DECENTRALIZED COLLEGE ADMISSIONS

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PRELIMINARY AND INCOMPLETE

Abstract. We develop a model of decentralized college admissions in which students’ preferences for colleges are uncertain, and colleges must incur costs when their enrollments exceed their capacities. Colleges’ admission decisions then become a tool for strategic yield management, because the enrollment at a college depends on not only students’ uncertain preferences but also other colleges’ admission decisions. We find that colleges’ equilibrium admission decisions exhibit “strategic targeting”—colleges may forgo admitting (even good) students likely sought after by the others and may admit (not as good) students likely overlooked by the others. Randomization in admissions may also emerge. The resulting assignment fails to be efficient (among students and among all parties including colleges and students) and leads to justified envy among students. Restricting the number of applications or allowing for wait-listing might alleviate colleges’ yield management problem, but the resulting assignments are still inefficient and admit justified envy. Centralized matching via Gale and Shapley’s Deferred Acceptance algorithm eliminates colleges’ yield management problem and justifies envy among students and attains efficiency. It also attains the outcome that is jointly optimal among colleges, but some colleges may be worse off relative to decentralized matching.

1. Introduction

The standard market design research on matching focuses on how best to design a centralized matching mechanism, taking the societal consensus on centralization as a given. While such a consensus exists in a number of markets (e.g., medical residency matching and public school matching), many markets remain decentralized (e.g., college admissions and graduate school admissions). Decentralized markets often exhibit congestion and do not operate efficiently (Roth and Xing, 1997). Although it is widely believed that these markets will benefit from improved coordination or centralization, it is not well understood why they remain decentralized and what welfare benefits would be gained by improving coordination or by centralizing them.
At least part of the problem is the lack of an analytical grasp of decentralized matching markets. Often treated as a black box, the equilibrium and welfare implications of decentralized matching markets have not been understood well in the literature. Indeed, we have yet to develop a workhorse model of decentralized matching that could serve as a useful benchmark for comparison with a centralized system.1

The current paper develops an analytical framework for understanding decentralized matching markets in the context of college admissions. In essence, college admissions are a case of two-sided, many-to-one matching, and much is understood about how best to organize such a market using a central clearinghouse.2 However, in many countries, such as the US, Korea and Japan, college admissions are organized similarly to decentralized labor markets, with exploding and binding admissions made by schools during a short window of time, among other things.3

With limited offers and acceptances to clear the markets, decentralized matching provides only a limited chance for colleges to learn students’ preferences and to condition their admission decisions on them. This presents a challenge for colleges in managing its yield. Inability to forecast yield accurately could result in too many or too few students enrolling a college relative to its capacity. Either mistake is costly. For instance, 1,415 freshmen accepted Yale’s invitation to join its incoming class in 1995-96, although the university had aimed for a class of 1,335. At the same year, Princeton also reported 1,100 entering students, the largest in its history. The college sets up mobile homes in fields and built new dorms to accommodate the students (Avery, Fairbanks and Zeckhauser, 2003).4

Importantly, the uncertainty facing a college with respect to a student’s enrollment depends not just on her preference but also on what other set of admissions she receives. This makes a college’s admission policy a strategic yield management decision. We provide a simple model of colleges’ strategic yield management problems and characterize the equilibrium outcomes of these strategic decisions. The explicit analysis of equilibrium allows us to evaluate the resulting assignment in terms of welfare and fairness and to compare this with outcomes that arise from other coordinated admissions and centralized matching.

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1The main exceptions are two excellent works by Chade and Smith (2006) and Chade, Lewis and Smith (2011). As we discuss more fully later, they focus on the portfolio decisions students face in application and colleges’ inference of students’ abilities based on imperfect signals. By contrast, the current paper focuses on the matching implications of college admissions, paying special attention to the yield management problem arising from (aggregately) uncertain students’ preferences.

2See Abdulkadiroğlu and Sönmez (2012) for an excellent survey.

3College admissions are centralized in varying degrees in Australia, China, Taiwan, Turkey and the UK.

4The cost may also take the form of an explicit sanction imposed on the admitting unit (e.g., department) by the government (as in Korea) or by the college (as in Australia).
In our baseline model, there are two colleges, each with limited capacity, and a unit mass of students with “scores” that are common for both colleges (e.g., high school GPA or SAT scores). Students apply to colleges at no cost. Colleges prefer students according to their scores, but they do not know students’ preferences toward them. This uncertainty takes an aggregate form: the mass of students preferring one college over the other varies across states that are unknown to the colleges. Over-enrollment is costly for a college in that it incurs a sufficiently high cost for each incremental enrollment in excess of its capacity. Our baseline model involves a simple time line: Initially, students simultaneously apply to colleges. Each college observes only the scores of those students who apply to them. Next, the two colleges simultaneously offer admissions to sets of students. Finally, the students who are admitted by either or both colleges decide on which admission they will accept.

Given that application is costless, students have a (weak) dominant strategy of applying to both colleges. Hence, the main focus of the analysis is the college’s admission decisions. Our main finding in this regard is characterized by “strategic targeting;” Since the students who attract competing admissions from the other college presents a greater enrollment uncertainty and add to a higher capacity cost, a college seeks to systematically avoid such students. Hence, in equilibrium, each college may forgo good students who are sought after by the other college and may admit less attractive students who appear overlooked by the other college. Randomization in admissions for students may also emerge. We then provide the existence of these equilibria. Next, we study the welfare and fairness properties of the equilibrium assignments and show that the assignment is typically unfair, that is, it entails justified envy among students, and fails to achieve efficiency among students and among all parties including colleges and students.

These results can be illustrated via a simple example. Suppose there are only two students, 1 and 2, applying to colleges A and B. Each college has one seat to fill and faces a prohibitively high cost of having two students. Student $i$ has score $v_i$, $i = 1, 2$, where $0 < v_2 < v_1 < 2v_2$. Each student has an equal probability of preferring either school, which is private information (unknown to the other student and to the colleges). Each college values having student $i$ at $v_i$. The applications are free of cost, and the timing is the same as that explained above.

Given the large cost of over-enrollment, each college admits only one of the students. Their payoffs are described as follow.

<table>
<thead>
<tr>
<th>$A$’s strategy</th>
<th>Admit 1</th>
<th>Admit 2</th>
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<tbody>
<tr>
<td>Admit 1</td>
<td>$\frac{1}{2}v_1, \frac{1}{2}v_1$</td>
<td>$v_1, v_2$</td>
</tr>
<tr>
<td>Admit 2</td>
<td>$v_2, v_1$</td>
<td>$\frac{1}{2}v_2, \frac{1}{2}v_2$</td>
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This game has a battle of the sexes’ structure (with asymmetric payoffs), so it is not
difficult to see that there are two different types of equilibria. First, there are two asymmetric
pure-strategy equilibria in which one college admits student 1 and the other admits student 2.
There is also a mixed-strategy equilibrium in which each college admits 1 with probability
\( \gamma := \frac{2v_1 - v_2}{v_1 + v_2} > 1/2 \) and admits 2 with probability \( 1 - \gamma \), where \( \gamma \) is chosen such that the other
college is indifferent. Both types of equilibria show the pattern of strategic targeting. In
the pure-strategy equilibria, colleges manage to avoid competition and thus randomness in
enrollment by targeting different students. In the mixed-strategy equilibrium, when a college
misses student 1 with probability \( 1 - \gamma \), the other college may admit the student. Thus, it
also exhibits targeting as colleges seek to avoid head-to-head competition, while it does not
result in perfect coordination.

This example, while extremely simple, suggests problems with decentralized matching in
terms of welfare and fairness. First, the student with high score (student 1) may be assigned
to a less preferred school (in both types of equilibria) even though both colleges prefer the
high scoring student; that is, justified envy arises. Second, it could be the case that student 1
prefers \( A \) and student 2 prefers \( B \), but the former is assigned to \( B \) and the latter is assigned
to \( A \), showing that the equilibrium outcome is inefficient among students. Lastly, the mixed-
strategy equilibrium is Pareto inefficient because both colleges may admit the same student,
in which case one college is unmatched and would rather match with the other student.

We next study two common ways for colleges to alleviate their yield management problem.
One common way is “self-targeting,” whereby colleges coordinate to restrict the number
of applications each student can submit. This form of coordination is observed in many
countries; for instance, students in the UK cannot apply to both Cambridge and Oxford,
students in Japan can apply to at most one public university, and students in Korea face
a similar restriction. Self-targeting reduces the enrollment uncertainty for the colleges, and
thus alleviates their yield management burden. Yet, we show that this method may not
completely eliminate the yield management problem and justified envy, and it may also fail
to achieve efficiency.

Another way to cope with the enrollment uncertainty is to employ a sequential admissions
strategy: Colleges admit some students and place others in the waiting lists in each of
multiple rounds and later extend further admissions to those in the waiting list when seats
open up from the previous round. This method is also observed in many countries, including
France and Korea. Sequential admissions may alleviate colleges’ yield management problem,
since colleges may adjust their admission offers based on the students’ acceptance behavior
and the information the colleges may learn over the course of the process. We show, however,
that colleges may still engage in strategic targeting under this mechanism, and the welfare and fairness problems still remain.

Finally, we consider a centralized matching via Gale and Shapley’s Deferred Acceptance algorithm (DA in short). We show that the DA eliminates colleges’ yield management problem and justified envy completely. It also attains efficiency, and the equilibrium outcome is “college efficient” in the sense that it is jointly optimal for the colleges. At the same time, it is possible for one college to be worse off relative to the decentralized matching. For instance, in the above example, suppose a pure-strategy equilibrium in which college \( i \) always gets student 1 is played. Then, that college will clearly be worse off from a switch to a centralization via DA because the college will not always attract student 1. This may explain a possible lack of consensus toward centralization and may underscore why college admissions remain decentralized in many countries.

The paper is organized as follows. Section 1.1 discusses the related literature. The model is introduced in Section 2. Equilibrium is characterized in Section 3. Section 3.1 establishes existence of equilibrium. Section 3.2 discusses welfare and fairness implications of equilibria. In Section 4, self-targeting via restriction on application is studied, and in Section 5, sequential admissions are studied. Centralized matching via DA is considered in Section 6. Section 7 concludes the paper. Proofs are provided in the Appendix unless stated otherwise. The Appendix also extends the baseline model to allow for more than two colleges and shows that our analysis in the two-college model carries over.

1.1. Related Literature. Several papers in the matching literature have considered decentralized matching markets. Roth and Xing (1997) study the entry-level market for clinical psychologists in which firms make offers to workers sequentially within a day and workers can accept, reject or hold an offer. They find that, mainly based on simulations, such a decentralized (but coordinated) market exhibits congestion, i.e., not enough offers and acceptances could be made to clear the market, and the resulting outcome is unstable. Neiderle and Yariv (2009) also study a decentralized (one-to-one matching) market in which firms make offers sequentially through multiple periods. They provide sufficient conditions under which such decentralized markets generate stable outcomes in equilibrium in the presence of market friction (namely, time discounting) and preference uncertainty. Like these models, our model concerns about the consequence of congestion arising from decentralized matching, but unlike Roth and Xing (1997), we have an analytical model that allows us to characterize both the equilibrium admission decisions and their welfare and fairness properties. In particular, the current framework develops a new theme of strategic targeting. Moreover, the explicit
analysis of equilibria permits a clear comparison with the outcome that would arise from a centralized matching.

The college admissions problem has recently received attention in the economics literature. Chade and Smith (2006) study students’ application decision as a portfolio choice problem. Chade, Lewis and Smith (2011) analyze colleges’ admission decisions together with the students’ application decisions. In their model, students with heterogeneous abilities make application decisions subject to application costs, and colleges set admission standards based on noisy signals on students’ abilities. Avery and Levin (2010) and Lee (2009) study early admissions. Unlike our model, these models have no aggregate uncertainty with respect to students’ preferences, which means that the colleges in their model do not face any enrollment uncertainty. Hence, colleges do not employ strategic targeting; they instead use cutoff strategies.

Some aspects of our equilibrium are related to political lobbying behavior studied by Lizzeri and Persico (2001, 2005). Just as colleges target students in our model, politicians in these models target voters for distributing their favors. In their models, voters are homogeneous, and a voter votes for the candidate that offers her the largest favor. In our model, however, students have heterogeneous abilities and preferences. Thus, colleges’ admission decisions are more complicated—admission probabilities vary according to students’ scores.

Our model also shares some similarities with directed search models, such as Montgomery (1991) and Burdett, Shi and Wright (2001). In these studies, each firm (seller) posts a wage (price), and each worker (buyer) decides which job to apply for. Firms have a fixed number of job openings and cannot hire more than the capacity, and workers can only apply to one firm. Workers’ inability to precisely coordinate their search decisions causes a “search friction,” so they randomize on application decisions. Just like the workers in these models, colleges in our model can be seen to engage in “directed searches” on students. The difference is that the colleges in our model offer admissions to many students, which raises a qualitatively novel problem.

2. Model

Our model is described as follows. There is a unit mass of students with score $v$ distributed on $[0,1]$ according to an absolutely continuous distribution $G(\cdot)$. There are two colleges, $A$ and $B$, each with capacity $\kappa < \frac{1}{2}$. (Appendix H will extend the model to include more than two colleges, showing that our main results carry over to that extension.) Each college values a student with score $v$ at $v$ and faces a cost $\lambda \geq 1$ for each incremental enrollment exceeding the quota. Each student has a preference over the two colleges, which is private information. A state of nature, $s \in [0,1]$, determines the fraction of students who prefer $A$
over \(B\): The state \(s\) is drawn from \([0,1]\) according to the uniform distribution. In state \(s\), a fraction \(\mu(s) \in [0,1]\) of students prefers \(A\) to \(B\), where \(\mu(\cdot)\) is strictly increasing and continuous in \(s\).\(^5\) While we shall consider a general environment with respect to \(\mu(\cdot)\), some result will consider a \textbf{symmetric environment} in which \(\mu(s) = 1 - \mu(1-s)\) for all \(s \in [0,1]\).

In a symmetric environment, the measure of students who prefer \(A\) over \(B\) is symmetric around \(s = \frac{1}{2}\).

The timing of the game is as follows. First, Nature draws the (aggregate-uncertainty) state \(s\). Next, all students simultaneously apply to college(s). Each college observes the scores of only those students who apply to it. Next, colleges simultaneously decide which applicant(s) to admit. Last, those students who have received at least one admission offer decide on which offer to accept.

We assume that there is no application cost for the students, so it is a weak dominant strategy for each student to apply to both colleges. Throughout this paper, we focus on a perfect Bayesian equilibrium in which students play the weak dominant strategy.\(^6\)

Colleges distribute admissions based on students’ scores. Let \(\alpha: [0,1] \to [0,1]\) and \(\beta: [0,1] \to [0,1]\) be college \(A\) and \(B\)’s admission strategies, respectively, in terms of the probability of offering an admission to each type \(v\).

For given \(\alpha(\cdot)\) and \(\beta(\cdot)\), let \(\mathcal{V}_A := \{v \in [0,1] | \alpha(v) > 0\}\) and \(\mathcal{V}_B := \{v \in [0,1] | \beta(v) > 0\}\) be the set of students to whom colleges \(A\) and \(B\), respectively, make an admission offer with positive probability. Let \(\mathcal{V}_{AB} := \mathcal{V}_A \cap \mathcal{V}_B\). If \(\mathcal{V}_{AB}\) has a positive measure in an equilibrium, this means that a positive measure of students has admissions from both colleges. We call such an equilibrium \textbf{competitive}. An equilibrium in which \(\mathcal{V}_{AB}\) has zero measure is called \textbf{non-competitive}.

Consider a student with score \(v\). The student will attend college \(A\) if either she is admitted only by \(A\), which happens with probability \(\alpha(v)[1 - \beta(v)]\), or admitted by both colleges but prefers \(A\) to \(B\), which happens with probability \(\mu(s)\alpha(v)\beta(v)\) in state \(s\). Thus, the mass of students who attend \(A\) in state \(s\), given strategies \(\alpha(\cdot)\) and \(\beta(\cdot)\), is

\[
m_A(s) := \int_0^1 \alpha(v)[1 - \beta(v) + \mu(s)\beta(v)] dG(v).
\]

Similarly, the mass of students who attend \(B\) in state \(s\) is

\[
m_B(s) := \int_0^1 \beta(v)[1 - \alpha(v) + (1 - \mu(s))\alpha(v)] dG(v).
\]

\(^5\)There is no loss of generality to assume the uniform distribution, because for a distribution \(F(\cdot)\) of \(s\), we can simply relabel \(s\) and the popularity of a college over the other is captured by \(\mu(\cdot)\).

\(^6\)The strategy of applying to both colleges can be made a strictly dominant strategy if students have some uncertainty about their scores, which is realistic in case the scores are either not publicly observable or depend on multiple dimensions of attributes, the weighting of which may be unknown to the students.
Each college realizes the scores of enrolled students as its gross payoff and incurs cost $\lambda$ for each increment beyond its capacity. Thus, college $A$ and $B$’s ex ante payoffs are, respectively

$$\pi_A := \mathbb{E}_s \left[ \int_0^1 v \alpha(v) [1 - \beta(v) + \mu(s) \beta(v)] dG(v) - \lambda \max \{m_A(s) - \kappa, 0\} \right]$$

and

$$\pi_B := \mathbb{E}_s \left[ \int_0^1 v \beta(v) [1 - \alpha(v) + (1 - \mu(s)) \alpha(v)] dG(v) - \lambda \max \{m_B(s) - \kappa, 0\} \right].$$

One immediate observation is that each college’s payoff is concave in its own admission strategy,\(^7\) that is, $\pi_A(\eta \alpha + (1 - \eta) \alpha') \geq \eta \pi_A(\alpha) + (1 - \eta) \pi_A(\alpha')$ for any feasible strategies $\alpha$ and $\alpha'$ and $\eta \in [0, 1]$. Therefore, mixing over $\alpha$’s is unprofitable for college $A$ (similarly $\beta$’s for college $B$). For this reason, any equilibrium is characterized by a pair $(\alpha, \beta)$. Of course, this does not mean that the equilibrium is in pure-strategies; the values of $\alpha$ and/or $\beta$ may be strictly interior, in which case the admission strategies would involve randomization.

In the following sections, we characterize different types of equilibria and establish their existence. We then provide welfare and fairness properties of equilibria.

3. Characterization of Equilibrium

We analyze colleges’ admission decisions in this section. To this end, we fix any equilibrium $(\alpha, \beta)$ and explore the properties the equilibrium satisfies. Later, we shall establish existence of the equilibria. We begin with the following observations, whose proofs are in Appendix A.

**Lemma 1.** In any equilibrium $(\alpha, \beta)$, the followings hold.

(i) $m_A(0) \leq \kappa \leq m_A(1)$ and $m_B(1) \leq \kappa \leq m_B(0)$.

(ii) $\mathcal{V}_A \cup \mathcal{V}_B$ is a connected interval with $\sup \{\mathcal{V}_A \cup \mathcal{V}_B\} = 1$ and $\inf \{\mathcal{V}_A \cup \mathcal{V}_B\} > 0$.

(iii) If the equilibrium is competitive (i.e., $\mathcal{V}_{AB}$ has a positive measure), then there exists a unique $(\hat{s}_A, \hat{s}_B) \in (0, 1)^2$ such that $m_A(\hat{s}_A) = \kappa$ and $m_B(\hat{s}_B) = \kappa$.

(iv) If the equilibrium is non-competitive (i.e., $\mathcal{V}_{AB}$ has zero measure), then $m_A(s) = m_B(s) = \kappa$ for all $s \in [0, 1]$. Further, almost every student with $v \geq G^{-1}(1 - 2\kappa)$ receives an admission offer from exactly one college. That is, $\alpha(v) = 1$ for almost every $v \in \mathcal{V}_A$ and $\beta(v) = 1$ for almost every $v \in \mathcal{V}_B$.

Part (i) of the lemma states that in equilibrium, colleges cannot have strict over-enrollment and/or strict under-enrollment in all states. This is obvious since if there were over-enrollment in all states for a college, then since $\lambda \geq 1$, it will profitably deviate by rejecting some students with $v < 1$, and if there were under-enrollment in all states, a college will likewise profitably deviate by accepting more students. Part (ii) suggests that if a student with

\(^7\)See Lemma D2 in Appendix D for the proof.
score \( v \) is admitted by either college, then all students with scores higher than such \( v \) must be admitted by some college at least with positive probability, and there is a positive mass of students in the low tail who are never admitted by either college. Parts (iii) suggests that in a competitive equilibrium, the colleges will suffer from under-enrollment in some states and over-enrollment in other states. This is intuitive since given (aggregately) uncertain preferences on the part of students, the presence of students who receive admissions from both colleges presents non-trivial enrollment uncertainty. Each college will deal with uncertainty by optimally trading off the cost of over-enrollment with the loss from under-enrollment, thus entailing both types of mistakes depending on the states. Part (iv) states that in a non-competitive equilibrium colleges avoid the over- and under-enrollment problems, and almost every top \( 2\kappa \) students receive admissions from only one college. This is, again, intuitive since the colleges in this case face no enrollment uncertainty, so they will fill their capacities exactly in all states with students who have the top \( 2\kappa \) scores.

In what follows, we shall focus on competitive equilibria. There are several reasons for this. It will be seen that competitive equilibria always exist (see Theorem 3). By contrast, non-competitive equilibria can be ruled out if either \( \lambda \geq 1 \) is not too large or \( \kappa < \frac{1}{2} \) is not too small (see Appendix B). Finally, even if a noncompetitive equilibrium exists, the characterization provided in Lemma 1-(iv) is sufficient for our welfare and fairness statements, as will be seen later.

