The Allocation of Future Business: Dynamic Relational Contracts with Multiple Agents

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Abstract

Consider a principal who motivates several agents using relational contracts. In a repeated game with imperfect private monitoring, we show that future business is allocated among agents to make promised bonuses credible. We calculate when first-best can be attained; for some parameters, no stationary relational contract can achieve first-best, but a non-stationary relational contract can do so by rewarding high output with additional future business. When players are impatient and the first-best unattainable, we consider relational contracts that reveal the true history to each player. An optimal contract in this class entails the eventual deterioration of some relationships into temporary or permanent low effort in order to allocate enough future business to other relationships to support high effort. When first-best cannot be attained, the principal can sometimes improve surplus by concealing information from the agents. We consider communication between agents, expand the model to include learning about agent productivity, and discuss the role of innovation in easing scarcity constraints.

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1 Introduction

1.1 Motivation

When formal contracts are unavailable, costly, or imperfect, individuals must rely on their relationships with one another to sustain cooperation. The resulting relational contracts are pervasive both within and between firms. For example, informal agreements typically complement incomplete formal contracts in the “just-in-time” supply chains used by Toyota, Chrysler, and others. Similarly, managers rely on long-term relationships to motivate employees and divisions. Regardless of the setting, a principal who uses relational contracts must have an incentive to follow through on her promises to the agents. If the parties interact repeatedly, then these promises are made credible by the understanding that if they are ever broken, the relationship sours and surplus is lost. In this way, future business serves as collateral to support current promises such as incentive pay.

When a principal interacts with several agents, how she chooses to allocate business among them critically determines the strength of her relational contracts with each of them. We consider a setting in which only a subset of agents can produce at any given time and the principal asks only one of them to produce. For example, a manager might source an essential input from one of the limited number of suppliers that can produce that input. Agents have limited information because they observe only their own output and pay. By promising additional future business to an agent, the principal increases the total surplus created by that relationship and hence can credibly promise a larger bonus if that agent performs well today. Ideally, the principal would promise enough future business to each agent to motivate him whenever he is called upon. However, the same future business cannot be promised to different agents; therefore, if the parties are sufficiently impatient, the principal must prudently allocate business over time to ensure that she can credibly reward high performance.

We show that the allocation of future business plays a fundamentally asymmetric role in a relational contract. To motivate an agent, the principal needs to reward him with a large

bonus following high output, but these bonuses are credible only if they are accompanied by the promise of additional future business. In contrast, the principal has no reason to withdraw business from a low-performing agent since she can simply give that agent a smaller payment today. As a result, we show that an optimal allocation rule for future business satisfies two intuitive properties. First, it rewards success: an agent who performs well today receives a larger share of future business. Second, it is tolerant of failure: a favored agent remains favored when he is allocated production but performs poorly, losing this status only when another agent is allocated production and performs well. More precisely, we characterize a relational contract that satisfies these two properties and induces first-best effort whenever any equilibrium does. When players are too impatient to attain first-best, the principal may ask one agent to shirk in order to promise sufficient future business to the other agent to motivate him. Restricting attention to relational contracts with two agents that provide incentives regardless of the agents’ beliefs, we show that one optimal allocation rule (among many) permanently favors an agent who performs well today, allowing the other relationship to deteriorate into persistent shirking.

Methodologically, we consider a repeated game with imperfect private monitoring—agents do not observe the output produced by or the payments made to their counterparts—so we cannot rely on standard recursive methods and instead develop alternative tools. This monitoring structure implies both that agents cannot coordinate to jointly punish deviations and that the principal may be able to exploit agents’ limited information about past play to better motivate effort. In our relational contract that attains first-best whenever any equilibrium does, the agents would be willing to work hard even if they learned the true history of past play. In contrast, if first-best is unattainable, we show that the principal typically finds it optimal to conceal information from the agents.

To put our results in context, consider a firm with two divisions: X and Y. Each division might learn of a profitable project in each period, but the headquarters of the company can support only one project at a time. If projects are very profitable, costs are low, or resources to launch projects are abundant, then we will show that the headquarters can prudently allocate resources to motivate both divisions, so that X or Y might be temporarily favored but neither becomes permanently dominant. When resources or profitable projects are scarce,
then headquarters still allocates resources to reward success, but optimally allows one division to become dominant for a long period of time or potentially permanently. The other division continues to operate, but does not work as hard\footnote{Interestingly, Workman (1998) notes that some companies favor research and development at the expense of other divisions.} Consistent with this example, Cyert and March (1963) argue that firms are composed of distinct coalitions that bargain and compete with one another, where payments between coalitions can take the form of “policy commitments” (35) like the allocation of future resources.

The allocation of future business also plays a pivotal role in many other real-world relational contracts. Managers assign tasks and promotions to reward their workers in addition to paying discretionary bonuses\footnote{See Baker, Jensen, and Murphy (1988) on the puzzle that promotions seem to be used as rewards instead of just cash.} Across firm boundaries, companies analogously allocate business among long-term suppliers to reward success. A survey by Krause et al (2000) finds that firms motivate their suppliers with “increased volumes of present business and priority consideration for future business,” while Farlow et al (1996) report that Sun Microsystems regularly divides a project among suppliers based in part on their past performance.

### 1.2 Outline of Paper

We formally state our game in the next section. In each period, the principal asks one of the available agents to privately choose effort that determines the quality of an output, and that output is then observed by both the principal and the chosen agent. No formal contracts are available, so the principal relies solely on relational incentives to induce high effort\footnote{More generally, imperfect formal contracts might also be available, but these typically do not eliminate the need for relational contracts. See Baker, Gibbons, and Murphy (1994) for more on the interaction between formal and relational incentives.} Because agents are able to observe only their own performance and bonuses, they cannot mutually punish a deviation by the principal.

In Section 3 we develop necessary and sufficient conditions for equilibrium and demonstrate that a producing agent is willing to work hard only if he believes that his contribution to total surplus following high output is sufficiently large. The principal’s decision of who should produce in each period determines the future surplus created by each relationship.
Agents decide whether to work hard based on their beliefs about past play, which are influenced by information disclosed by the principal, so private monitoring plays a crucial role in equilibrium.

Using these results, in Section 4 we fully characterize a non-stationary relational contract that attains first-best whenever any Perfect Bayesian Equilibrium does. The allocation rule in this equilibrium rewards high performance with favored status and additional future business, and is tolerant of failure in that the principal does not revoke this favored status until another agent performs well. This relational contract provides \textit{ex post} or \textit{belief-free} incentives in the sense that each agent is willing to work hard even if he learns the true history of the game, so the principal has no need to conceal information from the agents. Calculating when first-best can be attained, we highlight some comparative statics.

Section 5 turns to optimal relational contracts with two agents when first-best is unattainable. In this case, non-stationary allocation rules typically strictly dominate stationary ones. Moreover, the principal sometimes foregoes motivating one agent in order to promise enough business to the other agent to motivate him. Restricting attention to contracts in which the principal provides \textit{ex post} incentives to the agents, we characterize one (non-unique) optimal allocation rule that both rewards success and is tolerant of failure. In this rule, the principal’s relationship with one of the agents eventually becomes “perfunctory,” in the sense that the agent is allocated production infrequently and perpetually shirks. Which relationship breaks down depends on past performance: when one agent performs well, the other relationship is likely to suffer. A similar allocation rule can be optimal when agents have different productivities, leading to persistent dispersion in total surplus among \textit{ex ante} identical triads of a principal and two agents.

The relational contract in Section 5 is simple and provides \textit{ex post} incentives (in the sense described above), but the principal may be able to do better by taking advantage of agents’ limited information about past play. In Section 6, we show that the principal can benefit by concealing information from the agents, but only when first-best is unattainable. Thus,

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\textsuperscript{5}In this context, \textit{ex post} incentives are analogous to belief-free equilibria in the sense of Ely and Valimaki (2002) and Ely, Horner, and Olszewski (2005). In Section 4, we define full-disclosure relational contracts to formalize this notion of ex post incentives. In Appendix B, we define belief-free equilibria for this game and relate them to full-disclosure relational contracts.
While the contract from Section 5 is optimal among those that provide ex post incentives, such strong incentives come at a cost: the principal may be able to do even better by keeping the agents in the dark.

We extend the basic framework to examine some other settings in Section 7. First, we consider the role of the allocation rule when agents can communicate with one another to coordinate punishments. Although communication eases incentive constraints, under certain assumptions the principal’s allocation of future business remains a crucial motivational tool. Second, we show that relational concerns may lead a principal to optimally retain a low-productivity agent rather than replacing him. Finally, we observe that innovation can have a positive externality on the principal’s relationships by creating additional future business that can then be used to strengthen relational contracts. However, the innovating agent may not internalize this externality and so may underinvest in innovation.

1.3 Literature Review

The seminal papers on relational contracts by Bull (1987), MacLeod and Malcomson (1989), and Levin (2003) have spurred a large and growing literature; Malcomson (2012) has an extensive survey. Levin (2002) analyzes multilateral relational contracts, arguing that bilateral relationships may be more robust to misunderstandings between players. In our setting and unlike Levin (2002), bilateral relationships are the natural result of private monitoring and the allocation of business plays a central role.

In a model related to ours, Board (2011) exploits the notion that future business is a scarce resource to limit the number of agents with whom a principal optimally trades. He shows that the principal separates potential trading partners into “insiders” and “outsiders,” trades efficiently with the insiders, and is biased against outsiders. While agents in his model do not observe one another’s choices, the principal perfectly observes actions and so the dynamics are driven by neither moral hazard nor private monitoring, which are the central forces in our paper. In the context of repeated procurement auctions, Calzolari and Spagnolo (2010) similarly argue that an auctioneer might want to limit participation in the auction to bolster relational incentives with the eventual winner, and they discuss the interaction between relational incentives and bidder collusion. Li, Zhang, and Fine (2011) show that a
principal who is restricted to cost-plus contracts may use future business to motivate agents, but the dynamics of their equilibrium are not driven by relational concerns and are quite different from ours. In research related to the present paper, Barron (2012) considers how suppliers optimally invest to maximize the value of their relational contracts and shows that such suppliers might opt to be “generalists” rather than specializing in a narrow range of products.

Further afield, McAdams (2011) considers bilateral relational contracts in a matching market, which is related to our results on employee retention and turnover in Section 7.2. Our discussion of innovation builds upon Klein and Leffler’s (1981) observation that relationship-specific investments can strengthen relationships, and complements Halac’s (2012a) work on innovation in bilateral relational contracts. Unlike these papers, we focus on the effects of an agent’s innovation on the principal’s other relationships. Li (2002), Zhang (2002), and others consider the role of information in supply chains, typically in the context of static models with payoff-relevant private types. In contrast, private information arises endogenously in our setting due to the monitoring structure.

Our model is also related to a burgeoning applied literature involving private-monitoring games. Kandori (2002) provides a nice overview of the theory of such games. Fuchs (2007) considers a bilateral relational contracting problem and shows that efficiency wages are optimal if the principal privately observes the agent’s output. Wolitzky (2012a,b) considers enforcement in games where each agent observes an action either perfectly or not at all. In our model, output—which is an imperfect signal of an agent’s private effort—is similarly observed by some but not all of the players. Ali and Miller (2012) analyze what networks best sustain cooperation in a repeated game, but do not consider the allocation of business among players. Harrington and Skrzypacz (2011) discuss collusion when firms privately observe both prices and quantities, deriving an equilibrium that resembles the actions of real-world cartels.

Omitted proofs are in Appendix A and additional results referenced in the text are in Supplementary Appendices B and C available at [http://economics.mit.edu/grad/dbarron](http://economics.mit.edu/grad/dbarron)
2 Model and Assumptions

2.1 Model

Consider a repeated game with \( N + 1 \) players, denoted \( \{0, 1, ..., N\} \). We call player 0 the principal ("she"), while every other player \( i \in \{1, ..., N\} \) is an agent ("he"). In each round, the principal requires a single good that could be made by any one of a subset of the agents. This good can be thought of as a valuable input to the principal’s production process that only some of the agents have the capacity to make in each period. After observing which agents can produce the good, the principal allocates production to one of them, who chooses whether to accept or reject production and how much effort to exert. Critically, utility is transferable between the principal and each agent but not between two agents. At the very beginning of the game, the principal and each agent can “settle up” by transferring money to one another; these payments are observed only by the two parties involved.

Formally, we consider the infinite repetition \( t = 0, 1, ... \) of the following stage game with common discount factor \( \delta \):

1. A subset of available agents \( \mathcal{P}_t \in 2^{\{1, ..., N\}} \) is publicly drawn with probability \( F(\mathcal{P}_t) \).

2. The principal publicly chooses a single agent \( x_t \in \mathcal{P}_t \cup \{\emptyset\} \) as the producer.

3. \( \forall i \in \{1, ..., N\} \), the principal transfers \( w_{i,t}^A \geq 0 \) to agent \( i \), and agent \( i \) simultaneously transfers \( w_{i,t}^P \geq 0 \) to the principal. \( w_{i,t}^A \) and \( w_{i,t}^P \) are observed only by the principal and agent \( i \). Define \( w_{i,t} = w_{i,t}^A - w_{i,t}^P \) as the net transfer to agent \( i \).

4. Agent \( x_t \) accepts or rejects production, \( d_t \in \{0, 1\} \). \( d_t \) is observed only by the principal and \( x_t \). If \( d_t = 0 \), then \( y_t = 0 \).

5. If \( d_t = 1 \), agent \( x_t \) privately chooses an effort level \( e_t \in \{0, 1\} \) at cost \( c e_t \).

6. Output \( y_t \in \{0, y_H\} \) is realized and observed by agent \( x_t \) and the principal, with

\[
\text{Prob}\{y_t = y_H|e_t\} = p_e, \quad 1 \geq p_1 > p_0 \geq 0.
\]

\(^6\)Unless explicitly noted, \( w_{i,t}^A = 0 \) if \( w_{i,t} > 0 \) and \( w_{i,t}^A = 0 \) if \( w_{i,t} \leq 0 \). This convention holds for every result except Corollary 1.

\(^7\)Results analogous to those in Sections 3 - 6 hold for outside option \( \bar{u} > 0 \) so long as the principal can reject production by agent \( x_i \) in step 4 of the stage game without this rejection being observed by the other agents.
7. \( \forall i \in \{1, ..., N\} \), transfers \( \tau_{i,t}^A, \tau_{i,t}^P \) are made to and from agent \( i \), respectively. \( \tau_{i,t}^A, \tau_{i,t}^P \) are observed only by the principal and agent \( i \), with the net transfer to agent \( i \) denoted \( \tau_{i,t} = \tau_{i,t}^A - \tau_{i,t}^P \).

Let \( 1_{i,t} \) be the indicator function for the event \( \{ x_t = i \} \). Then discounted payoffs are

\[
U_0 = (1 - \delta) \sum_{t=0}^{\infty} \delta^t \left( d_t y_t - \sum_{i=1}^{N} (w_{i,t} + \tau_{i,t}) \right) \equiv \sum_{t=0}^{\infty} \delta^t (1 - \delta) u_0^i
\]

and

\[
U_i = (1 - \delta) \sum_{t=0}^{\infty} \delta^t (w_{i,t} + \tau_{i,t} - d_t c e_{i,t} 1_{i,t}) \equiv \sum_{t=0}^{\infty} \delta^t (1 - \delta) u_i^i
\]

for the principal and agent \( i \), respectively.

This monitoring structure implies that the principal observes every variable except effort, whereas agents do not see any of one another’s choices.\(^8\) Several features of this model allow us to cleanly discuss the role of future business in a relational contract. First, agents cannot communicate with one another. While stark, this assumption implies both that punishments are bilateral and that agents’ information plays an important role in equilibrium.\(^9\) In Section 7.1 we show that under stronger assumptions, some of our results hold when agents can communicate. Second, the wage is paid before the agent accepts or rejects production. One way to interpret \( d_t = 0 \) is as a form of shirking that guarantees low output, rather than explicit rejection.\(^10\) Third, we assume that some agents are unable to produce in each round. Between firms, suppliers might lack the time or appropriate capital to meet their downstream counterpart’s current needs\(^11\) within firms, a division might be unavailable because it has no new project that requires resources from headquarters. Finally, the principal cannot “multisource” by allocating production to several agents in each round. While this restriction

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\(^8\)In particular, agents cannot pay one another. In Appendix B we show that a stationary contract would typically be optimal if they could. Thus, our model may be best-suited for settings in which the agents primarily interact with one another via the principal.

\(^9\)In the absence of private monitoring, we could restrict attention to equilibria with (some definition of) bilateral punishments. It is non-trivial to formally define such a behavioral restriction in this model due to the role of the allocation rule.

\(^10\)Indeed, the results in Sections 3 - 6 hold if the agents accept or reject production before the wage is paid, but can costlessly choose effort “\( e_t = -1 \)” that guarantees \( y_t = 0 \).

\(^11\)For example, Jerez et al (2009) report that one of Infosys’ partners sources a product from the market only if Infosys “does not have the capability” to supply it.
substantially simplifies the analysis, the allocation of business would remain important with multisourcing so long as the principal profits from only one agent’s output in each period.\footnote{If multisourcing is possible, the choice of which agent’s output to use when multiple agents produce \( y_H \) plays a similar role to \( x_t \) in the baseline model.}

2.2 Histories, Strategies, and Continuation Payoffs

Define the set of histories \( h^T \) at the start of round \( T \) as

\[
\mathcal{H}_B^T = \{ P_t, x_t, \{ w_{i,t}^A \}, \{ w_{i,t}^P \}, d_t, e_t, y_t, \{ \tau_{i,t}^A \}, \{ \tau_{i,t}^P \} \}_{t=0}^T
\]

and denote by \( N \) the different nodes of the stage game. A partial history \( (h^{t-1}, n_t) \in \mathcal{H}_B^T \times N \) denotes that \( h^{t-1} \) occurred in the first \( t - 1 \) rounds and \( n_t \) is the node in period \( t \).\footnote{A partial history may also be written in terms of actions: \( (h^{t-1}, P_t, x_t) \) implies history \( h^{t-1} \) followed by \( P_t, x_t \).} The set of all histories is

\[
\mathcal{H} = \{ \emptyset \} \cup \left\{ \bigcup_{T=1}^\infty \{ (h^{T-1}, n_T) \mid h^{T-1} \in \mathcal{H}_B^T, n_T \in N \} \right\}.
\]

For each player \( i \in \{0, 1, ..., N\} \), \( I_i : N \rightarrow N \) describes \( i \)'s information sets in the stage game. Private histories at the start of each round for player \( i \) are \( h^T_i \in \mathcal{H}_{i,B}^T \) and player \( i \)'s private histories are

\[
\mathcal{H}_i = \{ \emptyset \} \cup \left\{ \bigcup_{T=1}^\infty \{ (h^{T-1}_i, I_i(n_T)) \mid h^{T-1}_i \in \mathcal{H}_{i,B}^T, n_T \in N \} \right\}.
\]

Two histories \( h^t, \hat{h}^t \) are indistinguishable by agent \( i \) if \( h^t_i = \hat{h}^t_i \).

Strategies for player \( i \) are denoted \( \sigma_i \in \Sigma_i \) with strategy profile \( \sigma = (\sigma_0, ..., \sigma_N) \in \Sigma = \Sigma_0 \times ... \times \Sigma_N \). Let \( u_i^t \) be player \( i \)'s stage-game payoff in round \( t \).
Definition 1 \( \forall i \in \{0, \ldots, N\} \), player \( i \)'s continuation surplus \( U_i : \Sigma \times \mathcal{H} \to \mathbb{R} \) given strategy \( \sigma \) and history \((h^{t-1}, n_t)\) is

\[
U_i(\sigma, (h^{t-1}, n_t)) = E_\sigma \left[ \sum_{t'=1}^{\infty} (1 - \delta) \delta^{t'-1} u_{i,t'} (h^{t-1}, n_t) \right] \tag{1}
\]

and the payoff to the principal from agent \( i \)'s production, \( U^i_0 : \Sigma \times \mathcal{H} \to \mathbb{R} \), is

\[
U^i_0(\sigma, (h^{t-1}, n_t)) = E_\sigma \left[ \sum_{t'=1}^{\infty} (1 - \delta) \delta^{t'-1} (I\{x_{t+t'} = i\} \delta t_{t+t'} y_{t+t'} - w_{i,t+t'} - \tau_{i,t+t'}) (h^{t-1}, n_t) \right]. \tag{2}
\]

Intuitively, \( U^i_0 \) is the “stake” the principal has in her relationship with agent \( i \): it is the expected output produced by agent \( i \) but not earned by him. Agents do not know the true history, so their beliefs about continuation surplus are expectations conditional on their private history: \( E_\sigma [U_i(\sigma, (h^{t-1}, n_t)) | h^{t-1}_i, \mathcal{I}_i(n_t)] \).

Definition 2 The \( i \)-dyad surplus \( S_i : \Sigma \times \mathcal{H} \to \mathbb{R} \) is the total expected surplus generated by agent \( i \):

\[
S_i(\sigma, (h^{t-1}, n_t)) = U^i_0(\sigma, (h^{t-1}, n_t)) + U_i(\sigma, (h^{t-1}, n_t)). \tag{3}
\]

Dyad surplus plays a critical role in the analysis. Intuitively, \( S_i \) is agent \( i \)'s contribution to total surplus—it is the surplus from those rounds when agent \( i \) is allocated production. We will show that \( S_i \) is a measure of the collateral that is available to support incentive pay for agent \( i \). We typically suppress the dependence of \( S_i, U_i, \) and \( U^i_0 \) on \( \sigma \).

A relational contract is a Perfect Bayesian Equilibrium (PBE) of this game, where the set of PBE payoffs is \( \text{PBE}(\delta) \) for discount factor \( \delta \in [0,1) \)[15]. We focus on the optimal relational contract, which maximizes \( \text{ex ante total} \) surplus: \( \max_{\sigma \in \text{PBE}(\delta)} \sum_{i=0}^{N} v_i \). This is equivalent to maximizing the principal’s surplus because utility is transferable between the principal and

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[15] A Perfect Bayesian Equilibrium is an assessment consisting of both a strategy \( \sigma^* \) and belief system for each player \( \mu_i^* = \{\mu_i^*\}_{i=0}^{N} \). Beliefs for player \( i \) \( \mu_i^* : \mathcal{H} \to \mathcal{H} \) assign a distribution over true histories \((h^{t-1}, n_t)\) at each of \( i \)'s information sets \((h_i^{t-1}, \mathcal{I}_i(n_t))\). Given these beliefs, \( \sigma_i^* \) is a best response at each information set. Bayes Rule is used to update beliefs \( \mu_i^* \) whenever possible. When Bayes Rule does not apply (i.e., the denominator is 0), \( \mu_i^* \) assigns positive weight only to histories that are consistent with \((h_i^{t-1}, \mathcal{I}_i(n_t))\), but is otherwise unconstrained.
each agent but it is not equivalent to maximizing agent $i$'s surplus.

Three assumptions are maintained throughout the paper, unless explicitly stated otherwise.

Assumption 1 Agents $\{1, \ldots, N\}$ are symmetric: $c$, $p_e$, and $y_H$ are identical for each agent, and $\forall P, P' \in 2^{\{1, \ldots, N\}}$, $F(P) = F(P')$ if $|P| = |P'|$.

Assumption 2 Full support: for any non-empty $P \subseteq \{1, \ldots, N\}$, $F(P) > 0$.

Assumption 3 High effort is costly but efficient: $y_H p_1 - c > y_H p_0$, $c > 0$.

3 The Role of Future Business in Equilibrium

In this section, we prove two essential lemmas that form the foundation of our analysis. The first result clarifies the punishments that can be used to deter deviations, and the second identifies the role of $S_i$ in the relational contract.

As a benchmark, Proposition 1 shows that when $y_t$ is contractible, the optimal formal contract generates first-best total surplus $V^{FB} = (1 - F(\emptyset))(y_H p_1 - c)$ regardless of the allocation rule.

Proposition 1 If output $y$ is contractible and the principal offers a take-it-or-leave-it contract after $P_t$ is realized, then $\exists$ a PBE with surplus $V^{FB}$.

Proof:

The following is an equilibrium contract of the stage game: $\tau_i(0) = 0$, $\tau_i(1) = \frac{c}{p_1 - p_0}$, and $w_i = c - p_1 \frac{c}{p_1 - p_0}$. Under this contract, agent $x_t$ accepts and chooses $e = 1$. Any allocation rule that satisfies $x_t \neq \emptyset$ if $P_t \neq \emptyset$ is incentive-compatible and generates $V^{FB}$. ■

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\(^{16}\)Proof sketch: Consider an optimal PBE $\sigma^*$. At the beginning of the game, agent $i$ pays a transfer equal to $U_i(\sigma^*, \emptyset)$ to the principal. The equilibrium proceeds as in $\sigma^*$ if agent $i$ pays this transfer. If agent $i$ does not pay, then in every future period $w_{i,t} = \tau_{i,t} = 0$, agent $i$ chooses $d_t = 0$, and the principal chooses an allocation rule that maximizes her continuation payoff given these actions. These strategies form an equilibrium if the principal believes $i$ choose $e_t = 0$ and agent $i$ believes the principal offers $w_{i,t} = \tau_{i,t} = 0$ in every future period following a deviation.
Since all players are risk-neutral, the principal can use a formal contract to costlessly induce first-best effort. This result stands in stark contrast to the rest of the analysis, in which future business is the critical determinant of a relationship’s strength. In this setting, the incompleteness of formal contracts is necessary for the allocation of future business to play an important role.

