Exam Schools, Ability, and the Effects of Affirmative Action: Latent Factor Extrapolation in the Regression Discontinuity Design∗

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Abstract

Selective school admissions give rise to a Regression Discontinuity (RD) design that non-parametrically identifies causal effects for marginal applicants. Without stronger assumptions nothing can be said about causal effects for inframarginal applicants. Estimates of causal effects for inframarginal applicants are valuable for many policy questions, such as affirmative action, that substantially alter admissions cutoffs. This paper develops a latent factor-based approach to RD extrapolation that is then used to estimate effects of Boston exam schools away from admissions cutoffs. Achievement gains from Boston exam schools are larger for applicants with lower English and Math abilities. I also use the model to predict the effects of introducing either minority or socioeconomic preferences in exam school admissions. Affirmative action has modest average effects on achievement, while increasing the achievement of the applicants who gain access to exam schools as a result.

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1 Introduction

Regression Discontinuity (RD) methods identify treatment effects for individuals at the cutoff value determining treatment assignment under relatively mild assumptions (Hahn, Todd, and van der Klaauw, 2001; Frandsen, Frolich, and Melly, 2012).\(^1\) Without stronger assumptions, however, nothing can be said about treatment effects for individuals away from the cutoff. Such effects may be valuable for predicting the effects of policies that change treatment assignments of a broader group. An important example of this are affirmative action policies that change cutoffs substantially.

Motivated by affirmative action considerations, this paper develops a strategy for the identification and estimation of causal effects for inframarginal applicants to Boston’s selective high schools, known as exam schools. The exam schools, spanning grades 7-12, are seen as the flagship of the Boston Public Schools (BPS) system. They offer higher-achieving peers and an advanced curriculum. Admissions to these schools are based on Grade Point Average (GPA) and the Independent School Entrance Exam (ISEE). The RD design generated by exam school admissions nonparametrically identifies causal effects of exam school attendance for marginal applicants at admissions cutoffs. Abdulkadiroglu, Angrist, and Pathak (forthcoming) use this strategy and find little evidence of effects for these applicants.\(^2\) Other applicants, however, may benefit or suffer as a consequence of exam school attendance.

Treatment effects away from RD cutoffs are especially important for discussions of affirmative action at exam schools. Boston exam schools have played an important role in the history of attempts to ameliorate racial imbalances in Boston. In 1974 a federal court ruling introduced the use of minority preferences in Boston exam school admissions as part of a city-wide desegregation plan. Court challenges later led the Boston school district to drop racial preferences. Similarly, Chicago switched from minority to socioeconomic preferences in exam school admissions following a federal court ruling in 2009.\(^3\)

This paper develops a latent factor-based approach to the identification and estimation of treatment effects away from the cutoff. I assume that the source of omitted variables bias in an RD design can be modeled using latent factors. The running variable is one of a number of

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\(^1\)Cook (2008) provides an extensive treatment of the history of RD. See also the surveys by Imbens and Lemieux (2008), van der Klaauw (2008), Imbens and Wooldridge (2009), Lee and Lemieux (2010), and DiNardo and Lee (2011).

\(^2\)Dobbie and Fryer (2013) find similar results in an RD study of New York City exam schools.

\(^3\)The use of affirmative action in exam school admissions is a highly contentious issue also in New York City where a federal complaint was filed in 2012 against the purely achievement-based exam school admissions process due to disproportionately low minority shares at these schools.
noisy measures of these factors. Assuming other noisy measures are available, causal effects for all values of the running variable are nonparametrically identified.\textsuperscript{4} In related work on the same problem, Angrist and Rokkanen (2013) postulate a strong conditional independence assumption that identifies causal effects away from RD cutoffs. The framework developed here relies on weaker assumptions and is likely to find wider application.\textsuperscript{5}

I use this framework to estimate causal effects of exam school attendance for the full population of applicants. These estimates suggest that the achievement gains from exam school attendance are larger among applicants with lower baseline measures of ability. I also use the latent factor framework to simulate effects of introducing either minority or socioeconomic preferences in Boston exam school admissions. These reforms change the admissions cutoffs faced by different applicant groups and affect the exam school assignment of 27-35\% of applicants. The simulations suggest that the reforms boost achievement among applicants. These effects are largely driven by achievement gains experienced by lower-achieving applicants who gain access to exam schools as a result.

In developing the latent factor-based approach to RD extrapolation I build on the literatures on measurement error models (Kotlarski, 1967; Hu and Schennach, 2008; Evdokimov and White, 2012) and (semi-)nonparametric instrumental variable models (Newey and Powell, 2003; Darolles, Fan, Florens, and Renault, 2011).\textsuperscript{6} Latent factor models have a long tradition in economics (Aigner, Hsiao, Kapteyn, and Wansbeek, 1984). In the program evaluation literature, for instance, latent factor models have been used to identify the joint distribution of potential outcomes (Carneiro, Hansen, and Heckman, 2001, 2003; Aakvik, Heckman, and Vytlacil, 2005; Cunha, Heckman, and Navarro, 2005; Battistin, Lamarche, and Rettore, 2013), time-varying treatment effects (Cooley Fruehewirth, Navarro, and Takahashi, 2011), and distributional treatment effects (Bonhomme and Sauder, 2011).

The educational consequences of affirmative action have mostly been studied in post-secondary schools with a focus on application and enrollment margins. Several papers have studied affirmative action bans in California and Texas as well as the introduction of the Texas 10\% plan (Long, 2004; Card and Krueger, 2005; Dickson, 2006; Andrews, Ranchhod, and Sathy, 2010; Cortes, 2010; Antonovics and Backes, 2013, forthcoming). Howell (2010) uses a structural model to simulate the

\textsuperscript{4}This is similar to ideas put forth by Lee (2008), Lee and Lemieux (2010), DiNardo and Lee (2011), and Bloom (2012). However, this is the first paper that discusses how this framework can be used in RD extrapolation.

\textsuperscript{5}For other approaches to RD extrapolation, see Angrist and Pischke (2009), Jackson (2010), DiNardo and Lee (2011), Dong and Lewbel (2013), Cook and Wing (2013), and Bargain and Doorley (2013).

\textsuperscript{6}See also the surveys by Hausman (2001), Blundell and Powell (2003), Chen, Hong, and Nekipelov (2011), and Horowitz (2011) as well as the references therein.
effects of a nation-wide elimination of affirmative action in college admissions, and Hinrichs (2012) studies the effects of various affirmative action bans around the United States. Only a few studies have looked at the effects of affirmative action in selective school admissions on later outcomes (Arcidiacono, 2005; Rothstein and Yoon, 2008; Bertrand, Hanna, and Mullainathan, 2010; Francis and Tanmuri-Pianto, 2012).

The rest of the paper is organized as follows. The next section outlines the econometric framework. Section 3 discusses extensions of this approach to fuzzy RD and settings with multiple latent factors. Section 4 discusses identification and estimation of the latent factor model in the Boston exam school setting. Section 5 reports latent factor estimates. Section 6 uses the model to analyse effects of affirmative action. Section 7 concludes.

2 Latent Factor Modeling in a Sharp RD Design

2.1 Framework

Suppose one is interested in the causal effect of a binary treatment $D \in \{0, 1\}$ on an outcome $Y \in \mathcal{Y}$ that can be either discreet or continuous. Each individual is associated with two potential outcomes: $Y(0)$ is the outcome of an individual if she is not exposed to the treatment ($D = 0$), and $Y(1)$ is the outcome of an individual if she is exposed to the treatment ($D = 1$). The observed outcome of an individual is

$$Y = Y(0)(1 - D) + Y(1)D.$$

In a sharp Regression Discontinuity (RD) design the treatment assignment is fully determined by whether the values of a continuous covariate $R \in \mathcal{R}$, often called the running variable, lies above or below a known cutoff $c$.\footnote{In a fuzzy RD the treatment assignment is only partially determined by the running variable. I discuss this extension in Section 3.1.} That is, the treatment assignment is given by

$$D = 1(R \geq c).$$

I ignore the presence of additional covariates to simplify the notation. It is possible to generalize all the results to allow for additional covariates by conditioning on them throughout.

The sharp RD design allows one to nonparametrically identify the Average Treatment Effect
(ATE) at the cutoff, $E[Y(1) - Y(0) \mid R = c]$, under the conditions listed in Assumption A. Assumption A.1 restricts the marginal density of $R$ to be strictly positive in a neighborhood of the cutoff $c$. Assumption A.2 restricts the conditional cumulative distribution functions of both $Y(0)$ and $Y(1)$ given $R$ to be continuous in $R$ at the cutoff $c$. Finally, Assumption A.3 requires that the conditional expectations of $Y(0)$ and $Y(1)$ exist at the cutoff $c$. Under these assumptions, the Average Treatment Effect at the cutoff is given by the discontinuity in the conditional expectation function of $Y$ given $R$ at the cutoff, as shown in Lemma 1 (Hahn, Todd, and van der Klaauw, 2001).

**Assumption A.**

1. $f_R(r) > 0$ in a neighborhood around $c$.
2. $F_{Y(0)|R}(y \mid r)$ and $F_{Y(1)|R}(y \mid r)$ are continuous in $r$ at $c$ for all $y \in \mathcal{Y}$.
3. $E[|Y(0)\mid R = c], E[|Y(1)\mid R = c] < \infty$.

