14.452 Economic Growth: Lecture 4, Foundations of Neoclassical Growth

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Foundations of Neoclassical Growth

- Solow model: constant saving rate.
- More satisfactory to specify the *preference orderings* of individuals and derive their decisions from these preferences.
- Enables better understanding of the factors that affect savings decisions.
- Enables to discuss the "optimality" of equilibria.
- Whether the (competitive) equilibria of growth models can be "improved upon".
- Notion of improvement: Pareto optimality.
Consider an economy consisting of a unit measure of infinitely-lived households.

I.e., an uncountable number of households: e.g., the set of households \( \mathcal{H} \) could be represented by the unit interval \([0, 1]\).

Emphasize that each household is infinitesimal and will have no effect on aggregates.

Can alternatively think of \( \mathcal{H} \) as a countable set of the form \( \mathcal{H} = \{1, 2, \ldots, M\} \) with \( M = \infty \), without any loss of generality.

Advantage of unit measure: averages and aggregates are the same

Simpler to have \( \mathcal{H} \) as a finite set in the form \( \{1, 2, \ldots, M\} \) with \( M \) large but finite.

Acceptable for many models, but with overlapping generations require the set of households to be infinite.
Time Separable Preferences

- Standard assumptions on preference orderings so that they can be represented by utility functions.
- In addition, **time separable preferences**: each household $i$ has an instantaneous (*Bernoulli*) utility function (or felicity function):

  \[ u_i(c_i(t)), \]

  \[ u_i : \mathbb{R}_+ \rightarrow \mathbb{R} \] is increasing and concave and $c_i(t)$ is the consumption of household $i$.

- Note instantaneous utility function is *not* specifying a complete preference ordering over all commodities—here consumption levels in all dates.
- Instead, household $i$ preferences at time $t = 0$ are obtained by combining this with exponential discounting.
Infinite Horizon and the Representative Household

- Thus given by the following von Neumann-Morgenstern expected utility function:

$$E_0^i \sum_{t=0}^{T} \beta^t_i u_i (c_i (t)),$$

where $\beta_i \in (0, 1)$ is the discount factor of household $i$, where $T < \infty$ or $T = \infty$ are the two cases to consider.

- To model households in infinite horizon, these two would then correspond to
  1. overlapping generations $\rightarrow$ finite planning horizon (generally...);
  2. “infinitely lived” or consisting of overlapping generations with full altruism linking generations $\rightarrow$ infinite planning horizon

- The second is often assumed because the standard approach in macroeconomics is to impose the existence of a representative household—costs of this to be discussed below.
Time Consistency

- Exponential discounting and time separability: ensure “time-consistent” behavior.
- A solution \( \{x(t)\}_{t=0}^{T} \) (possibly with \( T = \infty \)) is time consistent if:
  - whenever \( \{x(t)\}_{t=0}^{T} \) is an optimal solution starting at time \( t = 0 \), \( \{x(t)\}_{t=t'}^{T} \) is an optimal solution to the continuation dynamic optimization problem starting from time \( t = t' \in [0, T] \).
Challenges to the Representative Household

- An economy admits a representative household if preference side can be represented as if a single household made the aggregate consumption and saving decisions subject to a single budget constraint.
- This description concerning a representative household is purely positive.
- Stronger notion of “normative” representative household: if we can also use the utility function of the representative household for welfare comparisons.
- Simplest case that will lead to the existence of a representative household: suppose each household is identical.
Representative Household II

- I.e., same $\beta$, same sequence $\{e(t)\}_{t=0}^{\infty}$ and same $u(c_i(t))$

where $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is increasing and concave and $c_i(t)$ is the consumption of household $i$.

- Again ignoring uncertainty, preference side can be represented as the solution to
  \[\max_{\sum_{t=0}^{\infty} \beta^t u(c(t))},\] (2)

- $\beta \in (0, 1)$ is the common discount factor and $c(t)$ the consumption level of the representative household.

- Admits a representative household rather trivially.

- Representative household’s preferences, (2), can be used for positive and normative analysis.
Representative Household III

- If instead households are not identical but assume can model as if demand side generated by the optimization decision of a representative household:

- More realistic, but:
  1. The representative household will have positive, but not always a normative meaning.
  2. Models with heterogeneity: often not lead to behavior that can be represented as if generated by a representative household.