Therefore, fix any competitive equilibrium \((\alpha, \beta)\). It is convenient to rewrite \( A \)'s payoff at the equilibrium as follows:

\[
\pi_A = \int_0^1 v \alpha(v)[1 - \beta(v) + \bar{\mu}\beta(v)] dG(v) - \lambda \mathbb{E}_s[m_A(s) - \kappa | s > \hat{s}_A](1 - \hat{s}_A)
\]

\[
= \int_0^1 \alpha(v) H_{\alpha}(v, \beta(v)) dG(v) + \lambda (1 - \hat{s}_A) \kappa,
\]

where \( \bar{\mu} := \mathbb{E}_s[\mu(s)] \), \( \hat{s}_A \in (0, 1) \) is such that \( m_A(\hat{s}_A) = \kappa \) (as defined in Lemma 1-(iii)), and

\[
H_{\alpha}(v, \beta(v)) := v[1 - \beta(v) + \bar{\mu}\beta(v)] - \lambda \int_{\hat{s}_A}^1 [1 - \beta(v) + \mu(s)\beta(v)] ds \tag{3.1}
\]

is \( A \)'s marginal payoff from admitting a student with \( v \) for given \( \beta(\cdot) \) and \( \hat{s}_A \) in equilibrium. (We shall suppress its dependence on \( \hat{s}_A \) unless it is important.) This captures \( A \)'s local incentive, that is, what \( A \) gains by admitting \( v \), holding fixed its opponent’s decision and its own decisions for the rest of the students at \( \alpha(\cdot) \).

It is useful to examine the marginal payoff more closely, as it provides a key insight for strategic targeting. It can be decomposed as follow:

\[
H_{\alpha}(v, \beta(v)) = (1 - \beta(v)) H_{\alpha}(v, 0) + \beta(v) H_{\alpha}(v, 1),
\]
where
\[ H_\alpha(v, 0) = v - \lambda(1 - \hat{s}_A) \] (3.2)
and
\[ H_\alpha(v, 1) = \mu \left[ v - \lambda \int_{\hat{s}_A}^1 \mu(s) \frac{ds}{\mu} \right] \] (3.3)
are the marginal payoffs of college A from admitting type-v student when that student does not receive admission from B and when she does, respectively.

It is particularly instructive to consider the second terms of these expressions—the marginal cost to A of admitting a type-v student. Recall that the college incurs capacity cost only when there is over-enrollment. Suppose first that the student does not receive a competing offer. Then, she accepts A’s admission for sure. Hence, conditional on acceptance, over-enrollment occurs with probability \((1 - \hat{s}_A)\) for that college. So, in this case, the marginal cost of admitting type-v student is \(\lambda(1 - \hat{s}_A)\), which explains the second term of (3.2).

Suppose next that type-v student receives a competing offer from B. In this case, the student accepts college A only when she prefers A to B. Hence, conditional on accepting A’s offer, the over-enrollment arises with probability \(\int_{\hat{s}_A}^1 \mu(s) \frac{ds}{\mu}\), so its marginal cost (conditional on acceptance) is \(\lambda \int_{\hat{s}_A}^1 \mu(s) \frac{ds}{\mu}\), the second term inside the square brackets in (3.3).

It is important to notice that the likelihood of over-enrollment (conditional on acceptance) is higher when the student has a competing offer from B than she does not:
\[
\int_{\hat{s}_A}^1 \mu(s) \frac{ds}{\mu} > (1 - \hat{s}_A) \mathbb{E}[\mu(s)|s > \hat{s}_A] > (1 - \hat{s}_A),
\]
where the strict inequality follows since \(\hat{s}_A \in (0, 1)\) (by Lemma 1-(iii) and since \(\mathbb{E}[\mu(s)|s > \hat{s}_A] > \hat{s}_A > 0\)). The reason is as follows. When the student receives an offer from B, she is more likely to accept A’s offer when \(\mu(s)\) is high than when it is not. Thus, unlike the case where the student does not receive a competing offer, conditional on acceptance, the stat is more likely to be high when the admitted student has a competing offer than she does not. This explains why it is more costly to admit a student who is sought after by another college than a student who is not.

This observation implies that \(H_\alpha(v, \beta(v))\) partitions the students’ type space into three intervals, as depicted in Figure 3.1. For a student with \(v > \overline{v}_A := \lambda \int_{\hat{s}_A}^1 \mu(s) ds \frac{1}{\mu}\), we have \(H_\alpha(v, 1) > 0\), so college A admits such a student even if college B admits the student in equilibrium. For a student with \(v < \underline{v}_A := \lambda(1 - \hat{s}_A)\), we have \(H_\alpha(v, 0) < 0\), so college A has no incentive to admit such a student even if college B does not admit the student. For a student with \(v \in (\underline{v}_A, \overline{v}_A)\), we have \(H_\alpha(v, 0) > 0\) but \(H_\alpha(v, 1) < 0\). Hence, college A has an incentive to admit such a student if B does not admit the student, but not if college B
admits that student. Intuitively, each college considers the enrollment uncertainty worth taking on only when the student has a sufficiently high score, and for a student with a lower score (but above the lower cutoff), the college finds admission is worthwhile only when it is assured of facing no competition and thus no uncertainty in enrollment. As will be seen, the presence of this intermediate range of scores leads to non-cutoff equilibria.

The characterization of $B$’s admission strategy is completely symmetric. As before, $B$’s payoff from admitting a type-$v$ student is expressed as:

$$\pi_B = \int_0^1 \beta(v)H_\beta(v, \alpha(v))dG(v) + \lambda \hat{s}_B \kappa,$$

where

$$H_\beta(v, \alpha(v)) := v[1 - \alpha(v) + (1 - \mu)\alpha(v)] - \lambda \int_0^{\hat{s}_B} [1 - \alpha(v) + (1 - \mu(s))\alpha(v)] ds$$

is $B$’s marginal payoff from admitting a student with score $v$, holding fixed $A$’s admission decision and its own decisions for the remaining students at $\beta(\cdot)$. Just as before, $H_\beta(v, \alpha(v))$ partitions the students’ type space into three intervals separated by two threshold values $\underline{v}_B$ and $\overline{v}_B$, where $\underline{v}_B := \lambda \hat{s}_B < \overline{v}_B := \lambda \hat{s}_B \frac{\mathbb{E}[1-\mu(s)\mid s<\hat{s}_B]}{1-\mu}$, such that $B$ admits all students with $v > \overline{v}_B$ for sure, rejects all students with $v < \underline{v}_B$, and accepts students with $v \in (\underline{v}_B, \overline{v}_B)$ if they are not admitted by $A$ but rejects them when they are admitted by $A$.

Combining the two colleges’ admission decisions leads to the following characterization of equilibria.

**Theorem 1.** In any competitive equilibrium, there exist $\underline{v}_i < \overline{v}_i$, $i = A, B$, such that college $i$ admits students with $v > \overline{v}_i$ and students with $v \in [\underline{v}_i, \underline{v}_j]$ and rejects students with $v < \underline{v}_i$ and students $v \in [\overline{v}_j, \overline{v}_i]$, where $j \neq i$. Students with $v \in [\max \{\underline{v}_A, \underline{v}_B\}, \min \{\overline{v}_A, \overline{v}_B\}]$ are admitted by at least one college with positive probability.

Theorem 1 describes the structure of any competitive equilibrium. Figure 3.2 depicts a typical pure-strategy equilibrium. Here, top students with $v > \overline{v}_A = \max \{\overline{v}_A, \overline{v}_B\}$ receive offers from both colleges, because their scores are above the high cutoffs for both colleges. The next tier students with $v \in (\overline{v}_B, \overline{v}_A)$ receive offers only from $B$, since $A$ finds them admission-worthy only if $B$ does not admit them, but in this case, $B$ is interested in admitting them no matter what $A$ does. Each of the students in the intermediate range of scores, i.e., $[\underline{v}_A, \overline{v}_B]$, receives an admission offer from only one college. Obviously, how the two colleges
coordinate exactly on these students are indeterminate, and the figure depicts one possible coordination. The students with scores \( v \in \mathbb{[v_B, v_A]} \) receive offers only from \( B \), since it is the only college that finds them admission-worthy given that they are not admitted by \( A \). Finally, the students at the bottom below \( v_B = \min \{v_A, v_B\} \) do not receive any offers.

Clearly, strategic targeting occurs in this equilibrium: A college does not admit good students because they are sought after by the other college, and it admits less attractive students because they are not sought after by the other college. This feature stands in stark contrast with the cutoff strategy equilibrium found by the existing literature (see Chade, Lewis and Smith, 2011).

As noted, there may be many ways for colleges to coordinate their admissions for students with \( v \in [\underline{v}, \overline{v}] \), where \( \underline{v} := \max \{v_A, v_B\} \) and \( \overline{v} := \min \{v_A, v_B\} \). The range of different pure-strategy equilibria can be summarized by two extreme types of equilibria. We call a competitive equilibrium an \( A \)-priority equilibrium if \( \alpha(v) = 1 \) for all \( v \in [\underline{v}, \overline{v}] \), and a \( B \)-priority equilibrium if \( \beta(v) = 1 \) for all \( v \in [\underline{v}, \overline{v}] \). In words, in an \( i \)-priority equilibrium, the coordination is tilted in favor of college \( i \). Clearly, between these two equilibria, one can construct (infinitely) many equilibria.
In practice, it is implausible for colleges to achieve the kind of precise coordination described in the pure-strategy equilibria. It seems much more plausible for colleges to randomize over students with the intermediate range of scores $v \in [\underline{v}, \overline{v}]$. A typical mixed-strategy equilibrium is depicted in Figure 3.3.

Notice that the admission strategies outside the intermediate range is similar to that in the above pure-strategy equilibrium, as this is completely pinned down by Theorem 1. For the intermediate range of scores, interior-valued admissions strategies can be structured so as to keep each college indifferent, as follows. For each $v \in [\underline{v}, \overline{v}]$, let $\alpha(v) = \alpha_0(v)$ and $\beta(v) = \beta_0(v)$, where

$$H_\alpha(v, \beta_0(v)) = 0 \quad \text{and} \quad H_\beta(v, \alpha_0(v)) = 0,$$

or equivalently,

$$\alpha_0(v) := \frac{v - \lambda \hat{s}_B}{\overline{v} - \lambda \int_0^{\overline{s}_B} \mu(s) \, ds} \quad (3.4)$$

and

$$\beta_0(v) := \frac{v - \lambda (1 - \hat{s}_A)}{v(1 - \overline{\mu}) - \lambda \int_{\hat{s}_A}^{1} (1 - \mu(s)) \, ds}. \quad (3.5)$$

One can easily check that $\alpha_0(v), \beta_0(v) \in [0, 1]$ for $v \in [\underline{v}, \overline{v}]$. If college $B$ adopts $\beta_0(v)$ for a student $v$, then college $A$’s marginal gain from admitting that student is zero, so it is

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8It is important to note that the thresholds are not necessarily the same as in the pure-strategies, since different equilibria involve different cutoff states, $(\hat{s}_A, \hat{s}_B)$, which affect the marginal payoff functions $H_\alpha$ and $H_\beta$. 
indifferent about admitting that student. Here, it is college A’s best response to randomize according to $\alpha_0(\cdot)$. Since $H_\beta(v, \alpha_0(v)) = 0$, college B is indifferent, making its randomization a best response. Observe that both $\alpha_0(\cdot)$ and $\beta_0(\cdot)$ are increasing in $v$, which means that colleges admit students with higher scores with higher probabilities. This is intuitive: A higher score student is more valuable all else equal, so a high probability of admission for a higher score student is necessary to keep the opponent college indifferent. It is also interesting to observe discrete jumps in this figure — $\alpha_0(v_A) > 0$ and $\beta_0(v_B) < 1$. The former follows from the fact that $v_A > v_B$ implies $H_\beta(v_A, 0) > 0$, and the latter follows from $\overline{v}_A > \overline{v}_B$ which implies $H_\alpha(\overline{v}_B, 1) < 0$.

There could be many ways for colleges to play mixed-strategies: For instance, colleges could coordinate to use a pure-strategy for some students, say $[\hat{v}, \tilde{v}]$ for some $\hat{v} \in (\tilde{v}, \tilde{v})$, and use mixed-strategies for $v \in [\tilde{v}, \hat{v}]$. Consistent with our selection, we focus on the maximally mixed equilibrium (MME, in short) in which both colleges play mixed-strategies ($\alpha_0, \beta_0$) for students with $v \in [\tilde{v}, \hat{v}]$ and according to Theorem 1 for outside that range.

The characterization of equilibria has so far rested on the necessary conditions for competitive equilibria, particularly the “local” incentive compatibility with respect to each type of students. Whether the preceding characterizations based on MME and $i$-priority equilibria admit a well-defined strategy profile and, if so, whether they constitute competitive equilibria are not clear. We shall address these issues in the next subsection.

Before proceeding, though, it is important to recognize that the randomization by colleges results from their attempts to avoid competition for students in the intermediate range of scores. In this sense, as long as a competitive equilibrium admits the intermediate region, i.e., if $\underline{v} < \overline{v}$, one can say that equilibrium involves strategic targeting, regardless of whether the colleges play a mixed-strategy or a pure-strategy. Formally, we say an competitive equilibrium exhibits strategic targeting if $\underline{v} < \overline{v}$.

When do competitive equilibria exhibit strategic targeting and when do not? Certainly, Theorem 1 does not preclude a competitive equilibrium in which $\overline{v} = \min \{\overline{v}_A, \overline{v}_B\} < \max \{\underline{v}_A, \underline{v}_B\} = \underline{v}$. Figure 3.4 depicts such a possibility with $\underline{v}_B < \overline{v}_B < \underline{v}_A < \overline{v}_A$. As before, college $i$ admits students with $v > \overline{v}_i$ and rejects those with $v < \underline{v}_i$. Observe that college A does not admit any student with $v \in [\underline{v}_A, \overline{v}_A]$, since college B admits them for sure (because $\overline{v}_B < \underline{v}_A$). Even though colleges have targeting incentives in this example, the resulting equilibrium is indistinguishable from the cutoff equilibria featured in the existing research.

A natural question is when such an equilibrium can be ruled out. The exact condition for its existence appears difficult to find, but we show next that the symmetric environment is sufficient to guarantee strategic targeting behavior.
Theorem 2. If the environment is symmetric (i.e., \( \mu(s) = 1 - \mu(1 - s) \) for all \( s \)), then every competitive equilibrium exhibits strategic targeting.

Proof. See Appendix C. ■

3.1. Existence of MME and i-Priority Equilibrium. We now show that there exists an equilibrium in which \( \alpha(\cdot) \) and \( \beta(\cdot) \) involve maximal mixing, or \( A \)- or \( B \)-priority.\(^9\)

Theorem 3. There exists a competitive equilibrium with maximal mixing, or \( A \)- or \( B \)-priority.

Sketch of Proof. The proof involves three steps. The first step shows the existence of admission strategies that provide optimal local incentives for each other college. The second step shows that \( V_{AB} \) has a positive measure in the identified strategy profile. The last step shows that the identified strategies are indeed mutual (global) best responses.

\(^9\)Note that a general equilibrium existence follows from the Glicksberg-Fan theorem, since each college’s strategy space is compact and convex, and each college’s payoff function is concave in its own strategy. That is, if one does not insist on the particular structure of behavior we impose on MME (or \( A \)- or \( B \)-priority), it is easy to show the existence of an equilibrium admission strategies.
Step 1: For the first step, we prove that there exists \((\alpha, \beta) : [0, 1]^2 \to [0, 1]^2\) such that for each \(v \in [0, 1]\),

\[
\alpha(v; \hat{s}) = \begin{cases} 
1 & \text{if } H_a(v, 1; \hat{s}) > 0 \\
0 & \text{if } H_a(v, 1; \hat{s}) < 0, \ H_\beta(v, 1; \hat{s}) > 0 \\
\alpha_0(v; \hat{s}) & \text{if } H_a(v, 1; \hat{s}) < 0 < H_a(v, 0; \hat{s}), \ H_\beta(v, 1; \hat{s}) < 0 < H_\beta(v, 0; \hat{s}) \\
1 & \text{if } H_a(v, 0; \hat{s}) > 0, \ H_\beta(v, 0; \hat{s}) < 0 \\
0 & \text{if } H_a(v, 0; \hat{s}) < 0
\end{cases}
\]  

(3.6)

and

\[
\beta(v; \hat{s}) = \begin{cases} 
1 & \text{if } H_\beta(v, 1; \hat{s}) > 0 \\
0 & \text{if } H_\beta(v, 1; \hat{s}) < 0, \ H_a(v, 1) > 0 \\
\beta_0(v; \hat{s}) & \text{if } H_\beta(v, 1; \hat{s}) < 0 < H_\beta(v, 0; \hat{s}), \ H_a(v, 1; \hat{s}) < 0 < H_a(v, 0; \hat{s}) \\
1 & \text{if } H_\beta(v, 0; \hat{s}) > 0, \ H_a(v, 0; \hat{s}) < 0 \\
0 & \text{if } H_\beta(v, 0; \hat{s}) < 0
\end{cases}
\]  

(3.7)

where \(\alpha_0(\cdot)\) satisfies \(H_\beta(v, \alpha_0(v)) = 0\) for \(v \in [\underline{v}, \overline{v}]\), as given by (3.4), and \(\beta_0(\cdot)\) satisfies \(H_a(v, \beta_0(v)) = 0\) for \(v \in [\underline{v}, \overline{v}]\), as given by (3.5), and \(\hat{s} = (\hat{s}_A, \hat{s}_B)\) satisfies

\[
\hat{s}_A = \inf \{ s \in [0, 1] | m_A(s) - \kappa > 0 \}
\]

if the set in the RHS is nonempty, or else \(\hat{s}_A \equiv 1\), and

\[
\hat{s}_B = \sup \{ s \in [0, 1] | m_B(s) - \kappa > 0 \}
\]

if the set in the RHS is nonempty, or else \(\hat{s}_B \equiv 0\).

In other words, the strategy profile is required to satisfy the conditions of MME based on the local incentives \(H_a\) and \(H_\beta\). One can also easily structure the strategy profile to satisfy the requirements of an \(A\)-priority equilibrium by replacing \(\alpha_0(\cdot)\) and \(\beta_0(\cdot)\) with 1 and 0, respectively, and of a \(B\)-priority equilibrium by replacing them with 0 and 1, respectively.

To prove the existence of such a strategy profile, we construct a mapping \(T : S \to S\), where \(S := [0, 1]^2\), and show that its fixed point exists, and given its fixed point \((\hat{s}_A^*, \hat{s}_B^*)\), the profile \((\alpha(\cdot; \hat{s}_A^*, \hat{s}_B^*), \beta(\cdot; \hat{s}_A^*, \hat{s}_B^*))\) satisfies (3.6) and (3.7).

To begin, fix any \(\hat{s} = (\hat{s}_A, \hat{s}_B) \in S\), and consider the resulting profile \((\alpha(\cdot; \hat{s}), \beta(\cdot; \hat{s}))\). This strategy profile in turn induces the mass of students enrolling in colleges \(A\) and \(B\). Specifically, for college \(A\), we obtain

\[
m_A(s; \hat{s}) = \int_0^1 \alpha(v; \hat{s})[1 - \beta(v; \hat{s}) + \mu(s)\beta(v; \hat{s})] dG(v)
\]
and similarly for college $B$, we obtain

$$m_B(s; \hat{s}) = \int_0^1 \beta(v; \hat{s})[1 - \alpha(v; \hat{s}) + (1 - \mu(s))\alpha(v; \hat{s})] dG(v).$$

Observe that $m_A(s; \hat{s})$ and $m_B(s; \hat{s})$ in turn yield a new profile of cutoff states:

$$\hat{s}_A = \inf \{s \in [0, 1]|m_A(s; \hat{s}) - \kappa > 0\},$$

if the set in the RHS is nonempty, or else $\hat{s}_A \equiv 1$, and similarly $\hat{s}_B$ for college $B$.

$$\hat{s}_B = \sup \{s \in [0, 1]|m_B(s; \hat{s}) - \kappa > 0\},$$

if the set in the RHS is nonempty, or else $\hat{s}_B \equiv 0$.

We then define $T$ such that $T(\hat{s}) = \hat{s}$. We show in Appendix D that $T$ is a continuous map. Therefore, it has a fixed point by the Brouwer’s fixed point theorem. From the construction of $T$, it is immediate that, given the fixed point $\hat{s}^*$, $\hat{s} = \hat{s}^*$, the profile $(\alpha(\cdot; \hat{s}^*), \beta(\cdot; \hat{s}^*))$ satisfies (3.6) and (3.7).

**Step 2:** For the second step, we show that $V_{AB}$ has a positive measure in the strategy profile identified in Step 1. To prove, suppose to the contrary that $V_{AB}$ has measure zero. Then, $\hat{s}_B^* = 0$ and $\hat{s}_A^* = 1$. But in that case, $H_\alpha(v, 1) > 0$ and $H_\beta(v, 1) > 0$ for all $v$. Hence, $\bar{v}_A = \bar{v}_B = 0$. Thus, we cannot have a non-competitive equilibrium.

**Step 3:** Observe that the strategy profile $(\alpha, \beta)$ identified in Step 1 forms best responses but based on the local incentives of the colleges — namely, $(\alpha, \beta)$ entails no incentive for each college to unilaterally deviate in its admission decision on each student, holding constant its own admission strategies with respect to the other students. Hence, it does not rule out profitable deviation in its admission decisions on a mass of students. The third step shows that such deviation is not profitable; that is, the identified strategies are mutual (global) best responses.