Agents’ beliefs about the true history can evolve in complicated ways, so our next goal is to derive intuitive incentive constraints that depend on dyad-specific surplus $S_i$. In Appendix B, we show that any PBE payoff can be attained by an equilibrium in which agents do not condition on their past effort choices. Apart from effort, every variable is observed by the principal and at least one agent; deviations in these variables can be punished as harshly as possible.\footnote{Analogous to Abreu (1988).} Lemma 1 demonstrates that if the principal or agent $i$ reneges on a promised payment, the harshest punishment is the bilateral breakdown of their relationship.

**Lemma 1** Fix equilibrium $\sigma^*$ and an on-path history $(h^t, n_{t+1}) \in \mathcal{H}$. Consider two histories $h^{t+1}, \tilde{h}^{t+1} \in \mathcal{H}_{B}^{t+1}$ that are successors to $(h^t, n_{t+1})$ such that $h^{t+1} \in \text{supp}\{\sigma|h^t, n_t\}$ but $\tilde{h}^{t+1} \notin \text{supp}\{\sigma|h^t, n_t\}$, and suppose that $\tilde{h}_j^{t+1} = h_j^{t+1}$, $\forall j \notin \{0, i\}$. In the continuation game, the payoffs of the principal and agent $i$ satisfy

$$E_{\sigma^*}[U_i(\tilde{h}^{t+1})|\tilde{h}^{t+1}] \geq 0,$$

(4)

$$U_0(\tilde{h}^{t+1}) \geq \max_{\tilde{h}^{t+1} \in \mathcal{H}_i(\tilde{h}^{t+1})} U_0(\tilde{h}^{t+1}) - U^i_0(\tilde{h}^{t+1}).$$

(5)

**Proof:**

Let $\sigma^*$ be an equilibrium. If (4) is not satisfied, then agent $i$ could choose $w_{i,t}^P = \tau_{i,t}^P = 0$, $e_t = 0$, and $d_t = 0$ in each period, earning strictly higher surplus. Contradiction.

Define $\mathcal{H}_i(\tilde{h}^{t+1}) = \{\tilde{h}^{t+1} | \forall j \neq i, \tilde{h}_j^{t+1} = \tilde{h}_j^{t+1}\}$. If (5) is not satisfied, let

$$\tilde{h}^{t+1} \in \arg \max_{h^{t+1} \in \mathcal{H}_i(\tilde{h}^{t+1})} U_0(h^{t+1}) - U^i_0(h^{t+1}).$$

Recursively define a feasible continuation strategy for the principal from $\tilde{h}^{t+1}$. $\forall t' \geq 1$, the
principal plays $\sigma_0^*(\hat{h}^{t+t'})$, with the sole exception that $w_{i,t+t'}^A = \tau_{i,t+t'}^A = 0$. Let $h^{t+t'}$ be the observed history at the end of round $t+t'$. The principal chooses $\hat{h}^{t+t'+1}$ according to the distribution of length $t+t'+1$ histories induced by $\sigma_0^*(\hat{h}^{t+t'})$, conditional on the event $\hat{h}_{j}^{t+t'+1} = h_{j}^{t+t'+1}$ $\forall j \neq i$. This conditional distribution is well-defined because the specified strategy differs from $\sigma_0^*(\hat{h}^{t+t'})$ only in variables that are observed only by the principal and agent $i$, and $\hat{h}_{j}^{t+t'} = h_{j}^{t+t'}$ $\forall j \neq i$ by construction.

Under this strategy, agents $j \neq i$ cannot distinguish $\tilde{h}^{t+t'}$ and $\hat{h}^{t+t'}$ for any $t' \geq 1$, so the principal earns at least $E_{\sigma^*} \left[ \sum_{j \neq i} - (w_{j,t+t'} + \tau_{j,t+t'}) | \hat{h}^{t+t'} \right]$ if $x_{t+t'} = i$ and $E_{\sigma^*} \left[ u_{0}^{t+t'} + w_{i,t+t'} + \tau_{i,t+t'} | \hat{h}^{t+t'} \right]$ if $x_{t+t'} \neq i$. Agent $i$ is awarded production with the same probability as under $\sigma_0^*(\hat{h}^{t+1})$, so the principal’s payment is bounded from below by $U_0(\hat{h}^{t+1}) - U_i(\hat{h}^{t+1})$. ■

Intuitively, the harshest punishment following a deviation that is observed by only agent $i$ and the principal is the termination of trade in the $i^{th}$ dyad. Terminating trade holds agent $i$ at his outside option, which is his min-max continuation payoff. However, the principal remains free to trade with the other agents as the deviation is not publicly observed. In particular, the principal can act as if the true history is any $\hat{h}^t$ that is consistent with the beliefs of agents $j \neq i$. By choosing this $\hat{h}^t$ to maximize her surplus, the principal exploits the other agents’ ignorance of her deviation to ameliorate the punishment imposed on her by agent $i$.

Building on Lemma 1, Lemma 2 proves that agent $i$’s beliefs about dyad-specific surplus $S_i$ determine whether he works hard in equilibrium. High output can be credibly rewarded with a large bonus only if there is enough dyad-specific surplus $S_i$ to serve as collateral for this bonus. As a result, equilibrium conditions are summarized in a set of intuitive incentive constraints that must be satisfied when an agent works hard and produces high output.

Lemma 2 1. Let $\sigma^*$ be a PBE. Then

$$
(1 - \delta) \frac{c_t - p_{10}}{p_{1} - p_{0}} \leq \delta E_{\sigma^*} \left[ S_i(h^{t+1}) \mid (h^{t-1}_i, I_i(n_t)) \right],
\forall i \in \{1, ..., N\}, \forall h^{t-1} \in H, n_t \text{ on the equilibrium path immediately after } y_t
$$

such that $x_t = i$, $e_t = 1$, $y_t = y_H$. (6)
Let $\sigma$ be a strategy that generates total surplus $V$ and satisfies (6). Then $\exists$ a PBE $\sigma^*$ that generates the same total surplus $V$ and joint distribution of $\{P_t, x_t, d_t e_t, y_t\}_{t=1}^T$ as $\sigma$ for all $T$.

**Proof:**

We prove Statement 1 here and defer the proof of Statement 2 to Appendix A.

For all $t, h^{t+1} \in H_B^{t+1}$, define

$$D_i(h^{t+1}) = U_0(h^{t+1}) - \max_{\hat{h}^{t+1} \in \mathcal{H}^t \setminus \{h^{t+1}\}} \left[ U_0(\hat{h}^{t+1}) - U_i^{\hat{h}^{t+1}} \right].$$

(7)

By Lemma 1, the principal is punished by no more than $D_i(h^{t+1})$ if she deviates from a history $h^{t+1}$ to $\tilde{h}^{t+1}$ in round $t$ and it is only observed by agent $i$. Note that $D_i(h^{t+1}) \leq U_i(h^{t+1})$ by definition.

Fix history $(h^{t-1}, P_t, x_t = i, \{w_{i,t}\}, d_t = 1, e_t = 1, y_t) = (h^{t-1}, n_t)$, and let $\tau_i(y_t) = E_{\sigma^*}[\tau_{i,t} | h^{t-1}_i, I_i(n_t)]$, $U_i(y_t) = E_{\sigma^*}[U_i | h^{t-1}_i, I_i(n_t)]$ be agent $i$’s expected transfer and continuation payoff following output $y_t$. On the equilibrium path, agent $i$ chooses $e_t = 1$ only if

$$p_1[(1 - \delta)\tau_i(y_H) + \delta U_i(y_H)] + (1 - p_1)((1 - \delta)\tau_i(0) + \delta U_i(0)) - (1 - \delta)c \geq$$

$$p_0[(1 - \delta)\tau_i(y_H) + \delta U_i(y_H)] + (1 - p_0)((1 - \delta)\tau_i(0) + \delta U_i(0)).$$

(8)

Let $\tilde{h}^t$ be the history reached by choosing $\tau_{i,t} = 0$ rather than the scheduled transfer in round $t$. For bonus $\tau_{i,t}$ to be paid in equilibrium, we must have

$$(1 - \delta)E_\sigma[\tau_{i,t}|h^{t-1}, n_t, y_t] \leq \delta E_\sigma\left[ D_i(\tilde{h}^t)|h^{t-1}, n_t, y_t \right]$$

$$-(1 - \delta)E_\sigma[\tau_{i,t}|h^{t-1}, n_t, y_t] \leq \delta E_\sigma\left[ U_i(h^t) | h^{t-1}_i, I_i(n_t, y_t) \right]$$

(9)

or else the principal or agent $i$ would choose not to pay $\tau_{i,t}$. Plugging these constraints into (8) and applying the definition of $S_i$ results in (6). ■

Define $\tilde{S} = \frac{1 - \delta - c}{\delta p_1 - p_0}$ as the minimum dyad-specific surplus that agent $x_t$ must believe he receives following high effort and output. Statement 1 is analogous to MacLeod and
Malcomson’s (1989, Proposition 1) and Levin’s (2003, Theorem 3) arguments that reneging constraints on transfers (9) can be aggregated across principal and agent. To prove Statement 2, we construct an equilibrium $\sigma^*$ that replicates the allocation rule, accept/reject decisions, and effort choices of the strategy profile $\sigma$. In each period of this equilibrium, wage $w_{x_t,t} = d_t y_{H} p_{e_t} - E [\tau_{x_t,t} | h_t]$ is paid to the producing agent $x_t$. This payment ensures that the principal earns 0 in every period and so is willing to implement any allocation rule, while simultaneously revealing information about equilibrium effort and the true history $h_t$ to agent $x_t$. Following the agent’s accept or reject decision and effort, if $d_t = 0$, or $e_t = 0$, or both $e_t = 1$ and $y_t = y_H$, then $\tau_{x_t,t} = 0$ and no bonus is paid. If the agent is supposed to work hard but produces $y_t = 0$, then he pays his continuation surplus back to the principal, earning 0. So long as the original strategies satisfy (6), these payments communicate exactly enough information to ensure that $x_t$ is willing to both work hard when asked and repay the principal if he produces low output. The principal can effectively monitor the agent because she observes every variable but effort.

By promising production to an agent, the principal influences that agent’s beliefs about $S_i$ and hence whether he can be induced to work hard. Condition (6) makes clear that dyad-specific surplus $S_i$ and hence the principal’s allocation rule matters only when agent $i$ both works hard and produces high output. This asymmetry highlights the role of future business as collateral: intuitively, the agent is not rewarded for $y_t = 0$ and so no collateral is required, but $S_i$ must be large enough to make a substantial reward credible when $y_t = y_H$. Agent $i$’s beliefs about $S_i$—which are typically coarser than the true history—determine whether he is willing to work hard. Hence, the principal exploits both the allocation rule and agents’ expectations regarding the relationship in an optimal relational contract. Moreover, (6) does not depend on transfers, so any allocation rule, accept/reject decisions, and effort choices that satisfy (6) are part of a class of relational contracts with the same total surplus.

The principal has tremendous latitude to share or withhold information from each agent in a relational contract. In fact, Corollary shows that the principal can use transfer

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18 These transfers may seem unusual, since the principal pays a large up-front wage to the producing agent and is repaid if output is low. In the proof, these transfers ensure both that the principal is willing to implement the desired allocation rule, and that any deviations are immediately detected and punished as harshly as possible. Other transfer schemes may also work, but all relational contracts must contend with the fundamental collateral constraint (6).
payments to replicate private, costless messages to each agent.

**Corollary 1** Consider an augmented game in which the principal sends a private, costless message \( m_{i,t} \in \mathbb{R} \) to agent \( i \) at the same time as \( w_{i,t} \). Let \( \sigma \) be a strategy of this game that satisfies (6). Then \( \exists \) a PBE \( \sigma^* \) of the baseline game with same total payoff and joint distribution of \( \{P_t, x_t, d_t e_t, y_t\}_{t=1}^T \) as \( \sigma \) for all \( T \).

**Proof:**

By Lemma 2, it suffices to show that there exist strategies \( \hat{\sigma} \) in the baseline game that satisfy (6). There exists a surjective function from \( \mathbb{R} \) to \( \mathbb{R}^2 \), so every wage-message pair \( (w_{i,t}, m_{i,t}) \in \mathbb{R}^2 \) can be associated with a wage \( \hat{w}_{i,t} \in \mathbb{R} \). Fix such a mapping, and consider the strategy \( \hat{\sigma} \) of the baseline game that is identical to \( \sigma \) except that the wage at history \( h_t \) is the \( \hat{w}_{i,t} \) that maps to \( (w_{i,t}(h^t), m_{i,t}(h^t)) \). This strategy induces the same beliefs, joint distribution of \( \{P_t, x_t, d_t e_t, y_t\}_{t=1}^T \), and total surplus for all \( T \) as \( \sigma \) for every agent at each history. In particular, \( \hat{\sigma} \) satisfies (6) because \( \sigma \) does, so the result follows by Lemma 2.

The principal can influence agents’ expectations and incentives by concealing or revealing information and in particular could reveal the true history to each agent in every period. Together, Lemmas 1 and 2 underscore that allocating future business to agent \( i \) following high effort and output makes it possible to motivate him at the cost of reducing the future business that can be promised to other agents. The optimal allocation rule is shaped by this trade-off.

### 4 An Optimal Allocation Rule if First-Best is Possible

This section considers relational contracts that induce first-best effort. Stationary allocation rules are effective only when players are patient, but we introduce a simple non-stationary contract that attains first-best whenever any PBE does. In particular, this PBE can attain first-best for a strictly larger set of parameters than any stationary allocation rule.

Lemma 2 implies that \( S_t \) determines the bonuses that can be credibly promised to agent \( i \). If \( \delta \) is close to 1, players care tremendously about the future and so this collateral is abundant.
regardless of the allocation rule. Define a stationary allocation rule as one in which actions do not depend on previous rounds along the equilibrium path; then a relational contract with a stationary allocation attains first-best when players are patient.\footnote{Given a stationary allocation rule, the optimal contract within each dyad is very similar to that in Levin (2003), Theorem 6.}

**Proposition 2** There exists a stationary equilibrium with surplus $V^{FB}$ if and only if

\[
(1 - \delta) \frac{c}{p_1 - p_0} \leq \delta \frac{1}{N} V^{FB}. \tag{10}
\]

**Proof:**

Consider the following stationary allocation rule, accept/reject decision, and effort choice: in each round, $\text{Prob}_{x_t = i \mid P_t} = \begin{cases} \frac{1}{|P_t|} & \text{if } i \in P_t, \\ 0 & \text{otherwise} \end{cases}$, while agents choose $d_t = e_t = 1$. Agents are symmetric, so $x_t = i$ with ex ante probability $\frac{1}{N}(1 - F(\emptyset))$. By Lemma 2, these actions are part of an equilibrium that attains first-best if and only if

\[
(1 - \delta) \frac{c}{p_1 - p_0} \leq \delta \frac{1}{N} (1 - F(\emptyset))(y_Hp_1 - c),
\]

which proves that (10) is a sufficient condition to induce first-best with stationary contracts.

To prove that (10) is also necessary, note that for any stationary allocation rule there exists some $i$ such that $\text{Prob}\{x_{Stat} = i \mid P \neq \emptyset\} \leq \frac{1}{N}$ because $\sum_i \text{Prob}\{x_{Stat} = i \mid P \neq \emptyset\} \leq 1$. If (10) does not hold, then $i$ always chooses $e_t = 0$. But $F(\{i\}) > 0$, so first-best cannot be attained. ■

In a stationary relational contract, the allocation rule does not evolve in response to past performance. While stationary relational contracts attain $V^{FB}$ only if players are patient, a non-stationary allocation rule can induce first-best effort even when (10) does not hold. We introduce the **Favored Producer Allocation** and then prove that it induces first-best effort whenever any equilibrium does.

**Definition 3** Let $\phi$ be an arbitrary permutation of $\{1, ..., N\}$ and $\phi^I$ be the identity mapping. The **Favored Producer Allocation** is defined by:
1. Set \( \phi_1 = \phi^I \).

2. In each round \( t \), \( x_t = \arg\min_{i \in \{1, \ldots, N\}} \{ \phi_t(i) \mid i \in P_t \} \).

3. If \( y_t = 0 \): \( \phi_{t+1} = \phi_t \). If \( y_t = y_H \): \( \phi_{t+1}(i) = 1 \) if \( \phi_t(i) < \phi_t(x_t) \), and \( \phi_{t+1}(i) = \phi_t(i) \) otherwise.

4. On the equilibrium path, \( d_t = e_t = 1 \) iff (6) holds, and otherwise \( d_t = 1, e_t = 0 \).

In the Favored Producer Allocation, the principal ranks the agents and awards production to the “most favored” available agent—the \( i \in P_t \) with the lowest rank. The rankings remain unchanged if that agent produces low output, but he immediately becomes the most favored agent if he produces high output. This allocation rule rewards success because an agent who produces \( y_H \) is immediately promised a larger share of future business, and it is tolerant of failure in the sense that a favored agent remains so even if he performs poorly. Once an agent is favored, he loses that favor only when another agent performs well and replaces him.\(^{20}\) In each round, every agent is the sole available producer with positive probability and so every agent has the opportunity to become favored. The resulting dynamics resemble a tournament in which the most recent agent to perform well “wins” favored status.

Proposition 3 proves that a relational contract using the Favored Producer Allocation attains first-best whenever any equilibrium does.

**Proposition 3** Suppose \( V^{FB} \) is attainable in a PBE. Then the Favored Producer Allocation is part of a PBE that generates \( V^{FB} \). Moreover, \( \exists \) a non-empty, open \( \Delta \subseteq [0, 1] \) such that if \( \delta \in \Delta \), the Favored Producer Allocation attains first-best but no stationary equilibrium does.

**Proof:**

See Appendix A

Before discussing the proof, let’s consider why the Favored Producer Allocation rule is (a) tolerant of failure and (b) rewards success. For (a), consider a harsher allocation rule

\(^{20}\)Interestingly, there is some anecdotal evidence that downstream firms tend not to withdraw business from a poorly-performing supplier. For instance, Kulp and Narayanan (2004) report that one supplier “thought it unlikely that Metalcraft would pull business if a given supplier’s score dropped below acceptable levels.”
that withdraws business from agent $i$ whenever he produces $y_t = 0$. Typically, $i$ is favored because he produced high output at some point in the past. Thus, this harsher allocation rule would tighten (6) when agent $i$ produced $y_H$. At the same time, the harsher allocation rule would not relax this constraint for the other agents to the same extent because (6) depends only on histories following high output. An optimal allocation rule is tolerant of failure precisely because the promise of future business does not simply serve as a direct incentive for effort, but determines what incentive payments are credible. A favored agent works hard because he earns a lower transfer immediately following low output, but the allocation rule determines the strength of these incentives.

To show (b), compare the Favored Producer Allocation to a stationary relational contract. Fixing the total amount of continuation surplus at $V_{FB}$, the allocation rule determines what fraction of $V_{FB}$ is produced by each agent at a history. In a (symmetric) stationary equilibrium, an agent who performs well has $S_i = \frac{1}{N}V_{FB}$. In contrast, an agent who produces $y_H$ in the Favored Producer Allocation has $S_i > \frac{1}{N}V_{FB}$, since this agent is more likely to produce in each future period and so is likely to retain his favor.

The Favored Producer Allocation is simple enough that we can explicitly calculate when first-best can be attained. For two agents, Figure 1 plots this region as a function of the discount factor and the probability that both agents are available (agents are symmetric, so $F(\{1,2\})$ and $F(\emptyset) = 0$ are enough to pin down the entire distribution). The allocation of business plays a more important role in the relational contract and first best is easier to attain when the probability that both agents are available $F(\{1,2\})$ is large.\footnote{At $F(\{1,2\}) = 1$ (which is ruled out by Assumption 2), the Favored Producer Allocation collapses to a stationary equilibrium and so stationary equilibria are optimal.}

This comparative static has important implications for real-world relationships: for example, frequent (independent) disruptions in a supply chain decrease both total surplus and the probability that both agents are available, so a fortiori make it difficult to motivate effort.

We now turn to the proof of Proposition 3. Because PBE strategies depend on private histories—which grow increasingly complex over time—this game is not amenable to a recursive analysis. Instead, we develop a proof that uses the relatively easy-to-compute beliefs at the beginning of the game. Using this technique, we derive necessary conditions for first-best
Figure 1: Optimal equilibria for $y_H = 10$, $p_1 = 0.9$, $p_0 = 0$, $c = 5$, $F(\emptyset) = 0$. (A) a stationary equilibrium and the Favored Producer Allocation both attain first-best; (B) the Favored Producer Allocation attains first-best (but stationary equilibria do not); (C) first-best cannot be attained.

to be attained in a PBE and show that the Favored Producer Allocation attains first-best whenever these conditions hold.

By Lemma 2, agent $x_t$ must believe his dyad-specific surplus is at least $\tilde{S}$ whenever he chooses $e_t = 1$ and produces $y_t = y_H$. We define the expected obligation (denoted $\Omega^i_t$) owed to agent $i$ at time $t$ as the total expected dyad-specific surplus that must be promised to him to motivate first-best effort through period $t$.

**Definition 4** Given strategies $\sigma$, define

$$
\beta^L_{i,t} = \text{Prob}_{\sigma} \{ \exists t' < t \text{ s.t. } x_{t'} = i, y_{t'} = y_H \cap \{x_t = i\} \},
$$

$$
\beta^H_{i,t} = \text{Prob}_{\sigma} \{ \exists t' < t \text{ s.t. } x_{t'} = i, y_{t'} = y_H \cap \{x_t = i\} \}.
$$

Then the expected obligation owed to agent $i$ at time $t$, $\Omega^i_t$, is

$$
\Omega^i_t \equiv \frac{\Omega^{i-1}_t}{\delta} + \beta^L_{i,t}p_1\delta\tilde{S} - \beta^H_{i,t}(1 - \delta)(y_H p_1 - c) \tag{11}
$$
with initial condition \( \Omega_0^i \equiv 0 \).

\( \Omega_t^i \) is a state variable that tracks how much expected surplus is “owed” to \( i \) in round \( t \) of an equilibrium \( \sigma \) that induces first-best effort. Relaxing the true credibility constraint (6), suppose that an agent’s expected dyad-specific surplus must exceed \( \tilde{S} \) only the first time that he produces \( y_H \). At \( t = 1 \), agent \( i \) is allocated production with probability \( \beta_{i,1}^L \), produces \( y_H \) with probability \( p_1 \), and must be promised \( \delta \tilde{S} \) future surplus in expectation following \( y_H \) to satisfy (6). Therefore, \( \Omega_1^i \equiv \beta_{i,1}^L p_1 \delta \tilde{S} \) equals the expected continuation surplus that must be promised to agent \( i \) following the first period. At \( t = 2 \), agent \( i \) is still owed this initial amount—now worth \( \Omega_1^i / \delta \) due to discounting—and accumulates additional obligation \( \beta_{i,2}^L p_1 \delta \tilde{S} \) if he has not yet produced \( y_H \). If \( i \) already produced \( y_H \) and is allocated production at \( t = 2 \)—which occurs with probability \( \beta_{i,2}^H \)—then his obligation can be “paid off” at a rate equal to the expected surplus from production, \( (1 - \delta)(y_H p_1 - c) \). Putting these pieces together, the expected obligation promised to agent \( i \) in round 2 is

\[
\frac{\Omega_1^i}{\delta} + \beta_{i,2}^L p_1 \delta \tilde{S} - \beta_{i,2}^H (1 - \delta)(y_H p_1 - c),
\]

which equals \( \Omega_2^i \). A similar intuition applies for each \( t \).

Lemma 3 shows that if \( \Omega_t^i \) grows unboundedly as \( t \to \infty \), the corresponding strategy \( \sigma \) cannot be an equilibrium that attains first-best.

**Lemma 3** Consider strategies \( \sigma \) that attain the first-best total surplus \( V^{FB} \), and suppose \( \exists i \in \{1, ..., N\} \) such that

\[
\limsup_{t \to \infty} \Omega_t^i = \infty.
\]

Then \( \sigma \) is not an equilibrium.

**Proof:**

See Appendix A

If \( \limsup_{t \to \infty} \Omega_t^i \) is infinite, then the promises made to agent \( i \) exceed the total amount of surplus created in the game. Agent \( i \) can calculate \( \Omega_t^i \), so if \( \limsup_{t \to \infty} \Omega_t^i = \infty \) he knows that he will not always receive \( \tilde{S} \) expected surplus when he produces \( y_H \). While concealing
information from agent $i$ might alter the incentive constraint $[6]$, it cannot \textit{systematically} trick agent $i$ into expecting more surplus than is feasible.