**Theorem (Debreu-Mantel-Sonnenschein Theorem)** Let \( \epsilon > 0 \) be a scalar and \( N < \infty \) be a positive integer. Consider a set of prices \( P_\epsilon = \{ p \in \mathbb{R}_+^N : \frac{p_j}{p_{j'}} \geq \epsilon \text{ for all } j \text{ and } j' \} \) and any continuous function \( x : P_\epsilon \to \mathbb{R}_+^N \) that satisfies Walras’ Law and is homogeneous of degree 0. Then there exists an exchange economy with \( N \) commodities and \( H < \infty \) households, where the aggregate demand is given by \( x(p) \) over the set \( P_\epsilon \).
That excess demands come from optimizing behavior of households puts no restrictions on the form of these demands.

- E.g., $x(p)$ does not necessarily possess a negative-semi-definite Jacobian or satisfy the weak axiom of revealed preference (requirements of demands generated by individual households).

Hence without imposing further structure, impossible to derive specific $x(p)$’s from the maximization behavior of a single household.

Severe warning against the use of the representative household assumption.

Partly an outcome of very strong income effects:

- special but approximately realistic preference functions, and restrictions on distribution of income rule out arbitrary aggregate excess demand functions.
Gorman Aggregation

- Recall an indirect utility function for household \( i \), \( v_i (p, y^i) \), specifies (ordinal) utility as a function of the price vector \( p = (p_1, ..., p_N) \) and household’s income \( y^i \).
- \( v_i (p, y^i) \): homogeneous of degree 0 in \( p \) and \( y \).

**Theorem** *(Gorman’s Aggregation Theorem)* Consider an economy with a finite number \( N < \infty \) of commodities and a set \( \mathcal{H} \) of households. Suppose that the preferences of household \( i \in \mathcal{H} \) can be represented by an indirect utility function of the form

\[
v^i (p, y^i) = a^i (p) + b (p) y^i , \tag{3}
\]

then these preferences can be aggregated and represented by those of a representative household, with indirect utility

\[
v (p, y) = \int_{i \in \mathcal{H}} a^i (p) \, di + b (p) y ,
\]

where \( y \equiv \int_{i \in \mathcal{H}} y^i \, di \) is aggregate income.
Linear Engel Curves

- Demand for good $j$ (from Roy’s identity):

$$x_j^i (p, y^i) = - \frac{1}{b(p)} \frac{\partial a^i (p)}{\partial p_j} - \frac{1}{b(p)} \frac{\partial b(p)}{\partial p_j} y^i.$$  

- Thus linear Engel curves.
- “Indispensable” for the existence of a representative household.
- Let us say that there exists a strong representative household if redistribution of income or endowments across households does not affect the demand side.
- Gorman preferences are sufficient for a strong representative household.
- Moreover, they are also necessary (with the same $b(p)$ for all households) for the economy to admit a strong representative household.
  - The proof is easy by a simple variation argument.
Importance of Gorman Preferences

- Gorman Preferences limit the **extent of income effects** and enables the aggregation of individual behavior.
- Integral is “Lebesgue integral,” so when $\mathcal{H}$ is a finite or countable set, $\int_{i \in \mathcal{H}} y^i \, di$ is indeed equivalent to the summation $\sum_{i \in \mathcal{H}} y^i$.
- Stated for an economy with a finite number of commodities, but can be generalized for infinite or even a continuum of commodities.
- Note all we require is there exists a monotonic transformation of the indirect utility function that takes the form in (3)—as long as no uncertainty.
- Contains some commonly-used preferences in macroeconomics.
Example: Constant Elasticity of Substitution Preferences

- A very common class of preferences: constant elasticity of substitution (CES) preferences or Dixit-Stiglitz preferences.
- Suppose each household denoted by \( i \in \mathcal{H} \) has total income \( y^i \) and preferences defined over \( j = 1, \ldots, N \) goods

\[
U^i (x_1^i, \ldots, x_N^i) = \left[ \sum_{j=1}^{N} \left( x_j^i - \xi_j^i \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}},
\]

(4)

- \( \sigma \in (0, \infty) \) and \( \xi_j^i \in [-\bar{\xi}, \bar{\xi}] \) is a household specific term, which parameterizes whether the particular good is a necessity for the household.
- For example, \( \xi_j^i > 0 \) may mean that household \( i \) needs to consume a certain amount of good \( j \) to survive.
- Details: see recitation.
Gorman preferences also imply the existence of a normative representative household.