**Lemma 1.** (Hahn, Todd, and van der Klaauw, 2001) Suppose Assumption A holds. Then

$$E[Y(1) - Y(0) \mid R = c] = \lim_{\delta \downarrow 0} \{E[Y \mid R = c + \delta] - E[Y \mid R = c - \delta]\}.$$

The above lemma illustrates the power of sharp RD as it nonparametrically identifies the Average Treatment Effect at the cutoff under relatively mild assumptions. However, there is not much one can say about the Average Treatment Effect away from the cutoff without stronger assumptions. Figure 1 illustrates this extrapolation problem. To the left of the cutoff one observes $Y(0)$ as these individuals are not assigned to the treatment, and to the right of the cutoff one observes $Y(1)$ as these individuals are assigned to the treatment. The relevant counterfactual outcomes are unobservable.

To motivate the importance of extrapolation away from the cutoff, suppose one wanted to know the Average Treatment Effect for individuals with $R = r_0$ to the left of the cutoff. For these individuals one observes $E[Y(0) \mid R = r_0]$, but the counterfactual $E[Y(1) \mid R = r_0]$ is unobservable. Similarly, suppose one wanted to know the Average Treatment Effect for individuals with $R = r_1$ to the right of the cutoff. For these individuals one observes $E[Y(1) \mid R = r_1]$, but the counterfactual $E[Y(0) \mid R = r_1]$ is unobservable.

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8Continuity of the conditional expectation functions of the potential outcomes is enough for Lemma 1. However, continuity of the conditional cumulative distribution functions of the potential outcomes allows one to also identify distributional treatment effects (Frandsen, Frolich, and Melly, 2012).
In this paper I develop a latent factor-based solution to the extrapolation problem. Consider a setting in which $R$ is a function of a latent factor $\theta$ and disturbance $\nu_R$:

$$R = g_R(\theta, \nu_R)$$

where $g_R$ is an unknown function, and both $\theta$ and $\nu_R$ are potentially multidimensional. Suppose, for instance, that $R$ is an entrance exam score used in admissions to a selective school. Then, it is natural to interpret $R$ as a noisy measure of an applicant’s academic ability.

Figure 2 illustrates the latent factor framework when both $\theta$ and $\nu_R$ are scalars and $R = \theta + \nu_R$. Consider two types of individuals with low and high levels of $\theta$, $\theta_{\text{low}}$ and $\theta_{\text{high}}$. Furthermore suppose that $\theta_{\text{low}} < c$ and $\theta_{\text{high}} > c$. Then, if there was no noise in $R$, individuals with $\theta = \theta_{\text{low}}$ would not receive the treatment whereas individuals with $\theta = \theta_{\text{high}}$ would receive the treatment. However, because of the noise in $R$ some of the individuals with $\theta = \theta_{\text{low}}$ end up to the right of the cutoff, and similarly some of the individuals with $\theta = \theta_{\text{high}}$ end up to the left of the cutoff. Thus, both types of individuals are observed with and without the treatment.

I assume that the potential outcomes $Y(0)$ and $Y(1)$ are conditionally independent of $R$ given $\theta$, as stated in Assumption B. This means that any dependence between $(Y(0), Y(1))$ and $R$ is solely due to two factors: the dependence of $Y(0)$ and $Y(1)$ on $\theta$ and the dependence of $R$ on $\theta$.

**Assumption B.** $(Y(0), Y(1)) \perp \perp R | \theta$.

**Lemma 2.** Suppose that Assumption B holds. Then

$$E[Y(1) - Y(0) | R = r] = E\{E[Y(1) - Y(0) | \theta] | R = r\}$$

for all $r \in \mathcal{R}$.

Lemma 2 highlights the key implication of Assumption B. Under this assumption, the conditional Average Treatment Effect given $R = r$, $E[Y(1) - Y(0) | R = r]$, depends on two objects: the latent conditional Average Treatment Effect given $\theta$, $E[Y(1) - Y(0) | \theta]$, and the conditional distribution of $\theta$ given $R$, $f_{\theta | R}$.

Thus, the identification of the Average Treatment Effect away from the cutoff depends on one’s ability two identify these two objects. In the selective school

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9For all the results in this section it is enough to assume that $Y(0)$ and $Y(1)$ are conditionally mean independent of $R$ given $\theta$. However, the full conditional independence assumption stated here allows also for the extrapolation of distributional treatment effects away from the cutoff. I do not discuss this in detail as it is a straightforward extension of the results presented below.
admissions example Assumption B means that while \( Y(0) \) and \( Y(1) \) might depend on ability, they do not depend on the noise in the entrance exam score.

Figure 5 illustrates this by considering again the identification of the Average Treatment Effect for individuals with \( R = r_0 \) to the left of the cutoff and for individuals with \( R = r_1 \) to the right of the cutoff. As discussed above, the shap RD design allows one to observe \( E[Y(0) \mid R = r_0] \) and \( E[Y(1) \mid R = r_1] \), and the extrapolation problem arises from the unobservability of \( E[Y(1) \mid R = r_0] \) and \( E[Y(0) \mid R = r_1] \). Suppose the conditional expectation functions of \( Y(0) \) and \( Y(1) \) given \( \theta \), \( E[X \mid \theta] \) and \( E[Y(1) \mid \theta] \), depicted in Figure 3, are known. In addition, suppose the conditional densities of \( \theta \) given \( R = r_0 \) and \( R = r_1 \), \( f_{\theta \mid R}(\theta \mid r_0) \) and \( f_{\theta \mid R}(\theta \mid r_1) \), depicted in Figure 4, are known. Then, under Assumption B, the counterfactuals \( E[Y(1) \mid R = r_0] \) and \( E[Y(0) \mid R = r_1] \) are given by

\[
E[Y(1) \mid R = r_0] = E\{E[Y(1) \mid \theta] \mid R = r_0\}
\]

\[
E[Y(0) \mid R = r_1] = E\{E[Y(0) \mid \theta] \mid R = r_1\}.
\]

There is only one remaining issue: how does one identify the latent conditional Average Treatment Effect given \( \theta \), \( E[Y(1) − Y(0) \mid \theta] \), and the conditional distribution of \( \theta \) given \( R \), \( f_{\theta \mid R} \)? If \( \theta \) was observable, these objects could be identified using the covariate-based approach developed by Angrist and Rokkanen (2013). However, here \( \theta \) is an unobservable latent factor which complicates the identification of \( E[Y(1) − Y(0) \mid \theta] \) and \( f_{\theta \mid R} \). To achieve identification, I rely on the availability of multiple noisy measures of \( \theta \). To simplify the discussion, I consider in this section a setting in which \( \theta \) is unidimensional. I discuss an extension of the approach to settings with multidimensional latent factors in Section 3.2.

I assume that the data contains three noisy measures of \( \theta \), denoted by \( M_1 \), \( M_2 \), and \( M_3 \):

\[
M_1 = g_{M_1}(\theta, \nu_{M_1})
\]

\[
M_2 = g_{M_2}(\theta, \nu_{M_2})
\]

\[
M_3 = g_{M_3}(\theta, \nu_{M_3})
\]

where \( g_{M_1}, g_{M_2}, \) and \( g_{M_3} \) are unknown functions, and \( \nu_{M_1}, \nu_{M_2}, \) and \( \nu_{M_3} \) are potentially multidi-

\[10\] The covariate-based approach by Angrist and Rokkanen (2013) can be in certain cases used for extrapolation even if the conditional independence assumption holds only for a latent factor. For this assumption to work, one needs to assume that the running variable contains no additional information about the latent factor once one conditions on a set of covariates.
mensional disturbances. I focus on a setting in which is \( R \) is a deterministic function of at least one or potentially many of these measures, but it is possible to consider a more general setting that allows the relationship between \( R \) and \( M \) to be stochastic. Going back to the selective school example considered above, one might think of \( M_1 \) as the entrance exam score whereas \( M_2 \) and \( M_3 \) might be two pre-application baseline test scores.

I require \( \theta, M_1, \) and \( M_2 \) to be continuous but allow \( M_3 \) to be either continuous or discrete; even a binary \( M_3 \) is sufficient. I denote the supports of \( \theta, M_1, M_2, \) and \( M_3 \) by \( \Theta, M_1, M_2, \) and \( M_3 \). I occasionally also use the notation \( M = (M_1, M_2, M_3) \) and \( \mathcal{M} = M_1 \times M_2 \times M_3 \). I leave the properties of the latent factor \( \theta \), the unknown functions \( g_{M_1}, g_{M_2}, \) and \( g_{M_3} \) as well as the disturbances \( \nu_{M_1}, \nu_{M_2}, \) and \( \nu_{M_3} \) unspecified for now. I return to them below when discussing alternative sets of assumptions allowing for the identification of the measurement model.

2.2 Parametric Illustration

To provide a benchmark for the discussion about nonparametric identification of the latent factor model, I begin by considering the identification of a simple parametric model. I assume linearity and normality in the measurement models for \( M_1 \) and \( M_2 \) but leave the measurement model for \( M_3 \) flexible. In addition, I assume linearity in the latent outcome models \( E [Y (0) \mid \theta] \) and \( E [Y (1) \mid \theta] \).