To find conditions under which $\limsup_{t \to \infty} \Omega^i_t = \infty$, consider the problem $\min (\beta^H_{i,t}, \beta^L_{i,t}) : t' \leq t \Omega^i_t$, where the variables $\beta^H_{i,t}$ and $\beta^L_{i,t}$ are functions of the allocation rule. Lemma 4 solves this problem, demonstrating that it is without loss to assume that $\Omega^i_t = \Omega^j_t \forall i, j \in \{1, ..., N\}$ and that those agents who have already produced $y_H$ are granted production whenever possible.

**Lemma 4** Suppose $\exists$ strategies $\sigma$ that attain the first-best total surplus $V^{FB}$ such that $\forall i \in \{1, ..., N\}$, $\limsup_{t \to \infty} \Omega^i_t < \infty$. $\forall h^t$, let

$$E(h^t) = \{i \mid h^t, \exists t' < t \text{ s.t. } x_{t'} = i, y_{t'} = y_H\}$$

and $\forall t$,

$$E^i_t = \{h^t | i \in E(h^t)\}.$$

Then $\exists$ strategies $\hat{\sigma}$ that also attain $V^{FB}$ such that:

1. \textit{Obligation is finite:} $\forall i \in \{1, ..., N\}$, $\limsup_{t \to \infty} \hat{\Omega}^i_t < \infty$.

2. \textit{Ex ante, the allocation rule treats agents who have produced $y_H$ symmetrically:} $\forall t, i \in \{1, ..., N\}$

$$\text{Prob}\{x_t = i | E^i_t\} = \frac{1}{N} \sum_{j=1}^{N} \text{Prob}\{x_t = j | E^j_t\}$$

$$\text{Prob}\{x_t = i | (E^i_t)^C\} = \frac{1}{N} \sum_{j=1}^{N} \text{Prob}\{x_t = j | (E^j_t)^C\}.$$

3. \textit{Agents who have produced $y_H$ are favored:} $\forall t, h^t$, $\text{Prob}\{x_t \in E(h^t) | E(h^t) \cap P_t \neq \emptyset\} = 1$.

**Proof:**

See Appendix $A$.

$E(h^t)$ is the subset of players who have already produced $y_H$ in history $h^t$. Agents are \textit{ex ante} symmetric, so any equilibrium that attains first-best continues to do so if agents’ identities are randomized at the start of the game. Therefore, we need only consider strategies in which all $i$ who have already produced $y_H$ are treated symmetrically \textit{in expectation}, and similarly with all agents who have not produced $y_H$. Moreover, $i \in E(h^t)$ should be awarded
future production whenever possible, since doing so increases the rate at which \( i \)'s obligation is repaid while decreasing the obligation incurred by \( j \notin E(h^t) \).

Lemma 4 completely determines the obligation-minimizing \( \{\beta_{i,t}^H\} \) and \( \{\beta_{i,t}^L\} \), so we can characterize the parameter values at which \( \limsup_{t \to \infty} \Omega_{it}^H = \infty \) for all strategies. It turns out that minimized expected obligation diverges to infinity—and hence first-best is infeasible by Lemma 3—exactly when the Favored Producer Allocation does not attain first-best. This proves Proposition 3.

The Favored Producer Allocation gives an agent who has produced \( y_H \) the maximal fraction of future surplus, subject to the constraints that \( e_t = 1 \) in every period and (6) is satisfied. Given that agents cannot be tricked into believing that total surplus is larger than \( V_{FB} \), this allocation minimizes the probability that non-favored agents are awarded production, which maximizes the likelihood that a favored agent remains favored. Even when a favored agent is replaced, his punishment is mild because he retains priority whenever more-favored agents are unable to produce.

In one important sense, agents’ private information does not play a role in the Favored Producer Allocation: (6) is satisfied at the true history \( h^t \) on the equilibrium path, so each agent would be willing to follow his strategy even if he learned the true history. We refer to any equilibrium with this property as a full-disclosure equilibrium. Formally, full-disclosure equilibria provide ex post incentives to each player and so are belief-free.\(^{22}\) In Appendix B we show that a strategy profile generates the same total surplus as a belief-free equilibrium if and only if (6) holds at the true history, so we define full-disclosure equilibrium in terms of this condition.

**Definition 5** A PBE \( \sigma^* \) is a full-disclosure equilibrium (FDE) if \( \forall i \in \{1, ..., N\} \),

\[
(1 - \delta) \frac{c}{p_1 - p_0} \leq E[S_i(h^t)|h^{t-1}, n_t] \\
\forall i \in \{1, ..., N\}, \forall h^{t-1} \in \mathcal{H}, n_t \text{ on the equilibrium path immediately after } y_t
\]

such that \( x_t = i, e_t = 1, y_t = y_H \). \( \tag{12} \)

If (12) does not hold, then \( \sigma^* \) conceals information from the agents. The set of full disclosure

\(^{22}\)As introduced by Ely and Valimaki (2002) and Ely, Horner, and Olszewski (2005). We use a definition of belief-free equilibrium given in Appendix B
equilibrium payoffs is \( FD(\delta) \subseteq PBE(\delta) \subseteq \mathbb{R}^{N+1} \).

The sole difference between (12) and (6) is that (12) conditions on the true history rather than agent \( i \)'s (coarser) information set. Because full-disclosure relational contracts provide ex post incentives for effort, each agent is willing to follow the equilibrium even if he learns additional information about the true history. These equilibria are unaffected if the agents were to learn more information about past play in some (unmodeled) way, for instance by reading trade journals or the newspaper. Proposition 3 implies that if first-best is attainable, then there exists a PBE that does so and provides ex post incentives.

**Corollary 2** Suppose \( \exists \) a PBE \( \sigma^* \) that generates surplus \( V^{FB} \). Then \( \exists \) a full-disclosure equilibrium \( \sigma^{FD} \) that generates \( V^{FB} \).

**Proof:**

Condition (12) holds by direct computation in the Favored Producer Allocation. ■

In this section, we have explicitly characterized a relational contract that attains first-best whenever any Perfect Bayesian Equilibrium does. This allocation rule rewards success and resembles a tournament: agents compete for a temporary favored status that lasts until another agent produces high output. It is also tolerant of failure: rankings are unaffected when an agent produces low output. In this way, the principal ensures that she can credibly promise a large relational bonus to an agent that performs well.

5 Unsustainable Networks and Relationship Breakdown

We now turn to the case with two agents and consider relational contracts if first-best is unattainable. In this case, non-stationary relational contracts typically dominate stationary contracts, but the principal cannot always induce every agent to work hard. In particular, the principal might forego providing effective incentives to one agent to ensure that another agent is promised enough future business to motivate him. In the class of full-disclosure relational contracts with two agents, a variant of the Favored Producer Allocation turns out to be (non-uniquely) optimal: if an agent produces high output, he sometimes enters an
exclusive relationship in which the principal permanently favors him with future business. Once this occurs, the other agent shirks.

So long as first-best is unattainable but some cooperation is possible, Proposition 4 proves that every optimal relational contract tailors the allocation of business to past performance.

**Proposition 4** Suppose $V^{FB}$ cannot be attained in a PBE and $\exists V \in PBE(\delta)$ with $\sum_{i=0}^{N} V_i > (1 - F(\emptyset))y_{HP0}$. Then for any stationary equilibrium $\sigma^{Stat}$, there exists an equilibrium $\sigma^*$ that generates strictly higher surplus.

**Proof:**

Let $x_{Stat}$ be an optimal stationary allocation rule. Because $V^{FB}$ cannot be attained and $\sum_{i=0}^{N} V_i > (1 - F(\emptyset))y_{HP0}$, it must be that $\tilde{S} \leq \sum_{P|\epsilon \in P} F(P)(y_{HP1} - c)$, and so $\tilde{S} \leq (y_{HP1} - c)\text{Prob}\{x_{Stat} = i\}$ holds for a subset $M_{Stat}$ of $0 < m < N$ agents. Only this subset chooses $\epsilon = 1$ in equilibrium. Consider the non-stationary equilibrium that chooses a set of $m$ agents $M(P_1)$ so that $M(P_1) \cap P_1 \neq \emptyset$, then allocates production to the agents in $M(P_1)$ as in $M_{Stat}$. For $t > 1$, this non-stationary equilibrium generates the same surplus as the stationary equilibrium; for $t = 1$, it generates strictly higher surplus, since $\text{Prob}\{P_1 \cap M_{Stat} = \emptyset\} > 0$ by assumption 2. ■

How might agent 1 be promised $\tilde{S}$ dyad-specific surplus? Whenever agent 2 chooses $\epsilon = 1$ and produces $y_{H}$, he must be awarded enough future production to satisfy (6). As a result, 1 earns lower expected dyad-specific surplus when 2 chooses $\epsilon = 1$, and so one way to increase 1’s continuation payoff is to ask 2 to choose $\epsilon = 0$ in some histories. For this reason, shirking occurs on the equilibrium path if first-best cannot be attained.

For the rest of the section, we restrict attention to full-disclosure relational contracts from Definition 5. This is a substantial restriction, but it has the advantage that these relational contracts do not depend on subtle features of beliefs. For instance, an agent’s expectations about his future prospects might change if he were to learn unmodeled information about past play. As we have noted, a full-disclosure relational contract is robust to these signals, which might be difficult to rule out in a practical situation.\textsuperscript{23}

\textsuperscript{23}In the context of a collusion model with adverse selection rather than moral hazard, Miller (2012) argues that ex post incentives are natural for this reason.
It turns out that among full-disclosure relational contracts, a simple variant of the Favored Producer Allocation is non-uniquely optimal. In this relational contract, the principal’s relationship with an agent might eventually become perfunctory: while both agents work hard at the beginning of the game, as \( t \to \infty \) it is almost surely the case that one of the agents chooses \( e_t = 0 \) whenever he produces. In other words, the principal sacrifices one relationship in order to provide adequate incentives in the other. The principal continues to rely on this perfunctory relationship when no better alternatives exist because \( y_H p_0 \geq 0 \), but she offers no incentive pay and has low expectations about output.

**Definition 6** Let \( N = 2 \). The \((q_1, q_2)\)-Exclusive Dealing allocation rule is defined by:

1. **Begin the game in state** \( G_1 \). In state \( G_i \), \( \text{Prob}_\sigma \{ x_t = i | i \in P_t \} = 1 \), \( \text{Prob}_\sigma \{ x_t = -i | P_t = \{-i\} \} = 1 \), and both agents choose \( e_t = 1 \). If \( y_t = y_H \), transition to \( ED_{x_t} \) with probability \( q_{x_t} \geq 0 \), otherwise transition to state \( G_{x_t} \). If \( y_t = 0 \), stay in \( G_i \).

2. **In state** \( ED_i \), \( \text{Prob}_\sigma \{ x_t = i | i \in P_t \} = 1 \) and \( \text{Prob}_\sigma \{ x_t = -i | P_t = \{-i\} \} = 1 \). If \( x_t = i \), \( e_t = 1 \); otherwise, \( e_t = 0 \). Once in \( ED_i \), remain in \( ED_i \).

We refer to continuation play in state \( ED_i \) as exclusive dealing.

In \((q_1, q_2)\)-Exclusive Dealing, each agent faces the possibility that his relationship with the principal breaks down at some time in the future. Before breakdown occurs, the allocation rule is the same as in the Favored Producer Allocation and both agents are expected to choose \( e = 1 \). Once agent \( i \) enters exclusive dealing—which happens with probability \( q_i \) whenever \( i \) produces \( y_H \)—agent \(-i\) stops exerting effort and his relationship with the principal becomes perfunctory. Like the Favored Producer Allocation, \((q_1, q_2)\)-Exclusive Dealing rewards success and is tolerant of failure: high output is rewarded by both favored status and the possibility of a permanent relationship, while low output does not change the allocation rule but leaves the door open for the other agent to win exclusive dealing.

Proposition 5 shows that \((q^*, q^*)\)-Exclusive Dealing is optimal among full-disclosure equilibria for appropriate \( q^* \in [0, 1] \). A critical caveat is that it is not uniquely optimal: there exist other allocation rules that work, including many that do not involve any permanent
Proposition 5 Let \( N = 2 \). \( \exists q^* \in [0, 1] \) such that the \((q^*, q^*)\)-Exclusive Dealing equilibrium is an optimal full-disclosure equilibrium.

Proof:

See Appendix A.

While the proof is lengthy, the intuition for this result is straightforward. Consider the continuation game immediately after some history \( h_t \) in which agent 1 chooses \( e_t = 1 \) and produces \( y_t = y_H \). Total surplus is increasing in the probability that agent 2 works hard. However, if agent 2 works hard and produces high output, then he must be promised at least \( \bar{S} \) dyad-specific surplus, which makes it harder to satisfy agent 1’s constraint \( (6) \). It turns out that both the benefit of agent 2 working hard in round \( t' > t \) and the cost of providing incentives to him are discounted by \( \delta^{t'-t} \) and so scale at the same rate over time. Therefore, any relational contract that maximizes the sum of discounted probabilities that agent 2 works hard, subject to satisfying agent 1’s incentive constraint, is optimal. In particular, there exists a \( q^* \in [0, 1] \) such that \((q^*, q^*)\)-Exclusive Dealing solves this problem.

While exclusive dealing is certainly not uniquely optimal, it is interesting that an optimal relational contract might entail the break-down of one relationship, not as a punishment, but in order to credibly promise a large bonus to the other agent. Figure 2 illustrates the implications of Propositions 3, 4, and 5 for two agents.

Agents are ex ante identical in the baseline model, so the identity of the agent whose relationship sours does not affect total continuation surplus. If agent 1 is instead more productive than agent 2—so that high output for agent 1 is \( y_H + \Delta y > y_H \)—then which relationship breaks down influences long-run profitability. In Appendix C we show that the proof of Proposition 5 extends to this asymmetric case for some parameters, implying that

\[ ^{24} \] For example, an allocation rule that grants temporary exclusive dealing to a high performer for \( K \) periods immediately following \( y_H \) is also optimal (subject to integer constraints on \( K \)).

\[ ^{25} \] As in Proposition 1 of Board (2012), this result can be interpreted as saying it is optimal for the principal to separate agents into “insiders” and “outsiders” and be biased against the “outsiders.” Unlike Board, we focus on a moral hazard problem, which implies that the allocation rule is tolerant of failure and looks like a tournament between the agents.
Figure 2: Optimal equilibria for $y_H = 10$, $p_1 = 0.9$, $p_0 = 0$, $c = 5$, $F(\emptyset) = 0$. (A) the Favored Producer Allocation can attain first-best; (B) $(q^*, q^*)$-Exclusive Dealing is an optimal FDE, and non-stationary equilibria strictly dominate stationary equilibria; (C) no effort can be supported in equilibrium.

$(q_1, q_2)$-Exclusive dealing is optimal (albeit with $q_1 \neq q_2$). When $\Delta y$ is not too large, the principal might enter an exclusive relationship with either agent and so ex ante identical principal-agent networks exhibit persistent differences in long-run productivity. In a market with many identical principal-agent groups, some principals would seem to have strong relationships with their most productive agents while others would be stuck in perfunctory transactions. This equilibrium is not uniquely optimal, but it shows that persistent productivity differences are not incompatible with optimal relationships.

6 When is it Optimal to Conceal Information?

Proposition 3 shows that concealing information is not necessary when first-best can be achieved, and Proposition 5 shows that Exclusive Dealing is optimal among full-disclosure equilibria otherwise. In this section, we prove that providing full-disclosure incentives typi-

\[26\] Persistent performance differences among seemingly similar companies are discussed in Gibbons and Henderson (2012).
cally comes at a cost when first-best is unattainable: the principal might do even better by not providing ex post incentives and keeping agents in the dark\textsuperscript{27}

Full-disclosure relational contracts have the great advantage of being simple and providing strong incentives for effort. In contrast, the relational contract we construct to show that concealing information can be optimal is quite complicated and relies on relatively subtle features of the agents’ beliefs. As a result, a full-disclosure relational contract may be easier to implement in practice even if the principal could theoretically earn a higher surplus by concealing information. Nevertheless, some firms appear to be secretive about their relational scorecards—Farlow et al (1996) report that Sun Microsystems used to withhold the details of its relational scorecards from suppliers\textsuperscript{28}

Proposition\textsuperscript{6} proves that concealing information is optimal whenever first-best is unattainable but agents can be strictly motivated to exert effort.

**Proposition 6** Let \( N = 2 \) and suppose that first-best cannot be attained in a PBE but that

\[
\frac{c}{p_1 - p_0} < \frac{\delta}{1 - \delta} (F(\{1\}) + F(\{1, 2\}))(y_{HP_1} - c). \tag{13}
\]

Let \( \sigma^* \) be a full-disclosure equilibrium; then is it not an optimal PBE.

**Proof:**

See Appendix A

In the proof of Proposition\textsuperscript{6} we construct a modified version of \((q^*, q^*)\)-Exclusive Dealing that conceals information in \( t = 2 \). Intuitively, concealing information relaxes the full-disclosure incentive constraint \textsuperscript{12} so that it need hold only in expectation across histories. Continuation play following the event \( \{e_t = 1, y_t = y_H\} \) is inefficient if first-best is unattainable, so relaxing \textsuperscript{12} allows for more efficient continuation play. We illustrate this intuition in Figure 3, where the shape of the payoff frontier follows from Proposition\textsuperscript{5}.

\textsuperscript{27} Using the definition of belief-free equilibria given in Appendix B, the equilibrium used to prove this result is weakly belief-free in a sense analogous to Kandori (2011), but not belief-free.

\textsuperscript{28} Sun did eventually reveal the details of these scorecards, but only so that their suppliers could adopt the same scorecard to manage their own (second-tier) relationships.
Figure 3: Set of full-disclosure dyad-specific surpluses when $V^{FB} \notin PBE(\delta)$

Once agent $i$ produces high output in a full-disclosure relational contract, continuation play must be sub-optimal in order to give him $\tilde{S}$ surplus. In an optimal FDE, consider all histories such that $x_1 = 1$, $x_2 = 2$, and $e_1 = e_2 = 1$, and let $S^{FD}(y_1, y_2) \in FD(\delta)$ be the expected continuation surplus following outputs $y_1$, $y_2$. In the optimal full-disclosure equilibrium, $S^{FD}_2(y_1, y_H) \geq \tilde{S}, \forall y_1 \in \{0, y_H\}$ to induce $e_2 = 1$ as illustrated in Figure 3.

Now, consider the alternative strategy profile where agent 2 is informed of $y_1$ only after he chooses $e_2 = 1$. Let $S(y_1, y_2) \in FD(\delta)$ be the continuation surplus following round 2 in this alternative strategy profile. For $e_2 = 1$ to be incentive-compatible, we require only that

$$\phi S_2(y_H, y_H) + (1 - \phi)S_2(0, y_H) \geq \tilde{S},$$

where $\phi \in [0, 1]$ is the probability agent 1 assigns to $y_1 = y_H$ at the time he chooses $e_2 = 1$.

In round 1, agent 1’s constraint (6) includes only histories following $y_1 = y_H$, so it is a function of $S_1(y_H, y_H)$ but not $S_1(0, y_H)$. So long as $\phi \in (0, 1)$ and $\tilde{S}$ is on the interior of the line segment shown in Figure 3, we can choose $S_2(y_H, y_H) < \tilde{S}$ and $S_2(0, y_H) > \tilde{S}$ as in Figure 3 so that $\phi S(y_H, y_H) + (1 - \phi)S(0, y_H) = \tilde{S}$ and $S_1(y_H, y_H) > S^{FD}_1(y_H, y_H)$. Player 1’s constraint (6) is then slack under $S$ because it was satisfied under $S^{FD}$, and this slack
can be used to choose more efficient continuation payoffs when $y_l = y_H$.

Proposition 6 is related to the long literature on correlated equilibria such as Aumann (1974) and Myerson (1986), as well as to recent results by Rayo and Segal (2010), Kamenica and Gentzkow (2011), and Fong and Li (2010). These latter papers consider how one player can manipulate the signal observed by a second player to induce him to take a desired action. Similarly, Proposition 6 demonstrates that concealing information can be used to create slack in a player’s incentive constraints, which can then be used to induce high effort more frequently. Unlike Rayo and Segal or Kamenica and Gentzkow, our model is dynamic and the principal conceals information about the true history rather than a payoff-relevant state. Unlike Fong and Li, this information is about other agents.

7 Extensions

Finally, we extend the intuition of the baseline model to analyze several applications. Section 7.1 considers communication between agents and shows that the intuition underpinning Proposition 3 remains qualitatively valid in this case provided that agents earn a positive share of the profits they produce. Section 7.2 discusses how relational contracts might inhibit the principal from replacing inefficient agents. Section 7.3 argues that innovation can increase the stock of future business and so facilitate relational contracting, but also generates externalities that are not internalized by the innovator.

7.1 Communication

In this subsection, we show that the allocation of future business remains an important tool even if agents can communicate, so long as they also earn rents. The baseline model makes the stark assumption that agents cannot send messages to one another and so are unable to multilaterally punish deviations. To consider how joint punishment affects our results, we define an augmented game with communication between agents.

\footnote{For multilateral punishment to be feasible, the principal must be unable to stop the agents from communicating a deviation.}
**Definition 7** The augmented game with communication is identical to the baseline repeated game, except that each player simultaneously chooses a publicly-observed message $m_i \in M$ at the beginning of each round, where $|M|$ is large but finite.

Using the message space $M$, agents can share information with one another and coordinate to punish the principal. In this augmented game, the allocation rule would play no role in equilibrium if the principal were to earn all of the surplus in each period and $\tilde{S} \leq V^{FB}$. On the other hand, the allocation rule is an important motivational tool if agents keep some of the profits they produce, a notion formalized in Assumption 4.

**Assumption 4** Fix $\gamma \in [0, 1]$. An equilibrium satisfies $\gamma$-rent-seeking if at any history $(h^{t-1}, P_t, x_t)$ on the equilibrium path, agent $x_t$ earns

$$u^t_{x_t} = \gamma \sum_{i=0}^{N} u^t_i,$$

the principal earns $u^t_0 = (1 - \gamma) \sum_{i=0}^{N} u^t_i$, and all $i \notin \{0, x_t\}$ earn $u^t_i = 0$.

Assumption 4 is similar to an assumption made in Halac (2012b) and can be viewed as a reduced-form model of bargaining power. Agents do not pay bonuses to one another, so agent $i$’s rents can only be used as collateral in his relationship. Lemma 5 shows that agent $i$ is only willing to work hard if the sum of his and the principal’s surpluses exceeds $\tilde{S}$.

**Lemma 5** The following condition holds in any PBE $\sigma^*$:

$$(1 - \delta) \frac{c_{p_1 - p_0}}{c_{p_1 - p_0}} \leq \delta E_{\sigma^*} \left[ U_0(h^{t+1}) + U_i(h^{t+1}) \mid (h_i^{t-1}, I_i(n_t)) \right]$$

$\forall i \in \{1, ..., N\}, \forall h^{t-1} \in H, n_t$ on the equilibrium path immediately after $y_t$ such that $x_t = i, e_t = 1, y_t = y_H$. (14)

**Proof:**

This proof is similar to that of Statement 1 of Lemma 2 except that transfers must instead satisfy the conditions

$$\tau_i(y_t) \leq \frac{\delta}{1-\delta} E_{\sigma} \left[ U_0(h^{t+1}) \mid h_i^{t-1}, n_t, y_t \right]$$

$$-\tau_i(y_t) \leq \frac{\delta}{1-\delta} E_{\sigma} \left[ U_i(h^{t+1}) \mid h_i^{t}, I_i(n_t, y_t) \right]$$
where $n_t$ is a node following $e_t = 1$ and output $y_t$. If these conditions are not satisfied, then either the principal or agent $i$ would strictly prefer to deviate to $\tau_{i,t} = 0$ and be min-maxed. Plugging these expressions into (8) yields (14). ■

Condition (14) is similar to (6), except that the right-hand side includes the principal’s total expected continuation surplus rather than just her surplus from dyad $i$. The agents can coordinate to hold the principal at her outside option following a deviation, so her entire continuation surplus can be used to support incentive pay in each period. However, $U_0(h^{t+1}) + U_i(h^{t+1})$ does not include the continuation surplus for the other agents. As in the original game, promising future business to agent $i$ increases $U_i(h^{t+1})$ and relaxes (14) for $i$ while tightening this constraint for the other agents. Unlike Lemma 2, the principal may have an incentive to deviate from an optimal allocation rule, so (14) is only a necessary condition for equilibrium. Nevertheless, Lemma 6 shows that a version of the Favored Producer Allocation continues to be an equilibrium.

**Lemma 6** Let Assumption 4 hold. If (14) is satisfied under the Favored Producer Allocation for $d_t = e_t = 1$, $\forall t$, then there exists an equilibrium that uses the Favored Producer Allocation and generates $V^{FB}$ total surplus.

**Proof:**

See Appendix A.

In the proof of Lemma 6, every player simultaneously reports every variable except $e_t$ in each period. All of these variables are observed by at least two players, so any lie is immediately detected and punished by complete market breakdown. These messages effectively make the monitoring structure public; the Favored Producer Allocation remains a relational contract because it is a full-disclosure equilibrium.

Allocating business to $i$ relaxes his constraint (14) but tightens this constraint for the other agents. Using the same logic as in Proposition 3, Proposition 7 proves that a relational contract using the Favored Producer Allocation attains first-best whenever any PBE does.