Recall an allocation is *Pareto optimal* if no household can be made strictly better-off without some other household being made worse-off.
Theorem (Existence of a Normative Representative Household)

Consider an economy with a finite number \( N < \infty \) of commodities, a set \( \mathcal{H} \) of households and a convex aggregate production possibilities set \( Y \). Suppose that the preferences of each household \( i \in \mathcal{H} \) take the Gorman form,

\[
v^i(p, y^i) = a^i(p) + b(p)y^i.
\]

Then any allocation that maximizes the utility of the representative household,

\[
v(p, y) = \sum_{i \in \mathcal{H}} a^i(p) + b(p)y, \quad \text{with } y \equiv \sum_{i \in \mathcal{H}} y^i,
\]

is Pareto optimal.

Moreover, if \( a^i(p) = a^i \) for all \( p \) and all \( i \in \mathcal{H} \), then any Pareto optimal allocation maximizes the utility of the representative household.
Infinite Planning Horizon

Most growth and macro models assume that individuals have an infinite-planning horizon

Two reasonable microfoundations for this assumption

First: “Poisson death model” or the perpetual youth model: individuals are finitely-lived, but not aware of when they will die.

1. Strong simplifying assumption: likelihood of survival to the next age in reality is not a constant
2. But a good starting point, tractable and implies expected lifespan of $1/\nu < \infty$ periods, can be used to get a sense value of $\nu$.

Suppose each individual has a standard instantaneous utility function $u : \mathbb{R}_+ \to \mathbb{R}$, and a “true” or “pure” discount factor $\hat{\beta}$

Normalize $u(0) = 0$ to be the utility of death.

Consider an individual who plans to have a consumption sequence $\{c(t)\}_{t=0}^{\infty}$ (conditional on living).
Infinite Planning Horizon II

- Individual would have an expected utility at time $t = 0$ given by

$$U(0) = u(c(0)) + \hat{\beta}(1 - \nu)u(c(0)) + \hat{\beta}\nu u(0) + \hat{\beta}^2 (1 - \nu)^2 u(c(1)) + \hat{\beta}^2 (1 - \nu)\nu u(0) + \ldots$$

$$= \sum_{t=0}^{\infty} (\hat{\beta}(1 - \nu))^t u(c(t))$$

$$= \sum_{t=0}^{\infty} \beta^t u(c(t)), \quad (5)$$

- Second line collects terms and uses $u(0) = 0$, third line defines $\beta \equiv \hat{\beta}(1 - \nu)$ as “effective discount factor.”

- Isomorphic to model of infinitely-lived individuals, but values of $\beta$ may differ.

- Also equation (5) is already the expected utility; probabilities have been substituted.
Second: intergenerational altruism or from the “bequest” motive. Imagine an individual who lives for one period and has a single offspring (who will also live for a single period and beget a single offspring etc.).

Individual not only derives utility from his consumption but also from the bequest he leaves to his offspring.

For example, utility of an individual living at time $t$ is given by

$$u(c(t)) + U^b(b(t)),$$

$c(t)$ is his consumption and $b(t)$ denotes the bequest left to his offspring.

For concreteness, suppose that the individual has total income $y(t)$, so that his budget constraint is

$$c(t) + b(t) \leq y(t).$$
Infinite Planning Horizon IV

- $U^b(\cdot)$: how much the individual values bequests left to his offspring.
- Benchmark might be “purely altruistic:” cares about the utility of his offspring (with some discount factor).
- Let discount factor between generations be $\beta$.
- Assume offspring will have an income of $w$ without the bequest.
- Then the utility of the individual can be written as

$$u(c(t)) + \beta V(b(t) + w),$$

- $V(\cdot)$: continuation value, the utility that the offspring will obtain from receiving a bequest of $b(t)$ (plus his own $w$).
- Value of the individual at time $t$ can in turn be written as

$$V(y(t)) = \max_{c(t)+b(t)\leq y(t)} \left\{ u(c(t)) + \beta V(b(t) + w(t+1)) \right\},$$
Infinite Planning Horizon V

- Canonical form of a dynamic programming representation of an infinite-horizon maximization problem.
- Under some mild technical assumptions, this dynamic programming representation is equivalent to maximizing

$$\sum_{s=0}^{\infty} \beta^s u(c_{t+s})$$

at time $t$.
- Each individual internalizes utility of all future members of the “dynasty”.
- Fully altruistic behavior within a dynasty (“dynastic” preferences) will also lead to infinite planning horizon.
The Representative Firm I

While not all economies would admit a representative household, standard assumptions (in particular no production externalities and competitive markets) are sufficient to ensure a representative firm.