To this end, let $\tilde{\alpha}(v) \in [0, 1]$ be an arbitrary strategy for $v \in [0, 1]$, and consider a variation of $\alpha(\cdot)$ such that for any $t \in [0, 1],

$$\alpha(v; t) := t\tilde{\alpha}(v) + (1 - t)\alpha(v).$$

Define college $A$’s payoff function in terms of $\alpha(v; t)$,

$$V(t) := \int_0^1 v\alpha(v; t)[1 - \beta(v) + \bar{\mu}(v)] dG(v) - \lambda \int_{\hat{s}_A(t)}^1 \left[\int_0^1 \alpha(v; t)[1 - \beta(v) + \mu(s)\beta(v)] dG(v) - \kappa\right] ds,$$

where $\hat{s}_A(t)$ is the threshold state given by $\alpha(v; t)$. 


Observe that $\pi_A(\tilde{\alpha}) = V(1)$ and $\pi_A(\alpha) = V(0)$. Therefore, the proof is completed by showing that $V(1) \leq V(0)$. Because $\tilde{\alpha}(\cdot)$ is arbitrary, this will prove that $\alpha(\cdot)$ is a best response for a given $\beta(\cdot)$.

To see this, observe first that $V(\cdot)$ is concave in $t$ since $\pi_A$ is concave in $\alpha$ (which follows from the enrollment uncertainty) and $\alpha(v; t)$ is linear in $t$ (see Lemma D3). Therefore, we have

$$\pi_A(\tilde{\alpha}; \beta) = V(1) \leq V(0) + V'(0). \tag{3.8}$$

Next, using the local condition, (3.6), one can show that (see Lemma D4)

$$V'(0) = \int_0^1 [\tilde{\alpha}(v) - \alpha(v)]H_\alpha(v, \beta(v)) \, dG(v) \leq 0, \tag{3.9}$$

where the inequality holds since if $H_\alpha(v, \beta(v)) > 0$ for some $v$, then $\alpha(v) = 1$ and $\tilde{\alpha}(v) \leq 1$ for such $v$; if $H_\alpha(v, \beta(v)) < 0$ for some $v$, then $\alpha(v) = 0$ and $\tilde{\alpha}(v) \geq 0$ for such $v$; and $H_\alpha(v, \beta(v)) = 0$ otherwise.

Combining (3.8) and (3.9), we conclude that

$$\pi_A(\tilde{\alpha}) = V(1) \leq V(0) + V'(0) \leq V(0) = \pi_A(\alpha),$$

and this completes the proof.

3.2. Properties of Equilibria. We have seen that the equilibrium outcome involves strategic targeting. We now consider the implications of the equilibria in welfare and fairness.

Let us first define assignment and outcome. For a fixed $s$, an assignment is a mapping from $V \times \{A, B\}$ to $\{A, B\} \cup \{\emptyset\}$, where $V = [0, 1]$ is the students’ score space. That is, an assignment is an allocation of students to colleges where each student has a score and preference over $A$ and $B$ and no student is assigned to more than one college. An outcome is a mapping from a state to an assignment; that is, an outcome is the realized allocation in state $s$.

We say that a student has a justified envy if she prefers a college to the one she enrolls in, even though the former enrolls a student with a lower score. An outcome is said to be fair if for almost every state, the assignment it selects has no justified envy for almost all students.

Next, an outcome is Pareto efficient if for almost every state, the assignment it selects is not Pareto dominated, i.e., there is no other assignment in which both colleges and all students are weakly better off and either there is a college that is strictly better off or there is a positive measure of students who are strictly better off, relative to the initial assignment.

One may be interested in students’ welfare taking colleges as exogenous resources of the society. We say that an outcome if student efficient if for almost every state, there is
no other assignment in which all students are weakly better off and a positive measure of students are strictly better off relative to the initial assignment that the outcome selects.

Finally, we may consider college’s welfare only. We say that an outcome is college efficient if for almost every state, no other assignment can make both colleges weakly better off and at least one college strictly better off relative to the assignment that the outcome selects.

The next theorem states properties of equilibria that arise in decentralized matching.

**Theorem 4.** (i) Any non-competitive equilibrium is unfair, student inefficient, but college efficient.

(ii) Any non-competitive equilibrium is Pareto inefficient unless almost every student admitted by one college has higher score than those admitted by the other college.

(iii) Any competitive equilibrium is college inefficient and Pareto inefficient.

(iv) Any MME with \( v < \tilde{v} \) is unfair and student inefficient.

(v) Any competitive equilibrium with \( \tilde{v} < v \) is fair and student efficient.

**Proof.** Consider non-competitive equilibrium first.

**Proof of (i).** Consider any non-competitive equilibrium. For each state \( s \) except \( \mu(s) = 0 \) or 1, the equilibrium must admit a positive measure of students who prefer A but are assigned to B, and a positive measure of students who are assigned to A but have scores lower than those of the first group of students; that is, justified envy arises. Since justified envy arises for a positive measure of students for almost every state,

\[ \text{for almost every state, } \mu(s) \in (0,1) \] for almost every state.

Theorem 4.  (i) Any non-competitive equilibrium is unfair, student inefficient, but college efficient.

(ii) Any non-competitive equilibrium is Pareto inefficient unless almost every student admitted by one college has higher score than those admitted by the other college.

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(v) Any competitive equilibrium with \( \tilde{v} < v \) is fair and student efficient.

**Proof.** Consider non-competitive equilibrium first.

**Proof of (i).** Consider any non-competitive equilibrium. For each state \( s \) except \( \mu(s) = 0 \) or 1, the equilibrium must admit a positive measure of students who prefer A but are assigned to B, and a positive measure of students who are assigned to A but have scores lower than those of the first group of students; that is, justified envy arises. Since justified envy arises for a positive measure of students for almost every state, the outcome is unfair. Also, for almost every state, there must be a positive measure of students assigned to A but prefer B and a positive measure of students assigned to B but prefer A. Thus, the outcome is student inefficient. Next, the equilibrium is college efficient. To see this, observe first that in any non-competitive equilibrium, almost all top 2\( \kappa \) students are assigned to either college. Suppose now that for a given state, there is another assignment that makes both colleges weakly better off and at least one college strictly better off. Then, it must also admit almost all top 2\( \kappa \) students, or else at least one college is strictly worse off. Therefore, it is a reallocation of the initial assignment, hence if one college is strictly better off, then the other college must be strictly worse off. Thus, we reach a contraction. \( \Box \)

**Proof of (ii).** Suppose that almost all top \( \kappa \) students are assigned to one college, and the next top \( \kappa \) students are assigned to the other college. Then, any change of assignments by positive measure of students will leave the former college strictly worse off, hence it is Pareto efficient.

\[ ^{10} \text{since } \mu(\cdot) \text{ is strictly increasing and continuous in } s, \mu(s) \in (0,1) \text{ for almost every state.} \]
Suppose it is not the case in a non-competitive equilibrium. Note that for a fixed $s$, there are some $\mathcal{V}_i', \mathcal{V}_i'' \subset \mathcal{V}_i$ and $\mathcal{V}_j' \subset \mathcal{V}_j$, $i \neq j$, all with positive measures, such that $v' < v < v''$ whenever $v' \in \mathcal{V}_i', v'' \in \mathcal{V}_i''$ and $v \in \mathcal{V}_j'$. Let $i = A$ and $j = B$ without loss of generality. We can choose $\mathcal{V}_A', \mathcal{V}_A''$ and $\mathcal{V}_B'$ that satisfy

$$\frac{\int_{\mathcal{V}_A' \cup \mathcal{V}_A''} v \, dG(v)}{\int_{\mathcal{V}_A' \cup \mathcal{V}_A''} 1 \, dG(v)} = \frac{\int_{\mathcal{V}_B'} v \, dG(v)}{\int_{\mathcal{V}_B'} 1 \, dG(v)}$$

(3.10)

and

$$(1 - \mu(s)) \int_{\mathcal{V}_A' \cup \mathcal{V}_A''} 1 \, dG(v) = \mu(s) \int_{\mathcal{V}_B'} 1 \, dG(v).$$

(3.11)

(If either (3.10) or (3.11) is violated, we can adjust $\mathcal{V}_A', \mathcal{V}_A''$ and/or $\mathcal{V}_B'$ by adding or subtracting a positive mass of students.) Note that the LHS (resp. RHS) of (3.11) is the measure of students who prefer $B$ (resp. $A$) in $\mathcal{V}_A' \cup \mathcal{V}_A''$ (resp. $\mathcal{V}_B'$). From (3.10), we have

$$\frac{\int_{\mathcal{V}_A' \cup \mathcal{V}_A''} v \, dG(v)}{(1 - \mu(s)) \int_{\mathcal{V}_A' \cup \mathcal{V}_A''} 1 \, dG(v)} = \frac{\int_{\mathcal{V}_B'} v \, dG(v)}{(1 - \mu(s)) \int_{\mathcal{V}_B'} 1 \, dG(v)}$$

$$\Leftrightarrow \frac{\int_{\mathcal{V}_A' \cup \mathcal{V}_A''} v \, dG(v)}{\mu(s) \int_{\mathcal{V}_B'} 1 \, dG(v)} = \frac{\int_{\mathcal{V}_B'} v \, dG(v)}{(1 - \mu(s)) \int_{\mathcal{V}_B'} 1 \, dG(v)}$$

$$\Leftrightarrow (1 - \mu(s)) \int_{\mathcal{V}_A' \cup \mathcal{V}_A''} v \, dG(v) = \mu(s) \int_{\mathcal{V}_B'} v \, dG(v),$$

where the first equivalence follows from (3.11). The last equivalence shows that the average value of students who prefer $B$ in $\mathcal{V}_A' \cup \mathcal{V}_A''$ is the same as that of students who prefer $A$ in $\mathcal{V}_B'$. Thus, in state $s$, a fraction $1 - \mu(s)$ of students in $\mathcal{V}_A' \cup \mathcal{V}_A''$ who prefer $B$ to $A$ can be swapped with a fraction of $\mu(s)$ of students in $\mathcal{V}_B'$ who prefer $A$ to $B$. This reassignment leaves both colleges the same in welfare and makes all students weakly better off and some positive measure of students strictly better off. Since this argument holds for all $s$ except $\mu(s) = 0$ or 1, the outcome is Pareto inefficient. $\Box$

Consider now competitive equilibrium.

Proof of (iii). Recall that there are cutoff states $(\hat{s}_A, \hat{s}_B)$ such that colleges have a mass of unfilled seats in a positive measure of states, $[0, \hat{s}_A)$ for $A$ and $(\hat{s}_B, 1]$ for $B$, despite the fact that there are unmatched and acceptable students (inf $\{\mathcal{V}_A \cup \mathcal{V}_B\} > 0$ in Lemma 1-(ii)). Assigning those unmatched students to a college with excess capacity improves the social welfare. Thus, it is college inefficient and Pareto inefficient. $\Box$

Proof of (iv). Consider a MME with $\underline{v} < \bar{v}$. Fix a state $s$ such that $\mu(s) \neq 0, 1$. For those students in $[\underline{v}, \bar{v}]$, there is a positive measure of students who are assigned to a college,
say $B$, but prefer $A$, and their scores are higher than a positive measure of students who are assigned to $A$, even though both colleges prefer the high-score students. Moreover, students in $[\nu, \bar{\nu}]$ get zero admissions with positive probabilities even when their scores are high. Thus, it entails justified envy for a positive measure of states for almost every state. Student inefficiency follows from that for almost every state, there are two groups of positive measure of students in $[\nu, \bar{\nu}]$, one preferring $A$ but assigned to $B$ and the other preferring $B$ but assigned to $A$. $\square$

Proof of $(v)$. Consider a competitive equilibrium with $\bar{\nu} < \nu$. Let $\bar{\nu}_B < \nu_A$, as depicted in Figure 3.4, without loss of generality, so college $B$ alone admits students with scores in $[\bar{\nu}_B, \nu_A]$ and those with $\nu > \nu_A$ are admitted by both colleges.

Only the students who are not admitted by either college or admitted only by college $B$ may have envies. However, the students whom they envy have higher scores. So, no justified envy arises in every state $s$, making the outcome fair. For student efficiency, observe that those students who are admitted by both colleges choose their preferred college. Hence, they cannot be better off from swapping their assignments with others. Next, those who are admitted only by $B$ may prefer $A$, but there are no students assigned to $A$ who prefer $B$ to $A$. Thus, the outcome is student efficient. $\square$ 

4. Coordinated Matching: Self-Targeting

So far, we have characterized the pattern of colleges’ strategic targeting and provided existence and welfare and fairness properties of such equilibria. In the current and the following sections, we study two common ways for colleges to alleviate their yield management burden in decentralized matching.

We begin with students’ self-targeting: Colleges coordinate to limit the set of schools to which students can apply, thereby forcing students to “self-target” colleges. For instance, students cannot apply to both Cambridge and Oxford in the UK, and applicants in Japan can only apply to one public university.11 In Korea, all schools (more precisely, college-department pairs) are partitioned into three groups, and students are allowed to apply to only one in each group.

This method alleviates colleges’ yield management problem by improving the odds of enrollment for the colleges, since students apply only to those colleges that they are mostly

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11There is no restriction on the number of applications for private universities. However, public universities are generally more popular than private ones in Japan.
likely to accept when admitted.\footnote{Although there is no such restriction in the US, high application fees may serve this role. See Chade and Smith (2006) and Chade, Lewis and Smith (2011) for students application decisions subject to application costs, without aggregate uncertainty.} In our model with two colleges, if the number of applications is restricted to one, colleges face no enrollment uncertainty because no student admitted by a college will turn down its offer. However, students’ application behavior will be strategic; thus, the overall welfare effects are not clear a priori.

We now provide a simple model showing students’ application behavior when the students can apply to only one of the two colleges. To this end, we introduce students’ cardinal preferences for colleges.\footnote{Note that this does not alter the previous analyses, because even if students have cardinal preferences, it is still a weak dominant strategy for students to apply to both colleges in the previous model.} Each student has a taste \( y \in [0, 1] \), which is independent of score \( v \in [0, 1] \). A student with taste \( y \) obtains payoff \( y \) from attending college \( A \) and \( 1 - y \) from attending college \( B \). Thus, students with \( y \in [0, \frac{1}{2}] \) prefer \( B \) to \( A \), and those with \( y \in [\frac{1}{2}, 1] \) prefer \( A \) to \( B \). To facilitate the analysis, we assume that colleges observe an applicant’s score \( v \) but not her preference \( y \), while each student knows her preference \( y \) but not her score \( v \).\footnote{This also does not alter the previous analyses, because if students do not know their scores perfectly, then it is a strict dominant strategy for them to apply to both colleges when there is no restriction on the number of applications (see footnote 6).}

A student’s taste \( y \) is drawn according to a distribution that depends on the underlying state. For a given \( s \), let \( K(y|s) \) be the distribution of \( y \) with a density function \( k(y|s) \). Then, \( \mu(s) \equiv 1 - K(\frac{1}{2}|s) \) is the mass of students who prefer \( A \) to \( B \) in state \( s \). We assume that \( k(y|s) \) is continuous and obeys (strict) monotone likelihood ratio property (MLRP). That is, for any \( y' > y \) and \( s' > s \),

\[
\frac{k(y'|s')}{k(y|s')} > \frac{k(y'|s)}{k(y|s)},
\]

meaning that a student’s taste is more likely to be high in a high state. We further assume that there is \( \delta \) such that

\[
\left| \frac{k(y|s)}{k(y|s')} \right| < \delta \quad \text{for any } y \in [0, 1] \quad \text{and } s \in [0, 1],
\]

which means that students’ tastes changes moderately according to states. Each student with taste \( y \) forms a posterior belief about the states,

\[
l(s|y) := \frac{k(y|s)}{\int_0^1 k(y|s)ds}.
\]

Before proceeding, we make the following observations: First, for the students, applying to a school dominates not applying at all. Second, since students do not know their scores and their preferences are independent of the scores, students’ applications depend only on
their preferences. Third, since students’ preferences depend on states, the mass of students applying to each college varies across states. Let \( n_i(s) \) be the mass of students who apply to college \( i = A, B \) in state \( s \).

Consider colleges’ admissions decisions. Since a college faces no enrollment uncertainty, a cutoff strategy is optimal. If \( n_i(s) \geq \kappa \) in state \( s \), then college \( i \) will set its cutoff so as to admit students up to its capacity. Otherwise, it will admit all applicants. More precisely, the cutoff of college \( i \) in state \( s \), denoted by \( c_i(s) \), is given by
\[
c_i(s) := \inf \{ c \in [0, 1] \mid n_i(s)[1 - G(c)] \leq \kappa \}.
\]

Consider now students’ application decisions. Fix any \( \sigma : [0, 1] \rightarrow [0, 1] \) which maps from taste to a probability of applying to \( A \). Given the strategy \( \sigma \), the mass of students applying to \( A \) in state \( s \) equals
\[
n_A(s) := \int_0^1 \sigma(y)k(y|s) \, dy.
\]
Clearly, \( n_B(s) = 1 - n_A(s) \). A student with taste \( y \) has a probability of being admitted by \( i \)
\[
P_i(y) = E_s[1 - G(c_i(s)) \mid y] = \int_0^1 [1 - G(c_i(s))] l(s|y) \, ds.
\]
Note that a student with taste \( y \) will apply to \( A \) if and only if
\[
yP_A(y) \geq (1 - y)P_B(y) \geq 0.
\]

Lemma 2. For a small \( \delta \), there is a cutoff equilibrium in which students with taste \( y \geq \hat{y} \) apply to \( A \) and those with \( y < \hat{y} \) apply to \( B \).

Proof. To be added.

By the lemma, we can focus on cutoff equilibrium. Let \( \hat{y} \) be the cutoff. Since all students with \( y \geq \hat{y} \) apply to \( A \), the mass of students applying to \( A \) is \( n_A(s) = \int_{\hat{y}}^1 k(y|s) \, dy = 1 - K(\hat{y}|s) \), and similarly \( n_B(s) = K(\hat{y}|s) \).

Theorem 5. Suppose \( \mu(s) \geq \frac{1}{2} \) for all \( s \) Then, \( \hat{y} \geq \frac{1}{2} \), where \( \hat{y} \) is the equilibrium cutoff.

Proof. See Appendix E.

Theorem 5 shows students’ strategic applications when college \( A \) is more popular than the other for all states. Consider a student with taste \( y \) who expects that \( P_B(y) > P_A(y) \) since \( A \) is more popular than \( B \). If she prefers \( B \) \( (y < \frac{1}{2}) \), then it is optimal for her to apply to \( B \). If the student prefers \( A \) \( (y \geq \frac{1}{2}) \), then there is a trade-off since her payoff is higher if she can attend \( A \) over \( B \), but she believes that she has a higher chance of admission to \( B \). Thus, if she only mildly prefers \( A \), then she may apply to \( B \) instead of \( A \).
We provide a simple example with two states to illustrate the results. Figure 4.1 depicts the equilibrium assignments of the example.

Example 1. Suppose that there are two states $a$ and $b$ with equal probability. Let $K(y|a) = y^2$, $K(y|b) = y$ and $\kappa = 0.4$. Then, we have

<table>
<thead>
<tr>
<th>$\hat{y}$</th>
<th>$n_A(a)$</th>
<th>$n_B(a)$</th>
<th>$c_A(a)$</th>
<th>$c_B(a)$</th>
<th>$n_A(b)$</th>
<th>$n_B(b)$</th>
<th>$c_A(b)$</th>
<th>$c_B(b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.547$</td>
<td>$0.701$</td>
<td>$0.299$</td>
<td>$0.429$</td>
<td>$0$</td>
<td>$0.453$</td>
<td>$0.547$</td>
<td>$0.116$</td>
<td>$0.269$</td>
</tr>
</tbody>
</table>

Observe that if $n_i(s) \geq \kappa$ for all $s$ and all $i = A, B$, then the self-targeting eliminates colleges’ yield management problem, since each college fills its capacity with the best students among those who applied to it. However, it does not hold in general because there can be under-subscription to a college in some state. In the above example, for instance, the mass of applicants to college $B$ in state $a$ is smaller than its capacity ($n_B(a) = 0.299 < \kappa = 0.4$).

Let us now consider welfare and fairness properties of the equilibrium outcome. First, the equilibrium is unfair. That is, justified envy arises in that (i) students who happen to have applied to a more popular school for a given state may be unassigned even though their scores could have been good enough for the other school (see Figure 4.1(a)); and (ii) students who mildly prefer the popular school to the less popular one may be assigned to the latter college even though they could have been assigned to the popular one (see Figure 4.1(b)).

Second, there can be under-subscription to a college in equilibrium so that its capacity is not filled even though there are unassigned, acceptable students. By assigning those students to unfilled seats of a college, both the students and college will be better off. Thus, the equilibrium outcome is still Pareto, student and college inefficient.
In the next theorem, we provide conditions under which justified envy among students and/or under-subscription to a college arise. If there is a college that suffers from under-subscription, then the equilibrium outcome is not Pareto and student efficient as discussed above.

**Theorem 6.** Suppose $\mu(s) \geq \frac{1}{2}$ for all $s$. Then, there exists a positive mass of students who have justified envy. Suppose $K(\hat{y}|s) < \kappa$ for a positive measure of states $s$, where $\hat{y}$ is the cutoff defined in Theorem 5. Then, college $B$ suffers from under-subscription, and the assignment is Pareto, student and college inefficient.

The first part of the theorem provides a sufficient condition under which justified envy arises. Observe that justified envy arises whenever $c_A(s) \neq c_B(s)$. Suppose $c_A(s) > c_B(s)$. Then, the students with scores in $[c_B(s), c_A(s)]$ and tastes in $[\hat{y}, 1]$ are not assigned to any school, even though their scores are good enough to be assigned to college $B$. Suppose $c_A(s) < c_B(s)$. Then, the students with scores in $[c_B(s), 1]$ and tastes in $[\frac{1}{2}, \hat{y}]$ (whenever $\hat{y} > \frac{1}{2}$) are assigned to $B$, even though their scores are good enough to be assigned to college $A$; and those with scores in $[c_A(s), c_B(s)]$ and tastes in $[0, \hat{y}]$ are not assigned to any school, even though their scores are good enough for college $A$. We show in Appendix E that there is a positive measure of states in which $c_A(s) \neq c_B(s)$.