**Proposition 7** Consider the game in Definition 7 and suppose Assumption 4 holds. If any PBE has total surplus $V^{FB}$, then the equilibrium from Lemma 6 also has total surplus $V^{FB}$.

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If $\gamma > 0$, $\exists$ a non-empty interval $\Delta \in [0, 1]$ such that if $\delta \in \Delta$ this equilibrium attains $V^{FB}$ but stationary equilibria do not.

**Proof:**

See Appendix A.

Under Assumption 4, $U_0(h^t) = (1 - \gamma)V^{FB}$ at any on-path history $h^t$ in a relational contract that attains first-best. By (14), agent $i$ must expect to earn at least $\tilde{S} - (1 - \gamma)V^{FB}$ whenever he chooses $e_t = 1$. Using this intuition, we can define the residual obligation $\hat{\Omega}_i^t$ as the amount of expected dyad-specific surplus that must be given to agent $i$ for him to work hard:

$$\hat{\Omega}_i^t \equiv \frac{\hat{\Omega}_i^{t-1}}{\delta} + \beta_t^L p_1 \delta(\tilde{S} - (1 - \gamma)V^{FB}) - \beta_t^H [(1 - \delta)\gamma(y_{HP1} - c)].$$

(15)

Analogues of Lemmas 3 and 4 hold for $\hat{\Omega}_i^t$, so the proof follows the same lines as that of Proposition 3.

When the fraction of surplus $\gamma$ earned by an agent is small, the principal earns more surplus and thus (14) is easier to satisfy. Intuitively, the rent-seeking activities of one agent have a negative externality on the principal’s relationship with other agents. Rent-seeking by $i$ makes the principal more willing to renege on the other agents, since she loses less surplus in the punishment following a deviation. Agent $i$ does not internalize this negative externality because his relationship with the principal is determined by $U_0(h^t) + U_i(h^t)$ and so is only affected by how rent is shared in other dyads.

### 7.2 Symmetric Learning about Agent Productivity

In this extension, we explore how the scarcity of future business affects the principal’s decision to fire or retain agents. Suppose that each agent $i$ has an unknown productivity that is learned when he first produces high output. While efficient turnover requires that the principal replace low-productivity agents, the optimal relational contract may lead the principal to instead retain them. Definition 8 modifies the baseline model so that each agent $i$ has a permanent productivity $y_i$ that is unknown to all parties. Each odd-numbered agent has an even-numbered counterpart—identical in all respects save productivity—who serves as his
replacement if he is fired.

**Definition 8** Consider the following infinite-horizon dynamic game:

1. At the start of the game, \( y_i \in \{ \bar{y}, \bar{y} \} \) is drawn \( \forall i \in \{1, ..., N\} \), with \( \text{Prob}\{y = \bar{y}\} = \lambda \), \( \bar{y} > y \), and \( \bar{y}p_1 - c > 0 \). \( \{y_i\}_{i=1}^{N} \) is perfectly persistent and unknown to all players.

2. \( N \) is an even number, and \( \forall i \in \{1, 3, 5, ..., N-1\} \), \( F(P) = 0 \) unless \( \{i, i+1\} \in P \) or \( \{i, i+1\} \in \{1, ..., N\}\{P \).

3. In each round, the baseline stage game is played, except that following effort \( e \) by agent \( i \), \( \text{Prob}\{y = y_i|e\} = p_e \), and otherwise \( y = 0 \).

We focus on a very particular set of equilibria that capture a notion of “permanently firing” an agent. An equilibrium satisfies permanent replacement if whenever \( i \in \{1, 3, 5, ..., N-1\} \) satisfies \( i \in P_t \) but \( x_t = i + 1 \), then \( x_{t'} \neq i \) for any \( t' > t \). This restriction implies that if \( i \) is ever denied business in favor of \( i + 1 \), \( i \) is permanently fired and is never again allocated production. Once the principal learns that an odd-numbered agent has low productivity (which occurs as soon as that agent produces high output), she can replace him with a potentially more efficient even-numbered alternative. The next result shows that the principal always replaces unproductive workers when formal contracts are available, but may retain these workers when she relies on relational contracts.

**Proposition 8** If \( P_t \), \( x_t \), and \( y_t \) are contractible and the principal makes a take-it-or-leave-it contract offer in each period, then \( \exists \) an optimal PBE with permanent replacement such that \( \forall i \in \{1, 3, ..., N-1\} \), if \( \text{Prob}\{y_i = \bar{y}|h^t\} = 1 \) then \( x_{t'} \neq i \) \( \forall t' \geq t \). If no variables are contractible and \( p_0 = 0 \), then \( \exists \lambda^* > 0 \) such that if \( \lambda \leq \lambda^* \), \( \exists \) an open set \( \Delta(\lambda) \subseteq [0, 1] \) such that if \( \delta \in \Delta(\lambda) \), \( \exists \) an optimal PBE among those that satisfy permanent replacement such that \( \forall \) on-path \( h^t \) in which \( x_t = i \), \( y_t = \bar{y} \), and \( y_{t'} = 0 \) if \( x_{t'} = i \) \( \forall t' < t \), \( \text{Prob}_{\sigma^*}\{x_{t'} = i|h^t\} > 0 \) for any \( t' > t \).

**Proof:**

See Appendix A.
If the principal fires every low-productivity agent, then \( S_i \) is positive following high output from agent \( i \) only if \( y_i = \bar{y} \). If the probability \( \lambda \) that he has high productivity is small, then the agent’s dyad specific surplus might not be large enough to support incentive pay, which in turn inhibits learning because \( p_0 = 0 \). An effective relational contract must promise to retain an agent—even if he is inefficient—in order to motivate him to work hard, which reveals his type. As a result, optimal relational contracts exhibit *inertia*—ineffective workers are replaced infrequently or not at all.  

7.3 Innovation Can Relax Dynamic Enforcement Constraints

Relational contracts are frequently used when output takes the form of a new product, manufacturing method, or other hard-to-predict innovation. In this extension, we show that such innovation can *relax the scarcity of future business*, leading to more effective relationships between the principal and every agent. The innovating agent does not internalize this positive externality on other relationships and so underinvests. Definition 9 modifies the baseline model by giving agent 1 an opportunity to innovate at the beginning of the game, which creates an additional product that can then be allocated between the agents in each round.

**Definition 9** *The Game with Innovation with 2 agents satisfies:*

1. At \( t = 0 \), \( K \sim \rho \) is drawn from a finite set and publicly observed. Agent 1 publicly chooses whether to innovate \( \kappa \in \{0, 1\} \) at private cost \( \kappa K \). \( \forall i \in \{1, 2\} \), the principal and agent \( i \) can transfer utility: \( w_{i,0}^A \) and \( w_{i,0}^P \).

2. If \( \kappa = 0 \), the game is as in the baseline model.

3. If \( \kappa = 1 \), the game is as in the baseline model, but the principal also chooses a second agent \( x^\kappa_t \in \{1, 2\} \). This agent accepts or rejects \( d^\kappa_t \in \{0, 1\} \) at the same time as \( d_t \). If

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\[30\] McAdams (2011) demonstrates a complementary point: if both parties can immediately quit and be matched with identical partners, then they cannot be punished and so relational contracts become difficult to sustain. In our model, the agent is punished by termination, so the problem is instead guaranteeing these agents enough surplus if they work hard but turn out to have low productivity.

\[31\] A manager at Sun Microsystems notes that “the early development phases where Sun and a supplier work on product development are generally not covered by any contract...but are based on the good intentions of both sides.” (Farlow et al (1996))
\( d_t^\kappa = 1 \), the principal earns \((1 - \delta)y^\kappa > 0\), and otherwise the principal earns 0; these earnings are in addition to \((1 - \delta)y_t\).

If \( \kappa = 0 \), then this game is identical to the baseline model. If \( \kappa = 1 \), let \( U_t^\kappa(\sigma, h_t) \) be the continuation surplus for player \( i \) in the augmented game. In this model, innovation is represented by the extra surplus \( y^\kappa \) that can be produced by either the innovator or the other agent. We dramatically constrain the parameters of the model to simplify the analysis.

**Assumption 5** Let

\[
(1 - \delta)\frac{c - p_0}{p_1 - p_0} > \delta(1 - F(\emptyset))(y_Hp_1 - c),
\]
\[
(1 - \delta)\frac{c - p_0}{p_1 - p_0} \leq \frac{\delta}{2}(1 - F(\emptyset))(y_Hp_1 - c) + \frac{\delta}{2}y^\kappa.
\]

The first inequality in Assumption 5 ensures that it is impossible to sustain high effort in any relational contract without innovation, while the second inequality guarantees that first-best can be attained in a stationary contract following innovation. While these assumptions are extremely stark, they are used to illustrate the possible impact of innovation on relationships. Proposition 9 demonstrates that in the relational contract, an innovator invests more than when formal incentive contracts are available but less than is efficient.

**Proposition 9** Let Assumption 5 hold, and suppose \( p_0 = 0 \).

1. If \( \{\kappa, K, P_t, y_t\} \) are contractible and the principal makes a take-it-or-leave-it offer in each period, then \( \kappa = 1 \) iff \( K \leq K_C \equiv y^\kappa \).

2. If instead only \( \{\kappa, K\} \) are contractible, then \( \kappa = 1 \) iff \( K \leq K_{NC} \equiv K_C + (1 - F(\emptyset))(y_Hp_1 - c) \).

3. If no variables are contractible, then \( \kappa = 1 \) iff \( K \leq K_R \), where \( K_C < K_R < K_{NC} \).

**Proof:**

See Appendix A.

If formal contracts are available, first-best effort can be supported as in Proposition 1 and so the only benefit of innovation is the extra surplus \( y^\kappa \). In contrast, when only \( \kappa \) and
are contractible, innovation increases the stock of future business and facilitates stronger relationships. By Assumption 5, the principal can induce first-best effort from both agents if and only if agent 1 chooses to innovate. When innovation is not contractible, it still induces first-best effort but the relational contract must compensate agent 1 for the cost of his investment $K$. Agent 1’s compensation is limited by his expected dyad-specific surplus, which does not include the positive externality of his investment on other agent’s incentives. As a result, agent 1 underinvests in innovation.

This simple example highlights an interesting trade-off between the *ex post* efficient allocation rule and *ex ante* efficient investment in innovation. The intuition of Grossman and Hart (1986) and Hart and Moore (1990) suggests that the principal might be able to mitigate this underinvestment by cleverly choosing the *other products* manufactured by an innovator, so that the promise of future business is used to encourage *ex ante* investment.

We believe the interactions between investment, the allocation rule, and relational contracts are an interesting direction for further inquiry.

## 8 Conclusion

In the absence of formal contracts, a principal must promise future business to her agents in order to motivate them. If the principal has a limited amount of future business, then her allocation of that business critically constrains her relational contracts with each agent. We have shown that the key incentive constraint in this problem is that agents must be promised enough business following high output, and hence the promise of future business plays an asymmetric role in the optimal equilibrium. When first-best is attainable, an optimal relational contract is tolerant of failure and resembles a tournament, where the prize is a (temporary) larger share of future business and the winners are those agents who performed well in the recent past. In an optimal full-disclosure equilibrium when first-best cannot be attained, this reward may involve exclusive dealing with a high-performing agent, which entails the permanent deterioration of the principal’s relationship with other agents.

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32 This point is made by Halac (2012a). Innovation must take a particular form to strengthen relationships; for instance, would not be relaxed if innovation increased both $y_H$ and $c$ by the same multiplicative constant.
The principal can mitigate these inefficiencies by concealing information about the history of play from the agent. Thus, a downstream firm (or boss) who interacts with multiple suppliers (or workers, or divisions) must carefully consider both the rule she uses to allocate business and the amount of information she reveals.

Like much of the relational contracting literature, one shortcoming of our model is that competition does not pin down the division of rents between players. In some realistic cases, suppliers might be expected to compete away their rents, so that a downstream firm would opt to cultivate multiple sources in order to secure better prices. There are potentially interesting interactions between rent-seeking and relationship cultivation, since an agent’s incentives depend critically on his beliefs about his future surplus. Nevertheless, those companies with very close supplier relations tend to source from a small number of suppliers, who earn substantial rents. Toyota even goes so far as to enshrine “[carrying] out business....without switching to others” in their 1939 Purchasing Rules (as noted by Sako (2004)).

We have outlined several contexts in which the allocation of future business plays an important role—a manager who must motivate different divisions, a company that must provide incentives to its suppliers—but these are only a few of the many cases in which relational incentives matter. For example, a firm can effectively increase an employee’s future surplus by promising him a promotion if he performs well. We believe that treating the allocation of tasks, promotions, and future business as collateral in a relational contract sheds new light on relationships both within and between firms.

References


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A Proofs

A.1 Lemma 2

We prove Lemma 2 for games with asymmetric productivities and costs: agent $i$ chooses $c = 1$ at cost $c_i$, and produces high output $y_i > 0$ with probability $p_i$. For each $i \in \{1, ..., N\}$, $y_i p_i - c_i > y_i p_0 > 0$. $y_i$ and $c_i$ are common knowledge. The principal’s $i$-dyad surplus $U_i^t(h^t)$ and dyad-specific surplus $S_i(h^t)$ are defined analogously to (2) and (3), respectively.

A.1.1 Statement 2: Sufficiency

Fix some strategy profile $\sigma$ that satisfies (6). We will construct an equilibrium $\sigma^*$ that induces the same distribution over $\{P_t, x_t, d_t e_t, y_t\}_{t=1}^T$ as $\sigma$ for all $T < \infty$. A word about notation: let $\mathcal{H}(\sigma)$ be the set of on-path histories for generic strategies $\sigma$, with element $h^t(\sigma) \in \mathcal{H}(\sigma)$.

Define an augmented history as an element $(h^t; h^t) \in \mathcal{H}^\text{Aug} \subseteq \mathcal{H} \times \mathcal{H}$. $\mathcal{H}^\text{Aug}$ is endogenously determined; to preview, the principal will be asked to keep track of an augmented history in each round, which relates on-path histories under $\sigma^*$ to histories under $\sigma$.

To simplify notation, let $\mathcal{N}^e$ be the set of nodes in the extensive-form stage game in which $e$ was just chosen, before Nature chooses $y$.

Constructing Equilibrium Strategies: We recursively construct the set $\mathcal{H}^\text{Aug}$ and a candidate equilibrium $\sigma^*$ that is payoff-equivalent to the original strategy profile $\sigma$.

First, we construct $\mathcal{H}^\text{Aug}$:

1. If $t = 0$, then the initial augmented history is $(\emptyset, \emptyset) \in \mathcal{H}^\text{Aug}$. If $t > 0$, let $(h^{t-1}(\sigma); h^{t-1}(\sigma^*)) \in \mathcal{H}^\text{Aug}$.

2. For every history $(h^{t-1}(\sigma), n_t), n_t \in \mathcal{N}^e$, that is on-path for $\sigma$, define the augmented history

   $$(h^{t-1}(\sigma), n_t; (h^{t-1}(\sigma^*), n_t^*)) \in \mathcal{H}^\text{Aug}$$

where $n_t^* \in \mathcal{N}^e$ is defined as follows. Note that actions with * are part of $n_t^*$, while those without are part of $n_t$.

   (a) $P_t^* = P_t$.

   (b) If $p_0 > 0$ or $d_t > 0$, then $x_t^* = x_t$, $d_t^* = d_t$, $e_t^* = e_t$. If $p_0 = 0$ and $d_t = 0$, then $x_t^* = x_t$, $d_t^* = 1$, and $e_t^* = 0$.

   (c) If $d_t = 0$, then $w_{i,t}^* = 0 \forall i$.

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We frequently refer to “all histories on the equilibrium path” such that some condition holds. Formally, interpret “all histories on the equilibrium path” as “almost surely on the equilibrium path.”

Note: if $p_0 = 0$, then $\{d_t = 0\}$ and $\{d_t = 1, e_t = 0\}$ generate the same surplus for every player and $w_{i,t}^* = 0$ for both of these events. As a result, $d_t^* = 1$ and $e_t^* = 0$ in $\sigma^*$ whenever $d_t = 0$ in $\sigma$. In this case, an augmented history $(h^t(\sigma), h^t(\sigma^*))$ may have rounds in which $d_t^* = 1$ and $e_t^* = 0$ but $d_{t'} = 0$; however, $\sigma^*$ still forms an equilibrium that generates the same total surplus and distribution over $(P_t, x_t, d_t e_t, y_t)_{t=1}^T$ for all $T < \infty$ as $\sigma$. 

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(d) If \(d_t = 1\) and \(e_t = 0\), then \(w_{x,t}^* = y_H p_0 \geq 0\) and \(w_{i,t}^* = 0\), \(\forall i \neq x_t\);
(e) If \(d_t = 1\) and \(e_t = 1\), then

\[
 w_{x,t}^* = y_H^t p_1 + (1 - p_1) \frac{\delta}{1 - \delta} E_\sigma [S_{x_t}(\sigma, h^t(\sigma)) | h_{x_t}^{t-1}, \mathcal{P}_t, x_t, w_{x,t}, d_t, y_t = 0]
\]  

(so \(w_{x,t}^* > y_H p_0\)) and \(w_{i,t}^* = 0\), \(\forall i \neq x_t\)

3. For any successor history \(h^t(\sigma)\) to \((h^{t-1}(\sigma), n_t)\) described in step 2, where output \(y_t = y_t^*\) and transfers \(\{\tau_{i,t}\}\) in \(n_t\) in the support of \(\sigma\) given \((h^{t-1}(\sigma), n_t)\), define the augmented history

\[
 (h^t(\sigma); (h^t(\sigma^*)) \in \mathcal{H}^\text{Aug}
\]  
as follows:

(a) If \(y_t = y_H^t\) at the augmented history \([16]\), then \(\tau_{i,t}^* = 0\), \(\forall i\).
(b) If \(y_t = 0\) but \(w_{x,t} < y_H^t p_1\), then \(\tau_{i,t}^* = 0\), \(\forall i\).
(c) If \(y_t = 0\) and \(w_{x,t} \geq y_H^t p_1\), then \(\tau_{x,t}^* = -\frac{w_{x,t} - y_H^t p_1}{1 - p_1}\) and \(\tau_{i,t}^* = 0\), \(\forall i \neq x_t\).

The previous recursion defines the set of all augmented histories in the game \(\mathcal{H}^\text{Aug}\), given \(\sigma\), so our next goal is to recursively define the candidate equilibrium \(\sigma^*\). In \(\sigma^*\), the principal will be asked to keep track of an augmented history in each period.

1. At \(t = 0\), the principal chooses augmented history \((\emptyset, \emptyset) \in \mathcal{H}^\text{Aug}\). For any \(t > 0\), let \((h^{t-1}(\sigma), h^{t-1}(\sigma^*)) \in \mathcal{H}^\text{Aug}\) be the unique augmented history tracked by the principal at the beginning of period \(t\).

2. When \(\mathcal{P}_t\) is realized, the principal chooses \((h^{t-1}(\sigma), n_t)\) for \(n_t \in \mathcal{N}^e\) according to \(\Psi_\sigma(h^{t-1}(\sigma), \mathcal{P}_t)\), which is the distribution induced by \(\sigma\) over the nodes \(n \in \mathcal{N}^e\). The principal then constructs the augmented history \(((h^{t-1}(\sigma), n_t), (h^{t-1}(\sigma^*), n_t^*)) \in \mathcal{H}^\text{Aug}\) in the same way as in the recursion defining \(\mathcal{H}^\text{Aug}\).

3. Given augmented history \(((h^{t-1}(\sigma), n_t), (h^{t-1}(\sigma^*), n_t^*))\), the principal chooses allocation \(x_t^*\) and wages \(\{w_{x,t}^*\}\).

4. Agent \(x_t\) accepts production if and only if \(w_{x,t}^* \geq y_H^t p_0\), and chooses \(e = 1\) if and only if \(w_{x,t}^* \geq y_H^t p_1\).

5. Agent \(x_t\) pays transfer \(\tau_{x,t}^*\) following output \(y_t^*\).

6. Let \((h^{t-1}(\sigma), n_t, y_t^*)\) be the successor history to \((h^{t-1}(\sigma), n_t)\) following output \(y_t^*\), and define \(\Psi_\sigma^t(h^{t-1}(\sigma), n_t, y_t^*)\) as the distribution over length-\(t\) histories \(h^t(\sigma)\) induced by \(\sigma\), given history \((h^{t-1}(\sigma), n_t, y_t^*)\). The principal chooses a history \(h^t(\sigma)\) according to \(\Psi_\sigma^t(h^{t-1}(\sigma), n^e, y_t^*)\) and constructs augmented history \((h^t(\sigma), h^t(\sigma^*))\), where \(h^t(\sigma^*)\) is the realized history under \(\sigma^*\). Along the equilibrium path, \((h^t(\sigma), h^t(\sigma^*)) \in \mathcal{H}^\text{Aug}\) by construction.

7. If any player deviates from the specified strategies, the principal thereafter chooses \(x_t = \min\{i | i \in \mathcal{P}_t\}\) and \(w_{i,t}^A = \tau_{i,t}^A = 0\). Agents choose \(d_t = 0\) in each period.
Uniqueness of \((h^t(\sigma), h^t(\sigma^*))\) for each \(h^t(\sigma) \in \mathcal{H}(\sigma)\): By inspection of the recursive construction given above, if every \(h^{t-1}(\sigma) \in \mathcal{H}(\sigma)\) is linked to a unique augmented history \((h^{t-1}(\sigma), h^{t-1}(\sigma^*)) \in \mathcal{H}^{\text{Aug}}\), then every \(h^t(\sigma) \in \text{supp}\{\sigma|h^{t-1}(\sigma)\}\) is identified with a unique augmented history \((h^t(\sigma), h^t(\sigma^*))\) according to the algorithm given above. The initial history is \((\emptyset,\emptyset)\), which links the only on-path history of length 0, \(\emptyset \in \mathcal{H}(\sigma)\), to the unique history \(\emptyset \in \mathcal{H}(\sigma^*)\). Thus, the claim is proven by induction.

Payoff-Equivalence of Old and New Strategies: Next, we argue that the strategy \(\sigma^*\) that results from this recursive construction is payoff-equivalent to \(\sigma\). We will actually prove a stronger result: define \(\chi^t_\sigma(h^t(\sigma))\) as the ex ante distribution over \(h^t(\sigma)\) under strategies \(\sigma\), and likewise define \(\chi^t_{\sigma^*}(h^t(\sigma), h^t(\sigma^*))\) as the distribution over augmented histories \((h^t(\sigma), h^t(\sigma^*))\) induced by \(\sigma^*\). Then we claim two properties:

1. \(\forall t \geq 0, \forall h^t(\sigma) \in \mathcal{H}(\sigma), \text{if } h^t(\sigma^*) \in \mathcal{H}(\sigma^*) \text{ is the unique history such that } (h^t(\sigma), h^t(\sigma^*)) \in \mathcal{H}^{\text{Aug}}, \text{then} \)
   \[ \chi^t_{\sigma^*}(h^t(\sigma), h^t(\sigma^*)) = \chi^t_{\sigma}(h^t(\sigma)) \]
   which, by the uniqueness of \((h^t(\sigma), h^t(\sigma^*))\) for given \(h^t(\sigma)\) and the fact that \(h^t(\sigma)\) and \(h^t(\sigma^*)\) imply the same values for \(\{P_s, x_s, d_s e_s, y_s\}_{t=1}^T\), implies that \(\sigma\) and \(\sigma^*\) induce the same distribution for these variables.

2. The stage-game actions specified by \(\sigma\) at history \(h^t(\sigma)\) and those specified by \(\sigma^*\) at \((h^t(\sigma), h^t(\sigma^*))\) differ only in terms of transfers \(\{w_{i,t}\}, \{\tau_{i,t}\}\), and therefore generate the same total (and dyad-specific) surplus.

We tackle each claim separately.

Claim 1: We prove this claim by induction. At \(t = 1\), \(\mathcal{H} = \emptyset\), and so the result is trivial. Therefore, it suffices to show that the conditional distribution over \(h^t(\sigma)\) given \(h^{t-1}(\sigma)\) induced by \(\sigma\) is identical to the distribution over \((h^t(\sigma), h^t(\sigma^*))\) given \((h^{t-1}(\sigma), h^{t-1}(\sigma^*))\) induced by \(\sigma^*\). Fix some augmented history of length \(t-1\). The distribution \(F(P)\) is exogenous. Conditional on the realized \(P_t\), for \(p_0 > 0\) actions \((x^*_t, d^*_t, e^*_t) \sim \Psi_\sigma(h^{t-1}(\sigma), P_t)\), while for \(p_0 = 0\) the analogous statement holds for \((x^*_t, d^*_te^*_t)\). Further, since each history \((h^{t-1}(\sigma), P_t, n^e)\) is mapped to a unique augmented history \(((h^{t-1}(\sigma), P_t, n^e), (h^{t-1}(\sigma), P_t, n^e))\), the augmented history \(((h^{t-1}(\sigma), P_t, n^e), (h^{t-1}(\sigma), P_t, n^e))\) has the same distribution as \((h^{t-1}(\sigma), P_t, n^e)\) conditional on the \((t-1)\)-period history. Because the distribution of \(y_e\) depends only on \(d_t\) and \(e_t\), for \(p_0 > 0\) the distribution of \(y^*_t\) given this augmented history is the same as the distribution of \(y^*_t\) given \((h^{t-1}(\sigma), P_t, n^e)\). Likewise, for \(p_0 = 0\) the distribution of \(y^*_t\) again has the same distribution as \(y^*_t\) given \((h^{t-1}(\sigma), P_t, n^e)\). Given this, the \(t\)-period augmented history \((h^t(\sigma), h^t(\sigma^*)) \sim \Psi'_\sigma(h^{t-1}(\sigma), P_t, n^e, y^*_t)\), and thus is identical to the distribution of \(h^t(\sigma)\) by construction. This proves Claim 1.