The measurement model takes the following form:

\[
\begin{align*}
M_1 &= \theta + \nu_{M_1} \\
M_2 &= \mu_{M_2} + \lambda_{M_2} \theta + \nu_{M_2} \\
M_3 &= g(\theta, \nu_{M_3})
\end{align*}
\]

where \( \lambda_{M_2}, \text{Cov} (\theta, M_3) \neq 0, \) and

\[
\begin{bmatrix}
\theta \\
\nu_{M_1} \\
\nu_{M_2}
\end{bmatrix} \mid M_3 
\sim N\left(\begin{bmatrix} \mu_{\theta} (M_3) \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2_{\theta} (M_3) & 0 & 0 \\ 0 & \sigma^2_{\nu_{M_1}} (M_3) & 0 \\ 0 & 0 & \sigma^2_{\nu_{M_2}} (M_3) \end{bmatrix}\right).
\]

In order to pin down the location and scale of \( \theta \), I have normalized \( \mu_{M_1} = 0 \) and \( \lambda_{M_1} = 1. \)
The latent outcome model takes the following form:

\[
E[Y(0) | \theta] = \alpha_0 + \beta_0 \theta \\
E[Y(1) | \theta] = \alpha_1 + \beta_1 \theta
\]

I assume that the \(Y(0)\) and \(Y(1)\) are conditionally independent of \(M\) given \(\theta\). Formally,

\[
(Y(0), Y(1)) \perp \perp M | \theta.
\]

This means that the noisy measures of \(\theta\) are related to the potential outcomes only through \(\theta\). Consequently, they can be used as instruments to identify the relationship between the potential outcomes and \(\theta\).

Given the simple parametric specification, the identification of \(f_{\theta,M}\) depends on one’s ability to identify the unknown parameters \(\mu_{M_2}, \mu_{\theta}, \sigma_{\theta}^2, \sigma_{\nu_{M_1}}^2, \) and \(\sigma_{\nu_{M_2}}^2\). These can be obtained from the moments of the joint distribution of \(M_1, M_2,\) and \(M_3\) by noticing that

\[
E[M_1 | M_3] = \mu_{\theta}(M_3) \\
E[M_2] = \mu_{M_2} + \lambda_{M_2}E[\mu_{\theta}(M_3)] \\
Var[M_1 | M_3] = \sigma_{\theta}^2(M_3) + \sigma_{\nu_{M_1}}^2(M_3) \\
Var[M_2 | M_3] = \lambda_{M_2}^2\sigma_{\theta}^2(M_3) + \sigma_{\nu_{M_2}}^2(M_3) \\
Cov[M_1, M_2 | M_3] = \lambda_{M_2}\sigma_{\theta}^2(M_3) \\
Cov[M_1, M_3] = Cov[\theta, M_3] \\
Cov[M_2, M_3] = \lambda_{M_2}Cov[\theta, M_3].
\]
As long as $\lambda_{M_2}, \text{Cov} (\theta, M_3) \neq 0$, as was assumed above, the unknown parameters are given by

\[
\begin{align*}
\mu_\theta (M_3) &= E [M_1 \mid M_3] \\
\lambda_{M_2} &= \frac{\text{Cov} [M_2, M_3]}{\text{Cov} [M_1, M_3]} \\
\mu_{M_2} &= E [M_2] - \lambda_{M_2} E [\mu_\theta (M_3)] \\
\sigma_\theta^2 (M_3) &= \frac{\text{Cov} [M_1, M_2 \mid M_3]}{\lambda_{M_k}} \\
\sigma_{\nu M_1}^2 (M_3) &= \text{Var} [M_1 \mid M_3] - \sigma_\theta^2 (M_3) \\
\sigma_{\nu M_2}^2 (M_3) &= \text{Var} [M_2 \mid M_3] - \lambda_{M_2}^2 \sigma_\theta^2 (M_3).
\end{align*}
\]

These parameters fully characterize the conditional joint distribution of $\theta$, $M_1$, and $M_2$ given $M_3$. The joint distribution $f_{\theta, M}$ is then given by

\[
f_{\theta, M} (\theta, m) = f_{\theta, M_1, M_2 \mid M_3} (\theta, m_1, m_2 \mid m_3) f_{M_3} (m_3).
\]

Let us now turn to the identification of $E [Y(1) - Y(0) \mid \theta]$. Given the simple parametric specifications for $E [Y(0) \mid \theta]$ and $E [Y(1) \mid \theta]$, the identification of $E [Y(1) - Y(0) \mid \theta]$ depends on one’s ability to identify the unknown parameters $\alpha_0$, $\beta_0$, $\alpha_1$, and $\beta_1$. These can be obtained from the moments of the conditional distribution of $Y$ given $M$ and $D$ and the conditional distribution of $\theta$ given $M$ and $D$ by noticing that

\[
\begin{align*}
E [Y \mid M = m^0, D = 0] &= E [Y (0) \mid M = m^0, D = 0] \\
&= E \{ E [Y (0) \mid \theta] \mid M = m^0, D = 0 \} \\
&= \alpha_0 + \beta_0 E [\theta \mid M = m^0, D = 0] \\
E [Y \mid M = m^1, D = 1] &= E [Y (1) \mid M = m^1, D = 1] \\
&= E \{ E [Y (1) \mid \theta] \mid M = m^1, D = 1 \} \\
&= \alpha_1 + \beta_1 E [\theta \mid M = m^1, D = 1]
\end{align*}
\]

for all $m^0 \in \mathcal{M}^0$ and $m^1 \in \mathcal{M}^1$. 

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The unknown parameters are given by

\[
\begin{bmatrix}
\alpha_0 \\
\beta_0
\end{bmatrix}
= \begin{bmatrix}
1 & E[\theta \mid M = m^{0,1}, D = 0] \\
1 & E[\theta \mid M = m^{0,2}, D = 0]
\end{bmatrix}
^{-1}
\begin{bmatrix}
E[Y \mid M = m^{0,1}, D = 0] \\
E[Y \mid M = m^{0,2}, D = 0]
\end{bmatrix}
\]

\[
\begin{bmatrix}
\alpha_1 \\
\beta_1
\end{bmatrix}
= \begin{bmatrix}
1 & E[\theta \mid M = m^{1,1}, D = 1] \\
1 & E[\theta \mid M = m^{1,2}, D = 1]
\end{bmatrix}
^{-1}
\begin{bmatrix}
E[Y \mid M = m^{1,1}, D = 1] \\
E[Y \mid M = m^{1,2}, D = 1]
\end{bmatrix}
\]

where \( m^{0,1}, m^{0,2} \in \mathcal{M}^0 \) and \( m^{1,1}, m^{1,2} \in \mathcal{M}^1 \). These parameters fully characterize \( E[Y(0) \mid \theta] \), \( E[Y(1) \mid \theta] \), and consequently \( E[Y(1) - Y(0) \mid \theta] \). The above result requires that the matrices

\[
\begin{bmatrix}
1 & E[\theta \mid M = m^{0,1}, D = 0] \\
1 & E[\theta \mid M = m^{0,2}, D = 0]
\end{bmatrix}
\begin{bmatrix}
1 & E[\theta \mid M = m^{1,1}, D = 1] \\
1 & E[\theta \mid M = m^{1,2}, D = 1]
\end{bmatrix}
\]

are full rank which is implied by the assumptions on the measurement model.

Finally, the conditional expectation functions of \( Y(0) \) and \( Y(1) \) given \( R = r \), \( E[Y(0) \mid R = r] \) and \( E[Y(1) \mid R = r] \) as well as the conditional Average Treatment Effect given \( R = r \), \( E[Y(1) - Y(0) \mid R = r] \), are then given by

\[
E[Y(0) \mid R = r] = E\{E[Y(0) \mid \theta] \mid R = r\}
= \alpha_0 + \beta_0 E[\theta \mid R = r]
\]

\[
E[Y(1) \mid R = r] = E\{E[Y(1) \mid \theta] \mid R = r\}
= \alpha_1 + \beta_1 E[\theta \mid R = r]
\]

\[
E[Y(1) - Y(0) \mid R = r] = (\alpha_1 - \alpha_0) + (\beta_1 - \beta_0) E[\theta \mid R = r]
\]

for all \( r \in \mathcal{R} \).

In this example I have imposed strong parametric assumptions to illustrate the identification of the latent factor model. However, these assumptions are not necessary for identification. In the following sections I relax the distributional and functional form assumptions on the measurement model, as well as the functional form assumptions on the latent outcome model.
2.3 Identification of the Latent Factor Distribution

2.3.1 Linear Measurement Model

I continue to assume a linear measurement model for $M_1$ and $M_2$ but leave the measurement model for $M_3$ flexible. The measurement model takes the following form:

\[
\begin{align*}
M_1 &= \theta + \nu_{M_1} \\
M_2 &=\mu_{M_2} + \lambda_{M_2} \theta + \nu_{M_2} \\
M_3 &= g_{M_3}(\theta, \nu_{M_3})
\end{align*}
\]

where $\lambda_{M_2} \neq 0$, $E[\nu_{M_1} | \theta] = E[\nu_{M_2} | \theta] = 0$. Assumption C lists conditions under which one can obtain nonparametric identification of $f_{\theta,M}$ using standard results from the literature on latent factor models as well as an extension to Kotlarski’s Lemma (Kotlarski, 1967; Prakasa Rao, 1992) by Evdokimov and White (2012).

**Assumption C.**

1. The relationship between $M_1$, $M_2$, and $\theta$ is as given in equations (1) and (2).
2. $\theta$, $\nu_{M_1}$, and $\nu_{M_2}$ are jointly independent conditional on $M_3$.
3. $\text{Cov}[\theta, M_3] \neq 0$ and $E[[\theta] | M_3] < \infty$, $E[\nu_{M_1} | M_3] = E[\nu_{M_2} | M_3] = 0$ a.s.
4. One of the following conditions holds:
   
   (a) The real zeros of the conditional characteristic function of $\nu_{M_1}$ given $M_3$ and its derivative are disjoint, and the conditional characteristic function of $\nu_{M_2}$ given $M_3$ has only isolated real zeros.
   (b) The conditional characteristic function of $\nu_{M_1}$ given $M_3$ is analytic.

