. Section 5 discusses implications for welfare and efficiency. Section 6 concludes.

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Krishnamurthy (2010) or Brunnermeier (2009) provide a descriptive analysis of how debt markets malfunctioned in the recent crisis.
1.1 Relation to the Theoretical Literature

The key features of our dynamic economy include asymmetrically-informed traders, news arrival, and liquidity shocks. Daley and Green (2011) study the decision of a privately-informed seller facing a market of buyers where information is revealed gradually over time. In their model, the asset is traded only once. We build on this framework by introducing liquidity shocks, which give rise to potential gains from resale and generate new dynamics and implications.

A number of other papers have considered settings with asymmetric information and news arrival. For example, Kremer and Skrzypacz (2007) study a dynamic signaling model where a grade about the seller is revealed at some finite time $T$. They show how an endogenous lemons market develops and that trade is always delayed in equilibrium. Bar-Isaac (2003) considers a monopolist’s selling decision when the quality of its service is revealed to the market gradually. Korajczyk et al. (1992); Lucas and McDonald (1990) study the effect of information releases on equity issues in a setting with adverse selection.

The equilibrium we construct is of the signaling-barrier variety; a seller of a low-type asset sells probabilistically at some lower boundary, which prevents beliefs from dropping below the boundary. Thus, not selling at the boundary is a positive signal about the value of the asset. This equilibrium feature is also present in Bar-Isaac (2003); Gul and Pesendorfer (2011).

Our work contributes to two strands in the asset pricing literature. The first is the literature on asset pricing with asymmetric information pioneered by Grossman and Sitglitz (1980) in a setting where agents are price-takers, and by Kyle (1985); Glosten and Milgrom (1985) in a setting with strategic traders. One of our primary contributions to this literature is to introduce gradual information arrival and study its implications for asset prices and efficiency, as well as its interaction with liquidity. In a setting with risk-averse agents, Vayanos and Wang (2012) show that asymmetric information hampers risk sharing, increases ex-ante expected returns, and reduces liquidity. Another feature of our model is that agents possess information which is “long-lived.” Gårleanu and Pedersen (2003) study a model with private liquidity shocks and adverse selection. The private information about the asset in their model is short-lived: it pertains only to cash flows arriving next period. They show that allocation costs arise and affect an asset’s required return due to the combination of a trader’s private information about both his liquidity preference and the asset’s cash flows next period. Eisfeldt (2004) also studies a model where information is short-lived and liquidity is determined endogenously.

The second strand is a broad literature studying asset pricing and liquidity in the pres-
ence of other frictions. For example Amihud and Mendelson (1986); Constantinides (1986); Vayanos (1998); Acharya and Pedersen (2005) do so in the presence of exogenous trading costs. Lo et al. (2004) examine trading volume in such a setting. Our friction is non-institutional, in that nothing in the form of the environment prevents efficient trade. Duffie et al. (2005, 2007) study the implications of search and bargaining on asset prices and liquidity. In their model, search frictions (i.e., the lack of a trading partner at any given point in time) generate a liquidation cost, while intermediation leads to bid-ask spreads and novel dynamics. Vayanos and Wang (2007); Vayanos and Weill (2008) develop search-based models that derive liquidity premia due to endogenous concentration of traders in segmented markets. Our model is absent both search and intermediation; a potential seller can contract directly with potential buyers at any point in time. Rather, it is the informational asymmetry, together with its gradual (but stochastic) dissipation, that generates an endogenous liquidation cost and equilibrium dynamics.

2 The Model

Our model builds on the framework developed in Daley and Green (2011) (hereafter DG11). In the economy, there is a continuum of agents, indexed by $A \in [0,1]$, and a single indivisible asset. The asset has persistent type $\theta \in \{L, H\}$. At every moment in time $t \in [0,\infty)$, the asset is held by one of the agents in the economy. We refer to this agent as the owner at time $t$, formally denoted by $A_t$.\(^2\) The asset generates a cash flow to its current owner that depends on $\theta$ and the owner’s liquidity status: either constrained or unconstrained. An unconstrained owner of a type-$\theta$ asset obtains an instantaneous cash flow (or dividend) of $v_\theta$, whereas a constrained owner has positive holding costs and obtains only $k_\theta < v_\theta$.\(^3\) All agents are risk neutral and discount future payoffs at rate $r$. Let $V_\theta$ and $K_\theta$ equal $v_\theta r$ and $k_\theta r$, respectively, as they represent the values of the asset being held ad infinitum by an unconstrained or constrained agent. We assume that $K_H > V_L$, meaning there is the potential for a “lemons” problem a la Akerlof (1970). In other words, holding costs create gains from trade, but are not overly punitive, which preserves strategic concerns at the forefront of our analysis.\(^4\)

Foreshadowing their equilibrium behavior, we refer to $A_t$ as a seller if she is liquidity constrained, and as a holder otherwise. All agents are unconstrained initially. Publicly

\(^2\)We define $A$ to be left-continuous, meaning $A_t$ should be interpreted as the owner at the beginning of “period” $t$.

\(^3\)This accommodates both additive and proportional holding costs without imposing either. In addition, because agents are risk neutral, nothing substantive changes if the cash flow is random with mean $v_\theta$ or $k_\theta$, depending on the owner’s liquidity status.

\(^4\)Numerical analysis and previous work suggests that if $K_H \leq V_L$, then the equilibrium we construct exists if and only if news quality is sufficiently high relative to holding costs (DG11). In contrast, as holding costs grow to infinity, constrained agents effectively become noise traders and trade is efficient.
observable liquidity shocks arrive according to a Poisson process, \( N = \{ N_t : 0 \leq t \leq \infty \} \), with intensity \( \lambda \); the arrival of the first shock after time \( t \) induces a positive holding cost, which transforms \( A_t \) from a holder into a seller. For simplicity, we assume that subsequent arrivals have no effect on \( A_t \) (i.e., a seller maintains a positive holding cost indefinitely).

The game begins at \( t = 0 \), with the asset owned by an unconstrained agent \( A_0 \). \( A_0 \) knows the asset’s type, potential buyers do not. At every \( t \geq 0 \), multiple buyers (unconstrained agents) make offers to the current owner. If a buyer’s offer is accepted: he becomes the new owner, learns the asset’s type, and the previous owner exits the economy. If the owner rejects all offers: she retains the asset, receives the flow payoff, and can entertain future offers.

**Remark 2.1.** An alternative interpretation of the model is that the asset (e.g., firm, project, or security) generates a commonly known payoff of \( v_L \), and, therefore has a liquidation (or book) value of \( V_L \). In addition, there is a growth opportunity that pays cash flow \( v_H - v_L \), if it is a “high” growth opportunity, and zero otherwise. The only place in which this interpretation leads to a different implication is with regard to trade volume (see Section 4.6).

### 2.1 Public Information

A key feature of the model is that news about the asset’s type is continuously and publicly revealed via a Brownian diffusion process, \( X \), where for all \( t \geq 0 \)

\[
X_t = \mu t + \sigma B_t
\]

and \( B \) is a standard Brownian motion. Define the signal-to-noise ratio \( \phi = |\mu_H - \mu_L|/\sigma \), which we assume to be strictly positive. Larger values of \( \phi \) imply higher quality news; \( \phi = 0 \) corresponds to a model without news.

To formalize the information structure, we introduce the probability space \((\Omega, \mathcal{F}, \mathbb{Q})\) in which \( \theta, B, \) and \( N \) are mutually independent. The state space \( \Omega \) contains all possible \((\theta, B, N)\) and allows for a (private) randomization device. The public history at time \( t \), which also corresponds to the information set of a time-\( t \) buyer, contains:

- The arrival times of liquidity shocks: \( \{N_s : 0 \leq s \leq t \} \)
- The history of news: \( \{X_s : 0 \leq s \leq t \} \)
- All times (if any) at which the asset has been traded before time \( t \): \((t_1, t_2, ...)\)

Let \( \{\mathcal{F}_t\}_{t \geq 0} \) denote the filtration generated by the public history.

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5 The assumption that \( A_0 \) is unconstrained is purely for convenience. Because the equilibrium studied is stationary, nothing substantive is altered under the opposite assumption.

6 Notice that the public information does not contain the level of past offers. This is consistent with
2.2 Strategies

A strategy for a buyer at time $t$ is an $\mathcal{F}_t$-measurable function to offers in $\mathbb{R}$. Aggregating buyers’ strategies over time then yields a process $\tilde{W} = \{\tilde{W}_t, 0 \leq t \leq \infty\}$, progressively measurable with respect to $\mathcal{F}_t$, where $\tilde{W}_t(\omega)$ is the collection of offers at time $t$ given $\omega$. The identity of the buyer making each offer, as well as the non-maximal offers, are irrelevant. Hence, our analysis will focus only on identifying the process $W \equiv \{\max\{\tilde{W}_t\}, 0 \leq t \leq \infty\}$ that is consistent with buyers playing optimally.

The asset owner’s information set contains the public history, the asset type, and the collection of offers made since the owner acquired the asset. In addition, we allow an owner to mix by including a private randomization device. Let $\{G^t_h\}_{h \geq t}$ denote the filtration generated by the information set of an owner who acquires the asset at time $t$. Given $W$, the problem facing an owner who acquires a type-$\theta$ asset at time $t$, hereafter a “($\theta,t$)-owner,” is to solve:

$$\sup_{\tau \geq t} E^\theta_t \left[ \int_t^\tau e^{-r(s-t)} ((1-I_s)v_\theta + I_s k_\theta) ds + e^{-r(\tau-t)} W_\tau \right] \quad (SP_{\theta,t})$$

where $I_t$ is the indicator that is equal to one if and only if $A_t$ is a seller at time $t$.

A pure strategy for a ($\theta,t$)-owner is an $\mathcal{F}_h$-adapted stopping time greater than or equal to $t$. A mixed strategy is a stopping time adapted to $G^t_h$ (to allow for randomization) and denoted by $\tau_{\theta,t} \geq t$. For our analysis, it will be convenient to represent a seller’s (mixed) strategy by the distribution it induces over pure strategies: let $S_{\theta,t} = \{S^\theta_{\theta,t} h, t \leq h \leq \infty\}$, denote the progressively measurable process with respect to $\mathcal{F}_t$, where

$$S^\theta_{\theta,t} h(\omega) \equiv \Pr(\tau_{\theta,t} \leq h | F_h)$$

From the buyer’s perspective, $S^\theta_{\theta,t}$ keeps track of how much probability mass the owner has “used up” by time $h$ by assigning positive probability to accepting offers at times $s \in [t,h]$. An upward jump in $S^\theta_{\theta,t}$ corresponds to the ($\theta,t$)-owner accepting with an atom of mass.

For any given sample path, $S_h^{\theta,t}$ is a CDF over the ($\theta,t$)-owner’s acceptance time. Let $S_{\theta,t} = \text{supp}(S^{\theta,t})$. We say that $S^{\theta,t}$ solves $(SP_{\theta,t})$ if each $\tau \in S^{\theta,t}$ solves $(SP_{\theta,t})$.

an interpretation that buyers are short-lived and make private offers as in Swinkels (1999); Kremer and Skrzypacz (2007); DG11. However, our model is also consistent with the interpretation that buyers make offers that are publicly observable, and may be either short or long-lived. In game-theoretic settings with public offers the set of equilibria is much larger than under the short-lived-buyer/private-offer assumption (Nöldeke and van Damme 1990)—because buyers can condition beliefs on the level of unexpectedly-rejected offers. However, there still exist equilibria in which buyers do not condition on this added information. Our equilibrium concept (Definition 2.2) effectively imposes this feature, making it irrelevant whether past offers are included in $\mathcal{F}_t$, or not.
2.3 The Market Belief

Along the equilibrium path, the market belief about the asset type must be consistent with the public history and the equilibrium strategies. We begin by deriving the belief process that updates only based on news and then incorporate the information content from the public history due to strategic effects into a second component. Under a change of variables, the market belief can be represented by the sum of these two processes.

The market begins with a common prior \( \pi = \Pr(\theta = H) \). Let \( f_{\theta}^t \) denote the density function of type \( \theta \)'s news at time \( t \), which is normally distributed with mean \( \mu_{\theta t} \) and variance \( \sigma^2_t \). Define \( \hat{P} \) to be the belief process for a Bayesian who updates only based on news starting from the prior (i.e., \( \hat{P}_0 = \pi \)).

\[
\hat{P}_t \equiv \frac{\hat{P}_0 f_{H}^t(X_t)}{\hat{P}_0 f_{H}^t(X_t) + (1 - \hat{P}_0) f_{L}^t(X_t)}
\]

It is useful to define a new process \( \hat{Z} \equiv \ln(\hat{P} / (1 - \hat{P})) \), which represents the belief in terms of its log-likelihood ratio. Because the mapping from \( \hat{P} \) to \( \hat{Z} \) is injective, there is no loss in making this transformation. By definition,

\[
\hat{Z}_t = \ln \left( \frac{\hat{P}_t}{1 - \hat{P}_t} \right) = \underbrace{\ln \left( \frac{\hat{P}_0 f_{H}^t(X_t)}{f_{L}^t(X_t)} \right)}_{Z_0} + \ln \left( \frac{f_{H}^t(X_t)}{f_{L}^t(X_t)} \right) - \frac{\phi}{\sigma} \left( X_t - \frac{\mu_H + \mu_L}{2} \right)
\]

and thus,

\[
d\hat{Z}_t = -\frac{\phi}{2\sigma} (\mu_H + \mu_L) dt + \frac{\phi}{\sigma} dX_t \quad (2)
\]

Now define \( P = \{P_t, 0 \leq t < \infty\} \) to be the equilibrium market belief process. \( P_t \) differs from \( \hat{P}_t \) because it accounts for the possibility and realizations of trade before time \( t \). Define \( Z \equiv \ln(P / (1 - P)) \). As before, there is no loss in making this transformation. Because Bayes rule is linear in log-likelihoods, we can decompose \( Z \) as \( Z = \hat{Z} + Q \), where \( Q \) is the stochastic process that keeps track of the information conveyed by the history of past acceptances and rejections. For example, along the equilibrium path and for all \( h \in (t_i, t_{i+1}) \) (recall that \( t_i \) denotes the time of the \( i \)th trade)\(^7\)\(^8\)

\(^7\)We have implicitly assumed that there is one \( P \) process common to all buyers. Along the equilibrium path, this feature is an implication of the common prior and Bayesian updating. We maintain this assumption off the equilibrium path as well.