Claim 2: As noted in Claim 1, the distribution of actions that affect total surplus \((x^*_t, d^*_t, e^*_t)\) (or \((x^*_t, d^*_te^*_t)\) for \(p_0 = 0\)) induced by \(\sigma\) and \(\sigma^*\) is identical at \(h^{t-1}(\sigma)\) and \((h^{t-1}(\sigma), h^{t-1}(\sigma^*))\), respectively. Thus, the expected total surplus produced at these original and augmented histories is the same. This proves Claim 2.
**Optimality of Principal’s Actions:** Next, we show that the principal has no profitable deviation. Under the specified strategies, the principal can earn no more than 0 at every \(h^t\); thus, it suffices to show that a deviation does not yield strictly positive surplus. Consider a deviation by the principal at history \(h^t\). This deviation could occur in three actions: \(x_t, \{w_{i,t}\}\) or \(\{\tau_{i,t}\}\). Note that in rounds subsequent to the deviation, the principal earns 0 whenever she allocates production to an agent that observed the deviation. Thus, the principal has no profitable deviation in \(\{\tau_{i,t}\}\), since positive bonus payments are both myopically costly and do not increase continuation payoffs. Suppose the principal deviates in \(\{w_{i,t}\}\). A deviation in \(w_{i,t}\) for \(i \neq x_t\) is unprofitable for the same reasons as a deviation in \(\{\tau_{i,t}\}\). If the principal pays some \(w_{x,t}\) to \(x_t\), she earns \(d_t y_t - w_{x,t}\). If \(w^A_{x,t} < y^H_{i} p_0\), then \(d_t = 0\), so this deviation is not profitable. If \(w^A_{x,t} \in [y^i_{H} p_0, y^H_{i} p_1]\), then \(d_t = 1, e_t = 0\), and so the principal’s expected surplus is \(y^H_{i} p_0 - w^A_{x,t} \leq 0\). If \(w^A_{x,t} > y^H_{i} p_1\), then the maximal surplus earned by the principal is

\[
y^H_{i} p_1 - w^A_{i,t} + (1 - p_1) \frac{w^A_{x,t} - y^H_{i} p_1}{1 - p_1} = 0.
\]

Therefore, the principal has no profitable deviations in any round, regardless of whether or not those deviations are detected.

**Optimality of Agent’s Actions:** Finally, we argue that each agent has no profitable deviation. This part of the argument is tricky, because we must carefully account for the agent’s private history at every step. For ease of exposition, we break the proof into several subsections.

Let \(h^{t-1}(\sigma), \hat{h}^{t-1}(\sigma) \in \mathcal{H}(\sigma)\), and let \((h^{t-1}(\sigma), h^{t-1}(\sigma^*)), (\hat{h}^{t-1}(\sigma), \hat{h}^{t-1}(\sigma^*)) \in \mathcal{H}^{Aug}\).

**Claim 1:** Suppose that agent \(i\) cannot distinguish \((h^{t-1}(\sigma), n_t)\) and \((\hat{h}^{t-1}(\sigma), \hat{n}_t)\). Then \(i\) cannot distinguish \((\hat{h}^{t-1}(\sigma^*), n^*_t)\) and \((\hat{h}^{t-1}(\sigma^*), \hat{n}^*_t)\), where \(n^*_t\) is defined so that \(((h^{t-1}(\sigma), n_t), (\hat{h}^{t-1}(\sigma^*), n^*_t))\) is an augmented history that occurs in equilibrium, and likewise for \(\hat{n}^*_t\) and \(((\hat{h}^{t-1}(\sigma), \hat{n}_t), (\hat{h}^{t-1}(\sigma^*), \hat{n}^*_t))\). Proof by induction. For \(t = 0\), there is no private information, so the result holds trivially. Suppose the result holds for all \(t < T\), and consider a history of length \(T\). By contradiction; suppose \(i\) can distinguish \((h^{T-1}(\sigma^*), n^*_t)\) and \((\hat{h}^{T-1}(\sigma^*), \hat{n}^*_t)\). By the inductive step, there must be some variable in round \(T\) that has a different realization in \(\hat{h}^{T-1}(\sigma^*)\) and \(h^{T-1}(\sigma^*)\). That variable cannot be \(P_T\) or \(x_T\) by definition of an augmented history and because \(n_t\) and \(\hat{n}_t\) are not distinguishable, and \(d_T, e_T\) are chosen by the agent. Thus, it must be that \(w^*_{i,T} \neq \hat{w}^*_{i,T}\) or \(\tau^*_{i,T} \neq \hat{\tau}^*_{i,T}\). Because \(\hat{\tau}^*_{i,T}\) is determined entirely by \(\hat{w}^*_{i,T}\) in the constructed equilibrium, it must be that \(w^*_{i,T} \neq \hat{w}^*_{i,T}\). This is only possible if \(x^*_t = i\).

There are two possibilities: if \(d^*_T = 0\), then \(w^*_T = \hat{w}^*_T = y^H_{i} p_0\) (vacuously if \(p_0 = 0\), and if \(e^*_T = 0\), then \(w^*_T = \hat{w}^*_T = y^H_{i} p_0\), and so contradiction obtains. Suppose instead \(d^*_T = e^*_T = 1\). Because the original histories \((h^{t-1}, n_t)\) and \((\hat{h}^{t-1}(\sigma^*), \hat{n}_t)\) are indistinguishable, it must be that agent \(i\)’s beliefs about the future at each node \(n_t\) are identical at these two histories, and in particular this must hold for \(n^e \in \mathcal{N}^e\). As a result, \([17]\) is identical under the two histories by construction, and so contradiction obtains.
Claim 2: The agent has no profitable deviation from $\tau_i^*$. Define $\mathcal{H}_i(h^{t-1}, n_t) \subseteq \mathcal{H}$ as the set of histories that $i$ finds indistinguishable from $(h^{t-1}, n_t)$. With abuse of notation, let

$$S_i(\sigma, \mathcal{H}_i(h^{t-1}, n_t)) = E_\sigma \left[ S_i(\sigma, (\hat{h}^{t-1}, \tilde{n}_t)) \mid (\hat{h}^{t-1}, \tilde{n}_t) \in \mathcal{H}_i(h^{t-1}, n_t) \right]$$

be the expected continuation dyad-specific surplus at history $h^t$ conditional on $i$’s information set.

First, note that any deviation by agent $i$ is immediately detected and punished by the principal. Thus, we need only check a deviation from $\tau_i^* < 0$; that is, when $x_i^* = i$, $e_i^* = 1$, and $y_t = 0$.

Let $((h^{t-1}(\sigma), n_t), (h^{t-1}(\sigma^*), n_t^*)) \in \mathcal{H}^{Aug}$ be an augmented history with nodes $n_t$, $n_t^*$ immediately following $y_t = 0$. Consider agent $i$’s beliefs at the time of the transfer $\tau_i^*$ under $\sigma^*$. Agent $i$ knows the true history is in $\mathcal{H}_i(h^{t-1}(\sigma^*), n_t^*)$, and moreover infers

$$S_i(\sigma, \mathcal{H}_i(h^{t-1}(\sigma), n_t)) = \frac{w_{i,t}^* - y_{i,t}p_1}{1 - p_i}$$

from the wage payment $w_{i,t}^*$. By claim 1, if $((h^t(\sigma), n_t), (h^t(\sigma^*), n_t^*)) \in \mathcal{H}^{Aug}$, then

$$\left\{ (\hat{h}^{t-1}(\sigma^*), \tilde{n}_t^*) \mid (\hat{h}^{t-1}(\sigma), \tilde{n}_t), (\hat{h}^{t-1}(\sigma^*), \tilde{n}_t^*) \in \mathcal{H}^{Aug} \text{ for some } (\hat{h}^{t-1}(\sigma), \tilde{n}_t) \in \mathcal{H}_i(h^{t-1}(\sigma), n_t) \right\} \subseteq \mathcal{H}_i(h^{t-1}(\sigma^*), n_t^*)).$$

That is, the strategies $\sigma^*$ induce a coarser partition than $\sigma$ over the set of augmented histories.

By the proof that $\sigma$ and $\sigma^*$ are payoff equivalent, we know

$$S_i(\sigma, h^t(\sigma)) = S_i(\sigma^*, h^t(\sigma^*)). \quad (20)$$

Moreover, because the principal and agents $j \neq i$ earn 0 in any round where $x_t = i$, agent $i$ earns the entire surplus produced, and so

$$U_i(\sigma^*, (h^{t-1}(\sigma^*), n_t^*)) = E_{\sigma^*} \left[ \sum_{\nu=0}^\infty \delta^\nu (1 - \delta) 1_{i,t}(d_{i,t}y_t - c_{i,t}) \right] = S_i(\sigma^*, (h^{t-1}(\sigma^*), n_t^*)). \quad (21)$$

As a result, agent $i$ believes his continuation surplus to be

$$E_{\sigma^*} \left[ U_i(\sigma^*, (h^{t-1}(\sigma^*), n_t^*)) \mid h_i^{t-1}(\sigma^*), I_i(n_t^*) \right] = S_i(\sigma^*, \mathcal{H}_i(h^{t-1}(\sigma^*), n_t^*)) \quad (22)$$

and so agent $i$’s expected continuation utility at the time of transfer $\tau_i^* > 0$ is $S_i(\sigma^*, \mathcal{H}_i(h^{t-1}(\sigma^*), n_t^*))$.

Using (19) and (20), we can write (22) as an expectation over dyad-specific surpluses in the original histories:

$$S_i(\sigma^*, \mathcal{H}_i(h^{t-1}(\sigma^*), n_t^*)) = E_\sigma \left[ S_i(\sigma, \mathcal{H}_i(h^{t-1}(\sigma), \tilde{n}_t)) \mid \left( \mathcal{H}_i(h^{t-1}(\sigma), \tilde{n}_t) \times \mathcal{H}_i(h^{t-1}(\sigma^*), n_t^*) \right) \cap \mathcal{H}^{Aug} \neq \emptyset \right]. \quad (23)$$
But from (18), we know that we can rewrite (23) as
\[ S_i(\sigma^*, H_i(h^{t-1}(\sigma^*), n_i^*)) = E_{\sigma} \left[ \frac{w_{i,t}^* - y_H^t P_1}{1 - P_1} \mid \left( H_i(h^{t-1}(\sigma), \tilde{n}_i) \times H_i(h^{t-1}(\sigma^*), n_i^*) \right) \cap H^{Aug} \neq \emptyset \right]. \]  
(24)

Now, at every candidate history \((\tilde{h}^{t-1}(\sigma^*), \tilde{n}_i)\) in this expectation, \(w_{i,t}^*\) must be the same, since otherwise \((\tilde{h}^{t-1}(\sigma^*), \tilde{n}_i) \notin H_i(h^{t-1}(\sigma^*), n_i^*)\). Informally, we are simply saying that agent \(i\) can perfectly infer what the dyad-specific continuation surplus would have been under \(\sigma\) from the wage payment \(w_{i,t}^*\).

Therefore, we can eliminate the conditional expectation from (24) to yield
\[ \delta S_i(\sigma^*, H_i((h^{t-1}(\sigma^*), p_t, n^e, y_t = 0))) = (1 - \delta) \frac{w_{i,t}^* - y_H^t P_1}{1 - P_1} = (1 - \delta) \tau_{i,t}^*. \]  
(25)

Finally, using (25), the agent earns \(-(1 - \delta)\tau_{i,t}^* + (1 - \delta)\tau_{i,t}^* = 0\) by following \(\sigma^*\), while following any deviation in \(\tau\), the agent is immediately punished and so earns no more than 0. So there is no profitable deviation from \(\tau_{i,t}^*\).

**Claim 3:** Agent \(i\)'s effort IC constraint is satisfied. Using (8), (20), (21), and (25), in order for the agent to choose \(e = 1\), we must have
\[ \delta S_{x_i}(\sigma, H_i(h^{t-1}(\sigma), p_t, x_t, x_{x_i,t}, d_t, y_t = y_H^i)) \geq (1 - \delta) \frac{c}{P_1 - P_0}. \]  
(26)

Agent \(i\) is only transferred \(w_{i,t}^* \geq y_H^t P_1\) at an augmented history \((h^{t-1}(\sigma), h^{t-1}(\sigma^*))\) such that \(e = 1\) at \(h^{t-1}(\sigma)\) under \(\sigma\). But by the assumption (6) made in the lemma, (26) is satisfied at any \(h^{t-1}(\sigma)\) where the agent chooses \(e = 1\) under \(\sigma\), and hence also holds at \((h^{t-1}(\sigma), h^{t-1}(\sigma^*))\) under \(\sigma^*\) because continuation strategies are payoff-equivalent. Finally, notice that the agent will always choose \(d = 0\) or \(e = 0\) when \(w_{x_i,t}^* \leq y_H P_0\) or \(w_{x_i,t}^* \in [y_H P_0, y_H P_1]\), respectively, because the agent expects that any deviation will either trigger a punishment or not affect continuation payoffs.

To summarize, we have constructed an equilibrium \(\sigma^*\) that is equivalent to the strategy \(\sigma\) under the relation defined by \(H^{Aug}\), which proves Lemma 2. \(\blacksquare\)

### A.2 A Note About Proposition 3

Proposition 3 follows from Lemmas 3 and 4, which are discussed later in the text. Therefore, we first prove these Lemmas and then turn to the argument for the Proposition.

### A.3 Lemma 3

Suppose there exists a strategy profile \(\sigma\) that attains first-best and \(\limsup_{t \to \infty} \Omega_i^t = \infty\) for some \(i \in \{1, \ldots, N\}\). Towards contradiction, suppose that \(\sigma\) form an equilibrium. Because \(\sigma\) attains the first-best, production must be accepted whenever \(P_t \neq \emptyset\), and agent \(x_t\) must always choose \(e_t = 1\).
By Lemma 2, in order for $e_t = 1$ in equilibrium, it must be that (6) holds. In particular, this condition must be satisfied the first time that every player produces $y_H$. Consider agent $i$’s *ex ante* expectations about play. Let

$$b_i(h^t_i) = 1\{x_t = i, y_t = y_H\} \ast 1\{\forall t', x_{t'} = i \Rightarrow y_{t'} = 0\}$$

(27)

be the indicator function for the event that $x_t = i$, $y_t = y_H$ in period $t$, and $y_{t'} = 0$ $\forall t' < t$ with $x_t = i$. Then by Lemma 2, we must have

$$b_i(h^t_i)\delta S \leq (1 - \delta)b_i(h^t_i)E_{\sigma}\left[\sum_{\nu = 1}^{\infty} \delta'^{t} 1_{i,t+\nu}(y_{H}p_1 - c) | h^t_i\right].$$

(28)

This inequality must hold in expectation across $H^t_i$:

$$E_{\sigma}\left[b_i(h^t_i)\right] \ast \delta S \leq E_{\sigma}\left[(1 - \delta)b_i(h^t_i)E_{\sigma}\left[\sum_{\nu = 1}^{\infty} \delta'^{t} 1_{i,t+\nu}(y_{H}p_1 - c) | h^t_i\right]\right].$$

Because $b_i(h^t_i)$ is an indicator function, $E[b_i(h^t_i)] = \text{Prob}\{x_t = i, y_t = y_H, x_{t'} = i \rightarrow y_{t'} = 0\} = p_1\beta^L_{t'}$ in any equilibrium that attains first-best. Dividing by $\delta^{K-t}$ and summing across $t = 1, \ldots, K$, we have

$$\sum_{k=1}^{K} \frac{p_1\beta^L_{k}}{\delta^{K-k}} \delta S \leq \sum_{k=1}^{K} \frac{1}{\delta^{K-k}} E_{\sigma}\left[(1 - \delta)b_i(h^k_i)E_{\sigma}\left[\sum_{\nu = 1}^{\infty} \delta'^{t} 1_{i,k+\nu}(y_{H}p_1 - c) | h^k_i\right]\right].$$

(29)

which can be re-written as

$$\sum_{k=1}^{K} \frac{p_1\beta^L_{k}}{\delta^{K-k}} \delta S \leq (1 - \delta)V^{FB}_{t} \sum_{k=1}^{K} \frac{1}{\delta^{K-k}} E_{\sigma}\left[\sum_{\nu = 1}^{\infty} 1_{i,k+\nu} | b_i(h^k_{i+k} = 1)\right] \text{Prob}_{\sigma}\{b_i(h^k_{i}) = 1\}. $$

(30)

This infinite sum consists of positive terms and is dominated by $\sum_{\nu = 1}^{\infty} \delta'^{t}(y_{H}p_1 - c)$, and so converges absolutely. We switch the order of summation on the right-hand side of (30), then consider the terms with $\psi = k + t'$ held constant. Using $\psi$ as an index, we can rewrite (30)

$$\sum_{\psi = 1}^{\infty} \sum_{k=1}^{K} \frac{p_1\beta^L_{k}}{\delta^{K-k}} \delta S \leq \sum_{\psi = 1}^{\infty} \sum_{\psi = 1}^{\infty} \frac{1}{\delta^{K-\psi}} E_{\sigma}\left[1_{i,\psi} | b_i(h^k_{i}) = 1\right] \ast \text{Prob}_{\sigma}\{b_i(h^k_{i}) = 1\}. $$

(31)

Now, note that $\sum_{k=1}^{K} \text{Prob}\{b_i(h^k_{i}) = 1\} = \text{Prob}\{\exists k \leq K \text{ s.t. } x_k = i, y_k = y_H\}$ because the underlying events (27) are disjoint. Thus, changing the index on the left-hand side to $\psi$ as well, (31) can be written

$$\sum_{\psi = 1}^{\infty} \sum_{\psi = 1}^{\infty} \frac{1}{\delta^{K-\psi}} \text{Prob}_{\sigma}\{x_{\psi} = i, \exists t' \leq \min\{K, \psi - 1\} \text{ s.t. } x_{t'} = i, y_{t'} = y_H\}. $$

(32)

Now, the tail for $\psi > K$ on the right-hand side of (32) converges, because it is dominated
by \(\sum_{k=K}^{\infty} \frac{1}{\delta^k} = \frac{1}{1-\delta}\). Therefore, in order for this inequality to hold, there must exist some constant \(C \in \mathbb{R}\) such that

\[
\sum_{\psi=1}^{K} \frac{1}{\delta_{K-\psi}} \sum_{\psi=1}^{K} \frac{1}{\delta_{K-\psi}} \text{Prob}_{\sigma} \left\{ x_\psi = i, \exists t' \leq \min\{K, \psi - 1\} \text{ s.t. } x_{t'} = i, y_{t'} = y_H \right\} \leq C
\]

for all \(K\). By definition, \(\text{Prob}_{\sigma} \left\{ x_\psi = i, \exists t' \leq \min\{K, \psi - 1\} \text{ s.t. } x_{t'} = i, y_{t'} = y_H \right\} = \beta^H_\psi\), so we conclude that in any equilibrium that attains first-best, \(\exists C \in \mathbb{R}\) such that

\[
\sum_{\psi=1}^{K} \frac{1}{\delta_{K-\psi}} \sum_{\psi=1}^{K} \frac{1}{\delta_{K-\psi}} \beta^H_\psi \leq C
\]

for any \(K \in \mathbb{Z}\). This is exactly our notion of obligation written non-recursively. Therefore, we conclude that if the left-hand side of (33) is unbounded, \(\sigma\) cannot be an equilibrium that attains first-best.

\[\blacksquare\]

**A.4 Lemma 4**

Fix strategies \(\sigma\) and consider the following alternative strategies \(\tilde{\sigma}\):

1. At the beginning of the game, agent \(i\) is assigned title \(\rho(i)\), where \(\rho: \{1, \ldots, N\} \to \{1, \ldots, N\}\) is a permutation drawn uniformly at random from the set of permutations.

2. Play proceeds according to \(\sigma\), with agent \(i\) treated as agent \(\rho(i)\) in all respects.

Because agents are *ex ante* symmetric, \(\tilde{\sigma}\) also generates first-best surplus. Moreover, if \(\limsup_{t \to \infty} \max_i \Omega^t_i < \infty\) under \(\sigma\), then

\[
\limsup_{t \to \infty} \tilde{\Omega}^t_i = \limsup_{t \to \infty} \frac{1}{N} \sum_{i=1}^{N} \Omega^t_i < \infty
\]

\(\forall i \in \{1, \ldots, N\}\) under \(\tilde{\sigma}\), as well. Define \(E(h^t) = \{i \in \{1, \ldots, N\} | \exists t' < t \text{ s.t. } x_{t'} = i, y_{t'} = y_H\}\) as the set of agents who have produced \(y_H\) at some point in the past; then \(\forall i \in \{1, \ldots, N\}\),

\[
\text{Prob}_{\tilde{\sigma}} \{ x_t = i | h^t \text{ s.t. } i \in E(h^t) \} = \frac{1}{N} \sum_{j=1}^{N} \text{Prob}_{\sigma} \{ x_t = j | h^t \text{ s.t. } j \in E(h^t) \}
\]

by construction. Therefore, every \(i \in E(h^t)\) is treated symmetrically, as is every \(j \notin E(h^t)\).

Given strategies \(\tilde{\sigma}\), consider the following strategies \(\hat{\sigma}\):

1. In each round, the allocation rule is

   \[
   \hat{x}_t \in \left\{ i \mid \mathcal{P}_t \cap E(h^t) = \emptyset \to i = \arg \min_{i \in \mathcal{P}_t} \rho(i) \right\}
   \]

2. Agents always choose \(\hat{d}_t = \hat{e}_t = 1\). Transfers are \(\hat{w}_{i,t} = \hat{r}_{i,t} = 0\).
We claim that \( \hat{\sigma} \) generates weakly lower obligation than \( \tilde{\sigma} \). Because both strategies implement first-best surplus, it suffices to show that \( \hat{x}_t \) leads to a lower obligation than \( \tilde{x}_t \). Agent labels are randomized at the beginning of the game, so

\[
\text{Prob}_\beta \{ x_t = i | h^t \text{ s.t. } i \in E(h^t) \} = \text{Prob}_\beta \{ x_t = j | h^t \text{ s.t. } j \in E(h^t) \}
\]

(35)

for all \( i, j \); we will denote the obligation of a generic agent by \( \Omega_t \). Note that \( \hat{x}_t \in E(h^t) \) whenever possible subject to the requirement that first-best is attained. Denote by \( E^m_t \) the event that \( m \) agents have produced \( y_H \) by time \( t \); then

\[
\phi^m_t \equiv \frac{1}{m} \text{Prob}_\beta \{ x_t \in E(h^t) | E^m_t \} \leq \frac{1}{m} \text{Prob}_\beta \{ x_t \in E(h^t) | E^m_t \}. \tag{36}
\]

Because \( \hat{\sigma} \) and \( \tilde{\sigma} \) are symmetric, \( \text{Prob}\{ i \in E_t(h^t) | E^m_t \} = \frac{m}{N} \) for both strategies. Then we can explicitly write \( \beta^H_t \) and \( \beta^L_t \) for \( \tilde{\sigma} \):

\[
\beta^H_t = \sum_{m=1}^{N} \frac{m}{N} \text{Prob}_\beta \{ E^m_t \} \phi^m_t
\]

and

\[
\beta^L_t = \sum_{m=0}^{N-1} \text{Prob}_\beta \{ E^m_t \} \left( \frac{1 - F(\emptyset)}{N} - \frac{m}{N} \phi^m_t \right).
\]

We can write \( \text{Prob}_\beta \{ E^m_t \} \) in terms of the previous history of play. In particular,

\[
E^0_t = [F(\emptyset) + (1 - F(\emptyset))(1 - p_1)] E^0_{t-1}
\]

because production is \( x \neq \emptyset \) and \( e = 1 \) whenever possible in any equilibrium that attains first-best. Similarly,

\[
\text{Prob}_\beta \{ E^m_t \} = \phi^m_{t-1} \text{Prob}_\beta \{ E^m_{t-1} \} + (1 - F(\emptyset) - \phi^m_{t-1})(1 - p_1) \text{Prob}_\beta \{ E^m_{t-1} \} + (1 - F(\emptyset) - \phi^m_{t-1}) p_1 \text{Prob}_\beta \{ E^m_{t-1} - 1 \}.
\]

In order to show that \( \hat{\sigma} \) generates weakly smaller obligation than \( \tilde{\sigma} \), it suffices by (36) to show that obligation is decreasing in \( \phi^m_t \). Obligation as written in (33) is smooth in \( \beta^L_t \) and \( \beta^H_t \), which in turn are smooth in \( \{ \phi^m_t \}_{m,t} \), so we can demonstrate this by showing that \( \frac{\partial \Omega_t}{\partial \phi^m_t} \leq 0 \) \( \forall t, t' \in \mathbb{N}, m \in \{ 1, \ldots, N \} \).