Assumption C.1 imposes the linearity on the measurement models for $M_1$ and $M_2$ as discussed above. To pin down the location and scale of $\theta$, I again use the normalization $\mu_{M_1} = 0$ and $\lambda_{M_2} = 1$. In addition, I assume that $\lambda_{M_2} \neq 0$ to guarantee that $M_2$ contains information about $\theta$. Assumption C.2 restricts $\theta$, $\nu_{M_1}$, and $\nu_{M_3}$ to be jointly independent conditional on $M_3$. An important implication of this is that there cannot be heteroscedasticity in $\nu_{M_1}$ and $\nu_{M_2}$ with respect to $\theta$. Assumption C.3 requires that $M_3$ is correlated with $\theta$ and that both $\nu_{M_1}$ and $\nu_{M_3}$ are mean independent of $M_3$. In addition, I assume that the conditional mean of $\theta$ given $M_3$ exists, thus ruling out distributions with particularly fat tails (e.g. the Cauchy distribution).

Lastly, Assumption C.4 imposes restrictions on the conditional characteristic functions of $\nu_{M_1}$ and $\nu_{M_2}$ given $M_3$. This assumption is best understood by considering first the original Kotlarski’s
Lemma (Kotlarski, 1967; Prakasa Rao, 1992). This lemma requires that the conditional characteristic functions of $\theta$, $\nu_{M_1}$, and $\nu_{M_2}$ given $M_3$ not to have any real zeros (such characteristic functions are typically called nonvanishing).\footnote{A nonvanishing characteristic function is closely related to (bounded) completeness of a location family. Therefore, it can be seen as requiring that the distribution varies sufficiently as a function of the location parameter. See the related discussion and references in Section 2.3.2.} This is a common assumption in the measurement error literature (Chen, Hong, and Nekipelov, 2011). It is satisfied by most standard distributions, such as the normal, log-normal, Cauchy, Gamma, Laplace, $\chi^2$ and Student’s $t$-distribution. However, it is violated by, for instance, uniform, triangular, and truncated normal distributions as well as by many discrete distributions.

Assumption C.4 uses recent work by Evdokimov and White (2012) to relax the assumptions of Kotlarski’s Lemma. Condition (a) allows for real zeros in the conditional characteristic functions of $\nu_{M_1}$ and $\nu_{M_2}$ given $M_3$. This substantially expands the class of distributions that allow for the identification of $f_{\theta,M}$. Condition (b) requires the conditional characteristic function of $\nu_{M_1}$ given $M_3$ to be analytic while imposing no restrictions on the conditional characteristic function of $\nu_{M_2}$ given $M_3$. Analyticity is a property satisfied, for instance, by distributions with exponentially bounded tails, such as the normal and Gamma distribution as well as distributions with bounded support. Importantly, both these conditions impose no restrictions on the conditional characteristic function of $\theta$ given $M_3$.

Theorem 1 states the identification result. Given the normalizations imposed on the linear measurement model, the covariance and mean independence assumptions imply that $\lambda_{M_2}$ is given by the ratio of the covariance between $M_2$ and $M_3$ and the covariance between $M_1$ and $M_3$. This amounts to using $M_3$ as an instrument for $M_1$ in a regression of $M_2$ on $M_1$ as $M_1$ is an imperfect measure of $\theta$. The means of $M_1$ and $M_2$ can then be used to obtain $\mu_{M_2}$. Suppose for a moment that the conditional distributions of $\theta$, $\nu_{M_1}$, and $\nu_{M_2}$ given $M_3$ were known up to a finite number of parameters. In this case one could, subject to the relevant full rank condition, identify the distributional parameters from a finite number of moment conditions. The rest of Theorem 1 uses the assumptions on the conditional characteristic functions to generalize this strategy to an infinite-dimensional problem. Under these assumptions, the conditional distributions of $\nu_{M_1}$ and $\nu_{M_2}$ given $M_3$, $f_{\nu_{M_1}|M_3}$ and $f_{\nu_{M_2}|M_3}$, as well as the conditional distribution of $\theta$ given $M_3$, $f_{\theta|M_3}$ can be uniquely determined from the conditional distribution of $M_1$ and $M_2$ given $M_3$, $f_{M_1,M_2|M_3}$. Together with the marginal distribution of $M_3$, $f_{M_3}$, this allows one to then construct $f_{\theta,M}$. 

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Theorem 1. Suppose Assumption C holds. Then

\[
\begin{align*}
\mu_{M_2} &= E[M_2] - \lambda_{M_2} E[M_1] \\
\lambda_{M_2} &= \frac{\text{Cov}[M_2, M_3]}{\text{Cov}[M_1, M_3]}.
\end{align*}
\]

In addition, the equation

\[
\begin{align*}
\int_{\Theta} f_{M_1|M_3}(m_1, m_2 | m_3) f_{\nu_{M_1}|M_3}(m_1 - \theta | m_3) f_{\nu_{M_2}|M_3}(m_2 - \mu_{M_2} - \lambda_{M_2} \theta | m_3) f_{\theta|M_3}(\theta | m_3) d\theta
\end{align*}
\]

for all \(m_1 \in M_1, m_2 \in M_2,\) and \(m_3 \in M_3\) admits unique solutions for \(f_{\nu_{M_1}|M_3}, f_{\nu_{M_2}|M_3},\) and \(f_{\theta|M_3}.\) Consequently, the joint distribution \(f_{\theta,M}\) is identified.

2.3.2 Nonlinear Measurement Model

The linear measurement model considered in Section 2.3.1 provides a natural starting point for the discussion regarding the identification of \(f_{\theta,M}.\) However, this model imposes important, and potentially unsatisfactory, restrictions on the relationship between the latent factor \(\theta\) and the measures \(M_1\) and \(M_2.\) First, both \(M_1\) and \(M_2\) are assumed to depend linearly on \(\theta.\) Second, \(\theta, \nu_{M_1},\) and \(\nu_{M_2}\) are assumed to be jointly independent conditional on \(M_3.\) This rules out, for instance, heteroskedasticity in \(\nu_{M_1}\) and \(\nu_{M_2}\) with respect to \(\theta.\) In this section I discuss an alternative set of identifying assumptions that address these concerns. For this purpose I use results by Hu and Schennach (2008) who study nonparametric identification and estimation in the presence of nonclassical measurement error.

I return to the general measurement model that took the form

\[
\begin{align*}
M_1 &= g_{M_1}(\theta, \nu_{M_1}) \\
M_2 &= g_{M_2}(\theta, \nu_{M_2}) \\
M_3 &= g_{M_3}(\theta, \nu_{M_3}).
\end{align*}
\]

Assumption D lists the conditions under which the joint distribution \(f_{\theta,M}\) is nonparametrically identified in this setting (Hu and Schennach, 2008; Cunha, Heckman, and Schennach, 2010).

Assumption D.
1. $f_{\theta,M}(\theta,m)$ is bounded with respect to the product measure of the Lebesgue measure on $\Theta \times \mathcal{M}_1 \times \mathcal{M}_2$ and some dominating measure $\mu$ on $\mathcal{M}_3$. All the corresponding marginal and conditional densities are also bounded.

2. $M_1$, $M_2$, and $M_3$ are jointly independent conditional on $\theta$.

3. For all $\theta', \theta'' \in \Theta$, $f_{M_3|\theta}(m_3 | \theta')$ and $f_{M_3|\theta}(m_3 | \theta'')$ differ over a set of strictly positive probability whenever $\theta' \neq \theta''$.

4. There exists a known functional $H$ such that $H[f_{M_1|\theta}(\cdot | \theta)] = \theta$ for all $\theta \in \Theta$.

5. $f_{\theta|M_1}(\theta | m_1)$ and $f_{M_1|M_2}(m_1 | m_2)$ form (boundedly) complete families of distributions indexed by $m_1 \in \mathcal{M}_1$ and $m_2 \in \mathcal{M}_2$.

Assumption D.1 requires $\theta$, $M_1$, and $M_2$ to be continuous but allows $M_3$ to be either continuous or discrete. Furthermore, it restricts the joint, marginal and conditional densities of $\theta$, $M_1$, $M_2$, and $M_3$ to be bounded. The support of the joint distribution $f_{\theta,M}$, on the other hand, is allowed to be either rectangular or triangular. Assumption D.2 restricts $\nu_{M_1}$, $\nu_{M_2}$, and $\nu_{M_3}$ to be jointly independent conditional on $\theta$ while allowing for arbitrary dependence between $\theta$ and these disturbances. Importantly, this assumption allows for both heteroskedasticity and correlation in the measurement errors in $M_1$, $M_2$, and $M_3$. Assumption D.3 requires the conditional distribution of $M_3$ given $\theta$ to vary sufficiently as a function of $\theta$. This assumption can be satisfied, for instance, by assuming strict monotonicity of the conditional expectation of $M_3$ given $\theta$. More generally this assumption can be satisfied if there is heteroskedasticity in $M_3$ with respect to $\theta$. Assumption D.4 imposes a normalization on the conditional distribution of $M_1$ given $\theta$ in order to pin down the location and scale of $\theta$. This normalization can be achieved by, for instance, requiring the conditional mean, mode or median of $M_1$ to be equal to $\theta$.