Theorem  (The Representative Firm Theorem) Consider a competitive production economy with $N \in \mathbb{N} \cup \{+\infty\}$ commodities and a countable set $\mathcal{F}$ of firms, each with a convex production possibilities set $Y^f \subset \mathbb{R}^N$. Let $p \in \mathbb{R}_+^N$ be the price vector in this economy and denote the set of profit maximizing net supplies of firm $f \in \mathcal{F}$ by $\hat{Y}^f (p) \subset Y^f$ (so that for any $\hat{y}^f \in \hat{Y}^f (p)$, we have $p \cdot \hat{y}^f \geq p \cdot y^f$ for all $y^f \in Y^f$). Then there exists a representative firm with production possibilities set $Y \subset \mathbb{R}^N$ and set of profit maximizing net supplies $\hat{Y} (p)$ such that for any $p \in \mathbb{R}_+^N$, $\hat{y} \in \hat{Y} (p)$ if and only if $\hat{y} (p) = \sum_{f \in \mathcal{F}} \hat{y}^f$ for some $\hat{y}^f \in \hat{Y}^f (p)$ for each $f \in \mathcal{F}$. 
The Representative Firm II

- Why such a difference between representative household and representative firm assumptions? Income effects.

- Changes in prices create income effects, which affect different households differently.

- No income effects in producer theory, so the representative firm assumption is without loss of any generality.

- Does not mean that heterogeneity among firms is uninteresting or unimportant.

- Many models of endogenous technology feature productivity differences across firms, and firms’ attempts to increase their productivity relative to others will often be an engine of economic growth.
Problem Formulation I

Discrete time infinite-horizon economy and suppose that the economy admits a representative household.

Once again ignoring uncertainty, the representative household has the $t = 0$ objective function

$$
\sum_{t=0}^{\infty} \beta^t u(c(t)), \tag{6}
$$

with a discount factor of $\beta \in (0, 1)$.

In continuous time, this utility function of the representative household becomes

$$
\int_{0}^{\infty} \exp(-\rho t) u(c(t)) \, dt \tag{7}
$$

where $\rho > 0$ is now the discount rate of the individuals.
Welfare Theorems I

- There should be a close connection between Pareto optima and competitive equilibria.
- Start with models that have a finite number of consumers, so $\mathcal{H}$ is finite.
- However, allow an infinite number of commodities.
- Results here have analogs for economies with a continuum of commodities, but focus on countable number of commodities.
- Let commodities be indexed by $j \in \mathbb{N}$ and $x^i \equiv \left\{ x^i_j \right\}_{j=0}^{\infty}$ be the consumption bundle of household $i$, and $\omega^i \equiv \left\{ \omega^i_j \right\}_{j=0}^{\infty}$ be its endowment bundle.
- Assume feasible $x^i$’s must belong to some consumption set $X^i \subset \mathbb{R}_+^\infty$.
- Most relevant interpretation for us is that at each date $j = 0, 1, \ldots$, each individual consumes a finite dimensional vector of products.
Welfare Theorems II

- Thus $x_j^i \in X_j^i \subset \mathbb{R}_+^K$ for some integer $K$.
- Consumption set introduced to allow cases where individual may not have negative consumption of certain commodities.
- Let $X \equiv \prod_{i \in H} X^i$ be the Cartesian product of these consumption sets, the aggregate consumption set of the economy.
- Also use the notation $x \equiv \{x^i\}_{i \in H}$ and $\omega \equiv \{\omega^i\}_{i \in H}$ to describe the entire consumption allocation and endowments in the economy.
- Feasibility requires that $x \in X$.
- Each household in $H$ has a well defined preference ordering over consumption bundles.
- This preference ordering can be represented by a relationship $\succeq_i$ for household $i$, such that $x' \succeq_i x$ implies that household $i$ weakly prefers $x'$ to $x$. 
Suppose that preferences can be represented by $u^i : X^i \to \mathbb{R}$, such that whenever $x' \succeq_i x$, we have $u^i(x') \geq u^i(x)$.

The domain of this function is $X^i \subset \mathbb{R}^\infty_+$.

Let $u \equiv \{u^i\}_{i \in \mathcal{H}}$ be the set of utility functions.