Using the recursive formulation of obligation (11), we can write

\[
\frac{\partial}{\partial \phi^m_t} \Omega_t = \frac{1}{\delta} \left( \frac{\partial}{\partial \phi^m_{t-1}} \Omega_{t-1} + \frac{\partial}{\partial \phi^m_t} \beta^L_{t-1} p \delta \tilde{S} - \frac{\partial}{\partial \phi^m_t} \beta^H_{t-1} (1 - \delta) V^{FB} \right).
\]

Repeatedly substituting for \( \Omega_{t-1} \), we find that

\[
\frac{\partial \Omega_t}{\partial \phi^m_t} = \frac{1}{\delta^{t-t' - 1}} \frac{\partial}{\partial \phi^m_{t'}} \Omega_{t'} + \sum_{s=1}^{t - t' - 1} \frac{1}{\delta^s} \left( \frac{\partial}{\partial \phi^m_{t-s}} \beta^L_{t-s} p \delta \tilde{S} - \frac{\partial}{\partial \phi^m_{t-s}} \beta^H_{t-s} (1 - \delta) V^{FB} \right). \tag{37}
\]
Next, note that
\[
\frac{\partial \beta_t^L}{\partial \phi_{t'}^m} = \begin{cases} 
\sum_{k=1}^{N} \left( \frac{1-F(\emptyset)}{N} - \frac{k}{N} \phi_k^t \right) \frac{\partial \text{Prob}_s \{ E_t^k \}}{\partial \phi_{t'}^m} & \text{if } t' < t \\
-m \frac{\partial \text{Prob}_s \{ E_t^m \}}{\partial \phi_{t'}^m} & \text{if } t' = t \\
0 & \text{if } t' > t 
\end{cases}
\]
while differentiating the second gives us
\[
\frac{\partial \beta_t^H}{\partial \phi_{t'}^m} = \begin{cases} 
\sum_{k=1}^{N} \frac{k}{N} \phi_k^t \frac{\partial \text{Prob}_s \{ E_t^k \}}{\partial \phi_{t'}^m} & \text{if } t' < t \\
m \frac{\partial \text{Prob}_s \{ E_t^m \}}{\partial \phi_{t'}^m} & \text{if } t' = t \\
0 & \text{if } t' > t 
\end{cases}
\]

Because \( \tilde{\sigma} \) is symmetric and \( \lim_{t \to \infty} \text{Prob}_s \{ E_t^N \} = 1 \), \( \beta_t^H + \beta_t^L = \frac{1-F(\emptyset)}{N} \) and \( \sum_{t=1}^{\infty} \beta_t^L \equiv \frac{1}{p_1} \).

Differentiating the first identity gives us that
\[
\frac{\partial}{\partial \phi_{t'}^m} \beta_t^L = - \frac{\partial}{\partial \phi_{t'}^m} \beta_t^H \tag{38}
\]
while differentiating the second gives us \( \frac{\partial}{\partial \phi_{t'}^m} \beta_t^L = - \sum_{s=t+1}^{\infty} \frac{\partial}{\partial \phi_{t'}^m} \beta_s^L \). Hence
\[
\frac{\partial}{\partial \phi_{t'}^m} \Omega_t = \left( p_1 \delta \tilde{V} + (1 - \delta)V^{FB} \right) \frac{\partial}{\partial \phi_{t-1}^m} \beta_{t-1}^L. \tag{39}
\]

The fact that \( \frac{\partial}{\partial \phi_{t'}^m} \beta_s^L > 0 \) for \( s > t \) implies that
\[
\frac{\partial}{\partial \phi_{t'}^m} \beta_{t'}^L < - \sum_{s=t'+1}^{\infty} \delta^{s-t'} \frac{\partial}{\partial \phi_{t'}^m} \beta_{s}^L < - \sum_{s=t'+1}^{t} \delta^{s-t'} \frac{\partial}{\partial \phi_{t'}^m} \beta_{s}^L \tag{40}
\]
where we have used the fact that \( \frac{\partial}{\partial \phi_{t'}^m} \beta_{s}^L > 0 \) for \( s > t \). Together, (38) and (40) imply that
\[
\sum_{s=1}^{t'-1} \frac{1}{s} \left( \frac{\partial}{\partial \phi_{t'}^m} \beta_{s}^L - \frac{\partial}{\partial \phi_{t'}^m} \beta_{s}^H (1 - \delta)V^{FB} \right) = \left( p_1 \delta \tilde{S} + (1 - \delta)V^{FB} \right) \sum_{s=1}^{t'-1} \frac{1}{s} \frac{\partial}{\partial \phi_{t'}^m} \beta_{s}^L < - \left( p_1 \delta \tilde{S} + (1 - \delta)V^{FB} \right) \frac{\partial}{\partial \phi_{t'}^m} \beta_{t'}^L. \tag{41}
\]

If we plug this expression into (37) and apply (39), we finally conclude
\[
\frac{\partial \Omega_t}{\partial \phi_{t'}^m} = \left( p_1 \delta \tilde{S} + (1 - \delta)V^{FB} \right) \frac{\partial}{\partial \phi_{t'}^m} \beta_{t'}^L + \left( p_1 \delta \tilde{S} + (1 - \delta)V^{FB} \right) \sum_{s=1}^{t'-1} \frac{1}{s} \frac{\partial}{\partial \phi_{t'}^m} \beta_{s}^L < 0
\]
precisely as we wanted to show. ■

### A.5 Proposition 3

By Lemma 3, it suffices to show that whenever the Favored Producer Automaton does not attain first-best, then \( \lim_{t \to \infty} \Omega_t = \infty \). By Lemma 4, it suffices to consider a strategy \( \hat{\sigma} \)
that randomizes player labels and gives production to those who have already produced \( y_H \) whenever possible. Using symmetry, let

\[
F^m = \sum_{\mathcal{P} | \mathcal{P} \cap \{1, \ldots, m\} \neq \emptyset} F(\mathcal{P})
\]

be the probability that at least one of a set of \( m \) of obligation (33), we can plug in for in the past. At \( t \) rule (34), we can explicitly calculate \( \phi \) whenever possible. Using symmetry, let \( b \) where

\[
\forall \{ \minimizing \text{sequence of} \}
\]

that randomizes agent labels at the beginning of the first period generates this obligation-

\[
\Omega_t = \sum_{k=0}^{t-1} \frac{1}{\delta^k} \left( \sum_{m=0}^{N} \text{Prob} \{ E_{t-1-k}^m \} \left( p_1 \delta S \left( \frac{1 - F(\emptyset) - F^m}{N} \right) - (1 - \delta) V^{FB} F^m \right) \right).
\]

Notice that \( 1 - F(\emptyset) - F^m \geq 0 \) with equality iff \( m = N \), so

\[
\sum_{m=0}^{N} \text{Prob} \{ E_{t-1-k}^m \} \left( p \delta S \left( \frac{1 - F(\emptyset) - F^m}{N} \right) - (1 - \delta) V^{FB} F^m \right)
\]

is positive and strictly increasing in \( \tilde{S} \). Therefore, if \( \exists \tilde{S}^* \) such that \( \lim_{t \to \infty} \Omega_t = C \) for some \( C \in \mathbb{R} \), then \( \lim_{t \to \infty} \Omega_t = \infty \) for any \( \tilde{S} > \tilde{S}^* \).

Obligation \( \Omega_t \) is determined entirely by \( \{ \beta^L_t \}, \{ \beta^H_t \} \), so any two strategies that induce the same \( \{ \beta^L_t \} \) and \( \{ \beta^H_t \} \) also generate the same \( \{ \Omega_t \} \). The Favored Producer Automaton that randomizes agent labels at the beginning of the first period generates this obligation-minimizing sequence of \( \{ \beta^L_t \}, \{ \beta^H_t \} \). Let \( S_G \) agent \( i \)'s surplus immediately after he produced \( y_H \) in the Favored Producer Automaton; if obligation converges when \( \tilde{S} = S_G \), and production is allocated as in the Favored Producer Automaton, then it diverges for \( \tilde{S} > S_G \) and the proposition is proven.

In the Favored Producer Automaton, at an \( h^t \) where \( x_t = i \), \( y_t = y_H \), agent \( i \)'s future surplus is

\[
\delta S_G = E \left[ (1 - \delta) V^{FB} \sum_{t'=1}^{\infty} \delta^{t'} b_{t+t'} | h^t \right] \quad (42)
\]

where \( b_{t+t'} = 1 \{ x_{t+t'} = i, \exists t'' < t + t' \text{ s.t. } x_{t''} = i, y_{t''} = y_H \} \). Let \( \zeta_t \equiv 1 \{ x_t = i; \forall t' < t, x_{t'} =

\)

To see this, suppose \( \tilde{S} = \tilde{S}^* + \epsilon, \) some \( \epsilon > 0 \). Then obligation is

\[
C + \sum_{k=0}^{t-1} \frac{1}{\delta^k} \left( \sum_{m=0}^{N} \text{Prob} \{ E_{t-1-k}^m \} p_1 \delta \left( \frac{1 - F(\emptyset) - F^m}{N} \right) \right)
\]

Every term in this sum is weakly positive, and \( \forall t \) the final term is

\[
\frac{1}{\delta^{t-1}} p_1 \delta \frac{1 - F(\emptyset)}{N}
\]

which diverges to \( \infty \) as \( t \to \infty \).
\( i \to y_t = 0 \). Because (42) holds at every true history \( h^t \) where \( x_t = i \) and \( y_t = y_H \), we can write

\[
\delta S_G = E \left[ (1 - \delta) V^{FB} \sum_{t' = 1}^{\infty} \delta^{t'} b_{t+t'} h^t \text{ s.t. } y_t = y_H, \zeta_t = 1 \right]
\]  

(43)

for any \( h^t \) where \( x_t = i, y_t = y_H \). Because \( e = 1 \) at every on-path history, \( \text{Prob}\{y_t = y_H | \zeta_t = 1\} = p_1 \). The event \( \{\zeta_t = 1, y_t = y_H\} \) can occur only once, so we can multiply both sides of (43) by \( \delta^t \{\zeta_t = 1, y_t = y_H\} \), sum across \( t \), and take expectations to yield

\[
\delta S_G \sum_{t=1}^{\infty} \delta^t \text{Prob}_S \{\zeta_t = 1, y_t = y_H\} = E \left[ \sum_{t=1}^{\infty} \delta^t \{\zeta_t = 1, y_t = y_H\} \right] E \left[ (1 - \delta) V^{FB} \sum_{t'=1}^{\infty} \delta^{t'} b_{t+t'} h^t \text{ s.t. } y_t = y_H, \zeta_t = 1 \right]
\]

(44)

where \( \text{Prob}_S \{\zeta_t = 1, y_t = y_H\} = p_1 \beta^L_t \) by definition. Using the Law of Iterated Expectations and dominated convergence, the right-hand side of (44) equals

\[
(1 - \delta) V^{FB} E \left[ \sum_{t=1}^{\infty} \delta^t \sum_{s=1}^{\infty} \delta^{t+s} b_{t+s} \right] = (1 - \delta) V^{FB} E \left[ \sum_{t=1}^{\infty} \delta^t b_t \sum_{s=1}^{\infty} \delta^s \right].
\]

Next, note that \( b_t = b_t \sum_{s=1}^{\infty} \delta^s \), since \( \sum_{s=1}^{\infty} \delta^s = 1 \) and equals 1 whenever \( b_t = 1 \). Combining this fact with the preceding calculations, we can write (44) as

\[
\delta S_G \sum_{t=1}^{\infty} \delta^t \beta^L_t = (1 - \delta) V^{FB} E \left[ \sum_{t=1}^{\infty} \delta^t b_t \right].
\]

(45)

Using dominated convergence to interchange summation and integration and noting that

\[
E[b_t] = \beta^H_t
\]

we have

\[
p_1 \delta S_G \sum_{k=1}^{\infty} \delta^k \beta^L_k - (1 - \delta) V^{FB} \sum_{k=1}^{\infty} \delta^k \beta^H_k = 0.
\]

Hence, by definition of \( \Omega_t \), if production is allocated according to the Favored Producer Automaton, then \( \lim_{t \to \infty} \delta^t \Omega_t = 0 \) when \( \tilde{S} = S_G \).

In the final step of the argument, suppose that \( \tilde{S} = S_G \). Using the non-recursive expression for obligation (33)

\[
p_1 \delta S_G \sum_{k=1}^{\infty} \delta^k \beta^L_k - (1 - \delta) V^{FB} \sum_{k=1}^{\infty} \delta^k \beta^H_k = \delta^t \Omega_t + p_1 \delta S_G \sum_{k=t+1}^{\infty} \delta^k \beta^L_k - (1 - \delta) V^{FB} \sum_{k=t+1}^{\infty} \delta^k \beta^H_k
\]

and hence

\[
\Omega_t = \frac{1}{\delta^t} (1 - \delta) V^{FB} \sum_{k=t+1}^{\infty} \delta^k \beta^H_k - \frac{1}{\delta^t} p_1 \delta S_G \sum_{k=t+1}^{\infty} \delta^k \beta^L_k.
\]

In any first-best equilibrium, \( \beta^L_k \to 0 \) as \( t \to \infty \), so the second term in this expression
vanishes as \( t \rightarrow \infty \). \( \lim_{t \rightarrow \infty} \beta_k^H = \frac{1}{N} \) due to symmetry, so

\[
\lim_{t \rightarrow \infty} \Omega_t = V^{FB} \delta \frac{1}{N}
\]

so obligation converges at \( \tilde{S} = S_G \). Thus, for \( \tilde{S} > S_G \), obligation diverges to \( \infty \), which implies by Lemma 2 that first-best cannot be attained. But for \( \tilde{S} \leq S_G \), the Favored Producer Automaton attains first-best.

Finally, we argue that \( \exists \) open \( \Delta \subseteq [0,1] \) such that for \( \delta \in \Delta \), the Favored Producer Allocation attains first best but a stationary equilibrium does not. Let \( \delta_{Stat} \) solve \((1-\delta_{Stat}) \frac{c}{p_{t-1} - p_0} = \delta_{Stat} \frac{1}{N} V^{FB} \). Proposition 2 implies that a stationary equilibrium attains first-best iff \( \delta > \delta_{Stat} \), and Assumption 3 implies that \( \delta_{Stat} > 0 \). Since both sides of (46) are continuous in \( \delta \), by Lemma 2 it suffices to show that for \( \delta = \delta_{Stat} \), the Favored Producer Automaton satisfies \( E_{\sigma^*} [S_i(h^{t+1})|S_i(h^t)|r_i,t] \geq V^{FB} \) for all \( (h^t, n_t) \) immediately following \( y_t = y_H \) such that \( x_t = i \) and \( e_t = 1 \). At any such history, \( i \) knows with certainty that he will be assigned rank 1 next period. Further, if we take \( r_{i,t} \) to denote the rank of player \( i \) at the start of round \( t \) in the Favored Producer Automaton, then for any on-path history \( h^t \), \( E_{\sigma^*} [S_i(h^t)|h^t] = E_{\sigma^*} [S_i(h^t)|r_{i,t}] \). By Assumptions 1 and 2, \( E_{\sigma^*} [S_i(h^t)|r_{i,t}] \) is strictly decreasing in \( r_{i,t} \) since \( r_{i,t} < r_{i,t} \) implies both that \( \text{Prob} \{ x_t = i \} > \text{Prob} \{ x_t = j \} \) and that \( \text{Prob} \{ r_{i,t+1} < r_{j,t+1} \} > \frac{1}{2} \) for all \( t' \geq 0 \). Since \( \sum_{i=1}^N E_{\sigma^*} [S_i(h^{t+1})|r_{i,t+1}] = V^{FB} \), \( E_{\sigma^*} [S_i(h^{t+1})|r_{i,t+1} = 1] > \frac{1}{N} V^{FB} \), and so (46) is slack in the Favored Producer Allocation for \( \delta = \delta_{Stat} \). This proves the result.

### A.6 Proposition 5

\( FD(\delta) \) is recursive. Let \( FD^P(\delta) \) be the Pareto frontier of \( FD(\delta) \). By definition of a full-disclosure equilibrium, agent \( x_i \) chooses \( e_t = 1 \) and produces \( y_t = y_H \) immediately preceding history \( (h^{t-1}, n_t) \) if and only if

\[
(1-\delta) \frac{c}{p_t - p_0} \leq \delta E [S_{x_t}(h^t)|h^{t-1}, n_t].
\]

Let \( \sigma^* \) be an optimal full-disclosure equilibrium, and let \( V^{EFF} = S_1(\sigma^*, \emptyset) + S_2(\sigma^*, \emptyset) \) be the total surplus produced in \( \sigma^* \). Define \( \alpha \equiv F(\{i\}) + F(\{1, 2\}) \), and note that \( F(\{i\}) = F(\emptyset) - \alpha \).

#### A.6.1 Preliminaries

**Claim 1:** Let \( \sigma^* \) be an optimal full-disclosure equilibrium. Either \( V^{EFF} = y_H p_0 \), or the following condition holds: \( \forall h^t \) on the equilibrium path such that in each \( t' \leq t \), either \( e_{t'} = 0 \) or \( y_{t'} = 0 \), \( \text{Prob}_{\sigma^*} \{ e_{t+1} = 1|h^t \} = 1 \). Let \( h^t \) be a history such that such that \( y_{t'} = 0 \forall t' < t \). We first claim that continuation play at \( h^t \) is efficient: \( \sum_{i=1}^2 U_i(h^t) = \max_{v \in FD(\delta)} v_1 + v_2 \).

Suppose not; then there exists a full-disclosure equilibrium \( \tilde{\sigma} \) that generates strictly higher surplus than \( \sigma^*(h^t) \). Consider an alternative strategy \( \tilde{\sigma} \) that is identical to \( \sigma^* \) except at history \( h^t \), at which \( \sigma(h^t) = \tilde{\sigma} \). By construction, these strategies generate larger total
surplus than $\sigma^*$, and moreover generate the same payoff as some equilibrium because (46) is still satisfied at all relevant histories. Contradiction.

If

$$(1 - \delta) \frac{c}{p_1 - p_0} > \alpha(y_H p_1 - c) \quad (47)$$

then (6) cannot hold at any history in equilibrium, and so $V^{Eff} = y_H p_0$. If (47) does not hold, then the strategy constructed in Proposition 4 is an equilibrium and so $V^{Eff} > y_H p_0$. In this case, there exists some history $h^t$ such that $\forall t' \leq t$, either $e_{t'} = 0$ or $y_{t'} = 0$, and $\text{Prob}_{\pi^*} \{e_{t+1} = 1 | h^t, x_{t+1} = i \} > 0$ for some $i \in \{1, 2\}$. Suppose $e_t = 1$ is optimal in some history, and let $(h^{t-1}, n_t)$ be the history immediately following $e_t = 1$ and $y_t = y_H$. Then it must be that

$$(1 - \delta)(y_H p_1 - c) + \delta p_1 \sum_{j=1}^{2} E \left[ S_j(h^t) | h^{t-1}, n_t \right] + (1 - p_1)\delta V^{Eff} > (1 - \delta)y_H p_0 + \delta V^{Eff} \quad (48)$$

where the inequality is strict because otherwise $V^{Eff} = (1 - \delta)y_H p_0 + \delta V^{Eff}$, which contradicts $V^{Eff} > y_H p_0$. Let $S(h^t) = S^*$ be the vector of dyad payoffs following $e_t = 1$ and $y_t = y_H$.

Now, consider a strategy $\sigma$ that is identical to $\sigma^*$ except at on-path histories $h^t$ that satisfy $\forall t' \leq t$, either $e_{t'} = 0$ or $y_{t'} = 0$, and specifies $\text{Prob}_{\sigma^*} \{e_{t+1} = 1 | h^t \} = 1$ at such histories. If $y_{t+1} = 0$, $\sigma$ and $\sigma^*$ both have continuation surplus $V^{Eff}$; if $y_t = y_H$, continuation dyad surplus is $S^* \in FD(\delta)$ (which is feasible because full-disclosure equilibria are recursive). Then $\sigma$ is equivalent to an equilibrium: (6) continues to hold because $\sigma$ differs from $\sigma^*$ only at histories $h^t$ such that $\forall t' \leq t$, either $e_{t'} = 0$ or $y_{t'} = 0$. By (48), $\sigma$ generates strictly higher surplus than $\sigma^*$. Contradiction.

A.6.2 Set-Up

Consider a full-disclosure equilibrium strategy $\sigma$, and let $(h^{t-1}, n_t) \in H \times N$ be a history with $x_t = i$, $e_t = 1$, and $y_t = y_H$. Let $\hat{S} \in \mathbb{R}_+^2$ be a vector of promises: that is, a minimal amount of continuation dyad-specific surplus for each agent. Let $h^t$ be the successor history on the equilibrium path to $(h^{t-1}, n_t)$. As in the proof of Lemma 2, we can restrict attention to equilibria in which the principal’s continuation surplus is zero.

In such equilibria, we want to maximize total continuation surplus subject to keeping the promises made to the agents and satisfying (6). The agents earn the entire dyad-specific surplus in the construction of Lemma 2, so it suffices to solve

$$\max \sum_{t'=1}^{\infty} \delta^{t'} (1 - \delta) E \left[ u_{1,t+t'} + u_{2,t+t'} | h^{t-1}, n_t \right] \quad (49)$$

subject to the constraint that the chosen $E [u_{1,t+t'} + u_{2,t+t'}]_{t'=1}^{\infty}$ be feasible and $\forall i \in \{1, 2\}$,

$$\sum_{t'=1}^{\infty} \delta^{t'} (1 - \delta) E \left[ u_{i,t+t'} | h^{t-1}, n_t \right] \geq \hat{S}_i \quad (50)$$
\[
\sum_{s=t}^{\infty} \delta^s (1 - \delta) E \left[ u_{x_{t+s}, t+s} | h^{t+i-1}, n_{t+i} \right] \geq \hat{S}
\]

\forall \hat{t}, (h^{t+i-1}, n_{t+i}) \text{ s.t. } e_{t+i} = 1, y_{t+i} = y_H

(51)

Where (50) is by definition of \( \hat{S} \) and (51) follows from (12).