Lastly, Assumption D.5 requires that the conditional distributions $f_{\theta|M_1}$ and $f_{M_1|M_2}$ are either complete or boundedly complete.\(^{12}\) The concept of completeness, originally introduced in statistics by Lehmann and Scheffe (1950, 1955), arises regularly in econometrics, for instance, as a necessary condition for the identification of (semi-)nonparametric instrumental variable models (Newey and Powell, 2003; Blundell and Powell, 2003; Chernozhukov and Hansen, 2005; Blundell, Chen, and Kristensen, 2007; Chernozhukov, Imbens, and Newey, 2007).\(^{13}\) It can be seen as an infinite-
dimensional generalization of the full rank condition that is central in the identification of various parametric models, such as the Generalized Method of Moments (Hansen, 1982).\textsuperscript{14} Intuitively, the completeness condition requires that $f_{\theta|M_1}$ and $f_{M_1|M_2}$ vary sufficiently as functions of $M_1$ and $M_2$. One way to see this is to consider the assumption of $L^2$-completeness which lies in between completeness and bounded completeness in terms of its restrictiveness.\textsuperscript{15} It can be shown that the conditional distribution of $X$ given $Z$, where $X$ and $Z$ denote generic random variables, is $L^2$-complete if and only if every nondegenerate square-integrable function of $X$ is correlated with some square-integrable function of $Z$ (Severini and Tripathi, 2006; Andrews, 2011).

An unsatisfactory feature of completeness assumptions is that, unlike the full rank condition in finite-dimensional models, these assumptions are generally untestable (Canay, Santos, and Shaikh, forthcoming). However, there has been some work on providing sufficient conditions for various forms of completeness of certain classes of distributions, such as the location, scale and exponential families, in both statistics (Ghosh and Singh, 1966; Isenbeck and Ruschendorf, 1992; Mattner, 1992; Lehmann and Romano, 2005) and econometrics (D’Haultfoeuille, 2011; Hu and Shiu, 2012).\textsuperscript{16}

In addition, some papers in the literature have focused on characterizing classes of distributions that fail the completeness assumption but satisfy the weaker bounded completeness assumption (Hoeffding, 1977; Bar-Lev and Plachky, 1989; Mattner, 1993). More recently, Andrews (2011) and Chen, Chernozhukov, Lee, and Newey (2013) have provided genericity results that imply that $L^2$-completeness holds almost surely for large classes of nonparametric distributions.\textsuperscript{17}

Theorem 2 states the identification result by Hu and Schennach (2008). Given the conditional joint independence assumption, one can write down the integral equation given in the theorem relating the observed conditional joint distribution of $M_1$ and $M_3$ given $M_3$, $f_{M_1,M_3|M_2}$, to the unobserved distributions $f_{M_1|\theta}$, $f_{M_3|\theta}$, and $f_{\theta|M_2}$. Furthermore, this relationship can be expressed in terms of linear integral operators that makes the problem analogous to matrix diagonalization in linear algebra. Using invertibility of some of the operators, provided by the (bounded) completeness and Rolin, 1990).

\textsuperscript{14}Notice that if the generic random variables $X$ and $Z$ are both discrete with finite supports $X = \{x_1, \ldots, x_K\}$ and $Z = \{z_1, \ldots, z_L\}$, the completeness assumption becomes the full rank condition $P[\text{rank}(Q) = K] = 1$ where $Q_{kl} = P[X = x_k \mid Z = z_l]$ (Newey and Powell, 2003).

\textsuperscript{15}The definition of $L^2$-completeness is analogous to the definition given above for (bounded) completeness with the exception that the condition needs to hold for all measurable square-integrable real functions $h$ (Andrews, 2011). Thus, $L^2$-completeness lies in between completeness and bounded completeness in the sense that completeness implies $L^2$-completeness which in turn implies bounded completeness.

\textsuperscript{16}For instance, the assumption of a nonvanishing characteristic function discussed in Section 2.3.1 is a necessary condition for completeness and a necessary and sufficient condition for bounded completeness of a location family (Ghosh and Singh, 1966; Isenbeck and Ruschendorf, 1992; Mattner, 1992).

\textsuperscript{17}See also Santos (2012) for a related discussion on the uniform closeness of complete and incomplete distributions.
assumption, one can obtain an eigenvalue-eigenfunction decomposition of an integral operator that only depends on the observed $f_M$. Given the additional assumptions, this decomposition is unique, and the unknown densities $f_{M_1|\theta}$, $f_{M_3|\theta}$, and $f_{\theta|M_2}$ are given by the eigenfunctions and eigenvalues of this decomposition. This allows one to then construct $f_{\theta,M}$.

**Theorem 2.** Suppose Assumption D holds. Then the equation

$$f_{M_1,M_3|M_2} (m_1,m_3 | m_2) = \int f_{M_1|\theta} (m_1 | \theta) f_{M_3|\theta} (m_3 | \theta) f_{\theta|M_2} (\theta | m_2) d\theta$$

for all $m_1 \in M_1$, $m_2 \in M_2$ and $m_3 \in M_3$ admits unique solutions for $f_{M_1|\theta}$, $f_{M_3|\theta}$, and $f_{\theta|M_2}$. Consequently, the joint distribution $f_{\theta,M}$ is identified.

### 2.4 Identification of the Latent Conditional Average Treatment Effect

Having identified the joint distribution of $\theta$ and $M$, $f_{\theta,M}$, and the conditional distribution of $\theta$ given $R$, $f_{\theta|R}$, the only missing piece in the identification of $E[Y(1) - Y(0) | R]$ is the latent conditional Average Treatment Effect $E[Y(1) - Y(0) | \theta]$. The identification of this is based on the identification of the latent conditional expectation functions $E[Y(0) | \theta]$ and $E[Y(1) | \theta]$. These functions can be identified by relating the variation in the conditional expectation of $Y$ given $M$ and $D$, $E[Y | M,D]$, to the variation in the conditional distribution of $\theta$ given $M$ and $D$, $f_{\theta|M,D}$ to the left ($D = 0$) and right ($D = 1$) of the cutoff. This problem is analogous to the identification of separable (semi-)nonparametric instrumental variable models (Newey and Powell, 2003; Darolles, Fan, Florens, and Renault, 2011). Assumption E lists the conditions under which $E[Y(0) | \theta]$ and $E[Y(1) | \theta]$ are nonparametrically identified for all $\theta \in \Theta$.

**Assumption E.**

1. $(Y(0), Y(1)) \perp \perp M | \theta$.
2. $0 < P[D = 1 | \theta] < 1$ a.s.
3. $f_{\theta|M,D} (\theta | m^0, 0)$ and $f_{\theta|M,D} (\theta | m^1, 1)$ form (boundedly) complete families of distributions indexed by $m^0 \in M^0$ and $m^1 \in M^1$.

Assumption E.1 requires that the potential outcomes $Y(0)$ and $Y(1)$ are conditionally independent of the measures $M$ given the latent factor $\theta$. In other words, the measurement errors in $M$ are not allowed to affect $Y(0)$ and $Y(1)$.$^{18}$ Assumption E.2 states a common support condition

$^{18}$This condition is often referred to as nondifferential measurement error (Bound, Brown, and Mathiowetz, 2001; Carroll, Ruppert, Stefanski, and Crainiceanu, 2006).
that guarantees that the conditional supports of $\theta$ to the left and right of the cutoff $c$ coincide. This means that the subset of $M$ entering $R$ must be sufficiently noisy measures of $\theta$ so that for all $\theta \in \Theta$ the realized value of $R$ can lie on both sides of the cutoff with strictly positive probability.

Finally, Assumption E.3 imposes a similar (bounded) completeness condition as in Assumption D for the identification of the nonlinear measurement model. Here the vector $M$ is used as an instrument for $\theta$, and the (bounded) completeness conditions can be thought of as an infinite-dimensional first stage condition. A sufficient condition for this assumption to be implied by the (bounded) completeness conditions in Assumption D is that $M_1$ does not enter $R$, and that there exists some $(m_2^d, m_3^d) \in M_2^d \times M_3^d$ such that $f_{\theta, M_2, M_3 | d} \left( \theta, m_2^d, m_3^d | d \right) > 0$ for all $\theta \in \Theta$, $d = 0, 1$.

Theorem 3 states the identification result. The conditional independence assumption allows one to write down the integral equations given in the theorem. Under the (bounded) completeness assumption, $E[Y(0) | \theta]$ and $E[Y(1) | \theta]$ are unique solutions to these integral equations. Finally, the common support assumption ensures that both $E[Y(0) | \theta]$ and $E[Y(1) | \theta]$ are determined for all $\theta \in \Theta$.

**Theorem 3.** Suppose Assumption E holds. Then the equations

$$
E \left[ Y \mid M = m^0, D = 0 \right] = E \left\{ E \left[ Y(0) \mid \theta \right] \mid M = m^0, D = 0 \right\}
$$

$$
E \left[ Y \mid M = m^1, D = 1 \right] = E \left\{ E \left[ Y(1) \mid \theta \right] \mid M = m^1, D = 1 \right\}
$$

for all $m^0 \in M^0$ and $m^1 \in M^1$ admit unique solutions for (bounded) $E[Y(0) | \theta]$ and $E[Y(1) | \theta]$ for all $\theta \in \Theta$. Consequently, $E[Y(1) - Y(0)]$ is identified.