\(^8\)In general, \( Q \) is pinned down along the equilibrium path by Bayes rule, however, writing this process for arbitrary times, given arbitrary strategies, is a cumbersome exercise that provides little insight beyond that found in the simple example derived in (3).
\[ Z_h = Z_{t_i} + \ln \left( \frac{f^{H}_{h-t_i}(X_h - X_{t_i})}{f^{H}_{h-t_i}(X_h - X_{t_i})} \right) + \ln \left( \frac{1 - S_{h-t_i}^H}{1 - S_{h-t_i}^L} \right) \] (3)

The third term on the right-hand side of (3) shows how beliefs can update over time due to strategic effects, despite the fact that trade does not occur\footnote{For example, suppose that the equilibrium calls for the \((L, t_i)\)-owner to accept an offer at time \(s \in (t_i, h)\), given \(F_s\), with probability in \((0, 1)\) (i.e., \(dS_{h-t_i}^L \in (0, 1 - S_{h-t_i}^L)\)) and for the \((H, t_i)\)-owner to reject almost surely \((dS_{h-t_i}^H = 0)\). If trade does not occur then it is more likely that \(\theta = H, (dQ_s > 0)\), and buyers revise their posteriors upward as shown in (3). On the other hand, if trade occurs at time \(s\), then it is infinitely more likely that \(\theta = L, (dQ_s = -\infty)\) and the buyers’ posterior places probability zero on \(\theta = H\).}

## 2.4 Equilibrium

For any \((\theta, t, h, \omega)\) such that \(h \geq t\) and \(S_{h-t}^{\theta, t} < 1\), there exists \(\tau \in S^{\theta, t}\) such that \(\tau(\omega) \geq h\).

For any such \(\tau\), define

\[ F_{\theta, t}(h, \omega) \equiv E^\theta \left[ \int_h^\tau e^{-r(s-h)} k_\theta ds + e^{-r\tau} W_\tau | G_h^t \right] \] (4)

\[ G_{\theta, t}(h, \omega) \equiv E^\theta \left[ \int_h^\tau e^{-r(s-h)} ((1 - I_s) v_\theta + I_s k_\theta) ds + e^{-r\tau} W_\tau | G_h^t, I_h = 0 \right] \] (5)

Denote the expected payoff to the \((\theta, t)\)-holder starting from time \(h\).

**Definition 2.2.** An equilibrium consists of \(\{S^L_t, S^H_t\}_{t \in \mathbb{R}_+}, W, Z\) such that

1. **Owner Optimality:** Given \(W\), for all \((\theta, t)\), \(S_{h-t}^{\theta, t}\) solves \((SP_{\theta, t})\).

2. **Belief Consistency:** For any \(t\) and history such that \(F_t \neq \emptyset\), \(Z_t\) satisfies Bayes rule.

3. **Zero Profit:** For any positive integer \(i\), if \(F_t \cap \{t_i = t\} \neq \emptyset\), then

\[ W_t = E[G_{\theta, t}(t, \omega)|F_{t_i, t_i = t}] \]

4. **No Deals:** For any \(t\) and history such that \(I_t = 1\), there does not exist \(q \in \mathbb{R}\) such that

\[ E[G_{\theta, t}(h, \omega)|F_t, F_{\theta, t}(h, \omega)] - q > 0 \]
payoff, and beliefs must follow from Bayes rule along the equilibrium path (i.e., $\mathcal{F}_t \neq \emptyset$). The interpretation of the Zero Profit condition is clear—any executed trade must earn the purchasing buyer zero expected surplus—and is motivated by the interpretation of Bertrand competition among buyers. If the No Deals condition fails, then there exists an offer that will earn a buyer a positive expected payoff; hence, this condition reflects the equilibrium requirement that no buyer can profitably deviate by making an offer that a seller would be willing to accept with positive probability. Note that the No Deals condition pertains only to histories in which the owner is a seller. The analogous condition for histories in which the owner is a holder results in a tautology and is therefore omitted.

3 Equilibrium Analysis

We first describe and provide a formal definition for a class of candidate equilibria. We then derive necessary conditions for any candidate within the class to be an equilibrium and demonstrate that there exists an element of the class satisfying these conditions. Equilibrium existence is established by verifying that the conditions are also sufficient.\textsuperscript{10}

The candidate equilibria have a Markovian structure; both the market belief, $Z$, and the owner’s liquidity status, $I$, follow Markov processes, and strategies are stationary with respect to the current realization of these processes. We use $(z, i)$ when referring to the state variable, which should be interpreted as any $(t, \omega)$ such that $(Z_t(\omega), I_t(\omega)) = (z, i)$. In addition, references to generic $z$ should be understood as $z \in \mathbb{R}$, as opposed to the degenerate belief levels $z = \pm \infty$, unless otherwise stated.\textsuperscript{11}

Let $w(z, i)$ denote the maximum of all offers made in the state $(z, i)$. In each candidate, play can be characterized by a pair $(\alpha, \beta) \in \mathbb{R}^2$, $\alpha < \beta$, and an increasing function $B: \mathbb{R} \to [V_L, V_H]$. Both $\alpha$ and $\beta$ represent important belief thresholds, and $B(z)$ represents a buyer’s expected value for the asset given the belief $z$. To describe play, first consider states in which the owner is a seller $(i = 1)$:

- When $z < \alpha$, buyers are said to be pessimistic, and they offer $w(z, 1) = V_L$. The low-type seller accepts with positive probability, and the high type rejects with probability one. If trade occurs, the market belief jumps immediately to $z = -\infty$, but if trade does not occur it jumps to $\alpha$.

\textsuperscript{10}Numerical analysis and previous work suggest that the necessary conditions identify a unique candidate among the class and that this candidate is the unique stationary equilibrium satisfying belief monotonicity, a mild refinement of off-equilibrium path beliefs (see DG11).

\textsuperscript{11}Specify continuation play after reaching a degenerate belief at time $t$, $Z_t \in \{\pm \infty\}$, as follows. For all $h \geq t$, $Z_h = Z_t$, buyers make non-serious offers when $I_h = 0$, and $W_h = E[V_\theta|Z_h]$ when $I_h = 1$, holders always reject, and sellers accept with probability one if $W_h \geq K_\theta$ and reject otherwise. Other specifications of continuation play consistent with Definition 2.2 exist; each is consistent with the equilibrium we construct.
• When \( z > \beta \), buyers are said to be optimistic, and trade is immediate: \( w(z, 1) = B(z) \) and both type sellers accept with probability one.

• When beliefs are intermediate, \( z \in (\alpha, \beta) \), the asset is not traded. Buyers make non-serious offers, and both sides of the market wait for more information to be revealed.

For states in which the owner is a holder \((i = 0)\):

• There is no trade for all \( z \). Buyers make non-serious offers that are rejected with probability one by both types of holder.

Intuition for the trading dynamics is as follows. When beliefs are favorable (for \( z \) high), a high-type seller has little to gain and a high cost of delay, \( r(B(z) - K_H) \). Therefore, a high-type seller is willing to trade and buyers are willing to offer \( B(z) \); trade occurs immediately at buyers’ value. As beliefs become less favorable, the market shuts down and waits for more news before making serious offers. In this region, a high-type seller will not accept \( B(z) \) because the combination of her flow payoff and the option value of trading in the future is more attractive. A low type would be happy to accept \( B(z) \). However, because the high type is not willing to sell at this price, buyers are willing to offer at most \( V_L \). The combination of the low type’s flow payoff and the option to trade in the future is more attractive than such an offer. In this region (i.e., \( z \in (\alpha, \beta) \)), any offer that would be accepted would earn the buyer a negative expected payoff and thus trade must not occur in equilibrium.

As the belief decreases, so too does a low-type seller’s option value from waiting. The belief where she is just indifferent between accepting \( V_L \) and delaying trade is \( \alpha \). For \( z < \alpha \) the low-type seller mixes between accepting and rejecting \( V_L \) in a way such that, conditional on not observing trade, the market belief jumps instantaneously to \( \alpha \), which serves as a lower reflecting barrier for the belief process while \( i = 1 \). In economic terms, not selling when the owner is constrained, but the market is pessimistic, is an imperfect signal of high value.\(^{12}\)

Finally, when \( i = 0 \), there is no trade because the holder has superior information but the same value for the asset as all potential buyers.

The following definition formalizes the description provided above.

**Definition 3.1.** For any pair \((\alpha, \beta) \in \mathbb{R}^2, \alpha < \beta, \) and measurable \( B : \mathbb{R} \to \mathbb{R} \), define \( m_t = \sup\{s \leq t : I_s = 1\} \), \( Q_t^\alpha = \max\{\alpha - \inf_{s \leq m_t} \hat{Z}_s, 0\} \), and \( \Xi(\alpha, \beta, B) \) to be the belief

\(^{12}\) Under the alternative interpretation given in Remark 2.1, a “trade” below \( \alpha \) should be interpreted as liquidation.
process and strategy profile such that for all $t, h \geq 0$:

$$Z_t = \begin{cases} -\infty & \text{if there exists } s < t \text{ such that the asset sold, } Z_s \leq \alpha, \text{ and } A_s \text{ is a seller} \\ \dot{Z}_t + Q_t^\alpha & \text{otherwise} \end{cases}$$

$$S^H_{h,t} = \begin{cases} 1 & \text{if there exists } s \in (t, h] \text{ such that } Z_s \geq \beta \text{ and } A_s \text{ is a seller} \\ 0 & \text{otherwise} \end{cases}$$

$$S^L_{h,t} = \begin{cases} 1 & \text{if there exists } s \in (t, h] \text{ such that } Z_s \geq \beta \text{ and } A_s \text{ is a seller} \\ 1 - e^{-(Q^c_h - Q^c_t)} & \text{otherwise} \end{cases}$$

$$W_t = \begin{cases} B(Z_t) & \text{if } Z_t \geq \beta \text{ and } A_t \text{ is a seller} \\ V_L & \text{otherwise} \end{cases}$$

Under this profile of play, the value functions of buyers, holders, and sellers are intertwined. A buyer who purchases the asset immediately becomes a holder; hence, a buyer’s value depends on a holder’s value. A holder eventually becomes a seller; hence, the holder’s value depends on the seller’s value. Of course, a seller’s value depends on the price at which the asset can be sold (i.e., the buyer’s value). To characterize the equilibrium, we derive a system of interdependent differential equations for the value functions and specify boundary conditions using equilibrium arguments.

### 3.1 Asset Values in Equilibrium

Given $w$ and $Z$, the problem facing an asset owner is to find an optimal policy (i.e., a stopping rule) to maximize her expected payoff given any initial state $(z, i)$. Due to the stationary structure of the candidate equilibrium, we can now write the seller’s problem, as given by $[SP_{\theta,t}]$, recursively. We use $F_\theta$ to denote the value function for a seller of type $\theta$. The Bellman equation for the seller’s problem is

$$F_\theta(z) = \max \left\{ w(z, 1), k_\theta dt + e^{-rdt} E^\theta [F_\theta(z + dZ_t)] \right\}$$  \hspace{1cm} (6)

When $i = 0$, a holder faces a similar problem. The only difference is that by rejecting the current offer there is probability $\lambda dt$ that she will be hit with a liquidity shock and become a seller. We use $G_\theta$ to denote the value function for a holder of type $\theta$. The Bellman equation for the holder’s problem is

$$G_\theta(z) = \max \left\{ w(z, 0), v_\theta dt + e^{-rdt} E^\theta [(1 - \lambda dt)G_\theta(z + dZ_t) + \lambda dtF_\theta(z + dZ_t)] \right\}$$  \hspace{1cm} (7)
3.1.1 Necessary Conditions

Fix a candidate $B : \mathbb{R} \to [V_L, V_H]$ that is increasing and differentiable. In the no-trade region, the seller rejects $w$ and takes her continuation value. Applying Ito’s lemma to $F_\theta$, using the law of motion of $\hat{Z}$, and taking the expectation conditional on $\theta$, (6) implies a differential equation that $F_\theta$ must satisfy for all $z \in (\alpha, \beta)$. Namely, for a high-type seller

$$rF_H'(z) = k_H + \frac{\phi^2}{2} (F''_H(z) + F'H(z))$$

and for a low-type seller

$$rF_L'(z) = k_L + \frac{\phi^2}{2} (F''_L(z) - F'L(z))$$

The equilibrium specifies that for all $z \geq \beta$, both types of seller trade immediately at $w(z, 1) = B(z)$. Therefore,

$$F_H(z) = F_L(z) = B(z) \quad \forall z \geq \beta$$

For all states $(z, 1)$ such that $z \leq \alpha$, the low type mixes, and the equilibrium belief jumps instantaneously to $\alpha$ conditional on no trade. Therefore,

$$F_H(z) = F_H(\alpha), \quad F_L(z) = F_L(\alpha) = V_L, \quad \forall z \leq \alpha$$

There are six boundary conditions that help pin down the seller’s value function in the interior of the no-trade region. Three of these are value matching conditions. As $z$ approaches $\beta$ from below, both seller types will accept an offer of $w = B(\beta)$ with probability one. Hence,

$$F_L(\beta^-) = B(\beta) \quad (12)$$
$$F_H(\beta^-) = B(\beta) \quad (13)$$

where $g(x^+)$ ($g(x^-)$) is used to denote the right (left) limit of the function $g$ at $x$. Similarly, as $z$ approaches $\alpha$ from above, a low type’s value approaches $V_L$.

$$F_L(\alpha^+) = V_L \quad (14)$$

Next, for the high type, because the belief process reflects at $z = \alpha$,

$$F_H'(\alpha^+) = 0 \quad (15)$$
(see Harrison (1985, chap. 5)). The two remaining conditions are smooth pasting conditions required for low-type (high-type) seller indifference at the lower (upper) boundary. As \( z \) approaches \( \alpha \) from above, the low-type seller must be indifferent between accepting \( w = V_L \) or taking her continuation payoff at that point. The same is true of the high type as \( z \) approaches \( \beta \) from below:

\[
F_L'(\alpha^+) = 0 \quad (16)
\]

\[
F_H'(\beta^-) = B'(\beta) \quad (17)
\]

To understand the necessity of these conditions, consider the game in state \((z, i) = (\beta, 1)\). Suppose that \( F_H'(\beta) < B'(\beta) \) and consider the following deviation: reject at \( z = \beta \) and continue to reject until \( z = \beta + \epsilon \) for some arbitrarily small \( \epsilon > 0 \). Instead of accepting \( B(\beta) \), the high type attains a convex combination of \( B(\beta + \epsilon) \) and \( F_H(\beta - \epsilon) \), which lies strictly above \( B(\beta) \), implying the deviation is profitable. On the other hand, if \( F_H'(\beta^-) > B'(\beta) \), then the high type would prefer to accept sooner.\(^{13}\)