To proceed, define

\[
L_{i_i}^{t+\tau'} = \{ \forall 1 \leq i \leq t', x_{t+i} = -i \Rightarrow (e_{t+i} = 0 \text{ OR } y_{t+i} = 0), h^t \}
\]

as the event that \( \{ x_{t+i} = -i, e_{t+i} = 1, y_{t+i} = y_H \} \) has not yet occurred in any round after \( h^t \), and let \( L_{-i}^{t+\tau'} = \left\{ L_{i_i}^{t+\tau'} \right\}^C \) be the complement of this event. Define

\[
\lambda_{t+\tau'} = \text{Prob}_\sigma \left\{ e_{-i,t+\tau'} = 1 | x_{t+\tau'} = -i, L_{i_i}^{t+\tau'} \right\},
\]

\[
\beta_{t+\tau'} = \text{Prob}_\sigma \{ x_{t+\tau'} = i | L_{i}^{t+\tau'} \}, \text{ and } b_{t+\tau'} = \{ x_{t+\tau'} = -i, e_{-i,t+\tau'} = 1, L_{i}^{t+\tau'} \}, \text{ and note that }
\]

\[
\text{Prob}_\sigma \left\{ L_{-i}^{t+\tau'} \right\} = \sum_{k=1}^{t'-1} \text{Prob}_\sigma \{ b_{t+\tau'} \} p_1.
\]

(53)

By the Law of Iterated Expectations, we can write (49) as

\[
\sum_{t'=1}^{\infty} \delta^{t'} (1 - \delta) \sum_{j=1}^{2} \left[ E \left[ u_{j,t+\tau'} | L_{i}^{t+\tau'} \right] \text{Prob}_\sigma \left\{ L_{i}^{t+\tau'} \right\} + E \left[ u_{j,t+\tau'} | L_{-i}^{t+\tau'} \right] \text{Prob}_\sigma \left\{ L_{-i}^{t+\tau'} \right\} \right] \]

(54)

Define \( (S_{1,t+\tau'}, S_{2,t+\tau'}) = S_{t+\tau'}^* \) by

\[
S_{i,t+\tau'}^* = \sum_{s=t'}^{\infty} \delta^s (1 - \delta) E \left[ u_{x_{t+s},t+s} | h^t, b_{t+\tau'}, y_{t+\tau'} = y_H \right]
\]

and note that

\[
E \left[ u_{j,t+\tau'} | L_{-i}^{t+\tau'} \right] \text{Prob}_\sigma \left\{ L_{-i}^{t+\tau'} \right\} = \sum_{s=1}^{t'} E \left[ u_{j,t+\tau'} | b_{t+s}, y_{t+s} = y_H \right] \text{Prob}_\sigma \{ b_{t+s}, y_{t+s} = y_H \}
\]

because the events \( \{ b_{t+s}, y_{t+s} = y_H \} \) for \( s \in \{1, \ldots, t'\} \) partition the event \( L_{-i}^{t+\tau'} \). Therefore,

\[
\sum_{t'=1}^{\infty} \delta^{t'} (1 - \delta) E \left[ u_{j,t+\tau'} | L_{-i}^{t+\tau'} \right] \text{Prob}_\sigma \left\{ L_{-i}^{t+\tau'} \right\} = \sum_{t'=1}^{\infty} \delta^{t'} (1 - \delta) \sum_{s=1}^{t'} E \left[ u_{j,t+\tau'} | b_{t+s}, y_{t+s} = y_H \right] \text{Prob}_\sigma \{ b_{t+s}, y_{t+s} = y_H \}.
\]

Using the dominated convergence theorem, we rearrange this expression to obtain:

\[
\sum_{t'=1}^{\infty} \text{Prob}_\sigma \{ b_{t+s}, y_{t+s} = y_H \} \sum_{t'=s}^{\infty} \delta^{t'} (1 - \delta) E \left[ u_{j,t+\tau'} | b_{t+s}, y_{t+s} = y_H \right]
\]
Now, \( \sum_{t=0}^{\infty} \delta^t (1 - \delta) E [u_{j,i+t'}|b_{i,t+s}, y_{t+s} = y_H] = \delta^s S^*_i \) by definition. Therefore, we can rewrite the second term in (54) as
\[
\sum_{t'=1}^{\infty} \sum_{j=1}^{2} \delta^{t'} \text{Prob}_i \left\{ b_{i,t'}, y_{t+i} = y_H \right\} S^*_{i,t+t'} \tag{55}
\]

Consider a relaxation of the problem (49)-(51) that ignores constraint (51) for player \( i \), and replaces this constraint for \( -i \) with the weaker set of constraints
\[
\sum_{s=0}^{\infty} \delta^{-i} (1 - \delta) E \left[ u_{x_{i,t+s}, t+s} | h^{t-1}, b_{t+i}, y_{t+i} = y_H \right] = S^*_{-i,t+i} \geq 1
\]
\[S^*_{t+i} \in FD(\delta) \]

The first constraint is a relaxation of the true constraint (51), while the second is implied by (51). This relaxed problem may be written
\[
\max_{\{x_{t+i}, e_{t+i}\}} \sum_{t'=1}^{\infty} \delta^{t'} (1 - \delta) \sum_{j=1}^{2} E \left[ u_{j,i+t'} | L_i^{t+t'} \right] \text{Prob}_i \left\{ L_i^{t+t'} \right\} + \delta^{t'} \text{Prob}_i \left\{ b_{i+t'}, y_{t+i} = y_H \right\} S^*_{j,t+t'} \tag{56}
\]
subject to:

1. Promise-keeping constraints for agent \( j \in \{1, 2\} \)
\[
\sum_{t'=1}^{\infty} \delta^{t'} (1 - \delta) \left[ E \left[ u_{j,i+t'} | L_i^{t+t'} \right] \text{Prob}_i \left\{ L_i^{t+t'} \right\} + \delta^{t'} \text{Prob}_i \left\{ b_{i+t'}, y_{t+i} = y_H \right\} S^*_{j,t+t'} \right] \geq S_j^* \tag{57}
\]

2. Constraint (46) whenever event \( L_{-i}^{t+t'} \) occurs:
\[
S^*_{-i,t+t'} \geq S, \forall t' \geq 1 \tag{58}
\]

3. Continuation play after \( L_{-i}^{t+t'} \) must be a full-disclosure equilibrium:
\[
S^*_{t+t'} \in FD(\delta), \forall t' \geq 1 \tag{59}
\]

**Claim 2: Without loss of generality, \( S^*_{1,t+t'} = S^*_1 \) and \( S^*_{2,t+t'} = S^*_2 \), \( \forall t' \geq 1 \).** For any solution to (56)-(59) for which the supposition fails, consider identical strategies, except that whenever event \( b_{i+t'} \) occurs, continuation play generates surplus \( S^* = (S^*_1, S^*_2) \) with \( S^* \in \arg \max_{V \in FD(\delta)} \sum_{t=0}^{T} V_i \) subject to the constraint
\[
V_i = \frac{1}{\sum_{t'=0}^{\infty} \delta^{t'} \text{Prob}_i \left\{ b_{i+t'}, y_{t+i} = y_H \right\}} \sum_{t'=0}^{\infty} \delta^{t'} \text{Prob}_i \left\{ b_{i+t'}, y_{t+i} = y_H \right\} S^*_{i,t+t'}, i \in \{1, 2\}.
\]

\( S^*_i \) is the limit of a sequence of convex combinations as \( T \to \infty \)
\[
\frac{1}{\sum_{t'=0}^{T} \delta^{t'} \text{Prob}_i \left\{ b_{i+t'}, y_{t+i} = y_H \right\}} \sum_{t'=0}^{T} \delta^{t'} \text{Prob}_i \left\{ b_{i+t'}, y_{t+i} = y_H \right\} S^*_{i,t+t'}
\]
and \( \{S_{i,t+1}'\} \subseteq FD(\delta) \); because \( FD(\delta) \) is convex, bounded, and closed, \( S^* \in FD(\delta) \) as well. \( FD(\delta) \) is convex because the messages in Corollary \[\text{1} \] can replicate a public randomization device, so

\[
\sum_{t'=0}^{\infty} \delta^t \text{Prob}_\sigma \{ b_{t,t'} \cdot y_{t,t'} = y_H \} (S_{i}^* + S_{2}^*) \geq \sum_{t'=0}^{\infty} \delta^t \text{Prob}_\sigma \{ b_{t,t'} \cdot y_{t,t'} = y_H \} (S_{i,t}^* + S_{2,t}^*).
\]

Finally, if the original \( \{S_{i,t+1}'\} \) solved the relaxed problem (56)-(59), then so does the alternative, which proves the claim.

### A.6.3 Return to Derivations

Consider the first term of (64). Note that

\[
\sum_{j=1}^{2} E \left[ v_{j,t'} \mid L_{i}^{t'} \right] = (1 - \delta) \beta_{i} \ (y_H \ p_i - c) + (1 - \delta) (1 - \beta_{i}) \ [y_H \ p_0 + \lambda_{i} \ (y_H \ (p_i - p_0) - c)]
\]

and that \( \lambda_{i} \beta_{i} (1 - \beta_{i}) \ Prob_\sigma \left\{ L_{i}^{t'} \right\} = \text{Prob}_\sigma \{ b_{i,t'} \} \).

Combining this fact with (55) and (61), we can write total continuation payoff (56) as

\[
(1 - \delta) (y_H (p_i - p_0) - c) \sum_{t'=1}^{\infty} \delta^{t'} \beta_{i} \left( 1 - \left( \sum_{k=1}^{t'-1} \text{Prob}_\sigma \{ b_{k,t+k} \} p_i \right) \right) \]

\[
+(1 - \delta) y_H p_0 \sum_{t'=1}^{\infty} \delta^{t'} (1 - \beta_{i}) \left( 1 - \left( \sum_{k=1}^{t'-1} \text{Prob}_\sigma \{ b_{k,t+k} \} p_i \right) \right) \]

\[
+(1 - \delta) (y_H (p_i - p_0) - c) \sum_{t'=1}^{\infty} \delta^{t'} (1 - \beta_{i}) \lambda_{i} \left( 1 - \left( \sum_{k=1}^{t'-1} \text{Prob}_\sigma \{ b_{k,t+k} \} p_i \right) \right) \]

\[
+(1 - \delta) (S_{1}^* + S_{2}^*) \sum_{t'=1}^{\infty} \delta^{t'} \left( \sum_{k=1}^{t'-1} \text{Prob}_\sigma \{ b_{k,t+k} \} p_i \right).
\]  

Note that

\[
\sum_{t'=1}^{\infty} \delta^{t'} \left( \sum_{k=1}^{t'-1} \text{Prob}_\sigma \{ b_{k,t+k} \} \right) = \frac{\delta}{1 - \delta} \sum_{t'=1}^{\infty} \delta^{t'} \text{Prob}_\sigma \{ b_{t,t'} \}
\]

so that (62) can be simplified (after much manipulation) to

\[
(1 - \delta) (y_H (p_i - p_0) - c) \sum_{t'=1}^{\infty} \delta^{t'} \beta_{i} \left( 1 - \left( \sum_{k=1}^{t'-1} \text{Prob}_\sigma \{ b_{k,t+k} \} p_i \right) \right) + \delta y_H p_0 \]

\[
+((1 - \delta) (y_H (p_i - p_0) - c) + \delta p_1 (S_{1}^* + S_{2}^*) - \delta y_H p_0 p_1) \sum_{t=1}^{\infty} \delta^{t} \text{Prob}_\sigma \{ b_{t,t+k} \} p_1.
\]

Because \( x_i = i, e_i = 1, y_t = y_H \) in round \( t \) of \( h_{i}^{t+1} \), \( \hat{S}_i \geq \hat{S} \). Further relax the problem (56)-(59) by ignoring (57) for agent \(-i\). For agent \( i \), (57) may be rewritten using similar manipulations:

\[
\delta \hat{S}_i \leq (y_H p_i - c) (1 - \delta) \sum_{t'=1}^{\infty} \delta^{t'} \beta_{i} \left( 1 - \left( \sum_{k=1}^{t'-1} \text{Prob}_\sigma \{ b_{k,t+k} \} p_i \right) \right) + \delta S_{i}^* \sum_{t=1}^{\infty} \delta^{t} \text{Prob}_\sigma \{ b_{t,t+k} \} p_1.
\]

We seek a solution to the constrained maximization problem (63) subject to (64), (58), and (59) in terms of variables \( \{\beta_{t,t'}\}_{t'} \), \( \{\text{Prob}_\sigma \{b_{t,t'}\}\}_{t'} \), and \( S^* \). Formally, \( \{\beta_{t,t'}\}_{t'} \) and
\{\text{Prob}_\sigma\{b_{t+t'}\}\} \text{ are closely related to one another, but we relax the problem by ignoring these constraints. Both (63) and the right-hand side of (64) are weakly increasing in } \beta_{t+t'} \forall t'. \text{ Therefore, it is optimal to choose } \beta_{t+t'} \text{ to be its maximum possible value: } \beta_{t+t'} = \alpha.

The final version of the relaxed problem is
\[
\max_{\{\text{Prob}_\sigma\{b_{t+t'}\}\}, S_i} \alpha \delta (y_H(p_1 - p_0) - c) + \delta y_Hp_0 + \sum_{t'=1}^{\infty} \delta' \text{Prob}_\sigma\{b_{t+t'}\} - \sum_{t'=1}^{\infty} \delta' \text{Prob}_\sigma\{b_{t+t'}\}
\]
subject to the promise-keeping constraint
\[
\delta \tilde{S}_i \leq \alpha \delta (y_Hp_1 - c) + \delta p_1 (S_i^* - \alpha (y_Hp_1 - c)) \sum_{t'=1}^{\infty} \delta' \text{Prob}_\sigma\{b_{t+t'}\}
\]
and the equilibrium constraints
\[
S_{i-1}^* \geq \tilde{S}, \quad S^* \text{a full-disclosure eq'm}
\]
Effectively, this problem has three choice variables, \(\sum_{t'=1}^{\infty} \delta' \text{Prob}_\sigma\{b_{t+t'}\}\), \(S_i^*\), and \(S_2^*\).

\(S_i^* \leq \alpha(y_Hp_1 - c)\) because \(\alpha(y_Hp_1 - c)\) is the maximum continuation surplus that can be given to agent \(i\), and \(S^*\) is optimally on the Pareto frontier of \(FD(\delta)\). Therefore,
\[
\delta \alpha(y_H(p_1 - p_0) - c) + \delta y_Hp_0p_1 \leq \delta \alpha(y_Hp_1 - c) + \delta(1 - \alpha)y_Hp_0 \leq S_1^* + S_2^*
\]
because \(\delta \alpha(y_Hp_1 - c) + \delta(1 - \alpha)y_Hp_0\) is the value of exclusive dealing with a single agent and thus the smallest Pareto-efficient payoff. Thus,
\[
((1-\delta)(y_H(p_1 - p_0) - c) + \delta p_1(S_1^* + S_2^*) - \delta \alpha(y_H(p_1 - p_0) - c) - \delta y_Hp_0p_1) \geq 0.
\]
The maximal feasible value of \(\sum_{t'=1}^{\infty} \delta' \text{Prob}_\sigma\{b_{t+t'}\}\) that satisfies (66) solves this problem. Next, we show that (66) binds at the optimum. Some value of \(\sum_{t'=1}^{\infty} \delta' \text{Prob}_\sigma\{b_{t+t'}\}\) makes (66) bind, so it suffices to show that some strategy attains this value. Let \(S_G\) and \(S_B\) be dyadic-specific payoffs in the good and bad states of the Favored Producer Automaton, respectively. Because first-best cannot be attained, \(S > S_G\). \(S_{i-1}^* \geq \tilde{S}\), so \(S_{i-1}^* < V^{FB} - \tilde{S} < S_B\). Because \(\delta \tilde{S}_i \geq \tilde{S}\), (66) is violated when \(\beta_{t+t'} = \alpha\), \(\Psi_{t+t'} = 1\) always; otherwise, the Favored Producer Automaton would induce high effort and thus attain the first-best. Additionally, (66) must be satisfied when \(\sum \delta' \text{Prob}_\sigma\{b_{t+t'}\} = 0\), since otherwise \(\tilde{S}_i > \alpha(y_Hp_1 - c)\), which is not feasible.

Consider the following class of strategies. Agent \(i\) chooses \(e_{t+t'} = 1\). With probability \(q\), agent \(-i\) chooses \(e_{t+t'} = 0\), \(\forall t'\); we will henceforth refer to this continuation strategy as “exclusive dealing with agent \(i\).” Otherwise, agent \(-i\) chooses \(e_{t+t'} = 1\) \(\forall t'\) until \(b_{t+t'} = 1\). In either case, \(\beta_{t+t'} = \alpha\) until \(b_{t+t'} = 1\). Once \(b_{t+t'} = 1\), continuation play generates fixed surplus \(S^*\). For this class of strategies, agent \(i\)’s dyad-specific surplus \(V_i^q\) is given by
\[
S_i^q = \alpha [(1-\delta)(y_Hp_1 - c) + \delta(q\alpha(y_Hp_1 - c) + (1-q)S_i^q)] + (1 - \alpha) [\delta p_1 S_i^* + \delta(1 - p_1)S_i^q].
\]
Rearranging this expression, we immediately show that \(S_i^q\) is continuous in \(q\), with \(\lim_{q \to 1} S_i^q = \)
\[ \alpha(y_H p_1 - c) \text{ and } \lim_{q \to 0} S_i^q \leq V_G. \]  By the Intermediate Value Theorem, there exists a \( q^* \) such that \( S_i^{q^*} = \hat{S}_i \). \[ \beta_{t+t'} = \alpha \] in this class of strategies, so the value of \( \sum \delta^t \text{Prob}_\sigma \{ b_{t+t'} \} \) corresponding to \( q^* \) makes (66) bind.

Finally, the Pareto frontier is downward-sloping, so if (66) binds then the promise \( \hat{S}_{-i} \) must also be satisfied or else \( \hat{S} \) is infeasible. Fixing continuation play \( S^* \), the class of equilibria characterized by \((q_1, q_2)\)-Exclusive Dealing solves the relaxed problem. It remains to show that \( q_1 = q_2 = q^* \) at each relevant history \( h^t+1 \).

Claim 3: Fix the vector of dyad-specific surpluses \((S_1, S_2)\) at history \( h^t+1 \). \( \exists \) a solution \((\bar{S}_1, \bar{S}_2)\) to (65) subject to (66) and (67) such that \((S_1, S_2) \leq (\bar{S}_1, \bar{S}_2) \) and \( \bar{S}_{-i} = \bar{S} \). Define \( S^{Eff,i} = (S_1^{Eff,i}, S_2^{Eff,i}) \) by

\[
S^{Eff,i} = \arg \max_{V \in FD(\delta)} V_i
\]

subject to

\[
S_1^{Eff,i} + S_2^{Eff,i} = V^{Eff}.
\]

There are two cases to consider: (1) \( S_i^{Eff,i} \geq \bar{S} \), or (2) \( S_i^{Eff,i} < \bar{S} \).

Consider case (2). In this case, \( S_1^* + S_2^* \) is decreasing in \( S_{-i}^* \) because the payoff frontier is convex. Let

\[
\bar{S}^* = \arg \max_{v \in FD(\delta)} v_2
\]

subject to

\[
v_1 = \bar{S}.
\]

Define \( Q = \sum_{t'=1}^{\infty} \delta^t \text{Prob}_\sigma \{ b_{t+t'} \} \). Let \((Q, S^*)\) satisfy (66), and consider an alternative \((\bar{Q}, \bar{S}^*)\), where \( \bar{Q} \) chosen so that (66) binds. Now \( \bar{S} \leq S_{-i}^* \), so \( \bar{S}_1^* + \bar{S}_2^* \geq S_1^* + S_2^* \) because \( FD(\delta) \) is convex. Moreover, \( \bar{S}_i^* \geq S_i^* \) because the Pareto frontier of \( FD(\delta) \) is downward-sloping and \( S_{-i}^* \leq S_{-i}^* \), so (66) continues to be satisfied. If \( S_i^* = \bar{S}_1^* \), then \( S_{-i}^* \geq S_{-i}^* \), \( Q = \bar{Q} \), and \((S_1, S_2) \leq (\bar{S}_1, \bar{S}_2) \) follows immediately. If \( \bar{S}_i^* > S_i^* \), then \( \bar{Q} > Q \), which further increases total surplus (65). Moreover, because (66) binds in both solutions, \( S_i^* = \bar{S}_1^* = \hat{S}_i \), and so \( S_{-i}^* \geq \bar{S}_{-i}^* \) because \( S_i^* + S_{-i}^* \geq S_i^* + S_{-i}^* \). This proves the claim in case (2).

Consider case (1). I claim that case (1) contradicts the assumption that first-best cannot be attained. By claim 1, \( e = 1 \) in the first round, and so we must have

\[
V^{Eff} = (1 - \delta)(y_H p_1 - c) + \delta V^{Eff}
\]

which implies that \( V^{Eff} = y_H p_1 - c = V^{FB} \). Contradiction.

A.6.4 Final Steps

By claim 3, the optimal equilibrium awards player \( i \) with exactly \( \bar{S} \) surplus following \( (x_t = i, \ e_t = 1, \ y_t = y_H) \) regardless of the rest of the history \( h^t \). Together with previous results,
we conclude that \((q^*, q^*)\)-Exclusive Dealing with \(q^*\) chosen so \(6\) binds is a solution to the relaxed problem. This is an equilibrium, proving the claim.\[\square\]

A.7 Proposition 6

It suffices to show that \(\exists\) a relational contract that conceals information and is strictly better than the optimal full-disclosure equilibrium. As usual, define \(\alpha \equiv F\{(1)\} + F\{(1, 2)\}\).

By Proposition 5, \(\exists q^* \in [0, 1]\) such that \(q^*\)-Exclusive Dealing is an optimal full-disclosure equilibrium. By (13), \(q = 1\) would give players strict incentives to work hard; because first-best cannot be attained, \(q^* \in (0, 1)\). For any \(\epsilon > 0\) such that \(q^* - \epsilon \in (0, 1)\), define

\[
\phi(\epsilon) \equiv \frac{(1 - q^* + \epsilon)p_1}{(1 - q^* + \epsilon)p_1 + (1 - p_1)} < p_1.
\]

Note that \(\exists \bar{q}(\epsilon), q(\epsilon)\) such that \(1 \geq \bar{q}(\epsilon) > q(\epsilon) \geq 0\) and \(\phi(\epsilon)q(\epsilon) + (1 - \phi(\epsilon))\bar{q}(\epsilon) = q^*\).

We construct a relational contract that conceals information; by Lemma 2 and Corollary 1 it suffices to specify an information partition, allocation rule, accept/reject decision, and effort choice. In \(t = 1\), \(x_1 \in \mathcal{P}_1\) is chosen randomly and \(d_1 = e_1 = 1\). Without loss of generality, let \(x_1 = 1\). In \(t = 2\), agent 2 remains uninformed of \(y_1\) with probability 1 if \(y_1 = 0\) and with probability \(1 - q^* + \epsilon\) if \(y_1 = y_H\). If \(1 \in \mathcal{P}_2\) then \(x_2 = 1\), otherwise agent 2 is given production whenever possible. If \(x_2 = 1\), then \(d_2 = e_2 = 1\). If \(x_2 = 2\), then \(d_2 = 1\), and \(e_2 = 1\) if and only if agent \(x_2\) was not informed of \(y_1\). From \(t = 3\) onwards, the continuation equilibrium is chosen from the full-disclosure Pareto frontier. Let \(V^t \in FD(\delta)\) be the full-disclosure payoff that maximizes agent \(i\)'s payoff among all optimal full-disclosure payoffs (that is, those which maximize total surplus), and \(V^{Ex,i} \in FD(\delta)\) be the continuation payoff when \(i\) is given exclusive dealing: \(V^{Ex,i}_t = \alpha(y_Hp_1 - c)\) and \(V^{Ex,i}_t = (F(\emptyset) - \alpha)y_Hp_0\). If \(x_1 = 1\), then the continuation payoff \(V^{Ex,q_1}(y_1, y_2) \in FD(\delta)\) is chosen identically to the optimal full-disclosure equilibrium. If \(x_1 = 2\) and \(y_1 = y_H\) was revealed to 2, then the continuation payoff is \(V^{Neq,L}(y_1, y_2) = V^{Ex,x_1}\). If \(x_1 = 2\) and \(y_1\) was not revealed, then the continuation payoff is \(V^{Neq,H}(y_1, y_2)\), where \(V^{Neq,H}(0, 0) = V^{Neq,H}(y_H, 0) = V^{x_1}, V^{Neq,H}(y_H, y_H) = \bar{q}(\epsilon)V^{Ex,x_2} + (1 - \bar{q}(\epsilon))V^{x_2}\), and \(V^{Neq,H}(0, y_H) = \bar{q}(\epsilon)V^{Ex,x_2} + (1 - \bar{q}(\epsilon))V^{x_2}\).

By Lemma 2, we need only show that \(6\) holds for the agents at each history. For \(t \geq 3\), the continuation equilibrium is full-disclosure and so this inequality holds by definition. In round \(t = 2\) and \(x_2 = 1\), this constraint is

\[
\tilde{S} \leq (q^*V^{Ex,x_1} + (1 - q^*)V^{x_1}) \tag{68}
\]

which holds because \(q^*\)-Exclusive Dealing is an equilibrium. If \(x_2 = 2\) and \(y_1 = y_H\) is revealed, then \(e_2 = 0\) and so \(6\) does not apply. If \(y_1\) is concealed, then agent 2 believes \(\text{Prob}_s\{y_1 = y_H|h^2_2\} = \phi(\epsilon)\). Therefore, agent 2's expected continuation surplus following \(y_2 = y_H\) is

\[
\phi(\epsilon)V^{Neq,H}(y_H, y_H) + (1 - \phi(\epsilon))V^{Neq,L}(0, y_H) = q^*V^{Ex,x_2} + (1 - q^*)V^{x_2} \geq \tilde{S}
\]

where equality is follows from the definition of \(V^{Neq,H}, \bar{q}\), and \(q\) and the inequality is a result of agent symmetry and (68).
Next, we check 1’s incentives in $t = 1$. Following $y_1 = y_H$, 1 believes he receives continuation surplus

$$S_1(y_H, \epsilon) = (1 - q^* + \epsilon)\alpha\delta\left(p_1 \left[ (1 - q^*)V_1^1 + q^*V_1^{Ex,1} \right] + (1 - p_1)V_1^1 \right) + (1 - \alpha)(1 - q^* + \epsilon)\delta\left(p_1(q^*)V_1^{Ex,2} + (1 - q^*)(V_1^2) + (1 - p_1)V_1^1 \right).$$

(69)

Because $q(\epsilon) < q^*$ and $V_1^{Ex,2} < V_1^2$,

$$p_1(q(\epsilon)V_1^{Ex,2} + (1 - q(\epsilon)V_1^2) + (1 - p_1)V_1^1 > p_1(q^*V_1^{Ex,2} + (1 - q^*)(V_1^2) + (1 - p_1)V_1^1$$

so

$$S_1(y_H, \epsilon) > (1 - q^* + \epsilon)\alpha\delta\left(p_1 \left[ (1 - q^*)V_1^1 + q^*V_1^{Ex,1} \right] + (1 - p_1)V_1^1 \right) + (1 - \alpha)(1 - q^* + \epsilon)\delta\left(p_1(q^*V_1^{Ex,2} + (1 - q^*)(V_1^2) + (1 - p_1)V_1^1 \right).$$

The right-hand side of this expression is continuous in $\epsilon$ and equals $q^*V_1^{Ex,1} + (1 - q^*)(V_1^1$ when $\epsilon = 0$. Hence, $S_1(y_H, 0) > \bar{S}$ by (68), and so $\exists \epsilon > 0$ such that $S_1(y_H, \epsilon) \geq \bar{S}$.