To see the second part of the theorem, recall that for given $\hat{y}$ in equilibrium, the mass of students applying to $B$ is $K(\hat{y}|s)$. Thus, if there is a positive measure of states in which $K(\hat{y}|s) < \kappa$, college $B$ faces under-subscription in such states (as state $a$ in the above example). Therefore, the equilibrium outcome is inefficient.

5. Sequential Admissions

In this section, we consider sequential admissions using waiting lists, another way of alleviating enrollment uncertainty that colleges use in decentralized matching. In France and Korea, for instance, colleges initially make admission offers to students up to their capacity limits, filling any declined seats with offers to students on the waiting list.

Even though this method may generate more admission offers and acceptances than the baseline model or self-targeting, we show that it is not enough to eliminate congestion altogether, and colleges may still want to engage in strategic targeting.

The main intuition is as follows. Suppose a college, say $A$, wishes to make admission offers to the most preferred candidates up to its capacity, who are also sought after by other colleges, planning to approach the next best students in the case that some of those first group of students turn its offer down. Often, college $A$ is uncertain if the next best students are willing to wait for it when a less popular college approaches to them in the first round.
Each of those students may also be uncertain about the likelihood that college A finds her acceptable, and hence she may accept the less popular college’s admission offer immediately. This uncertainty means that when college A is turned down by some of the first group of students, it may not have the second best group of students available. In fact, the students who are available at that point are far worse than the second best group of students, hence college A may wish to directly offer admissions to some of those second group of students instead of some of the first group of students.

We provide a simple model to formalize the intuition. There are three colleges, A, B and C, each with a mass $\kappa < \frac{1}{3}$ capacity. There is a unit mass of students with score $v$, where $v$ is distributed over $[0, 1]$ according to $G(\cdot)$ as before. All students like A and B better than C, but C is sufficiently better than not attending any school. Colleges’ preferences are given by students’ scores, but for each student, there is a probability $\varepsilon$ that colleges A and B find that the student is unacceptable. College C simply likes students according to $v$’s.

There are two states, $a$ and $b$. In state $i$, $i = a, b$, a fraction $s_i$ of students gets utility $u$ from A and $u'$ from B, and the remaining $1 - s_i$ students have the opposite preference, where $s_a = 1 - s_b > \frac{1}{2}$. In either state, students get utility $u''$ from C, where $u > u' > u''(> 0)$ and $u'' > (1 - \varepsilon)u$. The latter assumption means that even for a small uncertainty about the students’ acceptability by the better school, the certain utility from college C of a student is greater than the uncertain utility from the better school. Note that in state $a$, the mass of students who prefer A to B is larger than that of those who prefer B to A ($s_a > \frac{1}{2} > 1 - s_a$), and in state $b$, the former is smaller than the latter ($s_b < \frac{1}{2} < 1 - s_b$). Note that A and B are ex ante symmetric.

Suppose the capacity cost is prohibitively high so that each time a college makes admission decisions, it must be sure that the capacity will never be violated. The sequential admissions game has the following feature. In each round, the colleges make admission offers to a set of students and wait-list the remaining. The students who received offer(s) from college(s) must decide to accept or reject the offer immediately. After the first round, colleges A and B learn the state, so the game effectively ends in two rounds.

We show that there is no symmetric equilibrium in which both colleges A and B use a cutoff strategy (i.e., admit the top $\kappa$ students among those who are acceptable) in the first round. We then consider a simpler example and analyze equilibrium admission strategies. This shows that colleges still engage in strategic targeting. The equilibrium outcome of the example shows that the sequential admissions may entail justified envy and inefficiency.

Theorem 7. There is no symmetric equilibrium in which A and B offer admissions to the top $\kappa$ students (excluding those whom they find unacceptable) in the first round.
Sketch of Proof. Suppose there is such an equilibrium to the contrary. Then, colleges $A$ and $B$ will admit all acceptable students with $v > \hat{v}$, where $\hat{v}$ is such that each of $A$ and $B$ fills its capacity in the popular state, i.e., $s_a(1-\varepsilon)[1-G(\hat{v})] = \kappa$ (or equivalently, $(1-s_b)(1-\varepsilon)[1-G(\hat{v})] = \kappa$), and wait-lists the remaining students. College $C$ will offer admissions to all of these students (i.e., those whose values are above $\hat{v}$), knowing that exactly $\varepsilon^2$ of them will accept its offer (since those are not acceptable for both $A$ and $B$). It will also offer $\kappa - \varepsilon^2$ admissions to all students with $v \in [\bar{v}, \hat{v}]$, where $\bar{v}$ is such that $G(\hat{v}) - G(\bar{v}) = \kappa - \varepsilon^2$.

The students in $[\bar{v}, \hat{v}]$ now have a choice to make. If a student accepts $C$, then she will get $u''$ for sure, but if she turns down $C$’s offer, then with probability $1-\varepsilon$ the less popular one between $A$ and $B$ will offer an admission to her (assuming all other students admitted by $C$ have accepted that offer), and the student will earn the payoff $u$ if she happens to like the college, or $u'$ otherwise. Since $u'' > (1-\varepsilon)u$, she will accept $C$ immediately.

Given this, consider now the incentive for deviation of $A$. If it does not deviate, there will be seats left, equal to $\kappa - s_b(1-\varepsilon)[1-G(\hat{v})]$, in the less popular state. Thus, $A$ will fill them with students whose scores are below $\bar{v}$ (since those with scores in $[\bar{v}, \hat{v}]$ are taken by $C$).

Suppose now that college $A$ admits a small fraction, say $\delta'$, of (acceptable) students just below $\hat{v}$ instead of admitting those who are acceptable and slightly above $\hat{v}$, say $[\hat{v}, \hat{v} + \delta]$, where $\delta$ and $\delta'$ are such that

$$G(\hat{v} + \delta) - G(\hat{v}) = G(\hat{v}) - G(\hat{v} - \delta').$$  \hspace{1cm} (5.1)

Then, college $A$ benefits from this deviation for sufficiently small $\delta$. The reason is as follows. The loss from this deviation is that when $A$ is popular, it will get a worse group; and even in the unpopular state, $s_b$ fraction of these students are replaced by the worse group. But, the gain is that when $A$ is unpopular, it will get a discretely better group of students for the $s_a - s_b$ fraction of the vacant seats. Thus, for sufficiently small $\delta$, the order of magnitude for the gain is greater than that for the loss. We relegate a formal proof for this to Appendix F.

We now consider equilibrium admission strategies in a simpler example.

**Example 2.** There are three students, 1, 2 and 3, with scores $v_1 > v_2 > v_3$, where $v_2 > \frac{1}{2}(v_1 + v_3)$. There are three colleges, $A$, $B$ and $C$, and each of them has one seat to fill. Colleges value students with score $v_i$ at $v_i$, but student 2 has an $\varepsilon$ chance of being unacceptable by either $A$ or $B$, where $\varepsilon < \frac{v_1-v_2}{v_2-v_3}$. All students like $A$ and $B$ better than $C$, as in the above model. Students have a uniform preference (i.e., their preference orderings for colleges are the same). They receive utility $u$ from $A$ and $u'$ from $B$, or $u'$ from $A$ and $u$ from $B$ with
equal probability, and $u''$ from $C$ for sure, where $u > u' > u''(> 0)$ and $u'' > (1-\varepsilon)(\frac{1}{2}u + \frac{1}{2}u')$.

The timing of the game is the same as before.

Before proceeding on what equilibrium may arise, it is useful to begin with a few observations. First, colleges never make an admission offer to student 3 before they offer admissions to 1 and 2 (when 2 is acceptable), since the worst case for them is to have 3, which is always possible. Second, the result of Theorem 7 still works in this example. That is, there is no equilibrium in which both colleges $A$ and $B$ make admission offers to student 1 for sure (i.e., they use a cutoff strategy).\textsuperscript{15}

Consider now the following equilibrium. In the first round, college $C$ offers admission to student 2 for sure, and both $A$ and $B$ admit student 1 with probability $p = \frac{v_1 - v_2 - \varepsilon(v_2 - v_3)}{(1-\varepsilon)(v_2 - v_3)}$ and student 2 with probability $1 - p$, whenever they find that 2 is acceptable. Each college then places the other students whom they do not admit on the waiting list.

Note that $p$ makes the other college indifferent. That is, when a college, say $B$, offers admission to student 1, its payoff is

\[
\left[(1-p)(1-\varepsilon) + \frac{1}{2}(1 - (1-p)(1-\varepsilon))\right]v_1 + \frac{1}{2}(1 - (1-p)(1-\varepsilon))v_3.
\]

Here, $(1-p)(1-\varepsilon)$ is the probability that $A$ offers admission to student 2 when she is acceptable. In this case, student 1 accepts $B$ immediately, or else she will get an offer from $C$ in the second round (because $A$ offers admission to student 2, and 2 immediately accepts it). $1 - (1-p)(1-\varepsilon)$ is the probability that $A$ offers admission to student 1 (either when 2 is unacceptable or when 2 is acceptable but $A$ offers admission to 1), in which case $B$ is accepted by student 1 with probability $1/2$. If $B$ is rejected, then it seeks student 3 in the second round because student 2 is not available (both $A$ and $B$ happen to make admission offers to student 1, and $C$ offers an admission to student 2 and is immediately accepted, see footnote 15).

Similarly, when $B$ offers admission to student 2 (when she is acceptable), its payoff is

\[
\left[\frac{1}{2}(1-p)(1-\varepsilon) + (1 - (1-p)(1-\varepsilon))\right]v_2 + \frac{1}{2}(1 - (1-p)(1-\varepsilon))v_1.
\]

If $A$ offers admission to student 1, which happens with probability $(1 - (1-p)(1-\varepsilon))$, then $B$ is immediately accepted by 2. If $A$ also offers admission to student 2, which happens with

\textsuperscript{15}Suppose both $A$ and $B$ seek student 1 in the first round. Then, college $C$ will seek student 2 and will be immediately accepted. (Student 2 gets $u''$ if she accepts $C$. When she declines $C$, her expected utility is $(1-\varepsilon)(\frac{1}{2}u + \frac{1}{2}u'')$ because the college that was rejected by 1 will offer admission to her in the second round only when she is acceptable. Since $u'' > (1-\varepsilon)(\frac{1}{2}u + \frac{1}{2}u'')$, it is optimal for her to accept $C$ immediately.) Given this, if $A$ and $B$ offer admissions to student 1, then each of them gets the payoff $\frac{1}{2}v_1 + \frac{1}{2}v_3$. This is optimal for each of $A$ and $B$ only when 2 is unacceptable. However, if 2 is acceptable for a college, then the college can deviate to admit her instead and get $v_2$, since $v_2 > \frac{1}{2}(v_1 + v_3)$.\]
probability \((1 - p)(1 - \varepsilon)\), then \(B\) will be accepted by 2 with probability \(1/2\). If it is rejected, then it will make an admission offer to student 1 in the next round and will be accepted for sure (both \(A\) and \(B\) happen to make admission offers to student 2, and \(C\) also offers admission to 2; thus, student 1 is available in the second round, and she prefers \(B\) to \(C\)).

Equating (5.2) and (5.3), we have

\[
p = \frac{v_1 - v_2 - \varepsilon(v_2 - v_3)}{(1 - \varepsilon)(v_2 - v_3)}.
\]

Note that \(p \in (0, 1)\) because \(\varepsilon < \frac{v_1 - v_2}{v_2 - v_3}\) and \(v_2 > \frac{1}{2}(v_1 + v_3)\).

Given this, it is clear that \(C\) will not make an admission offer to student 1 in the first round. Suppose that college \(C\) does so. Then, even when both \(A\) and \(B\) happen to make admission offers to student 2, \(C\) will be rejected by student 1, since she will be admitted by either \(A\) or \(B\) in the second round and those are preferred than \(C\). Therefore, it is optimal for \(C\) to offer an admission to student 2 with probability 1 in the first round.

This example shows that colleges still engage in strategic targeting in sequential admissions: \(A\) and \(B\) compete for a better student, and \(C\) admits student 2, who may be overlooked by the competing colleges. The equilibrium assignment may entail justified envy, since it may be the case that student 1 attends \(B\) but she likes \(A\), or that student 2 attends \(C\) and 3 attends \(B\). The assignment is also student inefficient, since it is possible that student 1 likes \(A\) and 2 likes \(B\), but they are assigned to \(B\) and \(A\), respectively.

6. Centralized Matching via Deferred Acceptance

In the last two sections, we have considered two common ways that colleges use to alleviate their yield management problem in the decentralized matching. In this section, we consider a centralized matching with a Gale and Shapley’s Deferred Acceptance algorithm (henceforth DA). Many markets, such as public school admissions and medical residency assignments, are centralized via such an algorithm. College admissions are also centralized in some countries, although it is organized in varying degrees.\(^\text{16}\) We consider the equilibrium allocation under DA and compare this with the outcome that arises from the decentralized matching.

Suppose that the matching is organized by a clearinghouse that applies Gale and Shapley’s student-proposing DA.\(^\text{17}\) The algorithm works as follows. Initially, students and colleges report their preference orderings to the clearinghouse. In the first round, students propose their favorite college (according to the reported order). If a college has more applicants than

\(^{16}\)See Chen and Kesten (2011) for Shanghai mechanism and Westkamp (forthcoming) for Germany medical school matchings.

\(^{17}\)The outcome of college-proposing DA is the same as that of student-proposing DA in our model, since colleges have a uniform rank on students.
its quota, it tentatively admits a set of students that maximizes its payoff staying within the quota, rejecting the rest (according to the reported order). In each round, unassigned students propose their favorite college among those that have not rejected them. Each college selects a set of students that maximizes its payoff within the quota among those who are tentatively admitted in the previous round and those who propose it in the current round. The process ends until no further proposals are made, in which case each student is assigned to a college that holds her proposal.\textsuperscript{18}

To see this more precisely, consider a state $s$ such that $\mu(s) \geq 1 - \mu(s)$. In the first round, a fraction $\mu(s)$ of students proposes college $A$, and the remaining students propose college $B$. Each college tentatively admits the top $\kappa$ students among the applicants. Thus, colleges’ cutoffs in this round, denoted by $\hat{v}_i(s), i = A, B$, satisfy $\mu(s)[1 - G(\hat{v}_A(s))] = \kappa$ and $(1 - \mu(s))[1 - G(\hat{v}_B(s))] = \kappa$. Unassigned students then propose another college at the second round, and again, each college admits the top $\kappa$ students from those who are admitted in the first round and those who propose it at the current round. Thus, colleges’ cutoffs in this round satisfy $\mu(s)[1 - G(\hat{v}_A(s))] = \kappa$ and $1 - G(\hat{v}_B(s)) = 2\kappa$. Since there are no more colleges to which unassigned students can apply, the assignment is finalized in the second round in our model. The process is depicted in Figure 6.1.

Consider now the equilibrium properties of the DA outcome. Under DA, the matching is strategy proof for the students, so the students have a dominant strategy of reporting their preferences truthfully (Dubins and Freedman, 1981; Roth, 1982). In addition, colleges in our model also report their rankings and capacities truthfully.

\textsuperscript{18}Abdulkadiroğlu, Che and Yasuda (2012) and Azevedo and Leshno (2012) provide a model of DA in which a continuum mass of students is matched to a finite number of schools.
Lemma 3. Given the common college preferences, it is an ex post equilibrium for colleges to report their rankings and capacities truthfully.

The proof in Appendix G shows that if one college, say $B$, truthfully reports its capacity and preference, it is a best response for $A$ to do the same. The reason is that when $A$ manipulates its capacity or ranking, the possible gain comes from generating a “rejection chain” (Kojima and Pathak, 2009), i.e., $A$ rejects a positive mass of students by the manipulation, and those students apply to $B$, causing $B$ to reject some other positive mass of students. Then, those student (who are rejected by $B$) will then apply to $A$. If those second group of students are preferred by $A$ over the first group of students, then $A$ could be better off. However, the common preference of $A$ and $B$ implies that the second group of students are worse than the first group of students, since $B$ would not otherwise reject the second group of students.

The matching in the equilibrium involves no justified envy (Gale and Shapley, 1962; Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003), and is efficient among students (because colleges’ preferences are acyclic in the sense of Ergin (2002)) and Pareto efficient (an implication of stability). It also eliminates colleges’ yield management problem completely. Colleges never exceed their quotas (because it is never allowed by the algorithm), and have no seats left unfilled in the presence of acceptable unmatched students (a consequence of stability).

In fact, given the homogeneous preferences of the colleges, there exists a single cutoff such that a student is assigned to a college under DA if and only if her score exceeds that cutoff. In order words, only those with the top $2\kappa$ scores are assigned. This outcome is jointly optimal for the two colleges. In contrast, recall that competitive equilibrium in decentralized matching entails unfilled seats for colleges in low-demand states and exceeded quotas in high-demand states, so the assignment is far from jointly optimal. This observation suggests that at least one college must be strictly better off from a shift from decentralized matching to centralized matching via the deferred acceptance algorithm. Despite the overall benefit from switching centralization via DA, it is possible for one college to be worse off. To see this, consider the following example.

Example 3. Let $v \sim U[0,1]$, $s \sim U[0,1]$, $\lambda = 1$, $\kappa = 0.45$ and $\mu(s) = \frac{1}{2} \sqrt{s} + \frac{1}{2}$. Then, in a decentralized admission, colleges’ payoffs in equilibrium are $\pi_A = 0.304$ and $\pi_B = 0.158$. Suppose now that the DA is in use. Then, their payoffs are $\pi_{DA}^A = 0.352$ and $\pi_{DA}^B = 0.143$. Notice that $\pi_{DA}^A + \pi_{DA}^B = 0.495 > \pi_A + \pi_B = 0.462$ (overall benefit for the two colleges), $\pi_{DA}^A > \pi_A$ (college $A$ is strictly better off), but $\pi_{DA}^B < \pi_B$ (college $B$ is worse off).
In this example, college $A$ is more popular than $B$ for any state. Under decentralized matching, college $B$ has some chance to have high score students (because of colleges’ strategic targeting as we have seen), while it does not under DA. This may explain why college admissions are remained decentralized in many countries unlike public school matchings. In the former, colleges are agent that have their own interests and preferences over students, whereas in the latter, schools are treated as “resources” of a society, in which case schools’ preferences are interpreted as school-priorities that students claim for the schools.

Equilibrium properties of the assignment under DA are summarized in the follow.

**Theorem 8.** Under DA, the equilibrium outcome is fair, Pareto and student efficient, and jointly optimal among the colleges. However, some college may be worse off relative to decentralized matching.

7. Conclusion

The current paper has introduced and analyzed a new model of decentralized college admissions. In the model, colleges make admission decisions subject to aggregate uncertainty about students’ preferences and linear costs for any enrollment exceeding the capacity. We find that colleges’ admission decisions become a tool for strategic yield management, and in equilibrium, colleges try to reduce their enrollment uncertainty by strategically targeting students.

We also obtain the welfare and fairness implications of the equilibrium outcomes. We show that the equilibrium outcome under decentralized matching entails justified envy and fails to attain efficiency. Our analytical model permits a clear comparison of the outcomes that would arise (i) when students are forced to self-target (by the limited set of schools they can apply to), (ii) when admissions are made sequentially, and (iii) when the market is centralized via DA. Both self-targeting and sequential admissions may alleviate colleges’ yield management problems but may not eliminate them completely, and inefficiency and justified envy are still entailed. Centralized matching via DA completely eliminates the yield management problem and justified envy, and it also achieves Pareto and student efficiency in our model. We show, however, that it is possible for one college to be worse off using centralized matching relative to decentralized matching. This observation may explain why college admissions remain decentralized in many countries.

The model reveals several aspects to be further investigated. As assumed in our model, it is often the case that colleges have largely homogeneous preferences/evaluations based on students’ high school GPA and their scores from a nation-wide test, such as those in Australia, Japan or Korea. However, colleges may prefer students who have enthusiasm
or loyalty towards them, because these qualities may predict future donations and alumni activities (see Avery and Levin, 2010), or colleges may seek diversity in the student body. Interestingly, students’ idiosyncratic preferences may be one way to coordinate strategic targeting. If colleges know that “legacy” admits (who have a family history with the school) are more likely to accept admission offers, they can coordinate on admitting different students by each admitting their own legacy students, even if these students are objectively lower quality.

As noted in the paper, colleges under decentralized matching find ways to alleviate their yield management burden. Although we have considered two common ways, self-targeting and sequential admissions, used for this purpose, early admissions may also alleviate the yield management problem. By admitting a fraction of students early, a college can reduce uncertainty in the enrollment. Avery, Fairbanks and Zeckhauser (2003) provide empirical evidence that matriculation rates in early admissions are much higher than those in regular admissions (especially in the case of early decisions). Moreover, students are forced to reveal their preferences during the early admission process, and thus, colleges may have better information about students’ preferences. This may help colleges to manage the final class size in regular admissions and may alleviate their yield management problems.

Finally, our model can be applied to a broad range of decentralized matching beyond college admissions. For instance, in the directed search literature, little is known about cases in which workers (corresponding to colleges in our model) can apply for multiple jobs, together with the aggregate uncertainty of firms’ (students’) preferences. This paper has provided a tractable model in this direction.
Appendix A. Proof of Lemma 1

We begin with a couple of claims. We then prove the lemma in the sequence of (i), (iii), (ii), and (iv).