Given a \( B \) satisfying appropriate conditions, (8)-(17) pin down \((\alpha, \beta)\) and \( F_\theta(z) \) for all \( z, \theta \). Of course, the buyer’s value function is also endogenous. To determine \( B \), we must first find the asset value to each type of holder. The candidate prescribes that a holder never trades. Therefore, from (7), we have that

\[
G_\theta(z) = v_\theta dt + e^{-rdt} E^\theta[(1 - \lambda dt)G_\theta(z + dZ_t) + \lambda dt F_\theta(z + dZ_t)]
\]

Using similar arguments as before, the above equation implies the following differential equation for the value of each type holder:

\[
rG_H(z) = v_H + \lambda(F_H(z) - G_H(z)) + \frac{\sigma^2}{2} (G_H'(z) + G_H''(z)) \quad (18)
\]

\[
rG_L(z) = v_L + \lambda(F_L(z) - G_L(z)) - \frac{\sigma^2}{2} (G_L'(z) - G_L''(z)) \quad (19)
\]

The next step is to determine the boundary conditions for \( G_L \) and \( G_H \). To do so, we make use of the fact that as \( z \to \pm \infty \), the belief becomes degenerate, and the effect of news on equilibrium beliefs goes to zero. A holder is simply waiting for the shock to come, at which point she has a seller’s value for the asset. In the limit, a holder’s value for the asset is

\(^{13}\)The necessity of high-type-seller indifference at \( \beta \), and therefore (17), hinges on the specification of off-equilibrium-path beliefs. Regardless of this specification, the weaker condition \( F_H'(\beta^-) \leq B'(\beta) \) is necessary. Equilibria in which \( F_H'(\beta^-) < B'(\beta) \) can be sustained only by imposing “threat beliefs” for off-equilibrium-path rejections (i.e., the probability assigned to a high type decreases following an unexpected rejection). A mild refinement on off-equilibrium-path beliefs, namely that beliefs cannot decrease following an unexpected rejection, makes (17) necessary.
a weighted average of the fundamental value and a seller’s value. The following boundary conditions complete the characterization of the holder’s value function:

\[
\lim_{z \to \infty} G_\theta(z) = \frac{rV_\theta + \lambda \lim_{z \to \infty} F_\theta(z)}{r + \lambda} \quad \theta \in \{L, H\} \quad (20)
\]

\[
\lim_{z \to -\infty} G_\theta(z) = \frac{rV_\theta + \lambda \lim_{z \to -\infty} F_\theta(z)}{r + \lambda} \quad \theta \in \{L, H\} \quad (21)
\]

Finally, we turn to characterizing a buyer’s value function. Upon purchasing the asset, the buyer immediately becomes a holder. Therefore,

\[
B(z) = E[G_\theta(z)|z] = p(z)G_H(z) + (1 - p(z))G_L(z) \quad (22)
\]

where \( p(z) \equiv \frac{e^{\frac{z}{1+\epsilon}}}{1+e^{\frac{z}{1+\epsilon}}} \) denotes the probability assigned to \( \theta = H \) in state \( z \). Note that \( B \) as defined by (22) is continuously differentiable (since \( G_L, G_H \) are) and for any \((\alpha, \beta)\):

\[
\lim_{z \to \infty} B(z) = \lim_{z \to \infty} G_H(z) = \frac{rV_H + \lambda \lim_{z \to \infty} F_H(z)}{r + \lambda} = \frac{rV_H + \lambda \lim_{z \to \infty} B(z)}{r + \lambda} \quad (23)
\]

Therefore, since \( \lim_{z \to \infty} B(z) \) must be finite, \( \lim_{z \to -\infty} B(z) = V_H \). Similarly,

\[
\lim_{z \to -\infty} B(z) = \lim_{z \to -\infty} G_L(z) = \frac{rV_L + \lambda \lim_{z \to -\infty} F_L(z)}{r + \lambda} = V_L \quad (24)
\]

which aligns with the properties that \( B \) was assumed to possess at the outset.

### 3.2 Equilibrium Existence

The main result of this section is that an equilibrium of the candidate form exists.

**Theorem 3.2.** There exists an \((\alpha^*, \beta^*, B^*)\), such that \( \Xi(\alpha^*, \beta^*, B^*) \) is an equilibrium.

All formal proofs are in the appendix. The result is established via two lemmas. First, that a solution to the necessary conditions exists. Second, that any solution to the necessary conditions constitutes an equilibrium. As a sketch of the argument, we first show that any candidate can be completely characterized by \((\alpha, \beta, B)\). We then demonstrate the existence of a fixed point \((\alpha^c, \beta^c, B^c)\) such that:

- Given \( B^c \), both types of seller optimally solve their stopping problem by following the strategies given by \( \Xi(\alpha^c, \beta^c, B^c) \), and

- The induced seller value functions from following these strategies, \( F^c_L, F^c_H \), imply holder value functions \( G^c_L, G^c_H \) such that \( B^c = E[G_\theta(z)|z] \).
Due to the nature of the free-boundary problem, standard fixed-point theorems (e.g., Schauder) are not applicable. Rather, we exploit the structure of the system through the closed-form solutions to the differential equations (up to constants) and the necessary analytic boundary conditions. Finally, as alluded to in footnote 10 in all numerical examples considered, a unique solution to (8)-(24) was found. In such cases, there is a unique equilibrium of the candidate form.

3.3 Benchmark Cases

The following notation will simplify future expressions:

**Definition 3.3.** For any $z$, denote the expected fundamental value of the asset by $\bar{V}(z) \equiv E[V_\theta | z] = V_L + p(z)(V_H - V_L)$. Similarly, let $\bar{K}(z) \equiv E[K_\theta | z]$.

**Benchmark 1: The Symmetric Information Model**

A useful benchmark is the economy without asymmetric information. For an asset owner to have no private information, it must be that the cash flows are public. One way to accomplish is to assume that cash flows and the news process are synonymous (the analysis is unchanged by stochastic cash flows, see footnote 3). In the symmetric information model, equilibrium behavior is straightforward: (i) $Z_t = \hat{Z}_t$ because nothing can be learned from trading behavior, (ii) buyers are always willing to pay the expected fundamental value for the asset, $\bar{V}$, and (iii) if $A_t$ is hit with a liquidity shock at time $t$, she sells immediately at a price of $\bar{V}(Z_t)$. Therefore, the value functions for sellers, holders, and buyers are all equal to $\bar{V}$. Because the asset spends zero time in the possession of constrained agents, the equilibrium is fully efficient and the price reflects fundamentals. This is true regardless of the value of $\lambda$; in isolation, liquidity shocks do not cause inefficiency or prices to deviate from fundamentals.

**Benchmark 2: The Model without Resale**

Restoring the information asymmetry, consider the case in which $\lambda = 0$. A holder is never shocked and, therefore, does not trade in equilibrium. Hence, a type-$\theta$ holder’s value is simply $V_\theta$. In Section 2 we arbitrarily fixed $A_0$ to be a holder, meaning if $\lambda = 0$, she retains

---

14 Most notably, that the set of $B$ needed to define an onto operator is not a convex (or otherwise well-behaved) set.

15 This means that the news process varies with the owner’s status; when the owner is a seller (resp. holder) $E[dX_t] = k_\theta$ (resp. $v_\theta$). In order for the signal-to-noise ratio to remain constant, as in Section 2.1, one would have to assume that either the liquidity shocks lead to a constant holding cost, $v_H - k_H = v_L - k_L$, or that the volatility term, $\sigma$, varies with the owner’s status. However, in this benchmark, nothing substantive is altered by allowing the news quality to vary with the owner’s status.
the asset forever. Suppose instead that the initial owner is a seller. In this case, the first trade transfers the asset to a holder who then retains the asset forever. In this setting, buyers do not face future liquidity concerns; correspondingly, their value for the asset is \( B = \bar{V} \). This is the situation considered in [DG11]. In summary, we have the following:

**Proposition 3.4.** Regardless of the initial status of \( A_0 \), if \( \lambda = 0 \), there exists a unique equilibrium of the form described in Definition 3.1. In either case the equilibrium has the following properties:

- \( G_\theta(z) = V_\theta \) for all \( z \).
- \( B(z) = \bar{V}(z) \) for all \( z \).

Equilibrium value functions for this case are shown in Figure 2. Without liquidity shocks, when the asset trades it does so at its expected fundamental value. Thus, liquidity concerns are necessary for the divergence of prices from fundamentals.

**Comparison to Benchmarks**

Notice two related features common to both benchmarks: 1) whenever the asset trades, the price equals the fundamental value, and 2) \( B(z) = \bar{V}(z) \) for all \( z \). Neither of these are true of the equilibrium of the model with both asymmetric information and liquidity shocks. This illustrates that it is the interaction between liquidity concerns and asymmetric information that creates price divergence from fundamentals.

**Proposition 3.5.** If \( \lambda > 0 \), then in any \( \Xi \)-equilibrium, \( B(z) < \bar{V}(z) \) for all \( z \).
The intuition for the result is clear. Because buyers are competitive, $B(z)$ coincides with the total expected discounted stream of cash flows that the asset endows to the economy starting from state $(z,0)$. In expectation, the asset will spend a positive amount of time inefficiently allocated, so $B$ lies below $\bar{V}$.

Due to the complexity of the system characterizing the equilibrium, most of our comparative static results are numerical (see Sections 4-5). We do present the following analytic results, which provide insight as to how the economy is affected by an increase in shock frequency.

**Proposition 3.6.** Fixing all other parameters except $\lambda$, let $\Xi(\alpha_0, \beta_0, \bar{V})$ denote the unique $\Xi$-equilibrium with $\lambda = 0$, and $\Xi(\alpha_1, \beta_1, B)$ be any $\Xi$-equilibrium with $\lambda > 0$. Then, $\beta_1 > \beta_0$, $B(\beta_1) \geq \bar{V}(\beta_0)$, and $\beta_1 - \alpha_1 \geq \beta_0 - \alpha_0$.

From Proposition 3.5 we know that $B < \bar{V}$ when $\lambda > 0$. This means that the asset is worth less to the buyers, which gives the high-type seller stronger incentive to hold out, increasing the upper bound of the no-trade region $\beta$. Surprisingly, it turns out that the price at the upper boundary is higher when $\lambda > 0$: $B(\beta_1) \geq \bar{V}(\beta_0)$. Because the low type always receives the same payoff, $V_L$, at the lower boundary, and the price is higher at the upper boundary, her indifference at $\alpha$ requires the size of the no-trade region to increase.

### 3.4 Discussion of Assumptions

The economy we study, involving a single asset, is suggestive of over-the-counter markets where assets are heterogeneous and often reside with a single owner (e.g., investment banks involved in structured finance often retain entire tranches of a particular transaction). Nevertheless, we believe the economic forces described here apply more broadly to settings with large and sophisticated traders, i.e., settings for which private information and publicly observable credit constraints are prevalent.

Observability of the shocks is an important feature of both the model and its application. Observable liquidity shocks corresponds to a marketplace dominated by large traders whose liquidity needs are transparent. Investors must be able to discern traders with a credible reason for trading from speculators. If shocks were unobservable, then, in favorable market conditions, a holder of a low-value asset would prefer to sell before being hit by a shock, breaking the equilibrium. The moral is that an observable shock provides the owner with a credible reason to liquidate. Without this, buyers face more severe exposure to the lemons problem.

A model with unobservable liquidity shocks corresponds to a marketplace either dominated by private firms (e.g., hedge funds) or one in which the identity of trading partners
remains anonymous (e.g., dark pools), where the motivation for trading is often unclear. Formal analysis of a model with unobservable shocks and a comparison to the results in this paper is left for future work.

The single-asset assumption facilitates a tractable analysis. Our results can be easily extended to a multi-asset economy with mutually independent types; the trade patterns for each will be as characterized above. If instead asset types are correlated, additional information about asset $j$’s type is revealed via the news process and trading behavior of other assets. Incorporating the added information from additional news processes is straightforward; however, the information content of the trading behavior of other assets adds a novel dimension. For example, a sale of asset $k \neq j$ at price $V_L$ can be bad news for asset $j$, and lead to a discrete drop in the market’s belief about $\theta^j$, which increases the probability that asset $j$ will sell at price $V_L$, . . . , creating a feedback loop that could result in a fire sale, i.e., trade of many assets at low prices over a short period of time.

Our results are derived from a setting with risk-neutral agents and binary asset types; yet, the key forces would persist in a more general environment. With more than two asset types, trade remains inefficient—as higher type sellers have incentive to wait for news when beliefs are not favorable—and thus prices remain below fundamentals. Incorporating risk-aversion to the model would have two off-setting effects. On one hand, risk aversion will incent sellers to trade more quickly, shrinking the no-trade region and reducing the effect of information asymmetry. On the other hand, sufficient good news becomes more valuable as it not only increases the mean of traders’ expectations, but also reduces the variance, providing more incentive for sellers to delay trade. Which of these forces dominates, and the implications for the interaction of risk premia and liquidity, seems a promising direction for subsequent research.

4 Implications for Asset Pricing and Trade Patterns

The equilibrium in our model generates a number of empirical implications for asset prices and trade patterns in financial markets, which we elaborate upon in this section. We derive a number of familiar measures: bid-ask spread, excess returns, return volatility, illiquidity discount, and volume. Each is state dependent, hence both time-varying and stochastic. We illustrate how each of these objects varies with the underlying state variables and then discuss the implications for their correlation. Along the way, we relate our findings to the empirical literature.

To embark on this exercise, we first establish equilibrium bids, asks, and prices. In states where trade occurs with probability one this exercise is trivial. Yet, a defining feature of the equilibrium is periods of no trade in which establishing prices is less obvious.
4.1 Bid, Ask, and Price

In states where trade occurs with positive probability, the bid is equal to the offer. Recall that in states where no trade occurs the offer function is not uniquely pinned down. For ease of exposition, $\Xi$ specifies an offer of $V_L$ in such states. However, for any equilibrium, a more appropriate notion for the bid price is as follows,

**Definition 4.1.** For any history, the bid price is the maximal offer consistent with the equilibrium.

That is, any offer higher than the bid price would result in negative expected profit. Letting $B(z,i)$ denote the bid price in state $(z,i)$ of a $\Xi$-equilibrium, we immediately have

$$B(z,i) = iF_L(z) + (1 - i)G_L(z)$$

Although owners do not submit limit orders, the minimal acceptable offer to an owner is, of course, just the owner’s value function: $F_\theta$ (if $i = 1$) or $G_\theta$ (if $i = 0$). Since $\theta$ is unobservable, we define the ask price as the expectation conditional on the public history:

**Definition 4.2.** For any history, the ask price is the market expectation of the minimum offer acceptable to the seller.