### 3 Extensions

#### 3.1 Extrapolation of Local Average Treatment Effect in Fuzzy RD

In fuzzy RD the treatment is only partly determined by whether the running variable falls above or below the cutoff $c$: some individuals assigned to the treatment may end up not receiving the treatment while some individuals not assigned to the treatment may end up receiving the treatment. Thus, in fuzzy RD the probability of receiving the treatment jumps when the running variable $R$ crosses the cutoff $c$ but by less than 1:

$$
\lim_{\delta \downarrow 0} P[D = 1 \mid R = r + \delta] > \lim_{\delta \downarrow 0} P[D = 1 \mid R = r - \delta].
$$
Let \( Z \) denote the treatment assignment that is a deterministic function of the running variable:

\[
Z = 1 (R \geq c)
\]

Each individual is associated with two potential treatment status: \( D(0) \) is the treatment status of an individual if she is not assigned to the treatment \((Z = 0)\), and \( D(1) \) is the treatment status of an individual if she is assigned to the treatment \((Z = 1)\). Using this notation, the observed outcome and the observed treatment status can be written as

\[
Y = Y(0) + (Y(1) - Y(0)) D
\]

\[
D = D(0) + (D(1) - D(0)) Z.
\]

It is possible to categorize individuals into four mutually exclusive groups according to their compliance with treatment assignment (Imbens and Angrist, 1994; Angrist, Imbens, and Rubin, 1996): (1) individuals who receive the treatment whether or not they are assigned to the treatment are called always-takers \((D(0) = D(1) = 1)\), (2) individuals who do not receive the treatment whether or not they are assigned to the treatment are called never-takers \((D(0) = D(1) = 0)\), (3) individuals who receive the treatment if they are assigned to the treatment and do not receive the treatment if they are not assigned to the treatment are called compliers \((D(0) = 0, D(1) = 1)\), and (4) individuals who receive the treatment if they are not assigned to the treatment and do not receive treatment if they are assigned to the treatment are called defiers \((D(0) = 1, D(1) = 0)\).

I rule out defiers by assuming that being assigned to the treatment can only make an individual more likely to receive the treatment. This corresponds to the monotonicity assumption in the instrumental variables literature (Imbens and Angrist, 1994; Angrist, Imbens, and Rubin, 1996). Once defiers have been ruled out, fuzzy RD allows one to nonparametrically identify the Local Average Treatment Effect (LATE) for the compliers at the cutoff, \( E [Y(1) - Y(0) \mid D(1) > D(0), R = c] \). This is the group of individuals whose treatment status changes at the cutoff as they become eligible to the treatment. Since the treatment status of never-takers and always-takers is independent of treatment assignment, fuzzy RD contains no information about the Average Treatment Effect for these two groups.

Assumption F lists conditions under which the Local Average Treatment Effect is nonparametrically identified. Assumption F.1 restricts the marginal density of \( R, f_R \), to be strictly positive in
a neighborhood around the cutoff. Assumption F.2 imposes the monotonicity assumption stating that crossing the cutoff can only make an individual more likely to receive the treatment. In addition, it requires that this relationship is strict for at least some individuals at the cutoff, ensuring that there is a first stage. Assumption F.3 requires the conditional expectations of the potential treatment status to be continuous in $R$ at the cutoff. Assumption F.4 imposes continuity on the conditional cumulative distribution functions of the potential outcomes for the compliers.$^{19}$ Finally, Assumption F.5 requires that the conditional expectations of $Y(0)$ and $Y(1)$ exist at the cutoff.

**Assumption F.**

1. $f_R(r) > 0$ in a neighborhood around $c$.
2. $\lim_{\delta \to 0} P[D(1) \geq D(0) \mid R = r + \delta] = 1$ and $\lim_{\delta \to 0} P[D(1) > D(0) \mid R = r + \delta] > 0$
3. $E[D(0) \mid R = r]$ and $E[D(1) \mid R = r]$ are continuous in $r$ at $c$.
4. $F_{Y(0),D(0),D(1),R}(y \mid 0,1,r)$ and $F_{Y(1),D(0),D(1),R}(y \mid 0,1,r)$ are continuous in $r$ at $c$ for all $y \in \mathcal{Y}$.
5. $E[Y(0) \mid D(1) > D(0), R = c], E[Y(0) \mid D(1) > D(0), R = c] < \infty$.

The identification result is given in Lemma 3 (Hahn, Todd, and van der Klaauw, 2001). Under Assumption F, the Local Average Treatment Effect can be obtained as the ratio of the difference in the limits of the conditional expectation of $Y$ given $R = r$, $E[Y \mid R = r]$, as $r$ approaches $c$ from right and left, and the difference in the limits of the conditional expectation of $D$ given $R = r$, $E[D \mid R = r]$, as $r$ approaches $c$ from right and left. In other words, any discontinuity observed at the cutoff in the conditional expectation of $Y$ given $R$ is accounted to the treatment through the corresponding discontinuity at the cutoff in the probability of receiving the treatment.

**Lemma 3.** (Hahn, Todd, and van der Klaauw, 2001) Suppose Assumption F holds. Then

$$E[Y(1) - Y(0) \mid D(1) > D(0), R = c] = \lim_{\delta \to 0} \frac{E[Y \mid R = c + \delta] - E[Y \mid R = c - \delta]}{E[D \mid R = c + \delta] - E[D \mid R = c - \delta]}.$$

Assumption G lists conditions under which the Local Average Treatment Effect for compliers at any point $r$ in the running variable distribution, $E[Y(1) - Y(0) \mid D(1) > D(0), R = r]$ is nonparametrically identified in the latent factor framework. Assumption G.1 requires that the potential outcomes $Y(0)$ and $Y(1)$ and the potential treatment status $D(0)$ and $D(1)$ are jointly independent of $R$ conditional on the latent factor $\theta$. Assumption G.2 imposes the monotonicity

---

$^{19}$Continuity of the conditional expectation functions of the potential outcomes for compliers is enough for Lemma 3. However, continuity of the conditional cumulative distribution functions allows one to also identify distributional treatment effects for the compliers (Frandsen, Frolich, and Melly, 2012).
assumption for all \( \theta \in \Theta \). Assumption G.3 imposes this relationship to be strict at least for some \( \theta \in \Theta \).

**Assumption G.**

1. \((Y(0), Y(1), D(0), D(1)) \perp \perp R \mid \theta\).
2. \(P[D(1) \geq D(0) \mid \theta] = 1 \) a.s.
3. \(P[D(1) > D(0) \mid \theta] > 0 \) a.s.

The identification result is stated in Lemma 4. Under Assumption G, the Local Average Treatment Effect for complier at \( R = r \) is given by the ratio of the reduced from effect of treatment assignment on the outcome and the first stage effect of treatment assignment on treatment status at \( R = r \).

**Lemma 4.** Suppose Assumption G holds. Then

\[
E \left[ Y(1) - Y(0) \mid D(1) > D(0), R = r \right] = \frac{E \{E \left[ Y(D(1)) - Y(D(0)) \mid \theta \right] \mid R = r \}}{E \{E \left[ D(1) - D(0) \mid \theta \right] \mid R = r \}}
\]

for all \( r \in \mathcal{R} \).

Assumption H lists the conditions under which the latent reduced form effect of treatment assignment on the outcome, \( E[Y(D(1)) - Y(D(0)) \mid \theta] \), and the latent first stage effect of treatment assignment on the probability of receiving the treatment, \( E[D(1) - D(0) \mid \theta] \), are nonparametrically identified from the conditional distribution of \( Y \) given \( M \) and \( Z \), \( f_{Y|M,Z} \), and the conditional distribution of \( D \) given \( M \) and \( Z \), \( f_{D|M,Z} \). Assumption H.1 requires that the potential outcomes \( Y(0) \) and \( Y(1) \) and the potential treatment status \( D(0) \) and \( D(1) \) are jointly independent of \( M \) given \( \theta \). In other words, the measurement errors in \( M \) are assumed to to be unrelated to \((Y(0), Y(1), D(0), D(1))\). Assumption H.2 repeats the common support assumption from Assumption E whereas Assumption H.3 is analogous to the (bounded) completeness condition in Assumption E.

**Assumption H.**

1. \((Y(0), Y(1), D(0), D(1)) \perp \perp M \mid \theta\).
2. \(0 < P[D = 1 \mid \theta] < 1 \) a.s.
3. \(f_{\theta|M,Z}(\theta \mid m^0, 0) \) and \(f_{\theta|M,Z}(\theta \mid m^1, 1) \) form (boundedly) complete families of distributions indexed by \( m^0 \in \mathcal{M}^0 \) and \( m^1 \in \mathcal{M}^1 \).
Theorem 4 states the identification result. Together with Lemma 4 this result can be used to nonparametrically identify the Local Average Treatment Effect for compliers at any point in the running variable distribution. The proof of Theorem 4 is analogous to the proof of Theorem 3.

**Theorem 4.** Suppose Assumption H holds. Then the equations

\[
E \left[ Y \mid M = m, D = 0 \right] = E \left\{ E \left[ Y(D(0)) \mid \theta \right] \mid M = m, D = 0 \right\}
\]

\[
E \left[ Y \mid M = m, D = 1 \right] = E \left\{ E \left[ Y(D(1)) \mid \theta \right] \mid M = m, D = 1 \right\}
\]

\[
E \left[ D \mid M = m, D = 0 \right] = E \left\{ E \left[ D(0) \mid \theta \right] \mid M = m, D = 0 \right\}
\]

\[
E \left[ D \mid M = m, D = 1 \right] = E \left\{ E \left[ D(1) \mid \theta \right] \mid M = m, D = 1 \right\}
\]

for all \( r_0 \in \mathcal{R}_0 \) and \( r_1 \in \mathcal{R}_1 \) admit unique solutions for (bounded) \( E \left[ Y(D(0)) \mid \theta \right], E \left[ Y(D(1)) \mid \theta \right], E \left[ D(0) \mid \theta \right], \) and \( E \left[ D(1) \mid \theta \right] \) for all \( \theta \in \Theta \).