**Claim 1.** Suppose $\mathcal{V}_{AB}$ has zero measure. Then, $m_A(s) = m_B(s) = \kappa$ for all $s \in [0, 1]$.

*Proof.* Since $\mathcal{V}_{AB}$ is a measure zero set, colleges do not put a positive probability on admitting the same students, that is $\alpha(v) = 0$ for $v \in \mathcal{V}_B$, and $\beta(v) = 0$ for $v \in \mathcal{V}_A$. Since almost no students get admissions from both colleges with positive probability, a student will attend a college for sure if she is admitted by that college. Therefore, it is optimal for each college to admit the best students up to the capacity among those who are not admitted by the opponent college. Thus, we have $\alpha(v) = 1$ for $v \in \mathcal{V}_A$, and $\beta(v) = 1$ for $v \in \mathcal{V}_B$. Therefore, we have $m_i(s) = \kappa$ for all $s \in [0, 1]$ and for all $i = A, B$. ■

**Claim 2.** Suppose $\mathcal{V}_{AB}$ has a positive measure. Then, $m_A(s) < m_A(s')$ and $m_B(s) > m_B(s')$ for any $s < s'$.

*Proof.* Since $\mu(\cdot)$ is increasing in $s$, $m_A(\cdot)$ is increasing and $m_B(\cdot)$ is decreasing in $s$. ■

**Proof of Part (i).** If $\mathcal{V}_{AB}$ has zero measure, the proof is immediate from Claim 1. Suppose now that $\mathcal{V}_{AB}$ has a positive measure. Suppose $m_A(1) < \kappa$. Then, $A$ can benefit by admitting a mass $\kappa - m_A(1)$ of students. Let $\tilde{m}_A(s)$ be the mass of students attending $A$ under such deviation. Then, we have for any $s < 1$,

$$m_A(s) < m_A(s) + \mu(s)[\kappa - m_A(1)] \leq \tilde{m}_A(s) \leq m_A(s) + [\kappa - m_A(1)] < \kappa,$$

where the first and the last inequalities hold since $m_A(s) < m_A(1)$ and $\mu(s) \geq 0$. The second inequality becomes equality if all of the newly admitted students has been admitted by $B$, and the third inequality becomes equality if all of them has not been admitted by $B$.

Observe that under this deviation, $A$ admits more students without paying costs for over-enrollment; hence $A$ benefits. Therefore, we have $m_A(1) \geq \kappa$ in equilibrium. Similarly, if $m_A(0) > \kappa$, then $A$ can benefit by rejecting a mass $m_A(0) - \kappa$ students. The proof is analogous to above, so we omit it. ■

**Proof of Part (iii).** Suppose $\mathcal{V}_{AB}$ has a positive measure. Since $m_A(\cdot)$ is continuous in $s$ (which follows from the continuity of $\mu(\cdot)$), there exists a unique $\hat{s}_A$ such that $m_A(\hat{s}_A) = \kappa$. Suppose $\hat{s}_A = 1$. Then, $A$ has strict over-enrollment for all states except $s = 1$, a measure zero state. Similarly, if $\hat{s}_A = 0$, then $A$ has strict under-enrollment for all states except $s = 0$. These contradict to Part (i). Thus, we have $\hat{s}_A \in (0, 1)$. The proof for $\hat{s}_B$ is similar, hence omitted. ■
**Proof of (ii).** We first show \( \sup \{ \mathcal{V}_A \cup \mathcal{V}_B \} = 1 \). We then show that \( \mathcal{V}_A \cup \mathcal{V}_B \) is a connected interval and \( \inf \{ \mathcal{V}_A \cup \mathcal{V}_B \} > 0 \).

**Step 1.** \( \sup \{ \mathcal{V}_A \cup \mathcal{V}_B \} = 1 \).

**Proof.** Suppose on the contrary \( \bar{c} := \sup \{ \mathcal{V}_A \cup \mathcal{V}_B \} < 1 \). We show that a college can benefit by rejecting some students in favor of those in \([\bar{c}, 1]\). Suppose that \( \mathcal{V}_{AB} \) has zero measure. Then, a college, say \( A \), can benefit by rejecting a positive mass of students from the bottom of \( \mathcal{V}_A \) and admits the same mass of students from 1.

Suppose now that \( \mathcal{V}_{AB} \) has a positive measure. Let \([\bar{c} - \varepsilon, \bar{c}] \subset \mathcal{V}_{AB}\).\(^\text{19}\) Now, let \( A \) reject students in \([\bar{c} - \varepsilon, \bar{c}]\) and admit those in \([\bar{c}, \bar{c} + \delta]\), where \( \varepsilon \) and \( \delta \) are such that

\[
G(\bar{c} + \delta) - G(\bar{c}) = \mu(\hat{s}_A)[G(\bar{c}) - G(\bar{c} - \varepsilon)].
\] (A.1)

Then, the mass of student attending \( A \) from the deviation in state \( s \), denoted by \( \tilde{m}_A(s) \), is

\[
\tilde{m}_A(s) = G(\bar{c} + \delta) - G(\bar{c}) + m_A(s) - \mu(s)[G(\bar{c}) - G(\bar{c} - \varepsilon)]
= (\mu(\hat{s}_A) - \mu(s))[G(\bar{c}) - G(\bar{c} - \varepsilon)] + m_A(s),
\] (A.2)

where the second equality follows from (A.1). Since \( \tilde{m}_A(\hat{s}_A) = m_A(\hat{s}_A) \), \( A \)'s payoff from the deviation is

\[
\tilde{\pi}_A = \int_{\bar{c}}^{\bar{c} + \delta} v \, dG(v) + \int_{\bar{c} - \varepsilon}^{\bar{c}} v \alpha(v)[1 - \beta(v) + \bar{\pi}\beta(v)] \, dG(v) - \int_{\bar{c} - \varepsilon}^{\bar{c}} v \bar{\pi}\alpha(v)\beta(v) \, dG(v) - \lambda \mathbb{E}_s[\tilde{m}_A(s) - \kappa | s > \hat{s}_A](1 - F(\hat{s}_A)),
\]

and hence,

\[
\tilde{\pi}_A - \pi_A = \int_{\bar{c}}^{\bar{c} + \delta} v \, dG(v) - \int_{\bar{c} - \varepsilon}^{\bar{c}} v \bar{\pi}\alpha(v)\beta(v) \, dG(v) - \lambda \mathbb{E}_s[\tilde{m}_A(s) - m_A(s) | s > \hat{s}_A](1 - F(\hat{s}_A))
\geq \int_{\bar{c}}^{\bar{c} + \delta} v \, dG(v) - \bar{\pi} \int_{\bar{c} - \varepsilon}^{\bar{c}} v \, dG(v) - \lambda [G(\bar{c}) - G(\bar{c} - \delta)] \int_{\hat{s}_A}^{1} [\mu(\hat{s}_A) - \mu(s)] \, dF(s).
\]

where the inequality follows from that the fact that \( \alpha(v), \beta(v) \leq 1 \) and (A.2).

Observe that the first two terms in the RHS of the inequality are

\[
\int_{\bar{c}}^{\bar{c} + \delta} v \, dG(v) - \bar{\pi} \int_{\bar{c} - \varepsilon}^{\bar{c}} v \, dG(v) = \left[ v G(\bar{c} + \delta) - G(\bar{c}) \right] - \bar{\pi} v [G(\bar{c}) - G(\bar{c} - \varepsilon)]
+ \left[ \delta G(\bar{c} + \delta) - \int_{\bar{c}}^{\bar{c} + \delta} G(v) \, dv \right] - \bar{\pi} \varepsilon [G(\bar{c} - \varepsilon) - \int_{\bar{c} - \varepsilon}^{\bar{c}} G(v) \, dv]
\geq v [G(\bar{c} + \delta) - G(\bar{c})] - \bar{\pi} v [G(\bar{c}) - G(\bar{c} - \varepsilon)]
= \varepsilon [G(\bar{c}) - G(\bar{c} - \varepsilon)](\mu(\hat{s}) - \bar{\pi}),
\]

\(^{19}\)If there is a positive mass of students in \( \mathcal{V}_i \setminus \mathcal{V}_j \), then college \( i \) can benefit by replacing these students those in \([\bar{c}, 1]\). Thus, there is no loss generality to assume that \([\bar{c} - \varepsilon, \bar{c}] \subset \mathcal{V}_{AB}\).
where the first equality follows from the integration by parts and after some arrangement, and the last equality follows from (A.1).

Therefore,

\[
\tilde{\pi}_A - \pi_A > \bar{\epsilon} [G(\bar{\epsilon}) - G(\bar{\epsilon} - \varepsilon)] (\mu(\hat{s}_A) - \bar{\mu}) - \lambda [G(\bar{\epsilon}) - G(\bar{\epsilon} - \delta)] \int_{\hat{s}_A}^{1} [\mu(\hat{s}_A) - \mu(s)] dF(s)
\]

\[
\geq \bar{\epsilon} [G(\bar{\epsilon}) - G(\bar{\epsilon} - \varepsilon)] \left[ \mu(\hat{s}_A) - \bar{\mu} + \int_{\hat{s}_A}^{1} [\mu(s) - \mu(\hat{s}_A)] dF(s) \right]
\]

\[
> 0,
\]

where the inequality holds since \( \int_{\hat{s}_A}^{1} [\mu(\hat{s}_A) - \mu(s)] dF(s) \leq 0 \), \( \lambda \geq 1 \), and

\[
\mu(\hat{s}_A) - \bar{\mu} + \int_{\hat{s}_A}^{1} [\mu(s) - \mu(\hat{s}_A)] dF(s) = \int_{0}^{\hat{s}_A} [\mu(\hat{s}_A) - \mu(s)] dF(s) \geq 0.
\]

This proves that \( A \) can benefit from such a deviation. \( \square \)

**Step 2.** \( \mathcal{V}_A \cup \mathcal{V}_B \) is a connected interval.

*Proof.* Suppose on the contrary that there is gap in \( \mathcal{V}_A \cup \mathcal{V}_B \). The proof is analogous to Step 1, where \([\bar{\epsilon}, 1]\) is now replaces by the gap in \( \mathcal{V}_A \cup \mathcal{V}_B \). We omit the details. \( \square \)

**Step 3.** \( \inf \{ \mathcal{V}_A \cup \mathcal{V}_B \} > 0 \)

*Proof.* Suppose \( \mathcal{V}_{AB} \) has zero measure in equilibrium. Then, we have \( \inf \{ \mathcal{V}_A \cup \mathcal{V}_B \} = \zeta \), where \( 1 - G(\zeta) = 2\kappa \) from Step 1, Step 2 and Claim 1. Since \( \kappa < \frac{1}{2} \), we have the desired result.

Suppose now that \( \mathcal{V}_{AB} \) has a positive measure in equilibrium. Suppose \( \inf \{ \mathcal{V}_A \cup \mathcal{V}_B \} = 0 \) to the contrary. Then, we have \( |\mathcal{V}_A \cup \mathcal{V}_B| = 1 \), where \( |X| \) is the cardinality of a set \( X \subseteq [0, 1] \), from Step 1 and 2. Next, note that \( m_A(s) + m_B(s) = |\mathcal{V}_A \cup \mathcal{V}_B| \) for all \( s \in [0, 1] \).

Observe that since \( m_A(0) \leq \kappa \leq m_A(1) \), we must have \( m_B(1) \leq 1 - \kappa \leq m_B(0) \). Adding this to \( m_B(1) \leq \kappa \leq m_B(0) \), we have that

\[
m_B(1) \leq \frac{1}{2} \leq m_B(0).
\]

Similarly, we also have

\[
m_A(0) \leq \frac{1}{2} \leq m_A(1).
\]

Since \( \inf \{ \mathcal{V}_A \cup \mathcal{V}_B \} = 0 \), at least one of \( \underline{\mathcal{V}}_A \) and \( \bar{\mathcal{V}}_B \) is zero. Suppose \( \underline{\mathcal{V}}_A = \lambda (1 - F(\hat{s}_A)) = 0 \). Then, this implies that \( \hat{s}_A = 1 \), and so \( m_A(1) = \kappa \). This contradicts to (A.3) since \( \kappa < \frac{1}{2} \).

Similarly, if \( \underline{\mathcal{V}}_B = 0 \), then \( \hat{s}_B = 0 \) and so \( m_B(0) = \kappa \), which contradicts to (A.4). \( \square \)

**Proof of Part (iv).** The proof follows from Claim 1 and the fact that \( \mathcal{V}_A \cup \mathcal{V}_B = [\zeta, 1] \) where \( \zeta = G^{-1}(1 - 2\kappa) \) (from Part (iii)). \( \square \)
In this section, we show that when $\kappa < \frac{1}{2}$ is not too small or $\lambda > 1$ is not too large, there does not exist a non-competitive equilibrium. The reason is as follows. Suppose, for instance, $\mu = \frac{1}{2}$, $\kappa > \frac{1}{4}$ and $v \sim U[0, 1]$. Suppose further that the worst type college $A$ has is less than $\frac{1}{2}$, and the best type of college $B$ has is 1. Then, regardless of $\lambda$, $A$ can benefit by rejecting sufficiently small mass of students from the bottom and accepting the same mass of students close to one, because those newly admitted students will accept $A$ with probability close to $\frac{1}{2}$, and the (average) value of those students is discretely better than that of students rejected by $A$. Furthermore, $A$ will never have over-enrollment from this deviation. The first part Lemma B1 generalize this observation and show that for $\kappa$ not too small it is profitable. (In the proof, we do not require the restrictive assumptions made in the above example.)

Of course, this is just one way to deviate. The other way of deviation is to accept more of those close to one than those $A$ rejects at the bottom. But, the profitability of such deviation now depends on $\lambda$. The second part of Lemma B1 shows that for $\lambda$ not too large, it is profitable.

**Lemma B1.** Suppose that $\mathcal{V}_{AB}$ has zero measure. Then, we have the followings:

(i) There is $\hat{\kappa} < \frac{1}{2}$ such that for any $\kappa > \hat{\kappa}$, one college has an incentive to deviate.

(ii) There is $\hat{\lambda} > 1$ such that for any $\lambda < \hat{\lambda}$, one college has an incentive to deviate.

**Proof.** Suppose $\mathcal{V}_{AB}$ has zero measure. Then, $m_i(s) = \kappa$ for all $s$ and

$$\pi_i = \int_{\mathcal{V}_i} v \, dG(v), \quad i = A, B.$$ 

Now, let $\underline{c}_i = \inf \{\mathcal{V}_i\}$ and $\overline{c}_i = \sup \{\mathcal{V}_i\}$.

*Proof of (i).* Let $\underline{c}_A = \inf \{\mathcal{V}_A \cup \mathcal{V}_B\}$, without loss of generality. Then, $\underline{c}_A = G^{-1}(1 - 2\kappa)$ by Lemma 1. We show that college $A$ has an incentive to deviate. Suppose that $A$ rejects students in $[\underline{c}_A, \underline{c}_A + \delta]$ but accepts those in $[\overline{c}_B - \varepsilon, \overline{c}_B]$, where $\varepsilon$ and $\delta$ are such that

$$G(\overline{c}_B) - G(\overline{c}_B - \varepsilon) = G(\underline{c}_A + \delta) - G(\underline{c}_A).$$  

(B.1)

Note that the mass of students who attend $A$ under this deviation is

$$\tilde{m}_A(s) = \int_{\overline{c}_B - \varepsilon}^{\overline{c}_B} \mu(s)g(v) \, dv + \int_{\mathcal{V}_A \setminus [\underline{c}_A, \underline{c}_A + \delta]} g(v) \, dv = \mu(s)[G(\overline{c}_B) - G(\overline{c}_B - \varepsilon)] + \kappa - [G(\underline{c}_A + \delta) - G(\underline{c}_A)],$$

where the second equality follows from that $m_A(s) = \kappa$ for all $s$.

Let $\tilde{s}_A$ be such that $\tilde{m}_A(\tilde{s}_A) = \kappa$ (provided $\tilde{s}_A \in [0, 1]$), i.e.,

$$\mu(\tilde{s}_A)[G(\overline{c}_B) - G(\overline{c}_B - \varepsilon)] = [G(\underline{c}_A + \delta) - G(\underline{c}_A)].$$
Since $\mu(\cdot)$ is strictly increasing, (B.1) implies $A$ is never over-demanded in a positive measure of states from the this deviation. Thus, $A$’s payoff from the deviation is

$$\pi_A = \overline{\mu} \int_{\tau_{B-\varepsilon}}^{\tau_B} v \, dG(v) + \int_{\mathcal{V}_A \setminus [\underline{c}_A, \underline{c}_A + \delta]} v \, dG(v) = \overline{\mu} \int_{\tau_{B-\varepsilon}}^{\tau_B} v \, dG(v) + \pi_A - \int_{\underline{c}_A}^{\underline{c}_A + \delta} v \, dG(v).$$

Therefore,

$$\pi_A - \pi_A = \overline{\mu} \int_{\tau_{B-\varepsilon}}^{\tau_B} v \, dG(v) - \int_{\underline{c}_A}^{\underline{c}_A + \delta} v \, dG(v)$$

$$= \overline{\mu} \left[ \tau_B G(\tau_B) - (\tau_B - \varepsilon)G(\tau_B - \varepsilon) - \int_{\tau_{B-\varepsilon}}^{\tau_B} G(v) \, dv \right] - \left[ (\underline{c}_A + \delta)G(\underline{c}_A + \delta) - \underline{c}_A G(\underline{c}_A) - \int_{\underline{c}_A}^{\underline{c}_A + \delta} G(v) \, dv \right]$$

$$> \overline{\mu} \left[ \tau_B G(\tau_B) - (\tau_B - \varepsilon)G(\tau_B - \varepsilon) - \varepsilon G(\tau_B) \right] - \left[ (\underline{c}_A + \delta)G(\underline{c}_A + \delta) - \underline{c}_A G(\underline{c}_A) - \delta G(\underline{c}_A) \right]$$

$$= [G(\tau_B) - G(\tau_B - \varepsilon)][\overline{\mu} \tau_B - \underline{c}_A - \overline{\mu} \varepsilon - \delta]. \quad \text{(B.2)}$$

where the first equality follows from the integration by parts, and the last equality follows from (B.1). Observe that if $\overline{\mu} > \frac{\underline{c}_A}{\tau_B}$, then (B.2) is strictly positive for sufficiently small $\varepsilon$ and $\delta$, hence $\pi_A > \pi_A$. Note that since $\underline{c}_A = G^{-1}(1 - 2\kappa)$ and $m_i(s) = \kappa$ for all $s$ and $i$, $\tau_B = \sup \{ \mathcal{V}_i \}$ satisfies $G(\tau_B) \geq 1 - \kappa$. That is, we have $\tau_B \geq G^{-1}(1 - \kappa)$. Therefore,

$$\frac{\underline{c}_A}{\tau_B} \leq \frac{G^{-1}(1 - 2\kappa)}{G^{-1}(1 - \kappa)}. \quad \text{(B.3)}$$

Since the RHS of (B.3) is continuous in $\kappa$ and converges to zero as $\kappa$ approaches to $\frac{1}{2}$, there is $\hat{\kappa} < \frac{1}{2}$ such that for any $\kappa > \hat{\kappa}$, $\overline{\mu} > \frac{\underline{c}_A}{\tau_B}$ for any given $\overline{\mu}$. 

**Proof of (ii)**. Let $\tau_B = \sup \{ \mathcal{V}_A \cup \mathcal{V}_B \}$, without loss of generality. Then, $\tau_B = 1$ by Lemma 1. We show that college $A$ has an incentive to deviate. Suppose $A$ rejects students in $[\underline{c}_A, \underline{c}_A + \delta]$ but admits students in $[1 - \varepsilon, 1]$, where $\varepsilon$ and $\delta$ are such that

$$\eta \left[ 1 - G(1 - \varepsilon) \right] = G(\underline{c}_A + \delta) - G(\underline{c}_A) \quad \text{(B.4)}$$

and $\eta = \mu(1 - \underline{c}_A)$. The mass of students who attend $A$ in state $s$ under the deviation is

$$\tilde{m}_A(s) = \int_{1-\varepsilon}^{1} \mu(s) g(v) \, dv + \int_{\mathcal{V}_A \setminus [\underline{c}_A, \underline{c}_A + \delta]} g(v) \, dv = \mu(s)[1 - G(1 - \varepsilon)] + \kappa - [G(\underline{c}_A + \delta) - G(\underline{c}_A)].$$

Let $\hat{s}_A$ be such that $\tilde{m}_A(\hat{s}_A) = \kappa$, i.e., $\mu(\hat{s}_A)[1 - G(1 - \varepsilon)] = [G(\underline{c}_A + \delta) - G(\underline{c}_A)]$. Then, $\mu(\hat{s}_A) = \eta$ by (B.4), i.e., $\hat{s}_A = \mu^{-1}(\eta) = 1 - \underline{c}_A$.

Thus, $A$’s payoff from the deviation is

$$\pi_A = \overline{\mu} \int_{\tau_{B-\varepsilon}}^{\tau_B} v \, dG(v) + \int_{\mathcal{V}_A \setminus [\underline{c}_A, \underline{c}_A + \delta]} v \, dG(v) - \lambda \int_{\hat{s}_A}^{1} [m(s) - \kappa] \, ds$$

$$= \overline{\mu} \int_{\tau_{B-\varepsilon}}^{\tau_B} v \, dG(v) + \pi_A - \int_{\underline{c}_A}^{\underline{c}_A + \delta} v \, dG(v) - \lambda \left[ (1 - G(1 - \varepsilon)) \int_{\hat{s}_A}^{1} \mu(s) \, ds - [G(\underline{c}_A + \delta) - G(\underline{c}_A)](1 - \hat{s}_A) \right].$$
and the net payoff from the deviation is

\[
\tilde{\pi}_A - \pi_A = \bar{\mu} \int_{\tau_B - \epsilon}^{\tau_B} v dG(v) - \int_{\xi_A}^{\xi_B + \delta} v dG(v) - \lambda \left[ (1 - G(1 - \epsilon)) \int_{\hat{s}_A}^{1} \mu(s) ds - [G(\xi_A + \delta) - G(\xi_A)](1 - \hat{s}_A) \right] \\
> \bar{\mu}(1 - \epsilon)[1 - G(1 - \epsilon)] - (\xi_A + \delta)[G(\xi_A + \delta) - G(\xi_A)] \\
- \lambda \left[ (1 - G(1 - \epsilon)) \int_{\hat{s}_A}^{1} \mu(s) ds - [G(\xi_A + \delta) - G(\xi_A)](1 - \hat{s}_A) \right] \\
= [1 - G(1 - \epsilon)] \left( \bar{\mu} - \eta \xi_A - \bar{\mu} \epsilon - \eta \delta - \lambda \int_{\hat{s}_A}^{1} \mu(s) ds - \eta(1 - \hat{s}_A) \right),
\]

(B.5)

where the second inequality follows from the integration by parts of the first two terms of the RHS of the first equality, and the last equality follows from (B.4).