Letting $A(z,i)$ denote the ask price in state $(z,i)$ of a $\Xi$-equilibrium, we have that

$$A(z,i) = iE[F_\theta(z)|z] + (1 - i)E[G_\theta(z)|z]$$

(25)

As mentioned earlier, a notion of the price in states where trade may not occur is less obvious. Clearly, any consistent notion of the equilibrium price should fall within the bid price and the ask price, and one could argue that any price process satisfying this condition is consistent with our model. However, imposing a condition from arbitrage pricing theory requires that price equals the expected (net) cash flows from the asset discounted at the risk-adjusted rate, $r^*$. Since all agents in the economy are risk neutral, $r^* = r$.

**Definition 4.3.** For any history, the equilibrium price is the market expectation of future cash flows from the asset discounted at $r$.

**Lemma 4.4.** For any $\Xi$-equilibrium, the price, denoted by $P(z,i)$, is equal to the ask price: $P(z,i) = A(z,i)$.

Because buyers make zero profit, the proof of the lemma follows immediately from the fact that the owner value functions satisfy the appropriate Bellman equations. Having established bid, ask and price we can turn to the implications; before doing so we discuss briefly our choice of parameters.

20
4.2 Parametrization

For the numerical results in the remainder of this section, we fix the interest rate, \( r = .05 \), normalize \( v_H = 1 \) and let \( v_L = 0.5 \). To motivate this parametrization, suppose that “type” corresponds to whether the security will default: low-type securities default, high-type securities do not. Then our parametrization is consistent with a 50% recovery rate on defaulted securities as estimated by Moody’s for Ba-A rated tranches of mortgage-backed CDO’s ([Gluck and Remeza, 2000]). It is also in line with loss given default estimates for high LTV residential mortgages ([Qi and Yang, 2009]).

We set \( k_H = 5/8 \) and \( k_L = 5/16 \), which corresponds to a 3% higher borrowing cost for liquidity-constrained agents. For each measure of interest, we illustrate the effect of increasing both shock frequency (\( \lambda \)) and news quality (\( \phi \)); hence, these two parameters vary across the figures.

4.3 Illiquidity

Illiquidity is measured using the (percentage) bid-ask spread:

\[
BAS(z,i) = \frac{A(z,i) - B(z,i)}{P(z,i)}
\]

This is akin Kyle’s lambda ([1985]) in that it also captures the price impact at any time of trade. A few properties are immediate. First, the bid-ask spread is strictly positive in states where trade does not always occur. It is zero in the extremes as the information friction disappears. Finally, \( BAS(z,1) = 0 \) for all \( z \geq \beta \), since trade is immediate and price impact is zero.

Figure 3 plots the bid-ask spread for the parameters discussed above and confirms that illiquidity is highest during periods of no-trade. Moving from 3(a) to 3(b) illustrates the comparative static effect of increasing \( \lambda \), from 3(a) to 3(c) illustrates the effect of higher \( \phi \). The bid-ask spread increases with \( \phi \) as higher quality news increases the relative value of high-type assets, driving a larger wedge between what a high-type owner is willing to accept and what a buyer is willing to offer. Notice that the bid-ask spread decreases with \( \lambda \) for high \( z \). The intuition is that when \( z \) is large, a higher \( \lambda \) means a low-type holder expects to be hit sooner by a shock, providing the opportunity to trade at \( B \). Buyers are less concerned about being stuck with a lemon, and the bid rises.
4.4 The Illiquidity Discount

The illiquidity discount of the asset is measured relative to the symmetric information benchmark in which, upon arrival of a liquidity shock, the asset trades immediately at a price equal to the expected fundamental value, $\bar{V}$ (see Section 3.3). Therefore, we measure the illiquidity discount, denoted by $D$, as the amount (in percentage terms) that prices deviate from fundamentals:

$$D(z, i) = \frac{\bar{V}(z) - P(z, i)}{\bar{V}(z)}$$

Illiquidity discount (or liquidity premium) is often used as a measure of market inefficiency. In Section 5, we will see that the illiquidity discount is equivalent to percentage market inefficiency within our model, lending credibility to its use. Figure 4 illustrates that the discount is highly correlated with the bid-ask spread: both are greatest during periods of no-trade and converge to zero as the uncertainty is resolved. However, unlike the bid-ask spread, the illiquidity discount decreases with $\phi$ in 4(c). Section 5 shows that higher $\phi$ can also increase the illiquidity discount, especially when starting from low $\phi$ and $z$ near $\beta$. Higher $\phi$ as two counter-acting effects. First, it increases the speed at which uncertainty is resolved, which reduces the discount. It also provides incentive to the high-type seller to hold out longer in order to get a better price, increasing the size of the no-trade region and hence the required discount. Which of these two effects dominates depends on both the current news quality and the initial state.
4.5 Excess Returns and Return Volatility

Again, consider the symmetric information benchmark in which the prices are equal to fundamentals, $\bar{V}$. The cash flow to the buyer is $v_\theta$ and hence the instantaneous return is

$$dR_t = \frac{d\bar{V}(Z_t) + v_\theta dt}{\bar{V}(Z_t)}$$

Since $\bar{V}(Z_t)$ is an $F_t$-martingale,

$$E[dR_t|F_t] = \frac{0 + r\bar{V}(Z_t)dt}{\bar{V}(Z_t)} = rdt$$

Thus, a buyer’s expected return of purchasing the asset in the symmetric information model is simply the market discount rate, and the excess expected return, $E[dR_t|F_t] - rdt$, is zero.

Returning to the model with asymmetric information, and letting $P_t = P(Z_t, I_t)$, instantaneous returns are given by

$$dR_t = \frac{dP_t + v_\theta dt}{P_t} \quad (26)$$

Taking the $F_t$-expectation gives the following result:

**Proposition 4.5.** Expected excess returns when $(Z_t, I_t) = (z, i)$ are given by

$$\frac{1}{dt}E[dR_t|F_t] - r = \begin{cases} 
    r \left[ \frac{\bar{V}(z) - \bar{K}(z)}{E[F_\theta(z)|z]} \right] & \text{if } i = 1 \text{ and } z \in (\alpha, \beta) \\
    r \left( \frac{p(z)}{p(\alpha)} \left[ \frac{\bar{V}(z) - \bar{K}(z)}{E[F_\theta(z)|z]} \right] + \frac{p(z)}{p(\alpha)} - 1 \right) & \text{if } i = 1 \text{ and } z \in (-\infty, \alpha] \\
    0 & \text{otherwise}
\end{cases}$$

Excess returns are strictly positive (in some states) despite the fact that all agents are risk neutral. This does not imply that buyers earn excess returns in equilibrium. In fact, expected excess returns are zero conditional on trade. It is the information friction that prevents
traders from buying despite the appearance of positive excess returns during periods of no trade. As seen in Figure 5, when \( i = 1 \), excess returns are highest in the no-trade region, but remain positive for \( z < \alpha \) due to the strictly positive probability that trade will be delayed.

We can also use (26) and the evolution of the state variables to calculate return volatility, which has both a diffusive component and a jump component. The diffusive component of return volatility is \( \phi P_z / \mathcal{P} \), which follows directly from Ito’s lemma and the volatility of the Brownian component of \( Z \). From Figure 6, notice that diffusive volatility discretely decreases when a trade occurs \( z = \beta \) (i.e., comparing state \((\beta, 1)\) to state \((\beta, 0)\)). Similarly, if trade occurs at \( z \leq \alpha \), subsequent volatility drops to zero. Thus, trade, absent a contemporaneous shock arrival, has a calming effect on volatility.

Jump volatility arises from two different types of events. A jump in the price occurs either because a liquidity shock arrives (\( i \) jumps from 0 to 1) or due to information contained in trade patterns (i.e., when \( z < \alpha, i = 1 \)). The latter is of a larger order than the former since there is a discrete jump in the price that occurs with probability one in such states. We therefore normalize this component in Figure 6 labeled “\( i = 1 \) (jump only)”; hence, the scaling on the vertical axis is not applicable for this measure.

According to the equilibrium dynamics, volatility and returns can jump (when \( i = 1 \)) at the boundaries of the no-trade region. This is consistent with Eraker et al. (2003), who find strong evidence for jumps in both volatility and returns. Our model further suggests that a jump in returns should coincide with a jump in volatility (e.g., at \((\beta, 1)\)), though not the converse (e.g., at \((\alpha, 1)\)). Last, the model predicts that diffusive volatility is highest in the no-trade region when excess returns and illiquidity are high, but jump volatility peaks during a sell-off.

\(^{16}\)For ease of exposition, we omit dividend volatility (which is equal to \( \text{Std}(v_\theta|z)/\mathcal{P} \)) in the figures as it is relatively small (less than 2% for the given parameters) and similar to, though slightly larger than, the symmetric information benchmark.
4.6 Trade Volume

To compute volume, we calculate the expected number of trades over an arbitrary length of time given an initial state. Let $\nu_t$ denote the counting process, which keeps track of the number of trades that occur in $[0, t]$, i.e.,

$$
d\nu_t = 1_{\{A_t \neq A_{t-}\}}, \text{ where } \nu_0 = 0
$$

We let $f$ and $g$ denote the functions mapping $(t, z)$ to expected trade volume conditional on $i = 1$ and $i = 0$ respectively. That is, $f(t, z) \equiv E[\nu_t|(Z_0, I_0) = (z, 1)]$ and $g(t, z) \equiv E[\nu_t|(Z_0, I_0) = (z, 0)]$. The following proposition characterizes these functions through a system of PDEs.

**Proposition 4.6.** For any $t > 0$, the expected trade volume satisfies

$$
f(t, z) : \begin{cases} 
  f = \frac{p(\alpha) - p(z)}{p(\alpha)}(1 + \lambda t) + \frac{p(z)}{p(\alpha)}f(t, \alpha) & \text{for } z \leq \alpha \\
  f_t = \frac{\phi^2}{2} [(2(p(z) - 1)f_z + f_{zz}] & \text{for } z \in (\alpha, \beta) \\
  f = 1 + g & \text{for } z \geq \beta 
\end{cases} \quad (27)
$$

$$
g(t, z) : \quad g_t = \lambda(f - g) + \frac{\phi^2}{2} [(2(p(z) - 1)g_z + g_{zz}] \quad \text{for all } z \quad (28)
$$

with the boundary conditions

$$
\lim_{z \to \pm \infty} f(t, z) = 1 + \lambda t \\
\lim_{z \to \pm \infty} g(t, z) = \lambda t
$$

Figure 6: Volatility
and the initial conditions

\[
f(0, z) = \begin{cases} 
    \frac{p(\alpha) - p(z)}{p(\alpha)} & z \leq \alpha \\
    0 & z \in (\alpha, \beta) \\
    1 & z \geq \beta 
\end{cases}
\]

\[
g(0, z) = 0
\]

Using this result, we solve the system of PDEs numerically to calculate for expected trade volume, which is illustrated in Figure 7.

In the symmetric information benchmark, volume is constant in \( z \) and equal to the \( \pm \infty \) boundaries of the asymmetric information model (i.e., \( i + \lambda t \)). As expected, more information sensitive assets will have lower volume. This is in line with Easley et al. (1996), who document the large number of infrequently traded stocks (averaging less than one trade per day) and find them more likely to be associated with information based trades and higher bid-ask spreads. Volume is relatively high when the market is sufficiently optimistic or sufficiently pessimistic. It is for intermediate beliefs that volume drops, which seems consistent with anecdotal evidence of traders waiting for uncertainty to be resolved before entering the market.

Remark 4.7. A “trade” at price \( V_L \) corresponds to liquidation under the alternative interpretion of the model given in Remark 2.7. In this case, \( g, f \to 0 \) as \( z \to -\infty \), trade volume is strictly monotonic in \( z \), and thus positively correlated with the price. This is consistent with both the time-series and cross-sectional results documented by Cochrane (2002). Further, higher prices correspond to higher market-to-book ratios (i.e., growth stocks), whereas lower prices correspond to value stocks. That is, the market is optimistic about growth stocks (high price-to-book) relative to value stocks (low price-to-book). Thus, our model provides a potential force driving the value premium (e.g., Fama and French (1993)).
Figure 8: Relationship of illiquidity, expected returns, volume and volatility.

Figure 8 illustrates the relationship between bid-ask spread, excess returns, volatility and volume. As mentioned earlier, all of these measures are time-varying and stochastic thus providing micro-foundations for empirically observed phenomenon, such as stochastic volatility and time variation in returns. While the literature has debated the relationship between returns and volatility, consistent with (e.g., French et al. (1987)), our model predicts a positive relationship. Our model also predicts that excess returns due to illiquidity move inversely to trade volume and liquidity. This is consistent with studies by Amihud and Mendelson (1986); Brennan and Subrahmanyam (1996); Amihud (2002).

Our model provides a number of additional predictions: for example, volume should be high when the market is sufficiently optimistic and price levels are relatively high, or when the market is sufficiently pessimistic and prices are relatively low; volume drops, liquidity dries up, and expected returns are highest during periods of greatest uncertainty. Developing an accurate proxy for market optimism (or pessimism) in the cross-section remains a potential challenge to testing some of these predictions, though prices or market-to-book ratios should serve well in the time-series.

5 Welfare and Efficiency

In this section we investigate the equilibrium welfare and efficiency properties of the economy and how they vary with key parameters.

Chordia et al. (2001) find evidence for both a positive relationship and time variation in liquidity and trading activity.
Welfare

We will explore welfare via the value functions $F_L, F_H, G_L, G_H$. By construction, the (expected) payoff to every agent in the economy, except for $A_0$, is zero; any agent not initially endowed with the asset either never obtains it or pays a price for it that exactly equals the expected value it generates for him. Hence, the welfare properties we describe can be thought of either strictly as the welfare of $A_0$, depending on the prior and her initial liquidity status, or, more loosely, as the welfare of any owner once in possession of the asset (her gross welfare).