### 3.2 Settings with a Multiple Latent Factors

Section 2 focused on the identification of the measurement and latent outcome models in the presence of a one-dimensional latent factor \( \theta \). However, it is possible to generalize the identification results to a setting with a \( K \)-dimensional latent factor \( \theta = (\theta_1, \ldots, \theta_K) \). Instead of three noisy measures required in the one-dimensional case, the \( K \)-dimensional case requires the availability of \( 2 \times K + 1 \) noisy measures. To be more exact, this setting requires one to observe two noisy measures for each latent factor \( \theta_k, k = 1, \ldots, K \), as well as one measure that is related to all \( K \) latent factors.\(^{20}\)

Formally, I assume that the data contains \( 2 \times K + 1 \) noisy measures given by

\[
M_k^1 = g_{M_k^1} \left( \theta_k, \nu_{M_k^1} \right), k = 1, \ldots, K
\]

\[
M_k^2 = g_{M_k^2} \left( \theta_k, \nu_{M_k^2} \right), k = 1, \ldots, K
\]

\[
M_3 = g_W (\theta_1, \ldots, \theta_K, \nu_{M_3})
\]

where \( g_{M_k^1}, g_{M_k^2}, k = 1, \ldots, K, \) and \( g_{M_3} \) are unknown functions, and \( \nu_{M_k^1}, \nu_{M_k^2}, k = 1, \ldots, K, \) and \( \nu_{M_3} \) are potentially multidimensional disturbances. I focus on a setting in which is \( R \) is a deterministic function of at least one but potentially many of the measures for each \( \theta_k, k = 1, \ldots, K \).

\(^{20}\)It is possible to allow for this measure to be multidimensional by, for instance, containing an additional \( K \) measures for each latent factor \( \theta_k, k = 1, \ldots, K \).
However, it is possible to consider a more general setting that allows the relationship between $R$ and $M$ to be stochastic. Going back to the selective school example of Section 2.1, one might think of $R$ as being the average score in two entrance exams in English and Math, $M_1$ and $M_2$, that are noisy measures of English and Math ability, $\theta_1$ and $\theta_2$. $M_1$, $M_2$, and $M_3$ might instead consist of pre-application baseline test scores in English and Math.

I require $\theta_k$, $M_1^k$, and $M_2^k$, $k = 1, \ldots, K$, to be continuous but allow $M_3$ to be either continuous or discrete; even a binary $M_3$ suffices for identification. I denote the supports of $\theta_k$, $M_1^k$, $M_2^k$, $k = 1, \ldots, K$, and $M_3$ by $\times_k$, $M_1^k$, $M_2^k$, $k = 1, \ldots, K$, and $M_3$. In addition, I use $\Theta = \Theta_1 \times \cdots \times \Theta_K$ to denote the support of $\theta = (\theta_1, \ldots, \theta_K)$, $M_1 = M_1^1 \times \cdots \times M_1^K$ and $M_2 = M_2^1 \times \cdots \times M_2^K$ to denote the supports of $M_1 = (M_1^1, \ldots, M_1^K)$ and $M_2 = (M_2^1, \ldots, M_2^K)$, and $M = M_1 \times M_2 \times M_3$ to denote the support of $M = (M_1, M_2, M_3)$.

The introduction of multiple latent factors only affects Assumption C and Theorem 1 regarding the identification $f_{\theta, M}$ in the linear measurement model. Assumption D and Theorem 2 regarding the identification of $f_{\theta, M}$ in the nonlinear measurement model apply instead directly to this setting as long as one interprets $\theta$ and $M$ as defined above (Hu and Schennach, 2008). The same holds for Assumption E and Theorem 3 regarding the identification of the Average Treatment Effect away from the cutoff in sharp RD as well as for Assumption H and Theorem 4 regarding the identification of the Local Average Treatment Effect away from the cutoff in fuzzy RD.

Thus, I focus here on the identification of $f_{\theta, M}$ in the linear measurement model

\begin{align*}
M_1^k &= \theta_k + \nu_{M_1^k}, k = 1, \ldots, K \quad (3) \\
M_2^k &= \mu_{M_2^k} + \lambda_{M_2^k} \theta_k + \nu_{M_2^k}, k = 1, \ldots, K \quad (4) \\
M_3 &= g_{M_3} (\theta_1, \ldots, \theta_K, \nu_{M_3})
\end{align*}

where $\lambda_{M_2^k} \neq 0$, $E[\nu_{M_1^k} | \theta_k] = E[\nu_{M_2^k} | \theta_k] = 0$, $k = 1, \ldots, K$. Assumption I lists modified conditions under which one can obtain nonparametric identification of $f_{\theta, M}$ in this setting.

**Assumption I.**

1. The relationship between $M_1^k$, $M_2^k$, and $\theta_k$, $k = 1, \ldots, K$, are as given in Equations (3) and (4).
2. $\theta_k$, $\nu_{R_k}$, and $\nu_{B_k}$ are jointly independent conditional on $M_3$ for all $k = 1, \ldots, K$.
3. $\text{Cov}[\theta_k, M_3] \neq 0$, $E[|\theta_k| | M_3] < \infty$, $E[\nu_{M_1^k} | M_3] = E[\nu_{M_2^k} | M_3] = 0$ for all $k = 1, \ldots, K$.
4. One of the following conditions holds:
   (a) The real zeros of the conditional characteristic functions of $\nu_{M_1^k}$, $k = 1, \ldots, K$, given $M_3$ and
their derivatives are disjoint, and the conditional characteristic functions of $\nu_{M_2^k}$, $k = 1, \ldots, K$, given $M_3$ have only isolated real zeros.

(b) The conditional characteristic functions of $\nu_{M_1^k}$, $k = 1, \ldots, K$, given $M_3$ are analytic.

5. The components of $\nu_{M_1} = (\nu_{M_1^1}, \ldots, \nu_{M_1^K})$ and $\nu_{M_2} = (\nu_{M_2^1}, \ldots, \nu_{M_2^K})$ are jointly independent conditional on $M_3$.

6. The real zeros of the conditional joint characteristic functions of $\nu_{M_1}$ and $\nu_{M_2}$ given $M_3$ are disjoint.

Theorem 5 states the identification result. The proof of this theorem is similar to the proof of Theorem 1.

**Theorem 5.** Suppose Assumption I holds. Then

\[
\mu_{M_2^k} = E \left[ M_{2^k}^k \right] - \lambda_{M_2^k} E \left[ M_{1^k}^k \right], \quad k = 1, \ldots, K
\]

\[
\lambda_{M_2^k} = \frac{Cov \left[ M_{2^k}^k, M_3 \right]}{Cov \left[ M_{1^k}^k, M_3 \right]}, \quad k = 1, \ldots, K.
\]

In addition, the equation

\[
f_{M_1, M_2 \mid M_3} (m_1, m_2 \mid m_3) = \int \prod_{k=1}^{K} f_{\nu_{M_1^k} \mid M_3} \left( m_{1^k}^k - \theta_k \mid m_3 \right) f_{\nu_{M_2^k} \mid M_3} \left( m_{2^k}^k - \mu_{M_2^k} - \lambda_{M_2^k} \theta_k \mid m_3 \right) \times f_{\theta \mid M_3} (\theta \mid m_3) d\theta
\]

for all $m_1 \in M_1$, $m_2 \in M_2$, $m_3 \in M_3$, admits unique solutions for $f_{\nu_{M_1^k} \mid M_3}$, $f_{\nu_{M_2^k} \mid M_3}$, $k = 1, \ldots, K$, and $f_{\theta \mid M_3}$. Consequently, the joint distribution $f_{\theta, M}$ is identified.

### 4 Boston Exam Schools

I use the latent factor-based approach to RD extrapolation developed in the previous two sections to study the causal effects of attending selective public schools, known as exam schools, in Boston for the full population of applicants. This section describes the empirical setting and data as well as the identification and estimation of the latent factor model in the empirical setting.
4.1 Setting

Boston Public Schools (BPS) includes three exam schools that span grades 7-12: Boston Latin School, Boston Latin Academy, and John D. O’Bryant High School of Mathematics and Science. Latin School, founded in 1635, is the oldest and most selective out of the three exam schools, and it is also the first public and oldest still existing school in the United States. Latin School enrolls about 2,400 student. Latin Academy, founded in 1877, is Boston’s second oldest and second most selective exam school in Boston. It enrolls about 1,700 students. O’Bryant, founded in 1893, is the youngest and least selective out of the three exam schools. It enrolls about 1,200 students.

The exam schools differ considerably from traditional Boston public schools in terms of student performance. In the U.S. News & World Report high school ranking in 2013, for instance, Latin School, Latin Academy and O’Bryant formed the three best high schools in BPS, and ranked as 2nd, 20th, and 15th in Massachusetts. Furthermore, in 2012, the exam schools were among the four best schools in BPS in terms of the share of students scoring at a Proficient or Advanced level in the Massacusess Comprehensive Assessement System (MCAS) tests in English, Math and Science.\footnote{MCAS is a state-mandated series of achievement tests introduced for the purposes of No Child Left Behind.} Similarly, the exam schools formed the three best schools in the BPS system in 2012 in terms of both the students’ average SAT scores and 4-year graduation rates.\footnote{All of the exam schools are also among the few schools in BPS that have won the Blue Ribbon Award from the US Department of Education. In addition, Latin School was listed as one the top 20 high schools in the US by U.S. News & World Report in 2007.}

The fact that exam school students, on average, outperform other BPS students in terms of MCAS/SAT scores and graduation rates is not surprising given the considerable differences in student composition between the exam schools and traditional Boston public schools. For instance, the exam schools enroll a considerably higher share of white and Asian students as well as a higher share of female students than BPS as a whole. Limited English proficiency, special education, and low income rates are also negligible among exam school students when compared to other Boston public schools.

In addition to student composition, the exam schools differ from traditional Boston public schools along several other dimensions. The exam schools have a higher share of teachers who are licensed to teach in the area in which they are teach, as well as a higher share of core academic classes that are taught by teachers who hold a valid Massachusetts license and have demonstrated subject matter competency in the areas they teach. Exam school teachers are also older than teachers at other Boston public schools. While the student/teacher ratio is much higher at the
exam schools, it is to a large extent explained by the lack of students requiring special education.