Production side: finite number of firms represented by $\mathcal{F}$.

Each firm $f \in \mathcal{F}$ is characterized by production set $Y^f$, specifies levels of output firm $f$ can produce from specified levels of inputs.

I.e., $y^f \equiv \left\{y^f_j\right\}_{j=0}^\infty$ is a feasible production plan for firm $f$ if $y^f \in Y^f$.

E.g., if there were only labor and a final good, $Y^f$ would include pairs $(-l, y)$ such that with labor input $l$ the firm can produce at most $y$. 
Welfare Theorems IV

- Let \( Y \equiv \prod_{f \in F} Y^f \) represent the aggregate production set and 
  \( y \equiv \{ y^f \}_{f \in F} \) such that \( y^f \in Y^f \) for all \( f \), or equivalently, \( y \in Y \).

- Ownership structure of firms: if firms make profits, they should be 
  distributed to some agents

- Assume there exists a sequence of numbers (profit shares) 
  \( \theta \equiv \{ \theta^i_f \}_{f \in F, i \in \mathcal{H}} \) such that \( \theta^i_f \geq 0 \) for all \( f \) and \( i \), and \( \sum_{i \in \mathcal{H}} \theta^i_f = 1 \) 
  for all \( f \in F \).

- \( \theta^i_f \) is the share of profits of firm \( f \) that will accrue to household \( i \).
Welfare Theorems V

- An economy $E$ is described by $E \equiv (\mathcal{H}, \mathcal{F}, u, \omega, Y, X, \theta)$.
- An allocation is $(x, y)$ such that $x$ and $y$ are feasible, that is, $x \in X$, $y \in Y$, and $\sum_{i \in \mathcal{H}} x^i_j \leq \sum_{i \in \mathcal{H}} \omega^i_j + \sum_{f \in \mathcal{F}} y^f_j$ for all $j \in \mathbb{N}$.
- A price system is a sequence $p \equiv \{p_j\}_{j=0}^{\infty}$, such that $p_j \geq 0$ for all $j$.
- We can choose one of these prices as the numeraire and normalize it to 1.
- Also define $p \cdot x$ as the inner product of $p$ and $x$, i.e., $p \cdot x \equiv \sum_{j=0}^{\infty} p_j x_j$.

**Definition** Household $i \in \mathcal{H}$ is *locally non-satiated* if at each $x^i$, $u^i(x^i)$ is strictly increasing in at least one of its arguments at $x^i$ and $u^i(x^i) < \infty$.

- Latter requirement already implied by the fact that $u^i : X^i \rightarrow \mathbb{R}$. Let us impose this assumption.
Welfare Theorems VI

**Definition** A competitive equilibrium for the economy \( \mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, u, \omega, Y, X, \theta) \) is given by an allocation 
\[
\left( \begin{array}{c}
x^* = \{x^*_i\}_{i \in \mathcal{H}}, \\
y^* = \{y^*_f\}_{f \in \mathcal{F}}
\end{array} \right)
\] 
and a price system \( p^* \) such that

1. The allocation \((x^*, y^*)\) is feasible and market clearing, 
i.e., \( x^*_i \in X^i \) for all \( i \in \mathcal{H} \), \( y^*_f \in Y^f \) for all \( f \in \mathcal{F} \) and 
\[
\sum_{i \in \mathcal{H}} x^*_i = \sum_{i \in \mathcal{H}} \omega^i + \sum_{f \in \mathcal{F}} y^*_f \text{ for all } j \in \mathbb{N}.
\]

2. For every firm \( f \in \mathcal{F} \), \( y^*_f \) maximizes profits, i.e., 
\[
p^* \cdot y^*_f \geq p^* \cdot y \text{ for all } y \in Y^f.
\]

3. For every consumer \( i \in \mathcal{H} \), \( x^*_i \) maximizes utility, i.e., 
\[
u^i (x^*_i) \geq u^i (x) \text{ for all } x \text{ s.t. } x \in X^i \text{ and } p^* \cdot x \leq p^* \cdot x^*_i.
\]
Establish existence of competitive equilibrium with finite number of commodities and standard convexity assumptions is straightforward.

With infinite number of commodities, somewhat more difficult and requires more sophisticated arguments.