Figure 9(a) shows value functions and $\bar{V}$ for the same parameters as used in Section 4.2. Notice that $B < \bar{V}$ (Proposition 3.5), and that both high-type owner value functions lie (weakly) above $B$, while both low-type value functions lie (weakly) below $B$. Not surprisingly, the value to the owner of a high-value asset is strictly higher before being hit by a liquidity shock. However, the same is not true for the low-type asset. When beliefs are favorable, a low-type holder would prefer to become a seller because the low-type seller is able to pool with the high-type seller, trading at a price well above $V_L$.

Efficiency

To explore the efficiency of the economy we will compare the discounted expected value the economy derives from the asset in equilibrium to the amount it would derive if the asset was always efficiently allocated (i.e., $\bar{V}$). Because all buyers earn zero profit, the discounted expected value to the economy is $E[F_\theta(z)|z]$ or $E[G_\theta(z)|z]$, depending on whether the owner is a seller or a holder. We look at the percentage efficiency loss by defining the following:

\[
\text{Efficiency Loss} = \frac{\bar{V} - \text{Discounted Expected Value}}{\bar{V}} \times 100
\]
Notice that this measure of inefficiency is identical to the illiquidity discount (Section 4), i.e., the divergence of prices relative to the symmetric information benchmark. The equivalence follows from: i) within each model, the price is the discounted expected cash flow, and ii) in the symmetric information case, there are no trading frictions, so the asset is always efficiently allocated (Section 3.3). Figure 9(b) shows both measures of efficiency loss. \( \mathcal{L}^F, \mathcal{L}^G \) are positive for all \( z \) (this follows from Proposition 3.5), single-peaked with maximal inefficiency occurring at some \( z \in (\alpha, \beta) \), and tend to zero as \( z \to \pm \infty \).

5.1 Comparative Statics

**Shock Frequency:** Figure 10(a,b) illustrates how the value functions depend on the arrival rate of the shocks, \( \lambda \). When \( \lambda \approx 0 \), as in panel (a), the economy approximates the \( \lambda = 0 \) benchmark of Section 3.3. A holder expects to retain the asset for a long duration, so \( G_\theta \approx V_\theta \). In turn, \( B(z) = E[G_\theta(z)\mid z] \approx \bar{V}(z) \), meaning buyers will offer approximately fundamental value when the owner is a seller and \( z \geq \beta \). The seller value functions are also roughly equal to the benchmark levels shown in Figure 2.

Endogenous liquidation costs arise due to the no-trade region as liquidating a position is often associated with holding the asset while being constrained. When \( \lambda \) increases, as in panel (b), traders face the prospect of costly liquidation more frequently (in expectation). Buyers correctly anticipate these higher future liquidation costs, which decrease their value for the asset and hence lowers \( B \). These two forces negatively impact the high-type holder, lowering \( G_H \). However, as we saw above, a low-type holder is anxious for the shock to arrive when it facilitates pooling with the high type; for high values of \( z \), \( G_L \) increases with \( \lambda \).
As depicted in Figure 10(c), inefficiency increases with shock frequency, providing a clean manifestation of the underlying economics. Recall that neither the efficient value, $\bar{V}$, nor the ability of a symmetrically-informed economy to achieve this value, is affected by $\lambda$. Hence, the result follows only because the equilibrium behavior of the agents in the economy introduces periods of inefficient asset allocation and such periods occur more frequently when $\lambda$ increases, causing efficiency to decrease.

**News Quality:** As the news quality, $\phi$, increases, the high-type seller has more incentive to wait, and the size of the no-trade region increases. However, as the Figure 11(a,b) illustrate, this does not necessarily imply lower value functions. The reason is that increasing $\phi$ “speeds things up.” Market beliefs move more quickly through the no-trade region, and the asset may spend less time inefficiently allocated. Not surprisingly, improved news quality benefits the high-type owner (both holder and seller) and hurts the low-type owner. As $\phi \to \infty$, $\beta \to \infty$, as the high-type seller waits to be almost perfectly identified before trading, but the expected time to trade after being hit by a shock goes to zero. Therefore, $G_\theta, F_\theta$ converge to $V_\theta$, and $B$ converges to $\bar{V}$. One might think that, because the market is fully efficient under symmetric information, increasing news quality will bring the market closer to the benchmark and improve efficiency. Figure 11(c) illustrates that this is not necessarily the case. As $\phi$ increases, the inefficiency decreases for lower states but increases in higher states. Again, the intuition is that as $\phi$ increases the high type has more incentive to delay trade, thus causing $\beta$ to increase and efficient trades to be delayed.

**Borrowing Costs:** In each of the figures thus far $k_\theta = \delta v_\theta$, where $\delta = 0.625$. That is, the shock induces a proportional holding cost. Let us maintain the proportionality assumption,

---

18 For clarity, Figures 10(c), 11(c), and 12(c) only depict $\mathcal{L}^G$. The results for $\mathcal{L}^E$ are similar.
19 The convergence is uniform for $G_H, G_L, F_H, B$, but only pointwise for $F_L$. 

30
and consider the effect of varying $\delta$. The following trade-off arises. An increased holding cost means sellers receive lower net cash flows whilst in possession of the asset, hurting their welfare. However, this implies that the high-type seller is more willing to sell, decreasing $\beta$, which in turn makes the equilibrium more efficient for high levels of $z$ (Figure 12(c)). Looking at Figure 12(a,b), $G_H$ and $F_H$ are (substantially) lower under the higher holding cost ($\delta = 0.6$) for low $z$, but (slightly) higher when $\delta = 0.6$ for high $z$. For the low type, however, the efficiency force completely trumps the change in cash flows as $F_L$ and $G_L$ are (at least weakly) higher under the higher holding cost for all $z$. Said succinctly, increased holding costs promote efficient trade and may result in a Pareto improvement, but if not, the welfare cost is borne solely by the high-type asset owners. These results imply that government policies aimed at “easing” liquidity constraints of distressed financial institutions can have detrimental effects on the economy. Policies that provide incentives for banks to quickly liquidate unwanted positions, rather than waiting for a better price, may be preferable.

**Time:** Over time the news process and the trading behavior reveal evermore information about $\theta$ to the economy. Consequently, the asset type is eventually learned (up to any arbitrary precision) and the long-run, steady-state distribution of the market belief is degenerate; either $\pm \infty$. It follows that inefficiency disappears as $t \to \infty$ with probability one. One might be tempted to conjecture a stronger claim: that expected inefficiency is decreasing in time. This is not true. One way to see this result is to transform the belief back into probabilities, $p$, rather than log-likelihoods, $z$, and plot the transformed $L^G, L^F$ as a function of $p$. This is shown in Figure 13. Notice that $L^G$ is convex for $p > p(\beta)$. Further, starting from $p > p(\beta)$, $P$ is a continuous martingale and $L^G = L^F$. By Jensen’s inequality, starting from a belief of

\[ k_\theta = v_\theta - \Delta. \]

\[ \text{Figure 12: Effect of holding costs: $\delta = 0.9$ should be interpreted as a liquidity shock increasing an agent’s discount rate by roughly 0.56 percentage points (from $r = 0.05$ to $r/\delta \approx 0.0556$). Similarly, $\delta = 0.6$ corresponds to an increase in the discount rate of roughly 3.33 percentage points. Shock frequency and news quality are fixed at $\lambda = 0.5$, $\phi = 0.5$.} \]
Figure 13: Efficiency, with belief as probability, $\phi = 0.5$, $\lambda = 0.25$.

$p$ at time $t$, the expected inefficiency at time $t' > t$ is higher if $t' - t$ is small enough.

An intuition for this is as follows: starting from such a $p > p(\beta)$, we know that there is probability $1 - p$ that the limit distribution will consist of the degenerate belief that $\theta = L$. That is, there is probably $(1 - p)$ that $\theta = L$, and if so, it will be found out eventually. However, doing so will be costly, in terms of efficiency. If indeed $\theta = L$, then the market belief will, in expectation, smoothly decrease, going first through the region where inefficiency is much higher before it gets lower again (and eventually disappears).

6 Conclusion

We have presented a model that features news arrival and liquidity shocks in a market with asymmetrically-informed traders. Our model provides a unified framework from which to draw implications for asset prices, volatility, illiquidity, trade volume, welfare, and efficiency.
References


A Proofs for Section 3

Theorem 3.2 is proved via the following lemmas.

Lemma A.1. There exists candidate \((\alpha^c, \beta^c)\), \(\alpha^c < \beta^c\), and functions \((B^c, F^c_L, F^c_H, G^c_H, G^c_L)\) that simultaneously solve \((8)-(24)\).

Lemma A.2. If \(\alpha^c, \beta^c\) and functions \((B^c, F^c_L, F^c_H, G^c_H, G^c_L)\) simultaneously solve \((8)-(24)\), then \(\Xi(\alpha^c, \beta^c, B^c)\) constitutes an equilibrium.

The proofs for Lemmas A.1 and A.2 are found in §A.1 and §A.2. The proofs all other Section 3 results are found in §A.3. It will sometimes be useful to invoke the following shorthand: \(\bar{F}(z) \equiv E[F_\theta(z)|z]\) and \(\bar{G}(z) = E[G_\theta(z)|z]\).

A.1 Proof of Lemma A.1

The proof of Lemma A.1 involves several steps, which we detail below. By way of overview, in §A.1.1 we derive the functional forms of the value functions and boundary conditions. Then, in §A.1.2 we reduce the problem of finding solutions to \((8)-(24)\) to solving a system of two analytic, non-linear equations. Finally, in §A.1.3 we demonstrate that a solution to the reduced system, and therefore \((8)-(24)\), exists.

A.1.1 Value Functions and Boundary Conditions

The Seller’s Value Function

The solutions to the differential equations in (8) and (9) are of the form

\[
F_\theta(z) = C^L_1 e^{q^L_1 z} + C^L_2 e^{q^L_2 z} + K_L
\]

where \((C^L_1, C^L_2, C^H_1, C^H_2)\) are unknown constants, \((q^L_1, q^L_2) = \left(\frac{1}{2} \left(1 \pm \sqrt{1 + \frac{8r}{\phi^2}}\right)\right)\) \cite{Polyanin2003}. The necessary boundary conditions on the seller’s value function \((12)-(17)\) become:

\[
\begin{align*}
C^L_1 e^{q^L_1 \alpha} + C^L_2 e^{q^L_2 \alpha} + K_L &= V_L \quad (30) \\
q^L_1 C^L_1 e^{q^L_1 \alpha} + q^L_2 C^L_2 e^{q^L_2 \alpha} &= 0 \quad (31) \\
C^L_1 e^{q^L_1 \beta} + C^L_2 e^{q^L_2 \beta} + K_L &= B(\beta) \quad (32) \\
C^H_1 e^{q^H_1 \beta} + C^H_2 e^{q^H_2 \beta} + K_H &= B(\beta) \quad (33) \\
q^H_1 C^H_1 e^{q^H_1 \beta} + q^H_2 C^H_2 e^{q^H_2 \beta} &= 0 \quad (34) \\
q^H_1 C^H_1 e^{q^H_1 \alpha} + q^H_2 C^H_2 e^{q^H_2 \alpha} &= B'(\beta) \quad (35)
\end{align*}
\]

The Buyers’ Value Function

Lemma A.3. Suppose that \(F_L, F_H\) solve \((8)-(9)\) and \(G_L, G_H\) solve \((18)-(19)\). Then \(B\) solves the differential
Substituting the above into (39) and rearranging gives equation:

\[ B''(z) + (2p(z) - 1)B'(z) - \frac{2(r + \lambda)}{\phi^2} B(z) = \]

\[ \frac{-2}{\phi^2} \left( p(z)(\lambda F_H(z) + v_H) + (1 - p(z))(\lambda F_L(z) + v_L) \right) \]  \hspace{1cm} (36)

Proof. Let \( \eta_2 \equiv \frac{2(r + \lambda)}{\phi^2} \). From (22) and omitting the function arguments, we have that

\[ B' = pG_H' + (1 - p)G_L' + p'(G_H - G_L) \]  \hspace{1cm} (37)

\[ B'' = pG_H'' + (1 - p)G_L'' + 2p'(G_H' - G_L') + p''(G_H - G_L) \]  \hspace{1cm} (38)

And therefore

\[ B'' + B' - \eta_2 B = p (G_H'' + G_H' - \eta_2 G_H) + (1 - p) (G_L'' + G_L' - \eta_2 G_L) \]
\[ + (p'' + p')(G_H - G_L) + 2p'(G_H' - G_L') \]
\[ = p (G_H'' + G_H' - \eta_2 G_H) + (1 - p) (G_L'' + G_L' - \eta_2 G_L) \]
\[ + (p'' + p')(G_H - G_L) + 2p'(G_H' - G_L') + 2(1 - p)G_L' \]  \hspace{1cm} (39)

Using the functional form of \( p \), the last line of the above can be simplified:

\[ (p'' + p')(G_H - G_L) + 2p'(G_H' - G_L') + 2(1 - p)G_L' \]
\[ = \frac{2}{1 + e^z} \left( p'(G_H - G_L) + pG_H + (1 - p)G_L \right) \]
\[ = \frac{2}{1 + e^z} B' = 2(1 - p)B' \]

Substituting the above into (39) and rearranging gives

\[ B'' + (2p - 1)B' - \eta_2 B = p (G_H'' + G_H' - \eta_2 G_H) + (1 - p) (G_L'' + G_L' - \eta_2 G_L) \]
\[ = -\frac{2}{\phi^2} \left( p(\lambda F_H + v_H) + (1 - p)(\lambda F_L + v_L) \right) \]

where the second equality follows from substituting in from (18)-(19) and completes the proof. \[ \square \]

The analytic expression for the buyers' value function can now be derived piecewise using Lemma A.3 as follows:

1. For \( z < \alpha \), the homogenous solution to (36) is of the form \( B_{h,1}(z) = C_{11}^B \frac{1}{1 + e^z} e^{q_1^B z} + C_{12}^B \frac{1}{1 + e^z} e^{q_2^B z} \), where \( (q_1^B, q_2^B) = \frac{1}{2} \left( 1 \pm \sqrt{1 + 8 \frac{\alpha}{\phi^2}} \right) \) and \( C_{11}^B, C_{12}^B \) are arbitrary constants. For all \( z \leq \alpha \), \( F_{\theta}(z) = F_{\theta}(\alpha) \) and thus \( \bar{F}(z) = p(z)F_{H}(\alpha) + (1 - p(z))F_{L}(\alpha) \). This leads to the particular solution \( B_{p,1}(z) = \frac{r}{r + \lambda} F_{\theta}(\alpha) \).