Observe that if \( \bar{\mu} - \eta \xi_A - \lambda \left[ \int_{\hat{s}_A}^{1} \mu(s) ds - \eta(1 - \hat{s}_A) \right] > 0 \), then (B.5) is strictly positive for sufficiently small \( \epsilon \) and \( \delta \). Note that

\[
\bar{\mu} - \eta \xi_A - \lambda \left[ \int_{\hat{s}_A}^{1} \mu(s) ds - \eta(1 - \hat{s}_A) \right] = \bar{\mu} - \lambda \int_{\hat{s}_A}^{1} \mu(s) ds + (\lambda - 1) \eta \xi_A,
\]

Note that \( \bar{\mu} = \int_{0}^{1} \mu(s) ds > \int_{\hat{s}_A}^{1} \mu(s) ds \) (since \( \hat{s}_A < 1 \)). Thus, there exists \( \hat{\lambda} > 1 \) such that for any \( \lambda < \hat{\lambda} \), \( \tilde{\pi}_A > \pi_A \).

\[\square\]

**APPENDIX C. PROOFS OF THEOREM 2**

Suppose on the contrary that \( \bar{\nu} \leq \nu_B \) in competitive equilibrium. Suppose further that

\[
\nu_B \leq \bar{\nu}_B \leq \nu_A \leq \bar{\nu}_A,
\]

(C.1)

without loss of generality. For given \( \hat{s}_A \) and \( \hat{s}_B \) in equilibrium, the mass of students attending each college is

\[
m_A(s) = \mu(s)[1 - G(\bar{\nu}_A)],
\]

\[
m_B(s) = (1 - \mu(s))[1 - G(\bar{\nu}_A)] + G(\bar{\nu}_A) - G(\bar{\nu}_B).
\]

Notice that we must have \( \bar{\nu}_A \in (0, 1) \) in equilibrium, since if \( \bar{\nu}_A = 1 \), then \( m_A(s) = 0 \) for any \( s \in [0, 1] \), and if \( \bar{\nu}_A = 0 \), then \( \nu_B = \bar{\nu}_B = \nu_A = \bar{\nu}_A = 0 \). This implies that \( \hat{s}_A < 1 \) (or else, \( \nu_A = \bar{\nu}_A = 0 \)) and \( \hat{s}_A > 0 \) (or else, \( \bar{\nu}_A = \lambda \geq 1 \)). Since \( \hat{s}_A < 1 \), (C.1) becomes

\[
\nu_B \leq \bar{\nu}_B \leq \nu_A < \bar{\nu}_A,
\]

i.e., the last inequality becomes strict, or equivalently,

\[
\lambda \hat{s}_B \leq \frac{\lambda}{1 - \bar{\nu}} \int_{0}^{\hat{s}_B} (1 - \mu(s)) ds \leq \lambda(1 - \hat{s}_A) < \frac{\lambda}{\bar{\nu}} \int_{\hat{s}_A}^{1} \mu(s) ds.
\]

(C.2)
In equilibrium, we must have that
\[ m_A(\hat{s}_A) = \mu(\hat{s}_A)[1 - G(\overline{v}_A)] = \kappa, \]  
(C.3)
\[ m_B(\hat{s}_B) = (1 - \mu(\hat{s}_B))[1 - G(\overline{v}_A)] + G(\overline{v}_A) - G(\overline{v}_B) = \kappa. \]  
(C.4)
From (C.3), \(1 - G(\overline{v}_A) = \frac{\kappa}{\mu(\hat{s}_A)}.\) Substituting this into (C.4), we have that
\[ G(\overline{v}_A) - G(\overline{v}_B) = \kappa \left( \frac{\mu(\hat{s}_A) + \mu(\hat{s}_B) - 1}{\mu(\hat{s}_A)} \right). \]
Since \(\overline{v}_A > \overline{v}_B\) by (C.2), we have that
\[ \mu(\hat{s}_A) + \mu(\hat{s}_B) > 1 \iff \mu(\hat{s}_B) > 1 - \mu(\hat{s}_A) = \mu(1 - \hat{s}_A), \]
where the last equality follows from the symmetry of \(\mu(\cdot)\). Since \(\mu(\cdot)\) is strictly increasing, this implies that \(\hat{s}_B > 1 - \hat{s}_A\). Therefore, \(\lambda \hat{s}_B > \lambda (1 - \hat{s}_A)\) which contradicts to (C.2).

Appendix D. Proofs of Theorem 3

Lemma D1. \(T\) is continuous on \(S\).

Proof. Fix any \(\hat{s} = (\hat{s}_A, \hat{s}_B) \in S\). Recall that for the given \(\hat{s}\), \(\alpha(\cdot; \hat{s})\) and \(\beta(\cdot; \hat{s})\) are given by (3.6) and (3.7), and
\[ \overline{v}_A = \lambda(1 - \hat{s}_A), \quad \underline{v}_A = \frac{\lambda}{\mu} \int_{\hat{s}_A}^{1} \mu(s)ds, \quad \overline{v}_B = \lambda \hat{s}_B, \quad \underline{v}_B = \frac{\lambda}{1 - \mu} \int_{0}^{\hat{s}_B} (1 - \mu(s))ds. \]
Note that \(\overline{v}_A\) and \(\underline{v}_A\) are continuous in \(\hat{s}_A\), and \(\overline{v}_B\) and \(\underline{v}_B\) are continuous in \(\hat{s}_B\). Now let
\[ \underline{v} := \min \{\underline{v}_A, \underline{v}_B\}, \quad \overline{v} := \max \{\overline{v}_A, \overline{v}_B\}, \quad \overline{v} := \min \{\overline{v}_A, \overline{v}_B\}. \]
For such \(\hat{s}\), \(T(\hat{s}) = \tilde{s}\), where \(\tilde{s} = (\tilde{s}_A, \tilde{s}_B) \in S\), satisfies that
\[ \tilde{s}_A = \inf \left\{ s \in [0, 1] \left| \int_{0}^{1} \alpha(v; \hat{s})[1 - \beta(v; \hat{s}) + \mu(s)\beta(v; \hat{s})]dG(v) - \kappa > 0 \right. \right\}, \]
if the set in the RHS is nonempty, or else \(\tilde{s}_A \equiv 1\), and
\[ \tilde{s}_B = \sup \left\{ s \in [0, 1] \left| \int_{0}^{1} \beta(v; \hat{s})[1 - \alpha(v; \hat{s}) + (1 - \mu(s))\alpha(v; \hat{s})]dG(v) - \kappa > 0 \right. \right\}, \]
if the set in the RHS is nonempty, or else \(\tilde{s}_B \equiv 0\).

Consider now any \(\hat{s}' = (\hat{s}'_A, \hat{s}'_B) \in S\). Then, for such \(\hat{s}'\), \(\alpha(\cdot; \hat{s}') \equiv \alpha'\) and \(\beta(\cdot; \hat{s}') \equiv \beta'\) are given by (3.6) and (3.7), and
\[ \underline{v}' := \min \{\underline{v}_A, \underline{v}_B\}, \quad \overline{v}' := \max \{\overline{v}_A, \overline{v}_B\}, \quad \overline{v}' := \min \{\overline{v}_A, \overline{v}_B\}. \]
Again, \(\tilde{s}' = (\tilde{s}'_A, \tilde{s}'_B) \in S\) is defined by \(T\).
Lastly, for any $v_i = 0, 1$, we have
\[ |\alpha'(v) - \alpha(v)| = \sum_{i=1}^{3} |\alpha'(v) - \alpha(v)| I_{\mathcal{V}_i}(v), \]
where $I_{\mathcal{V}_i}(v)$ is the indicator function, which is 1 if $v_i \in \mathcal{V}_i$ or 0 otherwise. Therefore,
\[ \int_0^1 |\alpha'(v) - \alpha(v)| \, dG(v) = \sum_{i=1}^{3} \int_0^1 |\alpha'(v) - \alpha(v)| I_{\mathcal{V}_i}(v) \, dG(v). \]

Observe, first, that by the continuity of $v_i$ and $\tau_i$, $i = A, B$, there is a $\delta_1 > 0$ such that for any $\varepsilon > 0$, if $\|s' - \hat{s}\| < \delta_1$, then
\[ \int_0^1 I_{\mathcal{V}_1}(v) \, dG(v) < \frac{\varepsilon}{6}. \] (D.1)

Second, for any $v \in \mathcal{V}_2$, the continuity of $\alpha_0(\cdot)$, given by (3.4), implies that there is $\delta_2$ such that $\|s' - \hat{s}\| < \delta_2$ implies
\[ |\alpha'(v) - \alpha(v)| = |\alpha'_0(v) - \alpha_0(v)| < \frac{\varepsilon}{6}; \] (D.2)

Lastly, for any $v \in \mathcal{V}_3$, $\alpha'(v)$ and $\alpha(v)$ are either 0 or 1 at the same time, hence we have that
\[ |\alpha'(v) - \alpha(v)| = 0. \] (D.3)

Now, let $\delta = \min_{i=1,2,3} \{\delta_i\}$ and suppose $\|s' - \hat{s}\| < \delta$. Then, we have
\[
\int_0^1 |\alpha'(v) - \alpha(v)| \, dG(v) = \sum_{i=1}^{3} \int_0^1 |\alpha'(v) - \alpha(v)| \, dG(v)
= \int_0^1 |\alpha'(v) - \alpha(v)| I_{\mathcal{V}_1}(v) \, dG(v) + \int_0^1 |\alpha'_0(v) - \alpha_0(v)| I_{\mathcal{V}_2}(v) \, dG(v)
\leq \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}, \quad (D.4)
\]
where the first equality follows from (D.3), the first inequality holds since $|\alpha'(v) - \alpha(v)| \leq 1$ and the last follows from (D.1) and (D.2).

Similarly, we also have that

$$\int_0^1 |\beta'(v) - \beta(v)| \, dG(v) < \frac{\varepsilon}{3}. \quad (D.5)$$

Observe that

$$\left| \int_0^1 \alpha'(v)[1 - \beta'(v) + \mu(s)\beta'(v)] \, dG(v) - \int_0^1 \alpha(v)[1 - \beta(v) + \mu(s)\beta(v)] \, dG(v) \right|$$

$$= \left| \int_0^1 \left[ (\alpha'(v) - \alpha(v)) - (1 - \mu(s))[\alpha'(v)\beta'(v) - \alpha(v)\beta(v)] \right] \, dG(v) \right|$$

$$\leq \int_0^1 |\alpha'(v) - \alpha(v)| \, dG(v) + (1 - \mu(s)) \int_0^1 |\alpha'(v)\beta'(v) - \alpha(v)\beta(v)| \, dG(v)$$

$$\leq \frac{2}{3} \varepsilon, \quad (D.6)$$

The first part of (D.6) is smaller than $3/\varepsilon$ by (D.4). The second part of (D.6) is

$$\int_0^1 |\alpha(v)\beta'(v) - \alpha(v)\beta(v)| \, dG(v) = \int_0^1 |\alpha'(v)\beta'(v) - \alpha'(v)\beta(v) + \alpha'(v)\beta(v) - \alpha(v)\beta(v)| \, dG(v)$$

$$\leq \int_0^1 |\beta'(v) - \beta(v)| \, dG(v) + \int_0^1 |\alpha'(v) - \alpha(v)| \, dG(v)$$

where the first inequality holds since $\alpha'(v), \beta(v) \leq 1$, and the last inequality follows from (D.4) and (D.5). Therefore, if $\|\hat{s}' - \hat{s}\| < \delta$, then

$$\left| \int_0^1 \alpha'(v)[1 - \beta'(v) + \mu(s)\beta'(v)] \, dG(v) - \int_0^1 \alpha(v)[1 - \beta(v) + \mu(s)\beta(v)] \, dG(v) \right| < \varepsilon. \quad (D.7)$$

Similarly, we also have

$$\left| \int_0^1 \beta'(v)[1 - \alpha'(v) + (1 - \mu(s))\alpha'(v)] \, dG(v) - \int_0^1 \beta(v)[1 - \alpha(v) + (1 - \mu(s))\alpha(v)] \, dG(v) \right| < \varepsilon. \quad (D.8)$$

Combining (D.7) and (D.8), we conclude that there is $\delta > 0$ such that for any $\varepsilon > 0$, if $\|\hat{s}' - \hat{s}\| < \delta$, then $\|\hat{s}' - \hat{s}\| < \varepsilon$. Since $\hat{s}$ is chosen arbitrary, $T$ is continuous on $S$. ■

**Lemma D2.** $\pi_A$ is concave in $\alpha$.

**Proof.** Recall that

$$\pi_A = \int_0^1 \left[ \int_0^1 \nu \alpha(v)[1 - \beta(v) + \mu(s)\beta(v)] \, dG(v) \right] ds - \lambda \int_0^1 \max \left\{ \int_0^1 \alpha(v)[1 - \beta(v) + \mu(s)\beta(v)] \, dG(v) - \kappa, 0 \right\} ds. \quad (D.9)$$
Consider any feasible $\alpha$ and $\alpha'$. Then, for $\eta \in [0,1]$, the first part of (D.9) is linear in $\alpha$,

$$\int_0^1 v[\eta \alpha(v) + (1 - \eta)\alpha'(v)][1 - \beta(v) + \mu(s)\beta(v)]dG(v)$$

$$= \eta \int_0^1 \alpha(v)[1 - \beta(v) + \mu(s)\beta(v)]dG(v) + (1 - \eta) \int_0^1 \alpha'(v)[1 - \beta(v) + \mu(s)\beta(v)]dG(v).$$

And the second part is convex in $\alpha$, since

$$\max \left\{ \int_0^1 [\eta \alpha(v) + (1 - \eta)\alpha'(v)][1 - \beta(v) + \mu(s)\beta(v)]dG(v) - \kappa, 0 \right\}$$

$$= \max \left\{ \eta \left[ \int_0^1 \alpha(v)[1 - \beta(v) + \mu(s)\beta(v)]dG(v) - \kappa \right] + (1 - \eta) \left[ \int_0^1 \alpha'(v)[1 - \beta(v) + \mu(s)\beta(v)]dG(v) - \kappa \right], 0 \right\}$$

$$\leq \eta \max \left\{ \int_0^1 \alpha(v)[1 - \beta(v) + \mu(s)\beta(v)]dG(v) - \kappa, 0 \right\}$$

$$+ (1 - \eta) \max \left\{ \int_0^1 \alpha'(v)[1 - \beta(v) + \mu(s)\beta(v)]dG(v) - \kappa, 0 \right\}.$$

Therefore, we have that

$$\pi_A(\eta \alpha + (1 - \eta)\alpha') \geq \eta \pi_A(\alpha) + (1 - \eta)\pi_A(\alpha'),$$

that is, $\pi_A$ is concave in $\alpha$. \hfill $\blacksquare$

**Lemma D3.** $V(\cdot)$ is concave in $t$ for any $t \in [0,1]$.

**Proof.** Observe that $\alpha(v; t)$ is linear in $t$, since for any $t, t' \in [0,1]$,

$$\alpha(v; \eta t + (1 - \eta) t') = (\eta t + (1 - \eta) t')\tilde{\alpha}(v) + (1 - (\eta t + (1 - \eta) t'))\alpha(v)$$

$$= [\eta t \tilde{\alpha}(v) + \eta (1 - t) \alpha(v)] + [(1 - \eta) t' \tilde{\alpha}(v) + (1 - \eta) (1 - t) \alpha(v)]$$

$$= \eta \alpha(v; t) + (1 - \eta) \alpha(v; t').$$

Therefore, we have

$$V(\eta t + (1 - \eta) t') = \pi_A(\alpha(v; \eta t + (1 - \eta) t'))$$

$$\geq \eta \pi_A(\alpha(v; t)) + (1 - \eta) \pi_A(\alpha(v; t'))$$

$$= \eta V(t) + (1 - \eta) V(t'),$$

where the second equality follows from (D.10) and the inequality follows from (D.11). \hfill $\blacksquare$

**Lemma D4.** $V'(0) \leq 0$.

**Proof.** Let

$$W(t, \tilde{s}_A(t)) := \int_0^1 v\alpha(v; t)[1 - \beta(v) + \tilde{\mu}\beta(v)]dG(v) - \lambda \int_{\tilde{s}_A(t)}^1 \left[ \int_0^1 \alpha(v; t)[1 - \beta(v) + \mu(s)\beta(v)]dG(v) - \kappa \right] ds$$
and denote it by

\[ V(t) := W(t, \hat{s}_A(t)). \]

Observe that

\[ V'(t) = W_1(t, \hat{s}_A(t)) + W_2(t, \hat{s}_A(t))\dot{s}'(t), \]

where

\[
W_1(t, \hat{s}_A(t)) = \int_0^1 (\hat{\alpha}(v) - \alpha(v))\left[v[1 - \beta(v) + \mu\beta(v)] - \lambda \int_{\hat{s}_A(t)}^1 [1 - \beta(v) + \mu(s)\beta(v)]ds\right]dG(v)
\]

and

\[
W_2(t, \hat{s}_A(t)) = \lambda \left[\int_0^1 (\alpha(v; t)[1 - \beta(v) + \mu(\hat{s}_A(t))\beta(v)]dG(v) - \kappa\right].
\]

Notice that \( W_2(0, \hat{s}_A(0)) = 0 \) (by definition of \( \hat{s}_A \)). Therefore, we have

\[
V'(0) = W_1(0, \hat{s}(0)) = \int_0^1 (\hat{\alpha}(v) - \alpha(v))H_\alpha(v, \beta(v))dG(v) \leq 0,
\]

where the inequality follows from local incentives. \( \blacksquare \)

**Appendix E. Proofs for Section 4**

**Proof for Theorem 5.** Suppose on the contrary that \( \hat{y} < \frac{1}{2} \). Then, \( K(\hat{y}|s) < 1 - K(\hat{y}|s) \) since \( \frac{1}{2} \leq \mu(s) = 1 - K(\frac{1}{2}|s) < 1 - K(\hat{y}|s) \). Therefore, we have that

\[
P_A(y) - P_B(\hat{y}) = \int_0^1 \min\left\{ \frac{\kappa}{1 - K(\hat{y}|s)}, 1 \right\}l(s|y)ds - \int_0^1 \min\left\{ \frac{\kappa}{K(\hat{y}|s)}, 1 \right\}l(s|y)ds \leq 0.
\]

Hence,

\[
T(\hat{y}) = \hat{y} - \psi(\hat{y}) = \hat{y}P_A(\hat{y}) - (1 - \hat{y})P_B(\hat{y}) < \frac{1}{2}[P_A(\hat{y}) - P_B(\hat{y})] \leq 0,
\]

where the first inequality holds since \( \hat{y} < \frac{1}{2} \). Thus, we reach a contradiction. \( \blacksquare \)

**Proof of Theorem 6.** We show that there is a positive measure of states in which \( c_A(s) \neq c_B(s) \). Suppose on the contrary \( c_A(s) = c_B(s) \) for almost all \( s \). Recall that equilibrium admission cutoff of each college satisfies

\[
G(c_A(s)) = \max\left\{ 1 - \frac{\kappa}{1 - K(\hat{y}|s)}, 0 \right\} \quad \text{and} \quad G(c_B(s)) = \max\left\{ 1 - \frac{\kappa}{K(\hat{y}|s)}, 0 \right\}.
\]

Since \( G(\cdot) \) is strictly increasing, if \( c_A(s) = c_B(s) \), then we must have either \( n_i(s) < \kappa \) for all \( i = A, B \) (so that \( c_A(s) = c_B(s) = 0 \)) or \( n_A(s) = n_B(s) \geq \kappa \).

First, we cannot have \( n_i(s) < \kappa \) for all \( i \) in equilibrium, since this means that all applicants are admitted by either college, and this contradicts to \( 2\kappa < 1 \). Second, suppose \( n_A(s) = n_B(s) \geq \kappa \). This implies that \( K(\hat{y}|s) = \frac{1}{2} \) for all \( s \) (recall that \( n_A(s) = 1 - K(\hat{y}|s) \) and
\[ n_B(s) = K(\hat{y}|s). \] However, by (4.1), we have \( K(\hat{y}|s') < K(\hat{y}|s) \) for all \( s' > s \). Therefore, we reach a contradiction again. 

\[ \Box \]

**Appendix F. Proof of Theorem 7**

Consider a college, say \( A \). Under the strategy suggested in the theorem, its payoff is

\[
\pi_A = \frac{1}{2} s_a (1 - \varepsilon) \int_{\hat{v}}^{1} v dG(v) + \frac{1}{2} \left[ s_b (1 - \varepsilon) \int_{\hat{v}}^{1} v dG(v) + (1 - \varepsilon) \int_{\tilde{v}}^{\hat{v}} v dG(v) \right] 
\]

\[
= \frac{1}{2} (1 - \varepsilon) \left[ \int_{\hat{v}}^{1} v dG(v) + \int_{\tilde{v}}^{\hat{v}} v dG(v) \right],
\]

where \( \tilde{v} \) is such that

\[
(1 - \varepsilon) [G(\tilde{v}) - G(v)] = \kappa - s_b (1 - \varepsilon) [1 - G(\hat{v})].
\tag{F.1}
\]

and the second equality follows from \( s_a = 1 - s_b \).