It remains to show that an equilibrium with such an $\epsilon$ generates strictly higher surplus than $q^*$-Exclusive Dealing. In $t = 1$, total surplus in both equilibria is $(1 - \delta)(y_Hp_1 - c)$. In $t = 2$, the equilibrium that conceals information generates strictly higher surplus, since

$$(\alpha + (1 - \alpha)(1 - p_1 + (1 - q^* + \epsilon)p_1)) (y_Hp_1 - c) > (\alpha + (1 - \alpha)(1 - p_1 + (1 - q^*)p_1)) (y_Hp_1 - c)$$

because $\epsilon > 0$. In the continuation game starting at $t = 3$, the equilibrium that conceals information has total surplus

$$\alpha E \left[ V_1^{Eq}(y_1, y_2) + V_2^{Eq}(y_1, y_2) | e_1 = e_2 = 1 \right] + (1 - \alpha)p_1(q^* - \epsilon)(V_1^{Ex,1} + V_2^{Ex,1}) + (1 - \alpha)(1 - p_1(q^* - \epsilon)) \left(q^*(V_1^{Ex,2} + V_2^{Ex,2}) + (1 - q^*)(V_1^2 + V_2^2) \right).$$

On the other hand, $q^*$-Exclusive dealing has continuation payoffs

$$\alpha E \left[ V_1^{Eq}(y_1, y_2) + V_2^{Eq}(y_1, y_2) | e_1 = e_2 = 1 \right] + (1 - \alpha)p_1q^*(V_1^{Ex,1} + V_2^{Ex,1}) + (1 - \alpha)(1 - p_1q^*) \left(q^*(V_1^{Ex,2} + V_2^{Ex,2}) + (1 - q^*)(V_1^2 + V_2^2) \right).$$

Comparing these payoffs, we find that the concealing-information equilibrium dominates the optimal full-disclosure equilibrium so long as

$$- \left(V_1^{Ex,1} + V_2^{Ex,1} \right) + \left(q^*(V_1^{Ex,2} + V_2^{Ex,2}) + (1 - q^*)(V_1^2 + V_2^2) \right) \geq 0$$

which holds because $V_1^{Ex,1} + V_2^{Ex,1} = V_1^{Ex,2} + V_2^{Ex,2} < V_1^2 + V_2^2$. Thus, we have found an equilibrium that is strictly better than the optimal full-disclosure equilibrium. ■
A.8 Lemma

It suffices to show that in each round, players’ strategies are best-responses in the game where all variables except $e$ are publicly observed, since messages reveal the true history along the equilibrium path. By construction, the agent is willing to choose $e = 1$ if the IC constraint holds. Given that agent surplus is $U_i = \gamma V_G$ following high output and the principal’s payoff is $(1 - \gamma)V^{FB}$ in any first-best equilibrium, we have

$$p_1[\delta(1 - \gamma)V^{FB} + \delta \gamma V_G] - (1 - \delta)c \geq p_0[\delta(1 - \gamma)V^{FB} + \delta \gamma V_G]$$

which can be rearranged to yield the condition

$$(1 - \delta) \frac{c}{p_1 - p_0} \leq \delta [(1 - \gamma)V^{FB} + \gamma V_G]$$

In each round, the agent awarded production earns $\gamma(1 - \delta)(y_Hp_1 - c)$ and the principal earns $(1 - \gamma)(1 - \delta)(y_Hp_1 - c)$, so Assumption [4] is satisfied. If players follow the message strategy, the principal is willing to follow the allocation rule and pay $\tau(y_H) = (1 - \gamma)\frac{\delta}{1 - \delta} V^{FB}$, since any deviation is immediately revealed by messages and mutually min-maxed, leaving the principal a payoff of 0. Similarly, agent $i$ is willing to pay $\frac{\delta}{1 - \delta} \gamma V_G$ if favored and $\frac{\delta}{1 - \gamma} V_B$ otherwise.

Finally, we must check that messages are incentive compatible. Any unilateral deviation is immediately observed because the producing agent and principal send identical messages in each period. Thus, any deviation is punished by mutual min-maxing, yielding a continuation payoff of 0. Hence, no player has an incentive to deviate from the prescribed message profile and these strategies form an equilibrium. ■

A.9 Proposition

A.9.1 Claim 1: Obligation is the Correct Notion

We closely follow the proof of Proposition [3]. Given strategy $\sigma$ that yields first-best surplus, define the residual expected obligation for player $i$ as

$$\hat{\Omega}_i^t = \frac{\Omega_i^{t-1}}{\delta} + \beta_t^H p_1 \delta(\hat{S} - (1 - \gamma)V^{FB}) - \beta_t^H [(1 - \delta)\gamma(y_H p_1 - c)]. \quad (70)$$

We first claim that if $\limsup_{t \to \infty} \hat{\Omega}_i^t = \infty$, then the strategy $\sigma$ cannot form an equilibrium that attains first-best.

Under Assumption [4], continuation surpluses for agent $i$ and the principal at on-path history $h^t$ are

$$U_i(h^t) = \gamma V^{FB} \mathbb{E}_\sigma \left[ \sum_{t'=0}^\infty \delta^{t'} (1 - \delta) 1_{i,t+t' \mid h^t} \right],$$

$$U_0(h^t) = (1 - \gamma)V^{FB}$$
respectively. Therefore, the necessary condition (14) can be rewritten

\[(1 - \delta) \frac{C}{p_1 - p_0} \leq \delta E_{\sigma} \left[ (1 - \gamma) V^{FB} + \gamma V^{FB} \sum_{t' = 0}^{\infty} \delta^{t'} (1 - \delta) 1_{i,t+t'} \mid (h_{x_t}, n_t, y_H) \right] \tag{71} \]

where \(n_t = (P_t, x_t, w_{i,t}, d_t = 1, e_t = 1) \in \mathcal{N}_e\). As before, consider the relaxed game where (71) must only hold the first time each agent \(i\) produces \(y_H\). Define \(b_i(h_t^i)\) as in (27); then we can write (71) as

\[b_i(h_t^i)p_1 \delta S \leq b_i(h_t^i)p_1 \delta E_{\sigma} \left[ (1 - \gamma) V^{FB} + \gamma V^{FB} \sum_{t' = 0}^{\infty} \delta^{t'} (1 - \delta) 1_{i,t+t'} \mid (h_{x_t}, n_t, y_H) \right] \]

which can be rearranged to yield

\[b_i(h_t^i)p_1 \delta (\bar{S} - (1 - \gamma)V^{FB}) \leq b_i(h_t^i)p_1 \delta E_{\sigma} \left[ \gamma V^{FB} \sum_{t' = 0}^{\infty} \delta^{t'} (1 - \delta) 1_{i,t+t'} \mid (h_{x_t}, n_t, y_H) \right]. \]

This expression is nearly identical to the corresponding inequality (28), with the sole exception that the obligation incurred upon production is \(\bar{S} - (1 - \gamma)V^{FB}\) rather than \(\bar{S}\). A stationary equilibrium attains first-best if \(\bar{S} - (1 - \gamma)V^{FB} \leq 0\). If \(\bar{V} - (1 - \gamma)V^{FB} > 0\), then an argument identical to Lemma 3 proves the desired result.

### A.9.2 Claim 2: Business Allocated to an Agent who has Already Produced

Next, we argue that Lemma 4 holds for \(\hat{\Omega}_t\): that is, \(\limsup_{t \to \infty} \hat{\Omega}_t < \infty\) for first-best strategy \(\sigma\) only if \(\limsup_{t \to \infty} \hat{\Omega}_t < \infty\) for first-best strategy \(\hat{\sigma}\) that (1) randomizes agent labels at the beginning of the game, and (2) awards production to an agent that has already produced \(y_H\) whenever possible. Result (1) follows by the same argument as in Lemma 4.

For Result (2), \(\bar{S} - (1 - \gamma)V^{FB} > 0\) is independent of the allocation rule, so the only difference between this argument and the proof of Lemma 4 is that the final term in (70) is multiplied by \(\gamma\). Performing the same set of manipulations, we can show

\[\frac{\partial \hat{\Omega}_t}{\partial \phi^m_{\nu}} = \left( p \delta (\bar{S} - (1 - \gamma)V^{FB}) + (1 - \delta) \gamma V^{FB} \right) \frac{\partial}{\partial \phi^m_{\nu}} \beta^L_L + \left( p \delta (\bar{S} - (1 - \gamma)V^{FB}) + (1 - \delta) \gamma V^{FB} \right) \sum_{s=1}^{t-\nu-1} \frac{1}{\delta^s} \frac{\partial}{\partial \phi^m_{\nu}} \beta_{1-s} \]

which is negative by (41), proving the desired claim.

### A.9.3 Claim 3: \(\lim_{t \to \infty} \hat{\Omega}_t = \infty\) if the Favored Producer Automaton does not attain first-best

Finally, we prove the proposition by arguing that \(\hat{\Omega}_t \to \infty\) whenever the Favored Producer Automaton does not generate first-best surplus. We follow Proposition 3: in this context,
the Favored Producer Automaton gives agent \( i \) continuation surplus

\[
\delta V_1 = \gamma E \left[ (1 - \delta)V^{FB} \sum_{t' = 1}^\infty \delta^{t'} b_{t + t'} | h^t \right]
\]
as in (42). Manipulating this expression to resemble (45), we have

\[
p_1 \delta S_G \sum_{k=1}^\infty \delta^k \beta^L_k - (1 - \delta) \gamma V^{FB} \sum_{k=1}^\infty \delta^k \beta^H_k = 0.
\]

In order for the Favored Producer Automaton to attain first-best, it must be that \( S_G \leq \tilde{S} - (1 - \gamma)V^{FB} \). Plugging \( S_G = (\tilde{S} - (1 - \gamma)V^{FB}) \) into a non-recursive expression for residual obligation, we yield

\[
\delta^t \hat{\Omega}_t + p_1 \delta (\tilde{S} - (1 - \gamma)V^{FB}) \sum_{k=t+1}^\infty \delta^k \beta^L_k - (1 - \delta) \gamma V^{FB} \sum_{k=t+1}^\infty \delta^k \beta^H_k = 0
\]

and hence

\[
\hat{\Omega}_t = \frac{1}{\delta} (1 - \delta) \gamma V^{FB} \sum_{k=t+1}^\infty \delta^k \beta^H_k - \frac{1}{\delta} p_1 \delta (\tilde{S} - (1 - \gamma)V^{FB}) \sum_{k=t+1}^\infty \delta^k \beta^L_k.
\]

In any first best equilibrium \( \beta^L_k \to 0 \) as \( t \to \infty \), so the second term in this expression vanishes as \( t \to \infty \). \( \lim_{t \to \infty} \beta^H_k = \frac{1}{N} \) due to symmetry, so

\[
\lim_{t \to \infty} \hat{\Omega}_t = V^{FB} \gamma \frac{1}{N}
\]

and this lower bound on obligation converges at \( \tilde{S} - (1 - \gamma)V^{FB} = S_G \), which proves that \( \hat{\Omega}_t \to \infty \) for any larger \( \tilde{S} \), as desired. ■

A.10 Proposition 8

First, suppose that \( y_t \) is contractible and the principal makes a take-it-or-leave-it contract offer at the beginning of each period. Using a contract identical to that in the proof of Proposition 1, first-best effort is attained in any equilibrium. Fix \( i \in \{1, 3, 5, ..., N - 1\} \), and consider an on-path history \( h^t \) that satisfies: \( \forall t' < t, x_{t'} \neq i + 1 \), and if \( x_{t'} = i \), then \( y_{t'} = 0 \) or \( y_t = \hat{y} \). We claim that in an optimal equilibrium \( \sigma^* \), \( \text{Prob}_{\sigma^*} \{ x_{t+1} = i + 1 | h^t \} = 0 \). Towards contradiction, suppose not. Consider the alternative strategy \( \hat{\sigma} \) that is identical to every history except successors to \( h^t \). For every \( \mathcal{P}_t \in 2^{\{1, ..., N\}} \) such that \( F(\mathcal{P}_t) > 0 \), if \( \text{Prob}_{\sigma^*} \{ x_{t+1} = i + 1 | (h^t, \mathcal{P}_{t+1}) \} > 0 \), then \( \text{Prob}_{\hat{\sigma}} \{ x_{t+1} = i + 1 | (h^t, \mathcal{P}_{t+1}) \} = 0 \) and \( x_{t+1} = i \) whenever \( \sigma^* \) would specify \( x_{t+1} = i + 1 \). Because \( i \in \mathcal{P}_t \) whenever \( i + 1 \in \mathcal{P}_t \), this alternative allocation rule is feasible. At any history \( \hat{h}^t \) such that \( \text{Prob} \{ y_t = \hat{y} | \hat{h}^t \} = 1 \), \( \text{Prob}_{\sigma^*} \{ x_{t+1} = i | \hat{h}^t \} = 0 \) and production is awarded to \( i + 1 \) whenever \( \sigma^* \) would specify \( x_{t+1} = i \). Consider continuation surplus under \( \hat{\sigma} \) and \( \sigma^* \) from \( (h^t, \mathcal{P}_{t+1}) \) such that production
is awarded to $i + 1$ under $\sigma^*$ and $i$ under $\hat{\sigma}$. Then under $\sigma^*$, production can never again be awarded to $i$, and so the sum of dyad-specific surpluses for agents $i$ and $i + 1$ in round $t' > t$ conditional on $h'$ is

$$\text{Prob}_{\sigma^*} \{ x_{t'} = i + 1 | y_{i+1} = y \} (1 - q)(yp_1 - c) + \text{Prob}_{\sigma^*} \{ x_{t'} = i + 1 | y_{i+1} = \bar{y} \} q(\bar{y}p_1 - c).$$

Under $\hat{\sigma}$ the sum of dyad-specific surpluses for $i$ and $i + 1$ in round $t'$ is

$$\text{Prob}_{\hat{\sigma}} \{ x_{t'} = i | y_i = \bar{y} \} q(\bar{y}p_1 - c) + \text{Prob}_{\hat{\sigma}} \{ x_{t'} = i, \forall t'' < t', \text{ if } x_{t''} = i, y_{t''} = 0 | y_i = \bar{y} \} (1 - q)(yp_1 - c) + \text{Prob}_{\hat{\sigma}} \{ x_{t'} = i + 1, \exists t'' < t' \text{ s.t. } x_{t''} = i, y_{t''} = y | y_i = \bar{y} \} (1 - q) ((q\bar{y} + (1 - q)y)p_1 - c).$$

By construction of $\hat{\sigma}$,

$$\text{Prob}_{\hat{\sigma}} \{ x_{t'} = i, \forall t'' < t', \text{ if } x_{t''} = i, y_{t''} = 0 | y_i = \bar{y} \} + \text{Prob}_{\hat{\sigma}} \{ x_{t'} = i + 1, \exists t'' < t' \text{ s.t. } x_{t''} = i, y_{t''} = y | y_i = \bar{y} \} = \text{Prob}_{\sigma^*} \{ x_{t'} = i + 1 | y_{i+1} = \bar{y} \},$$

so $\hat{\sigma}$ generates strictly larger dyad-specific surplus for agents $i$ and $i + 1$ following history $h'$. But $\hat{\sigma}$ and $\sigma^*$ are identical at every other history, and $\hat{\sigma}$ is an equilibrium because the principal is residual claimant when formal contracts are available. This proves the claim.

Suppose instead that $y_i$ is not contractible. Towards contradiction, consider an equilibrium $\sigma$ such that $\forall i \in \{1, 3, ..., N - 1\}$ and any history $h'$, if $\text{Prob}\{y_i = y | h'\} = 1$, then $\text{Prob}_{\sigma} \{ x_{t'} = i | h' \} = 0 \forall t' > 0$. First, note that $\text{Prob}\{y_i = y \} \in \{0, 1 - q, 1\} \forall h' \in H$. Because $p_0 = 0$, $\text{Prob}\{y_i = y | h^{t+1}\} = \text{Prob}\{y_i = y | h^t\}$ if $h^{t+1}$ is a successor to $h^t$ such that $x_{t+1} = i$, and $e_{t+1} = 0$. Since $yp_1 - c > 0, e = 1$ is efficient. If $x_{t+1} = i$ and $e_{t+1} = 1$ at history $h^t$ such that $\text{Prob}\{y_i = y | h^t\} = 1 - p$, (6) must hold at $(h^t, P_{t+1}, x_{t+1} = i, d_{t+1} = 1, e_{t+1} = 1)$, so a fortiori

$$(1 - \delta) \frac{c}{p_1 - p_0} \leq \delta q \left( \sum_{P \in P} F(P) \right) (yp_1 - c)$$

(72)

because $x_{t' + t} \neq i$ if $y_i = y$. Thus, $\forall \delta \in [0, 1)$, $\exists q^*(\delta)$ such that if $q \leq q^*(\delta)$, (72) is not satisfied, and so $\text{Prob}_{\sigma} \{ e_{t+1} = 1 | h^t, x_{t+1} = i \} = 0$ in any equilibrium such that $\text{Prob}_{\sigma} \{ x_{t'+t} = i | x_{t+1} = i, y_{t+1} = y, h^t \} = 0 \forall t' > 1$. Moreover, $q^*(\delta)$ is continuous and strictly decreasing in $\delta$, with $\lim_{\delta \rightarrow 1} q^*(\delta) = 0$ and $q^*(\delta) > 0$ for all $\delta < 1$.

It remains to show that some other equilibrium $\sigma^*$ dominates $\sigma$. Consider instead an alternative equilibrium such that $x_t \in P_t \cap \{1, 3, ..., N - 1\}$ is drawn uniformly at random from the odd-numbered agents who are able to produce. This allocation rule is stationary, and so a stationary contract (as in Levin (2003)) can be implemented. Such a stationary contract induces $e = 1$ from every odd agent so long as

$$(1 - \delta) \frac{c}{p_1 - p_0} \leq \delta \left( q(\bar{y}p_1 - c) + (1 - q)(yp_1 - c) \right) \left( \sum_{P \in P} \frac{F(P)}{|P|} \right)$$

(73)
and so a sufficient condition for high effort is
\[
(1 - \delta) \frac{c}{p_1 - p_0} \leq \delta(yp_1 - c) \left( \sum_{\mathcal{P} \in \mathcal{P}} \frac{F(\mathcal{P})}{|\mathcal{P}|} \right). \tag{74}
\]

Because \( yp_1 - c > 0 \), \( \exists \, \delta \in (0,1) \) such that (74) holds \( \forall \, \delta \geq \delta \). Choose some \( \bar{\delta} > \delta \); then for \( q < q^*(\bar{\delta}) \), (72) does not hold for \( \delta \in [\bar{\delta}, \delta) = \Delta \). Therefore, if \( q < q^*(\bar{\delta}) \) and \( \delta \in \Delta \), any equilibrium with efficient turnover generates surplus 0, while the stationary equilibrium with inefficient turnover generates surplus \( q(yp_1 - c) + (1 - q)(yp_1 - c) > 0 \). This proves the claim. ■

\[\text{A.11 Proposition [9]}\]

Suppose first that \( \{\kappa, K, \mathcal{P}, y_t\} \) are contractible. By the same argument as Proposition \[1\] \( e_t = 1 \) in every period. Therefore, total surplus is \((1 - F(\emptyset))(y_Hp_1 - c) + y^c \) if \( \kappa = 1 \), and \((1 - F(\emptyset))(y_Hp_1 - c) \) if \( \kappa = 0 \). Because \( K \) and \( \kappa \) are contractible, investment will occur whenever efficient, which is true so long as \( K \leq K_C = y^c \).

Next, consider the continuation game following the choice of \( \kappa \) when \( \{\mathcal{P}_t, y_t\} \) are not contractible. If \( \kappa = 0 \), Lemma \[2\] applies, so \( e = 0 \) at every history in the game because \((1 - \delta)\frac{\kappa}{p_1} > \delta(1 - F(\emptyset))(y_Hp_1 - c) \) by Assumption \[3\]. Since we’ve assumed \( p_0 = 0 \), this implies that total surplus in equilibrium is 0.

If instead \( \kappa = 1 \), we claim that there is an equilibrium that generates first-best surplus. Consider the following stationary strategies:

1. \( \text{Prob}_\sigma \{x_t = i | i \in \mathcal{P}_t\} = \frac{1}{|\mathcal{P}|}, \text{Prob}_\sigma \{x_t^c = i\} = \frac{1}{N} \).

2. Wages:
   
   (a) If \( x_t = x_t^c = i \), \( w_{i,t} = y_Hp_1 + y^c + (1 - p_1)(1 - F(\emptyset))(y_Hp_1 - c + y^c) \) and \( w_{j,t} = 0 \) for \( j \neq i \).
   
   (b) If \( x_t \neq x_t^c \), \( w_{i,x_t} = y_Hp_1 + (1 - p_1)\frac{\delta}{(1 - \delta)N}(1 - F(\emptyset))(y_Hp_1 - c + y^c) \) and \( w_{i,x_t^c} = y^c \).

3. \( d_t = d_t^c = e_t = 1 \) if wages are as in step 2, otherwise \( d_t = d_t^c = e_t = 0 \).

4. If \( y_t = 0 \) and \( x_t = i \), \( \tau_{i,t} = -\frac{\delta}{(1 - \delta)N}(1 - F(\emptyset))(y_Hp_1 - c + y^c) \). Otherwise \( \tau_{i,t} = 0 \).

5. At any off-path history \( h^t \) in which the principal and at least one agent has observed a deviation, the principal plays the same allocation rule. Whenever \( x_t = i \) for some \( i \) that has observed a deviation, \( d_t = d_t^c = e_t = w_{i,t} = \tau_{i,t} = 0 \).

Under these strategies, the principal earns 0 surplus at every on-path history and no more than that following a deviation, and so has no incentive to deviate. Using \[8\], agent \( i \) is willing to choose \( e_t = 1 \) so long as

\[
\frac{c}{p_1} \leq \frac{\delta}{(1 - \delta)N}(1 - F(\emptyset))(y_Hp_1 - c + y^c)
\]

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which is satisfied by Assumption 3. Agent \( i \)'s continuation surplus at the beginning of each round is \((1 - F(0))(y_{H}p_1 - c) + y^κ\) on the equilibrium path and he receives 0 off the equilibrium path, so he is willing to pay \( \tau_{i,t} \) so long as

\[-(1 - \delta)\tau_{i,t} \leq \delta \frac{1}{N} (1 - F(0))(y_{H}p_1 - c + y^κ)\]

which is satisfied by the definition of \( \tau_{i,t} \). \( d_t = d^κ_t = 1 \) because the agent earns no more than 0 following a deviation and earns positive continuation surplus on the equilibrium path. Off the equilibrium path, these strategies are mutual best-responses, and the principal is willing off following a deviation and earns positive continuation surplus on the equilibrium path. Therefore, \( \tau \) is satisfied by the definition of \( \tau \) which is satisfied by Assumption 5. Agent \( i \) be the \( w \) or \( τ \) which is satisfied by the definition of \( U \) of Lemma 1. Therefore, \( \kappa \) is an upper bound on the amount of extra surplus generated when \( \kappa = 1 \), \( K_R \leq K_{NC} \) immediately follows.

Next, we claim that \( K_R < K_{NC} \). Suppose first that \( \text{Prob}_{\sigma^*} \{ \epsilon_{t+1} = 1 | h^t, \mathcal{P}_{t+1} \neq \emptyset \} = 1 \) \( \forall h^t \) on the equilibrium path. By Assumption 3, 4 is only satisfied if \( \exists \) on-path \( h^t \) such that \( \text{Prob}_{\sigma^*} \{ x_{i,t}^κ = 1 | h^t \} > 0 \). Agent 1’s dyad-specific surplus is therefore

\[ S_1^κ = \sum_{t=0}^{∞} \delta^t (1 - \delta) [\text{Prob}_{\sigma^*} \{ x_t^κ = 1 \} y^κ + \text{Prob}_{\sigma^*} \{ x_t = 1 \}(y_{H}p_1 - c)] \]

and so

\[ S_1^κ < (1 - F(0))(y_{H}p_1 - c) + y^κ = K_{NC}. \]

Suppose instead that \( \sigma^* \) does not generate first-best effort provision. Then

\[ S_1^κ \leq \sum_{i=1}^{N} S_i^κ < (1 - F(0))(y_{H}p_1 - c) + y^κ = K_{NC} \]

which similarly proves the claim. Therefore, \( K_R < K_{NC} \).

Finally, we claim \( K_R > K_C \). Consider a stationary equilibrium that is identical to the one considered previously, except \( \forall t, \text{Prob}_{\sigma} \{ x_t^κ = 1 \} = 0, \text{Prob}_{\sigma} \{ x_t = 1 | l \in \mathcal{P}_t \} = 1, \epsilon_t = 1 \) if and only if \( x_t = 1 \), and \( w_{x_t,t} = 0 \) if \( x_t \neq 1 \). By an argument similar to earlier in this proof,
this is an equilibrium with payoff

\[ U^\kappa_1 = S^\kappa_1 = \sum_{\mathcal{P} \mid 1 \in \mathcal{P}} F(\mathcal{P})(y_{Hp_1} - c) + y^\kappa. \]

\( \kappa = 0 \) is immediately punished, so agent 1 chooses \( \kappa = 1 \) if

\[ K \leq K_C + \sum_{\mathcal{P} \mid 1 \in \mathcal{P}} F(\mathcal{P})(y_{Hp_1} - c) \]