2. For \( z \in (\alpha, \beta) \), the homogenous solution to (36) is the same as above (i.e., \( B_{h,2}(z) = C_{21}^B \frac{1}{1 + e^z} e^{q_1^B z} + C_{22}^B \frac{1}{1 + e^z} e^{q_2^B z} \)). \( F_{\theta} \) takes the form in (29), which gives a particular solution of \( B_{p,2}(z) = \bar{F}(z) + \frac{r}{r + \lambda} \bar{F}(\alpha) \).
3. For $z > \beta$, since $\bar{F}(z) = B(z)$, (36) becomes

$$B'' + (2p(z) - 1)B' - 2 \frac{r}{\partial^2} B(z) = -2 \frac{r}{\partial^2} V(z)$$

which has homogeneous solution of the form $B_h,3(z) = C_{31}^B \frac{1}{1 + e^z} e^{q_3 B z} + C_{32}^B \frac{1}{1 + e^z} e^{q_4 B z}$, where $(q_3^B, q_4^B) = \left( \frac{1}{2}(1 \pm \sqrt{1 + \frac{2p}{r}}) \right)$ and a particular $B_{3,p}(z) = V(z)$.

To summarize, for any $z \notin \{ \alpha, \beta \}$, $B(z) = B_1(z)I_{z<\alpha} + B_2(z)I_{z\in(\alpha,\beta)} + B_3(z)I_{z>\beta}$, where

$$B_1(z) = rV(z) + \lambda \bar{F}(z) \left( \frac{r}{r + \lambda} + C_{11}^B \frac{e^{q_1 B z}}{1 + e^z} + C_{12}^B \frac{e^{q_2 B z}}{1 + e^z} \right)$$

$$B_2(z) = \bar{F}(z) + \frac{r(\bar{V}(z) - \bar{K}(z))}{r + \lambda} + C_{21}^B \frac{e^{q_1 B z}}{1 + e^z} + C_{22}^B \frac{e^{q_2 B z}}{1 + e^z}$$

$$B_3(z) = V(z) + C_{31}^B \frac{e^{q_3 B z}}{1 + e^z} + C_{32}^B \frac{e^{q_4 B z}}{1 + e^z}$$

and $(C_{11}^B, C_{12}^B, C_{21}^B, C_{22}^B, C_{31}^B, C_{32}^B)$ are arbitrary constants to be pinned down by boundary conditions.

Because $G_H$ and $G_L$ are continuously differentiable, (24) implies that $B$ is also continuously differentiable, which leads to the following additional boundary conditions:

$$B_1(\alpha) = B_2(\alpha)$$

(41)

$$B_1'(\alpha) = B_2'(\alpha)$$

(42)

$$B_2(\beta) = B_3(\beta)$$

(43)

$$B_2'(\beta) = B_3'(\beta)$$

(44)

A.1.2 Reducing the System

For any fixed $(\alpha, \beta) \in \mathbb{R}^2, \alpha < \beta$, (30)-(44) constitutes a system of ten linear equations in the ten variables $(C_1^L, C_2^L, C_1^H, C_2^H, C_{11}^B, C_{12}^B, C_{21}^B, C_{22}^B, C_{31}^B, C_{32}^B)$. The system of equations is linearly independent and therefore has a unique solution parameterized by $(\alpha, \beta)$.

To pin down $(\alpha, \beta)$, two remaining boundary conditions must be satisfied. Letting $C_{12}^B(\alpha, \beta)$ and $C_{31}^B(\alpha, \beta)$ be the parameterized solution values obtained from the linear subsystem, (33) and (24) are satisfied if and only if

$$C_{12}^B(\alpha, \beta) = 0$$

(45)

$$C_{31}^B(\alpha, \beta) = 0$$

(46)

Therefore, in order to prove existence of a solution to the entire system, (3)-(24), it is sufficient to show that there exists a pair $\alpha < \beta$ such that (45) and (46) hold.

A.1.3 Existence of Solution of the Reduced System

Having reduced (8)-(24) to (45) and (46), we now prove the following lemma, giving the desired result, Lemma A.1, as an immediate corollary.

Lemma A.4. There exists $(\alpha, \beta) \in \mathbb{R}^2, \alpha < \beta$, such that (45) and (46) simultaneously hold.
Corollary A.5. There exists a solution to the system \([30]-[44]\).

The proof of Lemma [A.4] relies on several additional lemmas. Throughout, we will make use of the following change of variables. For any \(\alpha < \beta\), let \(A \equiv e^\alpha \in \mathbb{R}_{++}\), \(D \equiv e^\beta \in (1, \infty)\) and \(x \equiv \sqrt{1 + \frac{8r}{\delta^2}} > 1\), \(y \equiv \sqrt{1 + \frac{8(r+\lambda)}{\delta^2}} > x\). Finally, let \(\hat{C}_{12}^B(A, D)\) and \(\hat{C}_{31}^B(A, D)\) refer to the constants as functions of \((A, D)\) (e.g., \(\hat{C}_{12}^B(e^\alpha, e^{\beta-\alpha}) = C_{12}(\alpha, \beta)\)).

Definition A.6. Define the correspondence \(D_{12} : R_{++} \rightarrow (1, \infty)\) as, for all \(A > 0\), \(D_{12}(A) = \{D : \hat{C}_{12}^B(A, D) = 0\}\).

Definition A.7. Define the correspondence \(D_{31} : R_{++} \rightarrow (1, \infty)\) as, for all \(A > 0\), \(D_{31}(A) = \{D : \hat{C}_{31}^B(A, D) = 0\}\).

Lemma A.8. For all \(A > 0\), i) \(D_{12}(A) \neq \emptyset\), ii) \(\inf D_{12}(A) > 1\), and iii) \(\sup D_{12}(A) < \infty\).

Proof. Solving \([30]-[44]\) for \(C_{12}^B\) and making the change of variables gives

\[
\hat{C}_{12}^B(A, D) = A^{\frac{1}{2}(y-1)} (A \times T_1(D) + Q_1(D))
\]

where, for all \(D > 1\), \(T_1(D) = \frac{\sum_i \kappa_i D^{g_i}}{1 + l_1 D^\alpha} < 0\), \(Q_1(D) = \frac{\sum_i v_i D^{h_i} + m}{1 + l_1 D^\alpha} > 0\), and \(l_1, g_i, h_i, m > 0\), \(\max_i \{g_i\} = x + \frac{1}{2}(y + 1)\), \(\bar{\kappa} = \kappa_{\arg \max_i \{g_i\}} < 0\), \(\sum_i \kappa_i = -\left(\frac{x^2 - 1}{2y(1+y)}\right)\), \(\sum_i v_i + m = 0\), \(\max_i \{h_i\} = \frac{3}{2}x + \frac{1}{2}y > \max_i \{g_i\}\), \(\bar{v} = v_{\arg \max_i \{h_i\}} > 0\). From this, we have that

\[
\lim_{D \to 1} T_1(D) = \frac{\sum_i \kappa_i}{1 + l_1} < 0, \quad \lim_{D \to \infty} T_1(D) = -\infty
\]

\[
\lim_{D \to 1} Q_1(D) = \frac{\sum_i v_i + m}{1 + l_1} = 0, \quad \lim_{D \to \infty} Q_1(D) = \infty
\]

Fixing any \(A > 0\), \(\lim_{D \to \infty} \hat{C}_{12}^B(A, D) = \infty\) (since \(\max_i \{h_i\} > \max_i \{g_i\}\) and \(\bar{v} > 0\)) and \(\lim_{D \to 1} \hat{C}_{12}^B(A, D) = -\left(\frac{x^2 - 1}{2y(1+y)}\right) A^{\frac{1}{2}(1+y)} < 0\) (since \(\lim_{D \to 1} Q_1(D) = 0\)). Properties (i)–(iii) follow: (i) from the continuity of \(\hat{C}_{12}^B\) and the intermediate value theorem; (ii) and (iii) from the fact that both limits are bounded away from zero.

Lemma A.9. For all \(A > 0\), i) \(D_{31}(A) \neq \emptyset\), ii) \(\inf D_{31}(A) > 1\), and iii) \(\sup D_{31}(A) < \infty\).

Proof. Solving \([30]-[44]\) analytically for \(C_{31}^B\) and making the change of variables gives

\[
\hat{C}_{31}^B(A, D) = A^{\frac{1}{2}(x+1)} (A \times T_2(D) + Q_2(D))
\]

where \(T_2(D) = \frac{\sum_i \kappa_i D^{g_i}}{D^{(1+l_2)D^\beta}}\), \(Q_2(D) = \frac{\sum_i v_i D^{h_i} + m}{D^{(1+l_2)D^\beta}}\), and \(l_2, g_i, h_i, m > 0\) with \(\max_i \{g_i\} = 2x + 1\), \(\bar{\kappa} = \kappa_{\arg \max_i \{g_i\}} > 0\), \(\max_i \{h_i\} = 2x\), \(\bar{v} = v_{\arg \max_i \{h_i\}} > 0\), \(\sum_i v_i = 0\), \(\sum_i \kappa_i = -\left(\frac{x+1}{x-1}\right) < 0\). From this, we immediately have that

\[
\lim_{D \to 1} T_2(D) = -\left(\frac{1 + x}{2x}\right) < 0, \quad \lim_{D \to \infty} T_2(D) = \infty
\]

\[
\lim_{D \to 1} Q_2(D) = 0, \quad \lim_{D \to \infty} Q_2(D) = \left(\frac{V_L - K_L}{2x^2}\right) > 0
\]
Fixing any $A > 0$: $\lim_{D \to \infty} \hat{C}_{31}^B(A, D) = \infty$ and $\lim_{D \to 1} \hat{C}_{31}^R(A, D) = -\left(\frac{x + 1}{2\pi}\right) A^{1/(1-x)} < 0$. Properties (i)–(iii) follow: (i) from the continuity of $\hat{C}_{31}^R$ and the intermediate value theorem; (ii) and (iii) from the fact that both limits are bounded away from zero. \hfill \Box

**Lemma A.10.** Let $d_{12} : \mathbb{R}_{++} \to (1, \infty)$, be an arbitrary function such that, for all $A > 0$, $d_{12}(A) \in D_{12}(A)$. Then,

1. $\lim_{A \to 0} d_{12}(A) = 1$, and
2. $\lim_{A \to \infty} d_{12}(A) = \infty$

**Proof.** Recall that $\hat{C}_{12}^B(A, D) = A^{\frac{1}{2}(x-1)} (A \times T_1(D) + Q_1(D))$, and, therefore, $A \times T_1(d_{12}(A)) + Q_1(d_{12}(A)) = 0$ for all $A > 0$.

1. For any $D$, as $\lim_{A \to 0} A \times T_1(D) + Q_1(D) = Q_1(D)$. Since $Q_1(D) > 0$ for all $D > 1$ and $\lim_{D \to 1} Q_1(D) = 0$, it must be that $\lim_{A \to 0} d_{12}(A) = 1$.

2. Suppose $\lim_{A \to \infty} d_{12}(A) = 0$. Then $A \times T_1(d_{12}(A)) + Q_1(d_{12}(A)) \to -\infty$, a contradiction. Therefore, $\lim_{A \to \infty} d_{12}(A) > 0$. Since $T_1(D) < 0$ for all $D$, $\lim_{A \to \infty} A \times T_1(D) = -\infty$ and thus in order to have $A \times T_1(d_{12}(A)) + Q_1(d_{12}(A)) = 0$, it must be that $Q_1(d_{12}(A)) \to \infty$ which requires that $\lim_{A \to \infty} d_{12}(A) = \infty$. \hfill \Box

**Lemma A.11.** Let $d_{31} : \mathbb{R}_{++} \to (1, \infty)$, be an arbitrary function such that, for all $A > 0$, $d_{31}(A) \in D_{31}(A)$. Then,

1. $\liminf_{A \to 0} d_{31}(A) > 1$,
2. $\limsup_{A \to \infty} d_{31}(A) < \infty$

**Proof.** Recall that $\hat{C}_{31}^B(A, D) = A^{-\frac{1}{2}(x+1)} (A \times T_2(D) + Q_2(D))$, and, therefore, $A \times T_2(d_{31}(A)) + Q_2(d_{31}(A)) = 0$ for all $A > 0$.

1. For any $\epsilon > 0$, let $N_\epsilon = \{(A, D) \in \mathbb{R}_{++} \times (1, \infty) : \|(A, D) - (0, 1)\| < \epsilon\}$. Let $(A_\epsilon, D_\epsilon)$ denote an arbitrary point such that $(A_\epsilon, D_\epsilon) \in N_\epsilon$. To prove the first result, it suffices to show that there exists $\epsilon > 0$ such that $\hat{C}_{31}^B(A, D) < 0$ for any $(A_\epsilon, D_\epsilon)$. Recall that $\lim_{D \to 1} \hat{C}_{31}^B(A, D) = -\left(\frac{x + 1}{2\pi}\right) A^{\frac{1}{2}(1-x)} < 0$. Further $Q_2(D_\epsilon)$ is arbitrarily close to $-\left(\frac{x + 1}{2\pi}\right)$ implying that $\lim_{A \to 0} A \times T_2(D_\epsilon) + Q_2(D_\epsilon) = Q_2(D_\epsilon)$, and, therefore, that $\lim_{A \to 0} \hat{C}_{31}^B(A, D_\epsilon) = \lim_{A \to 0} \frac{Q_2(D_\epsilon)}{A^{\frac{1}{2}(1+x)}}$. Note that $Q_2$ is continuously differentiable in $D$. Taking the derivative of $Q_2$ evaluated at $D = 1$ gives $Q_2'(1) = \frac{(V_L - K_H)(\pi^2 - 1)}{4\pi} < 0$ (since $K_H > V_L$).

Hence $Q_2(D_\epsilon) < 0$. Therefore, $\nabla \hat{C}_{31}^B(A = 0, D = 1) = \left(\frac{\partial \hat{C}_{31}^B}{\partial A}, \frac{\partial \hat{C}_{31}^B}{\partial D}\right)|_{(A = 0, D = 1)} < 0$. Using a Taylor expansion, $\hat{C}_{31}^B(A, D) \approx \hat{C}_{31}^B(0, 1) + \nabla \hat{C}_{31}^B(0, 1) \cdot (A, D - 1) < 0$ for any $(A, D) \in N_\epsilon$ and $\epsilon$ sufficiently small.