Consider now its payoff under the deviation. Notice first that those students in \( [\hat{v} - \delta', \hat{v}] \) accept college \( A \), since they prefer it over \( C \). Therefore, \( A \)'s payoff under the deviation is

\[
\pi_A^d = (1 - \varepsilon) \int_{\hat{v} - \delta'}^{\hat{v}} v dG(v) + \frac{1}{2} s_a (1 - \varepsilon) \int_{\hat{v} + \delta}^{\hat{v}} v dG(v) + \frac{1}{2} \left[ s_b (1 - \varepsilon) \int_{\hat{v} + \delta}^{\hat{v}} v dG(v) + (1 - \varepsilon) \int_{\tilde{v}}^{\hat{v}} v dG(v) \right] 
\]

\[
= (1 - \varepsilon) \int_{\hat{v} - \delta'}^{\hat{v}} v dG(v) + \frac{1}{2} (1 - \varepsilon) \left[ \int_{\hat{v} + \delta}^{\hat{v}} v dG(v) + \int_{\tilde{v}}^{\hat{v}} v dG(v) \right],
\]

where \( \hat{v} \) satisfies

\[
(1 - \varepsilon) [G(\hat{v}) - G(\hat{v})] = \kappa - (1 - \varepsilon) [G(\hat{v}) - G(\hat{v} - \delta')] - s_b (1 - \varepsilon) [1 - G(\hat{v} + \delta)],
\]

that is, \( \hat{v} \) is set to meet the capacity in the less popular state given that the students in \( [\hat{v} - \delta', \hat{v}] \) will attend it for sure in any state. Observe that \( \hat{v} > \tilde{v} \), since

\[
(1 - \varepsilon) [G(\hat{v}) - G(\hat{v})] = \kappa - s_b (1 - \varepsilon) [1 - G(\hat{v})] - s_a (1 - \varepsilon) [G(\hat{v}) - G(\hat{v} - \delta')] 
\]

\[
= (1 - \varepsilon) [G(\hat{v}) - G(\tilde{v})] - s_a (1 - \varepsilon) [G(\hat{v}) - G(\hat{v} - \delta')].
\tag{F.2}
\]

where the first equality follows from the fact \( s_a = 1 - s_b \) and (5.1), and the last equality follows from (F.1). Thus, we have

\[
\frac{2(\pi_A^d - \pi_A)}{1 - \varepsilon} = 2 \int_{\hat{v} - \delta'}^{\hat{v}} v dG(v) - \left[ \int_{\hat{v} + \delta}^{\tilde{v}} v dG(v) + \int_{\tilde{v}}^{\hat{v}} v dG(v) \right] 
\]

\[
= 2 \left[ \hat{v} G(\hat{v}) - (\hat{v} - \delta) G(\hat{v} - \delta') - \int_{\hat{v} - \delta'}^{\hat{v}} G(v) dv \right].
\]
Therefore, for sufficiently small \( \delta \) response for \( A \) students (who could be assigned to \( A \) from this deviation comes from that it causes a “rejection chain,” that is, \( A \) apparently, \( A \) capacity brings about a cost higher than her value. Suppose \( \lambda \) has no incentive to over-report its quota, or else it will pay costs for over-enrollment, \( A \) students apply to \( B \) group of students are worse than the first group of students, since \( A \) could be better off. However, the common preference of \( A \) to some positive mass of students with scores below \( \hat{v} \) such that a positive mass of students with scores above \( \hat{v} \) reject the second group of students.

\[
- \left[ (\hat{v} + \delta)G(\hat{v} + \delta) - \hat{v} G(\hat{v}) - \int_{\hat{v}}^{\hat{v} + \delta} G(v)dv \right] - \left[ \hat{v} G(\hat{v}) - G(\hat{v}) - \int_{\lambda}^{\hat{v}} G(v)dv \right] \\
= \hat{v} \left[ G(\hat{v}) - G(\hat{v} - \delta') \right] + 2 \left[ \delta' G(\hat{v} - \delta') - \int_{\hat{v} - \delta'}^{\hat{v}} G(v)dv \right] \\
- \left[ \delta G(\hat{v} + \delta) - \int_{\hat{v}}^{\hat{v} + \delta} G(v)dv \right] - \left[ \hat{v} G(\hat{v}) - G(\hat{v}) - \int_{\lambda}^{\hat{v}} G(v)dv \right] \\
\geq (\hat{v} - 2\delta') \left[ G(\hat{v}) - G(\hat{v} - \delta') \right] - \delta \left[ G(\hat{v} + \delta) - G(\hat{v}) \right] - \hat{v} \left[ G(\hat{v}) - G(\hat{v}) \right]
\]

where the second equality follows from the integration by parts, and the third equality follows from (5.1). The inequality holds since \( \int_{\hat{v} - \delta'}^{\hat{v}} G(v) dv \leq \delta' G(\hat{v}) \), \( \int_{\hat{v}}^{\hat{v} + \delta} G(v) dv \geq \delta G(\hat{v}) \) and \( \int_{\lambda}^{\hat{v}} G(v) \geq (\hat{v} - \lambda) G(\lambda) \). Observe that by rearranging (F.2), we have \( G(\hat{v}) - G(\gamma) = s_a \left[ G(\hat{v}) - G(\hat{v} - \delta') \right] \). Hence, using (5.1) again, we get

\[
2 \left( \frac{\pi^d_A - \pi_A}{1 - \varepsilon} \right) \geq [G(\hat{v}) - G(\hat{v} - \delta')] (\hat{v} - 2\delta' - \delta - \hat{v} s_a) = [G(\hat{v}) - G(\hat{v} - \delta')] \left( s_a (\hat{v} - \hat{v}) + s_b \hat{v} - (2\delta' + \delta) \right)
\]

Therefore, for sufficiently small \( \delta \), we have \( \pi^d_A > \pi_A \).

**Appendix G. Proof of Lemma 3**

Suppose college \( B \) truthfully reports its ranking and capacity. We show that it is a best response for \( A \) to do the same.

Observe first that it is a dominant strategy for \( A \) to report its capacity truthfully. Clearly, \( A \) has no incentive to over-report its quota, or else it will pay costs for over-enrollment, which is not profitable since \( \lambda \geq 1 \) and \( \nu \in [0, 1] \); that is, each student beyond the (true) capacity brings about a cost higher than her value. Suppose \( A \) under-reports its capacity. Apparently, \( A \) will have a positive mass of unfilled seats, a part of loss. The possible gain from this deviation comes from that it causes a “rejection chain,” that is, \( A \) rejects some students (who could be assigned to \( A \) if \( A \) had not underreported its quota), and those students apply to \( B \), causing \( B \) to reject some other students, who then apply to \( A \). If those second group of students are more preferred by \( A \) over the first group of students, then \( A \) could be better off. However, the common preference of \( A \) and \( B \) implies that those second group of students are worse than the first group of students, since \( B \) would not otherwise reject the second group of students.

Suppose now that \( A \) has changed its rankings for students. Then, there exists some value \( \hat{v} \) such that a positive mass of students with scores above \( \hat{v} \) are reported to be less preferred to some positive mass of students with scores below \( \hat{v} \). If the first group of students were not able to be admitted by \( A \) for all states when \( A \) reports its ranking truthfully, this does
not change A’s payoff. However, when those first group of students were able to be assigned to A for a positive measure states under truthful report, A’s deviation causes a rejection chain. The common preferences of A and B, again, implies that this is not profitable for A.

APPENDIX H. EXTENSION: MORE THAN TWO COLLEGES

Our main model in Section 2 considers the case with two colleges. In this section, we show that our analysis extends to the case with more than two colleges. While the extension works for any arbitrary number of colleges, we provide the result for the three-college case for expositional simplicity. It will become clear that the method also extends to larger numbers.

Let \( \sigma_i : [0, 1] \to [0, 1] \) be college \( i \)'s admission strategy, where \( i = 1, 2, 3 \). In each state \( s \in [0, 1] \), let \( \mu_{ijk}(s) \), where \( i, j, k = 1, 2, 3 \), denote the mass of students whose preference ordering is \( i > j > k \). Define the following notations.

- \( \mu_{i\succ j}(s) := \mu_{ijk}(s) + \mu_{ikj}(s) + \mu_{kij}(s) \) (the mass of students who prefer \( i \) to \( j \) in state \( s \)),
- \( \mu_{i\succ j,k}(s) := \mu_{ijk}(s) + \mu_{ikj}(s) \) (the mass of students who prefer \( i \) the most strongly in state \( s \)),

and

\[
\begin{align*}
\bar{\mu}_{i\succ j} &:= \int_0^1 \mu_{i\succ j}(s) \, ds, \\
\bar{\mu}_{i\succ j,k} &:= \int_0^1 \mu_{i\succ j,k}(s) \, ds.
\end{align*}
\]

For given \( \sigma_i(\cdot) \), \( i = 1, 2, 3 \), let \( n_i(v) \) be the probability that a student with score \( v \) attends college \( i \) in state \( s \) when she is admitted by \( i \). That is,

\[
n_i(v) := \left[ \prod_{t=j,k} (1-\sigma_t(v)) + \mu_{i\succ j}(s)\sigma_j(v)(1-\sigma_k(v)) + \mu_{i\succ k}(s)\sigma_k(v)(1-\sigma_j(v)) + \mu_{i\succ j,k}(s)\sigma_j(v)\sigma_k(v) \right].
\]

The student will attend college \( i \) if she is admitted only by \( i \), which happens with probability \( (1-\sigma_j(v))(1-\sigma_k(v)) \); or is admitted by college \( i \) and one of the less preferred colleges, which happens with probability \( \mu_{i\succ j}(s)\sigma_j(v)(1-\sigma_k(v)) + \mu_{i\succ k}(s)\sigma_k(v)(1-\sigma_j(v)) \) in state \( s \); or is admitted by both of the other colleges but prefers \( i \) the most, which happens with probability \( \mu_{i\succ j,k}(s)\sigma_j(v)\sigma_k(v) \) in state \( s \).

Thus, for a given profile of admission strategies, \( \sigma = (\sigma_i)_{i=1,2,3} \), in equilibrium, the mass of students who attend college \( i \) in state \( s \) is

\[
m_i(s) := \int_0^1 \sigma_i(v) n_i(v) \, dG(v),
\]

and college \( i \)'s payoff is

\[
\pi_i = \int_0^1 v \, \sigma_i(v) \, \bar{\pi}_i(v) \, dG(v) - \lambda \int_0^1 \max \{m_i(s) - \kappa, 0\} \, ds, \quad (H.1)
\]
where
\[
\bar{\pi}_i(v) := \int_0^1 n_i(v) \, ds. \tag{H.2}
\]

Recall that in the two-school case, the monotonicity of \(\mu(\cdot)\) yields cutoff states \((\hat{s}_A, \hat{s}_B)\) that trigger over-enrollment for each college, and the set of over-demanded states for each of them is a connected interval, \((\hat{s}_A, 1]\) and \([0, \hat{s}_B)\). To establish the existence of a maximally mixed equilibrium (MME), we have projected the admission strategies to a simpler (state) space, which allows us to use the Brouwer’s fixed point theorem. However, with more than two colleges, we do not know the structure of the set of over-demanded states in general, so we cannot directly define a map from cutoff states to cutoff states. Nonetheless, the main idea of the proof can be carried over, although to do so requires us using a fixed point theorem (Schauder) in a functional space.

Define a subdistribution \(F_i : [0, 1] \to [0, 1], i = 1, 2, 3\), such that \(F_i(0) = 0\) and
\[
F_i(s) := \text{Prob}(m_i(t) > \kappa \text{ for } t < s). \tag{H.3}
\]
The subdistribution of college \(i\) places a positive mass only on the states in which college \(i\) is over-demanded. Observe that \(F_i(\cdot)\) is nondecreasing and
\[
0 \leq F_i(s') - F_i(s) \leq s' - s, \quad \forall s' \geq s. \tag{20}
\]
Let \(\mathcal{F}_i\) be the set of all such subdistributions and \(\mathcal{F} := \times_{i=1}^3 \mathcal{F}_i\). (It will become clear that these subdistributions will play a similar role to the cutoff states in the two-school case.)

Using the subdistributions, each college’s payoff is now given by
\[
\pi_i = \int_0^1 v \sigma_i(v) \bar{\pi}_i(v) \, dG(v) - \lambda \int_0^1 (m_i(s) - \kappa) \, dF_i(s) \tag{H.4}
\]
\[
= \int_0^1 \sigma_i(v) H_i(v, \sigma_j(v), \sigma_k(v)) \, dG(v) + \lambda \int_0^1 \kappa \, dF_i(s),
\]

\(20\) The second inequality holds because
\[
F_i(s') - F_i(s) = \text{Prob}(m_i(t) > \kappa \text{ for } t < s') - \text{Prob}(m_i(t) > \kappa \text{ for } t < s)
\]
\[
= \text{Prob}(m_i(t) > \kappa \text{ for } s < t < s')
\]
\[
\leq \text{Prob}(s < t < s')
\]
\[
= s' - s.
\]

\(21\) Note that since \(F_i\) is Lipschitz continuous, so it is absolute continuous. Thus, the integration is well defined. Observe also that (H.4) does not involve \(\max\{\cdot, \cdot\}\) in the cost (see (H.1) for comparison), as the subdistribution is defined for states where \(m_i(s) > \kappa\) by (H.3), and the college’s cost is evaluated by the subdistribution.
where

\[ H_i(v, \sigma_j(v), \sigma_k(v)) := v \bar{\pi}_i(v) - \lambda \int_0^1 n_i(v) dF_i(s) \]  \hspace{1cm} (H.5)

is college \( i \)'s marginal payoff from admitting a student with score \( v \). The first part of the RHS of (H.5) is college \( i \)'s expected benefit, and the second part is its the expected cost. Note that this marginal payoff depends on the subdistribution \( F_i \), as \( \bar{\pi}_i(v) \) is a constant for given admission strategies \( (\sigma_i)_{i=1,2,3} \) (by (H.2)) and \( n_i(v) \) is evaluated by the subdistribution.

Consider now college \( i \)'s marginal payoff. First, \( H_i(v, 0, 0) \) is its marginal payoff from admitting a student with score \( v \) who is refused by both of the other colleges. Second, \( H_i(v, 1, 0) \) and \( H_i(v, 0, 1) \) are college \( i \)'s marginal payoffs from admitting a student with score \( v \) who is admitted by college \( j \) (or \( k \)) but rejected by \( k \) (or \( j \), respectively). Lastly, \( H_i(v, 1, 1) \) is the marginal payoff from a student with \( v \) who is admitted by both of the other colleges.

Using what we have so far established, (H.5) is decomposed as follow:

\[
H_i(v, \sigma_j(v), \sigma_k(v)) = (1 - \sigma_j(v))(1 - \sigma_k(v))H_i(v, 0, 0) + \sigma_j(v)(1 - \sigma_k(v))H_i(v, 1, 0) + (1 - \sigma_j(v))\sigma_k(v)H_i(v, 0, 1) + \sigma_j(v)\sigma_k(v)H_i(v, 1, 1).
\]

Let us now define \( v_{i1}^{11}, v_{i1}^{10}, v_{i1}^{01} \) and \( v_{i1}^{00} \) such that

\[ H_i(v_{i1}^{11}, 1, 1) = 0, \quad H_i(v_{i1}^{10}, 1, 0) = 0, \quad H_i(v_{i1}^{01}, 0, 1) = 0, \quad H_i(v_{i1}^{00}, 0, 0) = 0. \]

That is, \( v_{i1}^{10} \) (or \( v_{i1}^{00} \)) is the threshold score that makes college \( i \) indifferent from admitting or not a student who was admitted (or refused) by both of the other colleges. Likewise, \( v_{i1}^{10} \) and \( v_{i1}^{01} \) are the threshold scores that make college \( i \) indifferent from admitting or not a student who was admitted by only one of the other colleges.

Similar to the two-school case, \( H_i(v, \sigma_j, \sigma_k) \) partitions the students' type space as depicted in Figure H.1.\(^{22}\) Each college admits a student with score \( v \) such that \( H_i(v, 1, 1) > 0 \) and rejects a student with score \( v \) such that \( H_i(v, 0, 0) < 0 \). That is, each college admits a student whose score is high enough so that its marginal payoff from admitting the student is positive even when she is definitely admitted by both of the other colleges; and each college

\(^{22}\)It can be the case that \( v_{i1}^{01} < v_{i1}^{10} \).
rejects a student whose score is low enough that its marginal payoffs from admitting the student is negative. Where \( H_i(v, 1, 1) < 0 < H_i(v, 0, 0) \), each college admits a student with \( v \) if \( H_i(v, 1, 0) > 0 \) or \( H_i(v, 0, 1) > 0 \); that is, the college’s marginal payoff from admitting the student is positive only when she is admitted by one of the other colleges. This shows that colleges engage in *strategic targeting* for those intermediate range of scores.

Randomization may emerge for some students. For students with \( v \) such that
\[
\max_{i=1,2,3} \{H_i(v, 1, 0), H_i(v, 0, 1)\} < 0 <\min_{i=1,2,3} \{H_i(v, 0, 0)\},
\]
all three colleges engage in mixed-strategies, where the mixed-strategies satisfy
\[
H_i(v, \sigma_j(v), \sigma_k(v)) = 0 \quad \forall i, j, k = 1, 2, 3.
\]
For students with \( v \) such that \( H_k(v, 0, 0) < 0 \) and
\[
\max \{H_i(v, 1, 0), H_j(v, 1, 0)\} < 0 < \min \{H_i(v, 0, 0), H_j(v, 0, 0)\},
\]
college \( k \) does not admit such students, but colleges \( i \) and \( j \) engage in mixed-strategies satisfying
\[
H_i(v, \sigma_j, 0) = 0 \quad \text{and} \quad H_j(v, \sigma_i, 0) = 0.
\]

A typical mixed-strategy equilibrium is depicted in Figure H.2 when, for instance,
\[
v_3^{00} < v_2^{00} < v_1^{00} < v_3^{01} < v_2^{01} < v_1^{01} < v_3^{10} < v_2^{10} < v_1^{10} < v_3^{11} < v_2^{11} < v_1^{11}.
\]

Note that, as in the two-school case, there are many ways that colleges could coordinate (even in a mixed-strategy equilibrium). Hence, we consider the maximally mixed-strategy as before and provide the existence of such equilibrium.

For a given profile of subdistributions \( (F_i)_{i=1}^3 \), let \( \sigma := (\sigma_i)_{i=1}^3 \) be the profile of admission strategies that satisfy the local conditions described above. Then, such \( \sigma \) in turn determines a new profile of subdistributions, \( (F_i)_{i=1}^3 \) via (H.3). Next, we define \( T : \mathcal{F} \to \mathcal{F} \), a self-map from the set of subdistributions to itself, where \( \mathcal{F} = \times_{i=1}^3 \mathcal{F}_i \). The existence of equilibrium is achieved when \( T \) has a fixed point (on the functional space of \( \mathcal{F} \)).

As mentioned earlier, the idea of proving the existence of equilibrium is similar to the idea of Theorem 3, projecting the strategy profile into a simpler space. The difference is that in the two-school case, the strategy profiles are projected into the state space, but in the general case, they are projected into the set of subdistributions \( \mathcal{F} \).

**Theorem 9.** There exists an equilibrium with maximally mixed-strategies.
We first show that $\mathcal{F}$ is a compact and convex subset of a normed linear space, and $T : \mathcal{F} \rightarrow \mathcal{F}$ is continuous. Then, $T$ has a fixed point by Schauder’s fixed point theorem. We then show that the identified strategies in the previous step do indeed constitute mutual (global) best responses. We provide a formal proof in the next subsection.

**H.1. Proof of Theorem 9.** For given $(F_i)_{i=1,2,3}$, consider colleges’ strategy profile $(\sigma_i)_{i=1,2,3}$ which satisfies the following local conditions:

- $\sigma_i(v) = 1$ if $H_1(v, 1, 1) > 0$, $i = 1, 2, 3$.