**Definition** A feasible allocation \((x, y)\) for economy \(E \equiv (\mathcal{H}, \mathcal{F}, u, \omega, Y, X, \theta)\) is *Pareto optimal* if there exists no other feasible allocation \((\hat{x}, \hat{y})\) such that \(\hat{x}^i \in X^i, \hat{y}^f \in Y^f\) for all \(f \in \mathcal{F}\),

\[
\sum_{i \in \mathcal{H}} \hat{x}^i_j \leq \sum_{i \in \mathcal{H}} \omega^i_j + \sum_{f \in \mathcal{F}} \hat{y}^f_j \quad \text{for all } j \in \mathbb{N},
\]

and

\[
u^i(\hat{x}^i) \geq \nu^i(x^i) \quad \text{for all } i \in \mathcal{H},
\]

with at least one strict inequality.
Theorem (First Welfare Theorem I) Suppose that \((x^*, y^*, p^*)\) is a competitive equilibrium of economy \(E \equiv (H, F, u, \omega, Y, X, \theta)\) with \(H\) finite. Assume that all households are locally non-satiated. Then \((x^*, y^*)\) is Pareto optimal.
Proof of First Welfare Theorem I

- To obtain a contradiction, suppose that there exists a feasible \((\hat{x}, \hat{y})\) such that \(u^i(\hat{x}^i) \geq u^i(x^i)\) for all \(i \in \mathcal{H}\) and \(u^i(\hat{x}^i) > u^i(x^i)\) for all \(i \in \mathcal{H}'\), where \(\mathcal{H}'\) is a non-empty subset of \(\mathcal{H}\).

- Since \((x^*, y^*, p^*)\) is a competitive equilibrium, it must be the case that for all \(i \in \mathcal{H}\),

\[
p^* \cdot \hat{x}^i \geq p^* \cdot x^{i*}
\]

\[
= p^* \cdot \left( \omega^i + \sum_{f \in \mathcal{F}} \theta_f^i y^{f*} \right)
\]  \hspace{1cm} (8)

and for all \(i \in \mathcal{H}'\),

\[
p^* \cdot \hat{x}^i > p^* \cdot \left( \omega^i + \sum_{f \in \mathcal{F}} \theta_f^i y^{f*} \right).
\]  \hspace{1cm} (9)
Proof of First Welfare Theorem II

- Second inequality follows immediately in view of the fact that \( x^i * \) is the utility maximizing choice for household \( i \), thus if \( \hat{\chi}^i \) is strictly preferred, then it cannot be in the budget set.

- First inequality follows with a similar reasoning. Suppose that it did not hold.

- Then by the hypothesis of local-satiation, \( u^i \) must be strictly increasing in at least one of its arguments, let us say the \( j \)th component of \( x \).

- Then construct \( \hat{\chi}^i (\varepsilon) \) such that \( \hat{\chi}^i_j (\varepsilon) = \hat{x}^i_j \) and \( \hat{\chi}^i_j (\varepsilon) = \hat{x}^i_j + \varepsilon \).

- For \( \varepsilon \downarrow 0 \), \( \hat{\chi}^i (\varepsilon) \) is in household \( i \)’s budget set and yields strictly greater utility than the original consumption bundle \( x^i \), contradicting the hypothesis that household \( i \) was maximizing utility.

- Note local non-satiation implies that \( u^i (x^i) < \infty \), and thus the right-hand sides of (8) and (9) are finite.
Proof of First Welfare Theorem III

- Now summing over (8) and (9), we have

\[ p^* \cdot \sum_{i \in \mathcal{H}} \hat{x}_i \geq p^* \cdot \sum_{i \in \mathcal{H}} \left( \omega^i + \sum_{f \in \mathcal{F}} \theta^i_f y^f \right), \quad (10) \]

\begin{align*}
&= p^* \cdot \left( \sum_{i \in \mathcal{H}} \omega^i + \sum_{f \in \mathcal{F}} y^f \right), \\
\end{align*}

- Second line uses the fact that the summations are finite, can change the order of summation, and that by definition of shares \( \sum_{i \in \mathcal{H}} \theta^i_f = 1 \) for all \( f \).