2. Since $\max \{g_i\} = 2x + 1 > \max \{h_i\} = 2x$, for $(A, D)$ arbitrarily large $|A \times T_2(D)| >> |Q_2(D)|$. If $T_2(D) \neq 0$, then $|A \times T_2(D)|$ becomes arbitrarily large with $A$. Hence $\lim_{A \to \infty} T_2(d_{31}(A)) = 0$. Therefore, $\limsup_{A \to \infty} d_{31}(A) < \infty$. \hfill \Box

**Lemma A.12.** For any two values $a < a'$ both in $\mathbb{R}_{++}$, there exist two continuous paths, $p_{12}, p_{31}$, such that for each $i \in \{12, 31\}$,

1. $p_i : [0, 1] \to [a, a'] \times (1, \infty)$
2. For all $t \in [0, 1]$, $\hat{C}_i^B(p_i(t)) = 0$

3. $p_i^j(0) = a$, and $p_i^j(1) = a'$, where $p_i^j(t)$ denotes the $j$th component of $p_i(t)$

**Proof.** Fix an $i \in \{12, 31\}$ and any two values $a < a'$, both in $\mathbb{R}_{+}$. Define the set $S = \{(A, D) : A \in [a, a'], D = D_i(A)\}$. Because $\hat{C}_i^B$ is a uniformly continuous co-Lipschitz mapping, $S$ is a finite collection of disjoint, closed connected components [Maleva 2005]—implying that there exists an $\epsilon > 0$ such that

$$\min_{s, s' \in S} \left( \min_{(A, D) \in s} ||(A, D) - (A', D')|| \right) > \epsilon$$

Now let $D$ be any value in $(1, \inf\{D_i(a)\})$ and $\overline{D}$ be any value in $(\sup\{D_i(a')\}, \infty)$. Hence, $\hat{C}_i^B(a, D) < 0$ and $\hat{C}_i^B(a', \overline{D}) > 0$ (from the proofs of Lemmas A.8 and A.9). Then, if $S$ does not contain a component connecting $(a, d)$ to $(a', d')$ for some values of $d \in D_i(a)$, $d' \in D_i(a')$, there exists a continuous path $q : [0, 1] \to [a, a'] \times (1, \infty)$ such that $q(0) = (a, D)$, $q(1) = (a', \overline{D})$, and $\{t : q(t) \in S\} = \emptyset$. However, this violates the intermediate value theorem (since $q$ is continuous with $q(0) < 0$, $q(1) > 0$) implying that for any two for any two values $a < a'$, there exists a continuous path $p_i$ satisfying (1)-(3) of the Lemma.

**Proof of Lemma A.2** This follows nearly immediately from Lemmas A.10 through A.12. Using the notation from Lemma A.12 as $a \to 0$ and $a' \to \infty$, from Lemmas A.10 and A.11 $p^2_{12}(0) < p^1_{31}(0)$ and $p^2_{12}(1) > p^1_{31}(1)$. Because the paths are continuous, they must intersect. By construction, any intersection is a solution to (15) and (16) simultaneously.

### A.2 Proof of Lemma A.2

To prove Lemma A.2 we begin with some preliminary results in §A.2.1 then provide the main verification argument in §A.2.2.

#### A.2.1 Preliminaries

**Fact A.13.** If $(\alpha, \beta, B)$ solve (8)-24, then, under $\Xi(\alpha, \beta, B)$,

$$F_{\theta}(z) = E^z_{\theta} \left[ \int_0^{T(\beta)} e^{-rt} rK_{\theta} dt + e^{-rT(\beta)} B(Z_{T(\beta)}) \right]$$

$$G_{\theta}(z) = E^z_{\theta} \left[ \int_0^\tau e^{-rt} \gamma \xi dt + e^{-r\tau_1} F_{\theta}(Z_{\tau_1}) \right]$$

where $T(\beta) = \inf \{t : Z_t \geq \beta\}$, $E^z_{\theta}$ is the expectation over the process $Z$ under the law $Q^z_{\theta}$, and $\tau_1$ is the first arrival time of the next shock.

**Proof.** By construction (See, for example, Harrison, Chapter 5, Section 3).

**Definition A.14.** For $C^2$ function $f : \mathbb{R} \to \mathbb{R}$, $MB_H(f(z)) = \frac{z^2}{2} (f'(z) + f''(z)) - r(f(z) - K_H)$.

**Lemma A.15.** If $(\alpha, \beta, B)$ solve (8)-24, then, under $\Xi(\alpha, \beta, B)$, $MB_H(B_{3}(z)) < 0$ for all $z \geq \beta$.

**Proof.** We first establish three inequalities for all $z \geq \beta$: i) $B_3'(z) > V'(z)$, ii) $B_3(z) < V(z)$, and iii) $q_1^H(B_3(\beta) - K_H) > B_3'(\beta)$.
To establish (i) and (ii), recall (40) that

\[ B_3(z) = V(z) + C_{32} e^{q^H_2 z} \]

where \( C_{32}, q^H_{32} < 0 \). Therefore,

\[ B'_3(z) = V'(z) + C_{32} \frac{e^{z q^H_2}}{(e^z + 1)^2} (q^H_2 (1 + e^z) - e^z) > V'(z) \]

Thus, \( B_3(z) < V(z) \) and \( B'_3(z) > V(z) \).

To establish (iii), for a given \( \beta \), we can solve boundary conditions (33) and (35), obtaining

\[
C^H_1(\beta) = \frac{B'_3(\beta) + q^H(K_H - B_3(\beta))}{q^H_1 - q^H_2} e^{-q^H_2 \beta}
\]
\[
C^H_2(\beta) = -\frac{(B'_3(\beta) + q^H(K_H - B_3(\beta)))}{q^H_1 - q^H_2} e^{-q^H_2 \beta}
\]  (47)

Next, using boundary condition (34), we arrive at the correspondence

\[ B_H(\alpha) = \{ \beta \in \mathbb{R} : \beta \geq \alpha, \alpha = A_H(\beta) \} \]

where \( A_H(\beta) = \frac{1}{q^H_1 - q^H_2} \ln \left( \frac{-q^H_1 C^H_1(\beta)}{q^H_2 C^H_1(\beta)} \right) \). Because \( \frac{-q^H_1}{q^H_2} > 0 \), any real solution requires \( \text{sgn}(C^H_1(\beta)) = \text{sgn}(C^H_2(\beta)) \). Since \( F'_H(\beta) = B'_3(\beta) > 0 \) and \( \text{sgn}(F'_H) = \text{sgn}(C^H) \), it must be that \( C^H(\beta), C^H_2(\beta) > 0 \). Finally, \( C^H_2(\beta) > 0 \) and (48) imply that \( q^H_1(B_3(\beta) - K_H) > B'_3(\beta) \).

Having established (i)-(iii), for any \( C^H_{32} > 0 \), because \( B_3 < V \) and \( B'_3 > V' \), if \( q^H_1(B_3(z) - K_H) > B'_3(z) \) then, \( q^H(V(\beta) - K_H) > V'(\beta) \). Therefore, since \( \beta \) satisfies \( q^H(B_3(z) - K_H) > B'_3(z) \), it must be that \( \beta > \beta_H \equiv \inf \{ x : q^H(V(z) - K_H) > V'(z), \forall z > x \} \). Lemma B.3 of [DG11] shows that \( MB_H(V(z)) < 0 \) for all \( z \geq \beta_H \). Lemma A.15 then follows from the fact that \( MB_H(B_3(z)) \leq MB_H(V(z)) \) for all \( z \), which can be seen by differentiating \( MB_H(B_3(z)) \) with respect to \( C^H_{32} \) to get \( -\frac{e^{x(1-x)/2(x-1+e^x(1+x))}}{(1+e^x)^2} < 0 \), where \( x = \sqrt{1+8r/\phi^2} > 1 \).

\[ \square \]

**A.2.2 Verification**

**Proof of Lemma** A.2. If \( (\alpha, \beta, B) \) solve (3)-24, then, under \( \Xi(\alpha, \beta, B) \), each of the four requirements from Definition 2.2 are satisfied. We provide separate proofs for each condition below.

**Proof of Condition 2 (Belief Consistency).** We fix an arbitrary on-path history up to time \( t \) and show that \( Z_t = \tilde{Z}_t + Q_t^\alpha \) is Bayesian consistent with the strategy profile in \( \Xi(\alpha, \beta, B) \). There are two cases: i) there exists \( s < t \) such that \( W_s = V_L \) and the asset sold, or ii) no such \( s < t \) exists. For the first case, notice that only the low type ever accepts an offer of \( V_L \), hence such an action perfectly reveals \( \theta = L \), and the belief correctly becomes degenerate for all future times.

For the second case, we argue by induction. For \( t = 0, m_t = \emptyset \), and \( Q_t^\alpha = 0 \), meaning \( Z_0 = \ln \left( \frac{z}{1-z} \right) \) as it should. Now let \( t > 0 \) and \( M_t \leq m_t \) be \( \text{inf}\{s \leq m_t : I_s = 0\} \), and assume that \( Z_s = \tilde{Z}_s + Q_s^\alpha \) for all
To demonstrate No Deals, we first demonstrate three inequalities: i) Proof of Condition 3 (Zero Profit). If $t = m_t$, the argument is concluded. If $t > m_t$, then $I_s = 0$ for all $s \in (m_t, t]$, meaning both types of holder reject over this period and Bayes rule mandates that $Z_t = Z_{m_t} + (\hat{Z}_t - \hat{Z}_{m_t})$. Equivalently, $Q_t$ must equal $Q_{m_t}$, which is precisely what $Q^\alpha$ prescribes.

Proof of Condition 3 (Zero Profit). By construction, $w(z, 1) = B(z) = E[G_\theta(z)|z] = \beta$, where both types of seller trade with probability one, and $w(z, 1) = V_L = E[G_\theta(-\infty)|z = -\infty]$ for $z < \alpha^*$, where only the low-type seller trades.

Proof of Condition 4 (No Deals). To demonstrate No Deals, we first demonstrate three inequalities: i) $F_L \geq V_L$, ii) $F_H \geq B$, and iii) $F_H \geq F_L$.

(i) Note that $F_L'(z) = 0$ and $F_L''(z) > 0$ for all $z \in [\alpha, \beta]$. Hence, $F_L'(z) > 0$ for all $z \in (\alpha, \beta^*)$. The result then follows from $F_L''(\alpha) = V_L$.

(ii) Clearly the statement is true for $z \geq \beta$. To see that the statement holds for $z < \beta$, take any $\beta$ and $C_{12}^H$ and solve (47) and (48) for $C_1^H$ and $C_2^H$. By direct calculation, the implied value function $C_1^H e^{\theta z} + C_2^H e^{\theta z} + K_H > B_2(z)$ for all $z < \beta$. In any solution, for all $z \leq \alpha$, in order to solve (41)-(42), $C_{12}^H$ must be strictly positive implying that $B_1$ is increasing, therefore $B_1(z) \leq B_1(\alpha) = B_2(\alpha) < C_1^H e^{\theta \alpha} + C_2^H e^{\theta \alpha} + K_H = F_H(z)$. Hence (ii) holds for all $z$.

(iii) This is implied by the following: 1) Fact A.13, 2) $K_H \geq K_L$, 3) $B(\beta) = F_H(\beta) > K_H$ (which follows from $C_{12}^H, C_2^H > 0$, see the proof of Lemma A.15 and 4) $E_Z^H[e^{-\tau T(\beta)}] \geq E_Z^L[e^{-\tau T(\beta)}]$, because, for any $t$ and $z$, the distribution of $Z_t$ under the law $Q^H_t$ weakly first-order stochastically dominates the analogous distribution under $Q^L_t$.

Therefore, in any state $(z, 1)$, No Deals is satisfied because,

- If $q \geq F_H(z)$, then $E[G_\theta(z)|z, F_\theta(z) \leq q] = B(z) \leq q$.
- If $q \in [F_L(z), F_H(z)]$, then $E[G_\theta(z)|z, F_\theta(z) \leq q] = V_L \leq q$.
- If $q < F_L(z)$, $\{\theta : F_\theta(z) \leq q\} = \emptyset$.

Proof of Condition 1 (Owner Optimality). First, for any state $(z, 0)$, from Fact A.13, $G_H(z), G_L(z) \geq V_L = w(z, 0)$, so it is always optimal to reject when the owner is a seller, as the strategies specify. To see that that the strategies are optimal when the owner is a seller, we need to show that starting from any $(t, Z_t)$ such that $I_t = 1$, $T(\beta) = \inf\{s \geq t : Z_s \geq \beta\}$ solves (SP_H,t) and that both $T(\beta)$ and $T(\alpha, \beta) = \inf\{s \geq t : Z_s \notin (\alpha, \beta)\}$ solve (SP_L,t). Due to stationarity, we can normalize $t$ to zero. Start with $\theta = H$, and define

\[
F_H^{**}(z) \equiv \sup_{\tau \geq 0} E^\theta_0 \left[ \int_0^\tau e^{-\tau r}K_0 dt + e^{-\tau r}F_H(Z_\tau) \right]
\]

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By (ii) in the proof of Condition 4, \( F^*_H(z) \geq B(z) \). Therefore

\[
F^*_H(z) \geq F_H(z) = \sup_{\tau \geq 0} E^2_z \left[ \int_0^\tau e^{-r\tau} \phi(t) dt + e^{-r\tau} B(Z_t) \right]
\]

Define \( f_\theta(t, z) = (1 - e^{-r\tau}) + e^{-r\tau} F_\theta(z) \), which is \( C^2 \) on \( U \equiv \mathbb{R}\setminus \{\alpha, \beta\} \). By Ito’s formula

\[
f_H(t, Z_t) = f_H(0, Z_0) + \int_0^t A^H f_H(s, Z_s) I(Z_s \in U) ds + \int_0^t \phi e^{-rs} F_H'(Z_s) dB_s + \int_0^t e^{-rs} F_H''(\alpha) dQ^\alpha_s
\]

Using that \( A^H f_H(t, z) = 0 \) for all \( z \in \{\alpha, \beta\} \) (by construction), and \( A^H f_H(t, z) = e^{-rt} MB_H(B_3(z)) < 0 \) for all \( z \geq \beta \) (by Lemma A.15), we can conclude that

\[
f_H(t, Z_t) \leq f_H(0, Z_0) + M_t = F_H(z) + M_t
\]

where \( M \) is a martingale given by \( \int_0^t \phi e^{-rs} F_H'(Z_s) dB_s \). Taking the \( Q^H_z \)-expectation, using that \( F^*_H \) is bounded (by construction), the optional stopping theorem gives \( F^*_H(z) \leq F_H(z) \). Since the high type can attain \( F_H(z) \) by following the strategy \( T(\beta) \), we can conclude that \( F^*_H(z) = F_H(z) \) and hence \( S^{H,t} \) solves \( (SP_{H,t}) \) for all \( t \).