---

23Schauder’s fixed point theorem is a generalization of Brouwer’s theorem on a normed linear space. It guarantees that every continuous self-map on a nonempty, compact, convex subset of a normed linear space has a fixed point (see Ok, 2007). In our framework, $\mathbf{CB}([0,1])$, the space of the continuous and bounded real maps on $[0,1]$, is a normed linear space, and $\mathcal{F}$ is a nonempty, compact, convex subset of it.
• $\sigma_1(v) = 0$ if $H_1(v, 1, 1) < 0$, $H_2(v, 1, 1) > 0$, $H_3(v, 1, 1) > 0$.
\[\sigma_2(v) = 0 \text{ if } H_1(v, 1, 1) > 0, \ H_2(v, 1, 1) < 0, \ H_3(v, 1, 1) > 0.\]
\[\sigma_3(v) = 0 \text{ if } H_1(v, 1, 1) > 0, \ H_2(v, 1, 1) > 0, \ H_3(v, 1, 1) < 0.\]

• $\sigma_1(v) = 0$, $\sigma_2(v) = 1$, $\sigma_3(v) = 1$ if
\[\begin{cases} H_1(v, 1, 1) < 0 \\ H_2(v, 1, 1) < 0, \ H_2(v, 0, 1) > 0 \\ H_3(v, 1, 1) < 0, \ H_3(v, 0, 1) > 0 \end{cases}\]

• $\sigma_1(v) = 1$, $\sigma_2(v) = 0$, $\sigma_3(v) = 1$ if
\[\begin{cases} H_1(v, 1, 1) > 0, \ H_1(v, 0) > 0 \\ H_2(v, 1, 1) < 0 \\ H_3(v, 1, 1) < 0, \ H_3(v, 1, 0) > 0 \end{cases}\]

• $\sigma_1(v) = 1$, $\sigma_2(v) = 1$, $\sigma_3(v) = 0$ if
\[\begin{cases} H_1(v, 1, 1) < 0, \ H_1(v, 0, 1) > 0 \\ H_2(v, 1, 1) < 0, \ H_2(v, 1, 0) > 0 \end{cases}\]

• $\sigma_1(v) = 1$, $\sigma_2(v) = 0$, $\sigma_3(v) = 0$ if
\[\begin{cases} H_1(v, 1, 1) < 0, \ H_1(v, 0, 0) > 0 \\ H_2(v, 1, 1) < 0, \ H_2(v, 1, 0) < 0 \end{cases}\]

• $\sigma_1(v) = 0$, $\sigma_2(v) = 1$, $\sigma_3(v) = 0$ if
\[\begin{cases} H_1(v, 1, 1) < 0, \ H_1(v, 1, 0) < 0 \\ H_2(v, 1, 1) < 0, \ H_2(v, 0, 0) > 0 \end{cases}\]

• $\sigma_1(v) = 0$, $\sigma_2(v) = 0$, $\sigma_3(v) = 1$ if
\[\begin{cases} H_1(v, 1, 1) < 0, \ H_1(v, 0, 1) < 0 \\ H_2(v, 1, 1) < 0, \ H_2(v, 0, 1) < 0 \end{cases}\]

• $\sigma_i(v) = 0$ if $H_i(v, 0, 0) < 0$, $i = 1, 2, 3$.

• $\sigma_i(v)$’s satisfy $H_1(v, \sigma_2(v), \sigma_3(v)) = H_2(v, \sigma_1(v), \sigma_3(v)) = H_3(v, \sigma_1(v), \sigma_2(v)) = 0$, if
\[\max_{i=1,2,3} \{H_i(v, 1, 0), H_i(v, 0, 1)\} < 0 < \min_{i=1,2,3} \{H_i(v, 0, 0)\}\]

• $\sigma_i(v)$ and $\sigma_j(v)$ satisfy $H_i(v, \sigma_j, 0) = 0$ and $H_j(v, \sigma_i, 0) = 0$ if $H_k(v, 0, 0) < 0$ and
\[\max \{H_i(v, 1, 0), H_j(v, 1, 0)\} < 0 < \min \{H_i(v, 0, 0), H_j(v, 0, 0)\}\]
Now, let $\mathbf{CB}([0,1])$ be the space of continuous and bounded real maps on $[0,1]$. Then, $\mathbf{CB}([0,1])$ is a normed linear space, with a sup norm $\|\cdot\|$, i.e., for any $F,F' \in \mathbf{CB}([0,1])$,

$$\|F - F'\| = \sup_{s \in [0,1]} |F(s) - F'(s)|.$$ 

**Lemma H1.** $\mathcal{F}$ is compact and convex.

**Proof.** We first show that $\mathcal{F}_i$, $i = 1, 2, 3$, is closed. To this end, consider any sequence $\{F_i^n\}$, where $F_i^n \in \mathcal{F}_i$ for each $n$, such that $\|F_i^n - F_i\| \to 0$ as $n \to \infty$. We prove that $F_i \in \mathcal{F}_i$.

Observe first that $F_i$ is nondecreasing. To see this, note that $F_i(s') - F_i(s) < 0$ for some $s' > s$ otherwise. But then,

$$\|F_i^n - F_i\| \geq \max \{|F_i^n(s') - F_i(s)|, |F_i(s) - F_i^n(s)|\}$$

$$\geq \frac{1}{2} (|F_i^n(s') - F_i(s)| + F_i(s) - F_i^n(s))$$

$$\geq \frac{1}{2} |F_i^n(s') - F_i(s) + F_i(s) - F_i^n(s)|$$

$$\geq \frac{1}{2} |F_i(s) - F_i(s')|$$

$$> 0$$

which is a contradiction. Likely, for $s' > s$, we must have that $F_i(s') - F_i(s) \leq s' - s$. To see this, suppose on the contrary that $F_i(s') - F_i(s) > s' - s$. Then,

$$\|F_i^n - F_i\| \geq \max \{|F_i(s') - F_i^n(s')|, |F_i^n(s) - F_i(s)|\}$$

$$\geq \frac{1}{2} (|F_i(s') - F_i^n(s')| + |F_i^n(s) - F_i(s)|)$$

$$\geq \frac{1}{2} |F_i(s') - F_i(s) + F_i^n(s) - F_i^n(s')|$$

$$\geq \frac{1}{2} |F_i(s') - F_i(s) - (s' - s)|$$

$$> 0,$$

which is a contradiction again. Combining these, $F_i \in \mathcal{F}_i$, proving that $\mathcal{F}_i$ is closed.

Next, we prove that $\mathcal{F}_i$ is compact. Note that for any $F_i \in \mathcal{F}_i$ and $s, s' \in [0,1]$,

$$|F_i(s') - F_i(s)| \leq |s' - s|,$$

Hence, $\mathcal{F}_i$ is Lipschitz continuous with Lipschitz constant $K$, and hence it is equicontinuous and bounded. By the Arzèla-Ascoli theorem, $^{24}$ $\mathcal{F}_i$ is compact.

---

$^{24}$Arzèla-Ascoli theorem gives conditions for a set of $\mathbf{C}(T)$ to be compact, where $\mathbf{C}(T)$ is the space of continuous maps on $T$ and $T$ is a compact metric space. A subset of $\mathbf{C}(T)$ is compact if and only if it is closed, bounded, and equicontinuous.
Next, we prove that $\mathcal{F}_i$ is convex. Observe that for any $F_i, F'_i \in \mathcal{F}$ and $s, s' \in [0, 1]$, for and $\eta \in (0, 1)$,

\[
(\eta F_i + (1 - \eta)F'_i)(s') - (\eta F_i + (1 - \eta)F'_i)(s) = \eta(F_i(s') - F_i(s)) + (1 - \eta)(F'_i(s') - F'_i(s)) \\
\leq \eta(s' - s) + (1 - \eta)(s' - s) \\
= s' - s,
\]

which proves that $\mathcal{F}_i$ is convex.

Since $\mathcal{F}_i$ is compact and closed, so is its Cartesian product $\mathcal{F} = \times_{i=1}^3 \mathcal{F}_i$ (with respect to the product topology).

**Lemma H2.** $T$ is continuous.

**Proof.** The proof involves several steps:

**Step 1.** $v_i^{jk}$’s are continuous on $F_1, F_2, F_3$.

**Proof.** We first show that $v_i^{jk}$’s are continuous in $F_i$. Fix any $F_i \in \mathcal{F}_i$ and $\varepsilon > 0$. Take $\delta = \varepsilon/2$. Then, for any $F_i, F'_i \in \mathcal{F}_i$ such that $\|F_i - F'_i\| < \delta$, we have that

\[
\left| v_i^{jk} - v_i^{jk'} \right| = \left| \frac{\lambda}{\mu} \int_0^1 \mu(s)v_{i,j,k}(f_i(s) - f'_i(s))ds \right| \\
= \left| \mu_{i,j,k}(1)[F_i(1) - F'_i(1)] - \int_0^1 \mu_{i,j,k}'(s)[F_i(s) - F'_i(s)]ds \right| \\
\leq 2 \|F_i(s) - F'_i(s)\| \\
< \varepsilon,
\]

where the third equality follows from the integration by parts and $F_i(0) = F'_i(0) = 0$, and the first inequality holds since $\int_0^1 \mu'(s)ds = \mu(1) - \mu(0) \leq 1$. \Box

**Step 2.** $\sigma_i$’s in mixed-strategies are continuous.

**Proof.** Consider, at first, students with score $v$ such that

\[
H_k(v, 0, 0) < 0, \quad (H.6) \\
H_i(v, 1, 0) < 0 < H_i(v, 0, 0), \quad (H.7) \\
H_j(v, 1, 0) < 0 < H_j(v, 0, 0). \quad (H.8)
\]

That is, college $k$ puts zero probability for those students (by (H.6)), and colleges $i$ and $j$ use mixed-strategies $\sigma_i$ and $\sigma_j$ which satisfy $H_i(v, \sigma_j, 0) = 0$ and $H_j(v, \sigma_i, 0) = 0$. 
Now, let $J_i : [0, 1]^2 \times [0, 1]^2 \rightarrow [0, 1]$ such that

\[
J_i(F_i, F_j, \sigma_i, \sigma_j) \equiv H_i(v, \sigma_j, 0) = v[(1 - \sigma_j) + \mu_{i\sigma_j} \sigma_j(v)] - \lambda \int_0^1 [(1 - \sigma_j) + \mu_{i\sigma_j}(s)\sigma_j(v)] dF_i(s),
\]

\[
J_j(F_i, F_j, \sigma_i, \sigma_j) \equiv H_j(v, \sigma_i, 0) = v[(1 - \sigma_i) + \mu_{j\sigma_i} \sigma_i(v)] - \lambda \int_0^1 [(1 - \sigma_i) + \mu_{j\sigma_i}(s)\sigma_i(v)] dF_j(s).
\]

Then, $\sigma_i$ and $\sigma_j$ are the solution to $J_i = 0$ and $J_j = 0$ in terms of $F_i$ and $F_j$. Observe that

\[
J_i = (1 - \sigma_j)H_i(v, 0, 0) + \sigma_j H_i(v, 1, 0).
\]

Hence,

\[
\frac{\partial J_i}{\partial \sigma_j} = -H_i(v, 0, 0) + H_i(v, 1, 0) < 0,
\]

where inequality follows from (H.7). Similarly, we also have by (H.8)

\[
\frac{\partial J_j}{\partial \sigma_i} = -H_j(v, 0, 0) + H_j(v, 1, 0) < 0.
\]

Therefore,

\[
\Delta_{ij} := \left| \begin{array}{cc}
\frac{\partial J_i}{\partial \sigma_i} & \frac{\partial J_j}{\partial \sigma_j} \\
\frac{\partial J_i}{\partial \sigma_j} & \frac{\partial J_j}{\partial \sigma_i}
\end{array} \right| = 0 \quad \frac{\partial J_i}{\partial \sigma_j} \frac{\partial J_j}{\partial \sigma_i} = - \frac{\partial J_i}{\partial \sigma_i} \frac{\partial J_j}{\partial \sigma_j} < 0.
\]

Since $\Delta_{ij} \neq 0$, the Implicit function theorem implies that there are unique $\sigma_i$ and $\sigma_j$ such that

\[
J_i(F_i, F_j, \sigma_i, \sigma_j) = 0 \quad \text{and} \quad J_j(F_i, F_j, \sigma_i, \sigma_j) = 0.
\]

Furthermore, such $\sigma_i$ and $\sigma_j$ are continuous.

Consider now the case that $H_1(v, \sigma_2, \sigma_3) = H_2(v, \sigma_1, \sigma_3) = H_3(v, \sigma_1, \sigma_2) = 0$ when

\[
\max_{i=1,2,3} \{H_i(v, 1, 0), H_i(v, 0, 1)\} < 0 < \min_{i=1,2,3} \{H_i(v, 0, 0)\}.
\]  \hspace{1cm} (H.9)

Similar as before, let

\[
J_1(F_1, F_2, F_3, \sigma_1, \sigma_2, \sigma_3) \equiv H_1(v, \sigma_2, \sigma_3) = 0,
\]

\[
J_2(F_1, F_2, F_3, \sigma_1, \sigma_2, \sigma_3) \equiv H_2(v, \sigma_1, \sigma_3) = 0,
\]

\[
J_3(F_1, F_2, F_3, \sigma_1, \sigma_2, \sigma_3) \equiv H_3(v, \sigma_1, \sigma_2) = 0.
\]

Observe that

\[
J_i = (1 - \sigma_j)(1 - \sigma_k)H_i(v, 0, 0) + \sigma_j(1 - \sigma_k)H_i(v, 1, 0) + (1 - \sigma_j)\sigma_k H_i(v, 0, 1) + \sigma_j\sigma_k H_i(v, 1, 1)
\]

\[
= (1 - \sigma_j)H_i(v, 0, 0) + \sigma_j(1 - \sigma_k)H_i(v, 1, 0) - (1 - \sigma_j)\sigma_k H_i(v, 1, 0) + \sigma_j\sigma_k H_i(v, 1, 1).
\]
where the second inequality holds after some rearrangement using the fact that \(1 - \mu_{i\succ k}(s) = \mu_{k\succ i}(s)\). Therefore,

\[
\frac{\partial J_i}{\partial \sigma_j} = -H_i(v, 0, 0) + (1 - \sigma_k)H_i(v, 1, 0) + \sigma_k H_k(v, 1, 0) + \sigma_k H_i(v, 1, 1) < 0,
\]

where the inequality holds since \(H_i(v, 0, 0) > 0\), \(H_i(v, 1, 0) < 0\), \(H_k(v, 1, 0) < 0\) and \(H_i(v, 1, 1) < 0\) by (H.9). This implies that

\[
\Delta := \begin{vmatrix}
\frac{\partial J_1}{\partial \sigma_1} & \frac{\partial J_1}{\partial \sigma_2} & \frac{\partial J_1}{\partial \sigma_3} \\
\frac{\partial J_2}{\partial \sigma_1} & \frac{\partial J_2}{\partial \sigma_2} & \frac{\partial J_2}{\partial \sigma_3} \\
\frac{\partial J_3}{\partial \sigma_1} & \frac{\partial J_3}{\partial \sigma_2} & \frac{\partial J_3}{\partial \sigma_3}
\end{vmatrix} = \begin{vmatrix}
0 & \frac{\partial J_1}{\partial \sigma_2} & \frac{\partial J_1}{\partial \sigma_3} \\
0 & \frac{\partial J_2}{\partial \sigma_2} & \frac{\partial J_2}{\partial \sigma_3} \\
0 & \frac{\partial J_3}{\partial \sigma_2} & \frac{\partial J_3}{\partial \sigma_3}
\end{vmatrix} = \frac{\partial J_1}{\partial \sigma_2} \frac{\partial J_2}{\partial \sigma_3} \frac{\partial J_3}{\partial \sigma_1} + \frac{\partial J_1}{\partial \sigma_3} \frac{\partial J_2}{\partial \sigma_1} \frac{\partial J_3}{\partial \sigma_2} < 0
\]

Using the Implicit function theorem again, we conclude that such \(\sigma_1, \sigma_2, \sigma_3\) exist and they are continuous. \(\square\)

Observe that from Step 1 and Step 2, \(H_i(v, \sigma, \sigma_k), i = 1, 2, 3\), is continuous on \((F_i)_{i=1,2,3}\) for a given \(s\) and fixed \(v\).

**Step 3.** \(m_i(s)\) is continuous.

**Proof.** Consider any \(F_i, F_i' \in \mathcal{F}_i\) such that \(||F_i - F_i'|| < \delta\) for all \(i = 1, 2, 3\). Let \(\sigma_i\) and \(\sigma_i'\) be admission strategies of college \(i\) which correspond to \(F_i\) and \(F_i'\), respectively. For a given \(s\), fix any \(v\) and let

\[
n_i(v) = \prod_{t=j,k} (1 - \sigma_i(t)) + \mu_{i\succ j}(s)\sigma_j(v)(1 - \sigma_k(v)) + \mu_{i\succ k}(s)\sigma_k(v)(1 - \sigma_j(v)) + \mu_{i\succ j,k}(s)\sigma_j(v)\sigma_k(v)
\]

and

\[
n'_i(v) = \prod_{t=j,k} (1 - \sigma'_i(t)) + \mu_{i\succ j}(s)\sigma'_j(v)(1 - \sigma'_k(v)) + \mu_{i\succ k}(s)\sigma'_k(v)(1 - \sigma'_j(v)) + \mu_{i\succ j,k}(s)\sigma'_j(v)\sigma'_k(v).
\]

Let \(X := \{v \in [0, 1]| |\sigma_i(v) - \sigma'_i(v)| \geq \varepsilon/2\}\). Clearly,

\[
|\sigma_i(v) - \sigma'_i(v)| = |\sigma_i(v) - \sigma'_i(v)| \mathbf{1}_X(v) + |\sigma_i(v) - \sigma'_i(v)| \mathbf{1}_{X^c}(v),
\]

where \(\mathbf{1}_X(v)\) is the indicator function which is 1 if \(v \in X\) or 0 otherwise.

Since \(v_i^{jk}\) are continuous by Step 1, we have

\[
\int_0^1 \mathbf{1}_X(v) \, dG(v) < \frac{\varepsilon}{2}. \tag{H.10}
\]

For \(v \in X^c\), it must be the case that either \(\sigma_i = \sigma'_i\), or \(\sigma_i\) and \(\sigma'_i\) are the mixed-strategies. Thus, we have for \(v \in X^c\),

\[
|\sigma_i(v) - \sigma'_i(v)| < \frac{\varepsilon}{2}. \tag{H.11}
\]
Observe that
\[
\int_0^1 |\sigma_i(v) - \sigma'_i(v)| \, dG(v) = \int_0^1 |\sigma_i(v) - \sigma'_i(v)| \mathbb{1}_X(v) \, dG(v) + \int_0^1 |\sigma_i(v) - \sigma'_i(v)| \mathbb{1}_{X^c}(v) \, dG(v)
\]
\[
< \int_0^1 \mathbb{1}_X(v) \, dG(v) + \int_0^1 |\sigma_i(v) - \sigma'_i(v)| \mathbb{1}_{X^c}(v) \, dG(v)
\]
\[
< \varepsilon,
\]
where the first inequality holds since $\sigma_i, \sigma'_i \leq 1$, and the last inequality follows from (H.10) and (H.11).

This implies that there exist $\delta_i$ such that $\|F_i - F'_i\| < \delta_i$, for all $i, i' = 1, 2, 3$, implies
\[
\int_0^1 |\sigma_i(1 - \sigma_j)(1 - \sigma_k) - \sigma'_i(1 - \sigma'_j)(1 - \sigma'_k)| \, dG(v)
\]
\[
\leq \int_0^1 \left[ |\sigma_i - \sigma'_i|(1 - \sigma_j)(1 - \sigma_k) + |\sigma_j - \sigma'_j| \sigma'_i(1 - \sigma_k) + |\sigma_k - \sigma'_k| \sigma'_l(1 - \sigma'_j) \right] \, dG(v)
\]
\[
< \frac{\varepsilon}{4}
\]
Similarly, there are $\delta_i, t = 2, 3, 4$, such that $\|F_i - F'_i\| < \delta_t$ respectively imply that
\[
|\sigma_i \sigma_j (1 - \sigma_k) - \sigma'_i \sigma'_j (1 - \sigma'_k)| < \frac{\varepsilon}{4}, \quad |\sigma_i \sigma_k (1 - \sigma_j) - \sigma'_i \sigma'_k (1 - \sigma'_j)| < \frac{\varepsilon}{4}, \quad |\sigma_i \sigma_j \sigma_k - \sigma'_i \sigma'_j \sigma'_k| < \frac{\varepsilon}{4}.
\]

Now, let $\delta = \min_{t=1,2,3,4} \{\delta_t\}$. We have that $\|F_i - F'_i\| < \delta$ implies
\[
|m_i(s) - m'_i(s)| \equiv \left| \int_0^1 \sigma_i(v) n_i(v) \, dG(v) - \int_0^1 \sigma'_i(v) n'_i(v) \, dG(v) \right| < \varepsilon.
\]
That is, $m_i(s)$ is continuous on $(F_i)_{i=1,2,3}$. □

Lemma H2 proves the existence admission strategies that satisfy the local conditions. The proof that those strategies are mutual (global) best responses is analogous to that of the two college case. We briefly summarize it below:

Consider a college, say $i$. For given $\sigma_j(\cdot)$ and $\sigma_k(\cdot)$, let $\tilde{\sigma}_i(v) \in [0, 1]$ be an arbitrary strategy for $v \in [0, 1]$. Let $\tilde{\sigma}_i(v; t)$ be a variation of $\sigma_i(\cdot)$ such that for any $t \in [0, 1],$
\[
\sigma_i(v; t) := t\tilde{\sigma}_i(v) + (1 - t)\sigma_i(v).
\]
Define $i$'s payoff function in terms of $\sigma_i(v; t),$
\[
V(t) := \int_0^1 v \sigma_i(v; t) \tilde{n}_i(v) \, dG(v) - \lambda \int_0^1 \max \left\{ \int_0^1 \sigma_i(v; t) n_i(v) \, dG(v) - \kappa, 0 \right\} \, ds.
\]
Observe that $V(\cdot)$ is continuous and concave in $t$. Therefore, we have
\[
\pi_i(\tilde{\sigma}_i) = V(1) \leq V(0) + V'(0) \leq V(0) = \pi_i(\sigma_i),
\]
where the second inequality holds since

\[ V'(0) = \int_0^1 [\tilde{\sigma}_i(v) - \sigma_i(v)]H_i(v, \sigma_j(v), \sigma_k(v)) dG(v) \leq 0 \quad \text{(H.12)} \]

because if \( H_i(v, \sigma_j(v), \sigma_k(v)) \geq 0 \) for some \( v \), then \( \sigma_i(v) = 1 \geq \tilde{\sigma}_i(v) \); and if \( H_i(v, \sigma_j(v), \sigma_k(v)) \leq 0 \), then \( \sigma_i(v) = 0 \leq \tilde{\sigma}_i(v) \); and \( H_i(v, \sigma_j(v), \sigma_k(v)) = 0 \) otherwise.

**References**