- Finally, since \( y^* \) is profit-maximizing at prices \( p^* \), we have that

\[ p^* \cdot \sum_{f \in \mathcal{F}} y^f \geq p^* \cdot \sum_{f \in \mathcal{F}} y^f \text{ for any } \{y^f\}_{f \in \mathcal{F}} \text{ with } y^f \in Y^f \text{ for all } f \in \mathcal{F}. \quad (11) \]
Proof of First Welfare Theorem IV

- However, by market clearing of \( \hat{x}^i \) (Definition above, part 1), we have

\[
\sum_{i \in \mathcal{H}} \hat{x}^i_j = \sum_{i \in \mathcal{H}} \omega^i_j + \sum_{f \in \mathcal{F}} \hat{y}^f_j,
\]

- Therefore, by multiplying both sides by \( p^* \) and exploiting (11),

\[
p^* \cdot \sum_{i \in \mathcal{H}} \hat{x}^i_j \leq p^* \cdot \left( \sum_{i \in \mathcal{H}} \omega^i_j + \sum_{f \in \mathcal{F}} \hat{y}^f_j \right)
\leq p^* \cdot \left( \sum_{i \in \mathcal{H}} \omega^i_j + \sum_{f \in \mathcal{F}} y^f_j \right),
\]

- Contradicts (10), establishing that any competitive equilibrium allocation \((x^*, y^*)\) is Pareto optimal.
Welfare Theorems IX

- Proof of the First Welfare Theorem based on two intuitive ideas.
  1. If another allocation Pareto dominates the competitive equilibrium, then it must be non-affordable in the competitive equilibrium.
  2. Profit-maximization implies that any competitive equilibrium already contains the maximal set of affordable allocations.

- Note it makes no convexity assumption.

- Also highlights the importance of the feature that the relevant sums exist and are finite.
  - Otherwise, the last step would lead to the conclusion that “∞ < ∞”.

- That these sums exist followed from two assumptions: finiteness of the number of individuals and non-satiation.
Welfare Theorems

**Theorem** (First Welfare Theorem II) Suppose that \( (x^*, y^*, p^*) \) is a competitive equilibrium of the economy \( E \equiv (\mathcal{H}, \mathcal{F}, u, \omega, Y, X, \theta) \) with \( \mathcal{H} \) countably infinite. Assume that all households are locally non-satiated and that
\[
p^* \cdot \omega^* = \sum_{i \in \mathcal{H}} \sum_{j=0}^{\infty} p_j^* \omega_j^i < \infty.
\]
Then \( (x^*, y^*, p^*) \) is Pareto optimal.

**Proof:**

- Same as before but now local non-satiation does not guarantee summations are finite (10), since we sum over an infinite number of households.
- But since endowments are finite, the assumption that
\[
\sum_{i \in \mathcal{H}} \sum_{j=0}^{\infty} p_j^* \omega_j^i < \infty
\]
ensures that the sums in (10) are indeed finite.
Welfare Theorems X

- Second Welfare Theorem (converse to First): whether or not $\mathcal{H}$ is finite is not as important as for the First Welfare Theorem.

- But requires assumptions such as the convexity of consumption and production sets and preferences, and additional requirements because it contains an “existence of equilibrium argument”.

- Recall that the consumption set of each individual $i \in \mathcal{H}$ is $X^i \subset \mathbb{R}_+^\infty$.

- A typical element of $X^i$ is $x^i = (x^i_1, x^i_2, ...)$, where $x^i_t$ can be interpreted as the vector of consumption of individual $i$ at time $t$.

- Similarly, a typical element of the production set of firm $f \in \mathcal{F}$, $Y^f$, is $y^f = (y^f_1, y^f_2, ...)$.

- Let us define $x^i \left[ T \right] = (x^i_0, x^i_1, x^i_2, ..., x^i_T, 0, 0, ...)$ and $y^f \left[ T \right] = (y^f_0, y^f_1, y^f_2, ..., y^f_T, 0, 0, ...)$.

- It can be verified that $\lim_{T \to \infty} x^i \left[ T \right] = x^i$ and $\lim_{T \to \infty} y^f \left[ T \right] = y^f$ in the product topology.
Second Welfare Theorem I

Theorem

Consider a Pareto optimal allocation \((x^{**}, y^{**})\) in an economy described by \(\omega\), \(\{Y^f\}_{f \in F}\), \(\{X^i\}_{i \in H}\), and \(\{u^i(\cdot)\}_{i \in H}\). Suppose all production and consumption sets are convex, all production sets are cones, and all \(u^i(\cdot)\) are continuous and quasi-concave and satisfy local non-satiation. Suppose also that \(0 \in X^i\), that for each \(x, x' \in X^i\) with \(u^i(x) > u^i(x')\) for all \(i \in H\), there exists \(\bar{T}\) such that \(u^i(x[T]) > u^i(x')\) for all \(T \geq \bar{T}\) and for all \(i \in H\), and that for each \(y \in Y^f\), there exists \(\tilde{T}\) such that \(y[T] \in Y^f\) for all \(T \geq \tilde{T}\) and for all \(f \in F\). Then this allocation can be decentralized as a competitive equilibrium.
Second Welfare Theorem II