For the low type, we first demonstrate that both: 1) \( T(\beta) \), and 2) \( T(\alpha, \beta) \) achieve an expected payoff equal to \( F_L(z) \) starting from any initial \( Z_0 = z \). Let \( F_{L,j}(z) \) denote the expected payoff from playing according to the pure strategy \( (j) \) for \( j = 1, 2 \) starting from \( Z_0 = z \). The case for \( j = 1 \) is covered by Fact A.13. For \( z \in \{\alpha, \beta\} \), \( F_{L,2}(z) \) must solve [9] and therefore is of the form [29]. Clearly, \( F_{L,2} \) must satisfy value-matching at both \( \alpha \) and \( \beta \), i.e., [30] and [32], implying the constants are uniquely pinned down and \( F_{L,2}(z) = F_L(z) \) for all \( z \in \{\alpha, \beta\} \). Verifying that \( F_{L,2}(z) = F_L(z) \) for \( z \notin \{\alpha, \beta\} \) is immediate.

That \( F^*_L(z) = F_L(z) \) follows the same steps as the case for \( \theta = H \) after noting that [9] implies that \( A^L f_L = 0 \) for all \( z \in \{\alpha, \beta\} \) and \( A^L f_L < A^H f_H < 0 \) for all \( z > \beta \), which completes the proof. \( \square \)

### A.3 Remainder of Proofs for Section 3

**Proof of Theorem 3.2** The theorem is an immediate consequence of Lemmas A.1 and A.2. \( \square \)

**Proof of Proposition 3.4** Let \( t^0 = \inf\{t \geq 0 : I_t = 0\} \). Definition 3.1 specifies that the asset is never sold by a holder. Because \( \lambda = 0 \), a holder never transitions to a seller, meaning that the asset is held in perpetuity by a holder after \( t_0 \). Hence, i) \( G_\theta(z) = \int_0^\infty e^{-r\tau} v_\theta dt = \frac{v_\theta}{r} = V_\theta \), ii) \( B(z) = E[G_\theta(z)|z] = \bar{V}(z) \), for all \( z \) (from [22]), and iii) for all \( t > t^0 \), \( W_t = V_L \) and \( dQ_s = 0 \). If \( A_0 \) is a holder, then \( t^0 = 0 \) and it is trivial to verify that the profile satisfies the requirements of Definition 2.2. Finally, if \( A_0 \) is a seller, then \( B = V \) implies that model is identical to that of [DG11] and the result follows from Lemma 3.1 and Theorem 3.1 found therein. \( \square \)

**Proof of Proposition 3.5** Define \( \Pi(z,i) \) to be the \( F_t \)-expected discounted sum of all agents’ utilities starting from state \((Z_t, I_t) = (z,i)\). Because agents’ utilities are quasi-linear in money, transfers have no affect on \( \Pi \).
and
\[
\Pi(z, i) = E \left[ \int_0^\infty e^{-rt}(I_k - k_0 + (1 - I_i)v_0) dt | Z_0 = z, I_0 = i \right]
= E \left[ \int_0^\infty e^{-rt}v_0 dt | Z_0 = z \right] + E \left[ \int_0^\infty e^{-rt}(I_k - v_0) dt | Z_0 = z, I_0 = i \right]
= \bar{V}(z) - (v_0 - k_0)E \left[ \int_0^\infty e^{-rt}I_k dt | Z_0 = z, I_0 = i \right]
\]

In addition, because all buyers earn zero expected profit, all of this value goes to the current owner.

\[
\Pi(z, i) = iE[F_0(z)|z] + (1 - i)E[G_0(z)|z]
\]

From (22),
\[
B(z) = E[G_0(z)|z] = \Pi(z, 0)
\]

Because \(v_0 > k_0\), to prove that \(B(z) < \bar{V}(z)\), it is sufficient to argue that

\[
E \left[ \int_0^\infty e^{-rt}I_k dt | Z_0 = z, I_0 = 0 \right] > 0
\]

But this is nearly immediate from the structure of the equilibrium. Let \(t^1\) be the arrival of the first shock, and \(t^2 \geq t^1\) be the time of the first sale thereafter. Hence, if \(Z_{t^1} \in (\alpha, \beta)\), then \(\text{Prob}(t^2 > t^1) = 1\). Finally, because the shock arrives in finite time with probability 1, and because \(Z\) follows a diffusion while \(I = 0\), there is positive probability that \(Z_{t^1} \in (\alpha, \beta)\), giving

\[
0 < E \left[ \int_{t^1}^{t^2} e^{-rt} dt | Z_0 = z, I_0 = 0 \right] \leq E \left[ \int_0^\infty e^{-rt}I_k dt | Z_0 = z, I_0 = 0 \right]
\]

**Proof of Proposition 3.6.** Algegraic manipulation of Fact A.13 yields that, for any equilibrium \(\Xi(\alpha, \beta, B)\), \(F_L(\alpha) = K_L + E_A \left[ e^{-rT(\beta)} \right](B(\beta) - K_L)\). Condition (14) then implies \(E_A \left[ e^{-rT(\beta)} \right] = \frac{V_0 - K_L}{V_0 - K_L - K_L}\). In addition, direct calculation yields that \(E_A \left[ e^{-rT(\beta)} \right] = \frac{q^{1-q^{2(\beta - \alpha)}} - q^{1-q^{2(\beta - \alpha)}}}{q^{1-q^{2(\beta - \alpha)}} - q^{1-q^{2(\beta - \alpha)}}} D_{G11}\). Therefore, \(B(\beta) \geq \bar{V}(\beta_0) \iff (\beta_1 - \alpha_1) \geq (\beta_0 - \alpha_0)\). In addition, Proposition 3.5 shows that \(B < V\), meaning \(B(\beta_1) \geq \bar{V}(\beta_0) \implies \beta_1 \geq \beta_0\).

For the purpose of contradiction, suppose that \(B(\beta_1) < \bar{V}(\beta_0)\), and therefore \((\beta_1 - \alpha_1) < (\beta_0 - \alpha_0)\). Recalling the functional form of \(B_3\) from (40), \(B(\beta_1) < \bar{V}(\beta_0)\) implies that \(\bar{V}(\beta_1) + C_{B} \frac{e^{q^2(\beta_1)}}{1+e^{q^2(\beta_1)}} \beta_1 < \bar{V}(\beta_0)\), where \(C_{B}^{2} \beta_1 < 0\). This yields

\[
\frac{B'(\beta_1)}{B(\beta_1) - K_H} = \frac{\bar{V}'(\beta_1)}{\bar{V}(\beta_1) + C_{B} \frac{e^{q^2(\beta_1)}}{1+e^{q^2(\beta_1)}} - K_H} + \left( \frac{-C_{B}^{2} \frac{e^{q^2(\beta_1)}}{1+e^{q^2(\beta_1)}}}{(1-q)^2(1-q)^2(1+e^{q^2(\beta_1)})} \right) \frac{\bar{V}'(\beta_1)}{\bar{V}(\beta_1) + C_{B} \frac{e^{q^2(\beta_1)}}{1+e^{q^2(\beta_1)}} - K_H} > \frac{\bar{V}'(\beta_0)}{\bar{V}(\beta_0) - K_H}
\]

(49)

However, using Fact A.13 as above, condition (17) rearranges to

\[
\frac{d}{dz} E_{z = \beta} \left[ e^{-rT(\beta)} \right] = \frac{B'(\beta)}{B(\beta) - K_H}
\]

(50)

If \((\beta_1 - \alpha_1) < (\beta_0 - \alpha_0)\), then by direct calculation \(\frac{d}{dz} E_{z = \beta} \left[ e^{-rT(\beta)} \right] < \frac{d}{dz} E_{z = \beta_0} \left[ e^{-rT(\beta_0)} \right] = \frac{\bar{V}'(\beta_0)}{\bar{V}(\beta_0) - K_H} < \)
Proof of Lemma 4.4. The proof of Proposition 3.5 shows that the \( \mathcal{F}_t \)-expected discounted cash flow is given by \( \Pi(z, i) = i E[F_0(z)|z] + (1-i) E[G_0(z)|z] \). By Definition 4.3, \( \mathcal{P}(z, i) = \Pi(z, i) \), which from (25) is also equal to \( \mathcal{A}(z, i) \).

The next two propositions will require the use of the following result

**Lemma B.1.** Let \( f : \mathbb{R} \times \{0, 1\} \rightarrow \mathbb{R} \) denote an arbitrary function that is twice differentiable in its first argument almost everywhere. And let \( A \) denote (infinitesimal) generator of \((I_t, Z_t)\) under \( \mathbb{Q} \) (i.e., the public measure). For all states such that \( z > \alpha \) or \( i = 0 \), we have that

\[
Af(z, i) = \frac{\partial^2}{2} \left( (2p(z) - 1) f_z(z, i) + f_{zz}(z, i) \right) + (1 - i) \lambda (f(z, 1) - f(z, 0))
\] (51)

**Proof.** For all such states: (i) \( dZ_t = d\hat{Z}_t \) and equation (2) gives that \( E[dZ_t|\mathcal{F}_t] = \frac{\lambda}{2} (2p(Z_t) - 1) dt \), and (ii) \( I_t \) follows a jump process for all such states with arrival \( \lambda \) and fixed jump size \( (1 - i) \). The result then follows from Applebaum (2004, Theorem 3.3.3).

**Proof of Proposition 4.5.** We break the proof into the three cases:

1. For \( i = 0 \), using (51) we have that

\[
\frac{1}{dt} E[dP_t|\mathcal{F}_t] = \lambda (P(Z_t, 1) - P(Z_t, 0)) + \frac{\partial^2}{2} \left( (2p - 1) P_z(Z_t, 0) + P_{zz}(Z_t, 0) \right)
\]
\[
= \lambda (\hat{G} - \hat{V}) + (1 - p) \left[ (r + \lambda) L - (\lambda F_{L} + r F_{L}) \right] + (1 - p) \left[ (r + \lambda) H - (\lambda F_{H} + r F_{H}) \right]
\]
\[
= r (\hat{G} - \hat{V})
\]

and \( E[v_\theta|\mathcal{F}_t] = r \hat{V}(z) \), thus

\[
\frac{1}{dt} \left( E[dR_t|\mathcal{F}_t] - r \right) = \frac{E[dP_t|\mathcal{F}_t] + r \hat{V}(z)}{P_t dt} - r = \frac{r (\hat{G} - \hat{V}) + r \hat{V} - r = 0}{G}
\]

2. For \( i = 1 \) and \( z > \alpha \), using (51), we have that

\[
E[dP_t|\mathcal{F}_t] = \frac{\partial^2}{2} \left( (2p - 1) P_z + P_{zz} \right) dt
\]
\[
= \frac{\partial^2}{2} \left( p (F'_{H} + F''_{H}) + (1 - p) (F'_{L} - F'_{L}) \right) dt
\]

For \( z \in (\alpha, \beta) \), \( F_{H} \) and \( F_{L} \) satisfy (8) and (9). Substituting this in gives:

\[
E[dP_t|\mathcal{F}_t] = r (\hat{F}(z) - \hat{K}(z)) dt
\]

taking the \( \mathcal{F}_t \)-expectation we get that

\[
\frac{1}{dt} \left( E[dR_t|\mathcal{F}_t] - r \right) = r \left( \frac{\hat{F}(z) - \hat{K}(z) + \hat{V}(z)}{F(z) - 1} \right) = \frac{r (\hat{V}(z) - \hat{K}(z))}{F(z)}
\]

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3. For \( i = 1 \) and \( z \leq \alpha \), with probability \( \frac{p(\alpha) - p(z)}{p(\alpha)} \) the price jumps down to \( V_L \). With probability \( \frac{p(z)}{p(\alpha)} \) the price jumps up to \( \bar{F}(\alpha + dZ_t) \). Thus,

\[
\frac{1}{dt} E[dP_t|\mathcal{F}_t] = \frac{p(\alpha) - p(z)}{p(\alpha)}(V_L - P_t) + \frac{p(z)}{p(\alpha)} \left( \mathcal{P}(\alpha, 1) + E[dP_t|(Z_t, I_t) = (\alpha, 1)] - \mathcal{P}(z, 1) \right)
\]

\[
= \frac{p(z)}{p(\alpha)} r(\bar{F}(z) - \bar{K}(z))
\]

where the second equality use the fact that \( F_L(z) = F_L(\alpha) = V_L \). The expression given in the proposition for this case follows immediately.

\[
\square
\]

**Proof of Proposition 4.6.** The boundary conditions are uniquely pinned down by equilibrium play; when beliefs are degenerate, trade occurs immediately upon arrival of a shock. For the remainder of the proof, we break the state space into four different regions enumerated below.

1. For \( z \leq \alpha, i = 1 \): with probability \( \frac{p(\alpha) - p(z)}{p(\alpha)} \) trade occurs \( (dv_t = 1) \) and the state transitions to \( (-\infty, 0) \). With probability \( \frac{p(z)}{p(\alpha)} \) trade does not occur and the state transitions to \( (\alpha, 1) \). Therefore,

\[
f(t, z) = \frac{p(\alpha) - p(z)}{p(\alpha)} \left( 1 + \lim_{z \to -\infty} g(t, z) \right) + \frac{p(z)}{p(\alpha)} f(t, \alpha) = \frac{p(\alpha) - p(z)}{p(\alpha)} \left( 1 + \lambda t \right) + \frac{p(z)}{p(\alpha)} f(t, \alpha)
\]

where the second inequality follows from the boundary condition on \( g \).

2. For \( z \in (\alpha, \beta), i = 1 \): \( dv_t = 0 \) w.p.1. \( (f(0, z) = 0) \) Applying the Kolmogorov backward equation (e.g., Applebaum (2004, p. 164)) using the generator from (51) gives \( f_i \) in (27).

3. For \( z \geq \beta, i = 1 \): \( dv_t = 1 \) w.p.1. (thus \( f(0, z) = 1 \)) and the new owner is a holder. Thus \( f(t, z) = 1 + g(t, z) \).

4. For \( i = 0 \) and all \( z \): \( dv_t = 0 \) w.p.1. (thus \( g(0, z) = 0 \)). Again, applying the Kolmogorov backward equation using the generator from (51) gives \( g_t \) in (27).

\[
\square
\]