Theorem

(continued) In particular, there exist $p^{**}$ and $(\omega^{**}, \theta^{**})$ such that

1. $\omega^{**}$ satisfies $\omega = \sum_{i \in \mathcal{H}} \omega^{i**};$

2. for all $f \in \mathcal{F},$

$$p^{**} \cdot y^{f**} \leq p^{**} \cdot y \text{ for all } y \in \mathcal{Y}^f;$$

3. for all $i \in \mathcal{H},$

if $x^i \in X^i$ involves $u^i (x^i) > u^i (x^{i**})$, then $p^{**} \cdot x^i \geq p^{**} \cdot w^{i**},$

where $w^{i**} \equiv \omega^{i**} + \sum_{f \in \mathcal{F}} \theta^{i**}_f y^{f**}.$

Moreover, if $p^{**} \cdot w^{**} > 0$ [i.e., $p^{**} \cdot w^{i**} > 0$ for each $i \in \mathcal{H}$], then economy $\mathcal{E}$ has a competitive equilibrium $(x^{**}, y^{**}, p^{**}).$
Welfare Theorems XII

- Notice:
  - if instead if we had a finite commodity space, say with $K$ commodities, then the hypothesis that $0 \in X^i$ for each $i \in \mathcal{H}$ and $x, x' \in X^i$ with $u^i(x) > u^i(x')$, there exists $\tilde{T}$ such that $u^i(x[T]) > u^i(x'[T])$ for all $T \geq \tilde{T}$ and all $i \in \mathcal{H}$ (and also that there exists $\tilde{T}$ such that if $y \in Y^f$, then $y[T] \in Y^f$ for all $T \geq \tilde{T}$ and all $f \in \mathcal{F}$) would be satisfied automatically, by taking $\tilde{T} = \tilde{T} = K$.
  - Condition not imposed in Second Welfare Theorem in economies with a finite number of commodities.
  - In dynamic economies, its role is changes in allocations at very far in the future should not have a large effect.

- The conditions for the Second Welfare Theorem are more difficult to satisfy than those for the First.
- Also the more important of the two theorems: stronger results that any Pareto optimal allocation can be decentralized.
Welfare Theorems XIII

- Immediate corollary is an existence result: a competitive equilibrium must exist.
- Motivates many to look for the set of Pareto optimal allocations instead of explicitly characterizing competitive equilibria.
- Real power of the Theorem in dynamic macro models comes when we combine it with models that admit a representative household.
- Enables us to characterize the optimal growth allocation that maximizes the utility of the representative household and assert that this will correspond to a competitive equilibrium.
Sequential Trading I

- Standard general equilibrium models assume all commodities are traded at a given point in time—and once and for all.
- When trading same good in different time periods or states of nature, trading once and for all less reasonable.
- In models of economic growth, typically assume trading takes place at different points in time.
- But with complete markets, sequential trading gives the same result as trading at a single point in time.
- *Arrow-Debreu equilibrium* of dynamic general equilibrium model: all households trading at $t = 0$ and purchasing and selling irrevocable claims to commodities indexed by date and state of nature.
- Sequential trading: separate markets at each $t$, households trading labor, capital and consumption goods in each such market.
- With complete markets (and time consistent preferences), both are equivalent.
Sequential Trading II

- **(Basic) Arrow Securities**: means of transferring resources across different dates and different states of nature.
- Households can trade Arrow securities and then use these securities to purchase goods at different dates or after different states of nature.
- Reason why both are equivalent:
  - by definition of competitive equilibrium, households correctly anticipate all the prices and purchase sufficient Arrow securities to cover the expenses that they will incur.
- Instead of buying claims at time $t = 0$ for $x_{i,t'}^h$ units of commodity $i = 1, ..., N$ at date $t'$ at prices $(p_1,t', ..., p_N,t')$, sufficient for household $h$ to have an income of $\sum_{i=1}^{N} p_{i,t'} x_{i,t'}^h$ and know that it can purchase as many units of each commodity as it wishes at time $t'$ at the price vector $(p_1,t', ..., p_N,t')$.
- Details to come later.