There are also considerable differences between the exam schools and traditional Boston public schools in terms of their curricula. The curriculum at both Latin Academy and Latin School emphasizes the classics, and students at these schools take mandatory Latin classes, whereas the curriculum at O’Bryant focuses on Math and Science. Moreover, the exam schools offer a rich array of college preparatory classes and extracurricular activities, and enjoy to a varying extent additional funding that comes from alumni contributions.

All of the exam schools admit new students for grades 7 and 9, but in addition to this O’Bryant also admits some students for grade 10. In order to be admitted to one of the exam schools, a student is required to be a Boston resident. Students can apply to the exam schools from both inside and outside BPS. Each applicant submits a preference ordering of the exam schools to which they are applying. The admissions decisions are based on the applicants’ Grade Point Averages (GPA) in English and Math from the previous school year and the fall term of the ongoing school year as well as the applicants’ scores on the Independent School Entrance Examination (ISEE) administered during the fall term of the ongoing school year. The ISEE is an entrance exam used by several selective schools in the United States. It consists of five sections: Reading Comprehension, Verbal Reasoning, Mathematics Achievement, Quantitative Reasoning, and a 30-minute essay. Exam school admissions only use the first four sections of the ISEE.

Each applicant receives an offer from at most one exam school, and waitlists are not used. The assignment of exam school offers is based on the student-proposing Deferred Acceptance (DA) algorithm by Gale and Shapley (1962). The algorithm takes as inputs each exam school’s predetermined capacity, the applicants’ preferences over the exam schools, and the exam schools’ rankings of the applicants based on a weighted average of their standardized GPA and ISEE scores. These rankings differ slightly across the exam schools as for each school the standardization and ranking is done only within the pool of applicants to that school.

The DA algorithm produces exam school-specific admissions cutoffs that are given by the lowest rank among the applicants admitted to a given exam school. Since applicants receive an offer from at most one exam school, there is not a direct link between the exam school-specific running variables (an applicant’s rank among applicants to a given exam school) and the exam school offers. However, as in Abdulkadiroglu, Angrist, and Pathak (forthcoming), it is possible to construct a sharp sample for each exam school that consists of applicants who receive an offer if and only if their running variable is above the admissions cutoff for the exam school in question. Appendix B describes in
detail the DA algorithm and the construction of the sharp samples.

4.2 Data

The main data for this paper comes from three sources provided by the BPS: (1) an exam school application file, (2) a BPS registration and demographic file, and (3) an MCAS file. These files can be merged together using a unique BPS student identification number. In addition, I use the students’ home addresses to merge the BPS data with Census tract-level information from the American Community Survey (ACS) 5-year summary file for 2006-2011.²³

The exam school application file consists of the records for all exam school applications in 1995-2009. It provides me with information on each applicant’s application year and grade, application preferences, GPA in English and Math, ISEE scores, exam school-specific ranks, and the admissions decision. This allows me to reproduce the exam school-specific admissions cutoffs (the lowest rank among applicants admitted to a given exam school). I transform the exam school-specific ranks into percentiles, ranging from 0 to 100, within application year and grade. I then center these running variables to be 0 at the admissions cutoff for the exam school in question. Thus, the running variables give an applicant’s distance from the admissions cutoff in percentile units. Finally, I standardize the ISEE scores and GPA to have a mean of 0 and a standard deviation of 1 in the applicant population within each year and grade.²⁴

The BPS registration and demographic file consists of the records for all BPS students in 1996-2012. It provides me with information on each student’s home address, school, grade, gender, race, limited English proficiency (LEP) status, bilingual status, special education (SPED) status, and free or reduced price lunch (FRLP) status.

The MCAS file consists of the records for all MCAS tests taken by BPS students in 1997-2008. It provides me with information on 4th, 7th, and 10th grade MCAS scores in English, and 4th, 8th, and 10th grade MCAS scores in Math. In the case of retakes I only consider the first time a student took the test. I construct middle school and high school MCAS composites as the average MCAS scores in 7th grade English and 8th grade Math and 10th grade English and Math. I standardize the 4th grade MCAS scores in English and Math as well as the middle school and high school MCAS composite scores to have a mean 0 and a standard deviation of 1 in the BPS population within each year and grade.

²³See Abdulkadiroglu, Angrist, and Pathak (forthcoming) for a more detailed description of the BPS data.
²⁴The exam school application file only contains a combined index of GPA in English and Math.
Lastly, I use the ACS 5-year summary file for 2006-2011 to obtain information on the median family income, percent of households occupied by the owner, percent of families headed by a single parent, percent of households where a language other than English is spoken, the distribution of educational attainment, and the number of school-aged children in each Census tract in Boston. I use this information to divide the Census tracts into socioeconomic tiers as described in Section 6.1.

I restrict the sample to students who applied to the exam schools for 7th grade in 2000-2004. I focus on 7th grade applicants as most students enter the exam schools in 7th grade, and their exposure to the exam school treatment is longer. This is also the applicant group for which the covariate-based RD extrapolation approach by Angrist and Rokkanen (2013) fails. The restriction to application years 2000-2004 is done in order to have both 4th grade MCAS scores and middle/high school MCAS composite scores for the applicants. I exclude students who apply to the exam schools from outside BPS as these applicants are more likely to remain outside BPS and thus not have follow up information in the data. In addition, I exclude students with missing covariate or 4th grade MCAS score information.

Table 1 reports descriptive statistics for all BPS students as well as the exam school applicants in the estimation sample. Column (1) includes all BPS students enrolled in 6th grade in 2000-2004. Column (2) includes the subset of students who apply to the exam schools. Columns (3)-(6) include the subsets of applicants who receive no exam school offer or an offer from a given exam school. Exam school applicants are a highly selected group of students, with markedly higher 4th grade MCAS scores and lower shares of blacks and Hispanics, limited English proficiency, and special education than BPS students as a whole. Similarly, there is considerable selection even within exam school applicants according to their exam school assignment, with applicants admitted to a more selective exam school having higher 4th grade MCAS scores and lower shares of blacks and Hispanics, limited English proficiency, and students eligible for free or reduced price lunch.

4.3 Identification and Estimation

Throughout the rest of the paper I use $Z \in \{0, 1, 2, 3\}$ to denote the exam school assignment of an applicant where 0 stands for no offer, 1 for O’Bryant, 2 for Latin Academy and 3 for Latin School. Similarly, I use $S \in \{0, 1, 2, 3\}$ to denote the enrollment decision of an applicant in the fall following exam school application where 0 stands for traditional Boston public school, 1 for O’Bryant, 2 for Latin Academy and 3 for Latin School. Lastly, I use $R_1$, $R_2$, and $R_3$ to denote the running variables
for O’Bryant, Latin Academy, and Latin School.

As discussed in Section 4.1, each applicant receives at most one exam school offer that is determined by the DA algorithm. The exam school assignment of an applicant is a deterministic function of her running variables and application preferences, denoted by \( P \),

\[
Z = g_Z(R_1, R_2, R_3, P).
\]

The running variables are deterministic functions of the applicant’s scores in the Reading Comprehension, Verbal Reasoning, Mathematics Achievement, and Quantitative Reasoning sections of the ISEE, denoted by \( M_2^E, M_3^E, M_2^M, \) and \( M_3^M, \) as well as her GPA in English and Math, denoted by \( G, \)

\[
R_s = g_{R_s}(M_2^E, M_3^E, M_2^M, M_3^M, G), \quad s = 1, 2, 3.
\]

In addition, the data contains 4th grade MCAS scores in English and Math, denoted by \( M_1^E \) and \( M_1^M. \)

I treat the 4th grade MCAS score in English and the scores in the Reading Comprehension and Verbal Reasoning sections of the ISEE as noisy measures of an applicant’s English ability, denoted by \( \theta_E, \)

\[
M_k^E = g_{M_k^E}(\theta_E, \nu_{M_k^E}), \quad k = 1, 2, 3.
\]

I treat the 4th grade MCAS score in Math and the scores in the Mathematics Achievement and Quantitative Reasoning sections of the ISEE instead as noisy measures of an applicant’s Math ability, denoted by \( \theta_M, \)

\[
M_k^M = g_{M_k^M}(\theta_M, \nu_{M_k^M}), \quad k = 1, 2, 3.
\]

The test scores are strongly correlated with each other, but this correlation is far from perfect: some of the applicants scoring well in one test score perform relatively worse in another test. This can be seen from the scatterplots and correlation in Figure 8 and Table 4. Consistent with the latent factor structure specified above, test scores measuring English ability are more highly correlated with each other than with test scores measuring Math ability, and vice versa. The only exception
to this is 4th grade MCAS score in English that is most highly correlated with 4th grade MCAS score in Math. There is also a clear time-pattern within test scores measuring a given ability: the ISEE scores measuring the same ability are more highly correlated with each other than with the 4th grade MCAS score measuring the same ability.

Let $Y(s), s = 0, 1, 2, 3,$ denote potential outcomes under different enrollment decisions, and let $S(z), z = 0, 1, 2, 3,$ denote potential enrollment decisions under different exam school assignments. I assume that the potential outcomes and enrollment decisions are jointly independent of the test scores conditional on English and Math abilities and a set of covariates, denoted by $X$. Formally,

$$\left( \{Y(s)\}_{s=0}^{3}, \{S(z)\}_{z=0}^{3} \right) \perp \perp M \mid \theta, X,$$

where $M = (M_M^E, M_M^E, M_M^E, M_M^E, M_M^M, M_M^M, M_M^M)$. The covariates included in $X$ are GPA, application preferences, application year, race, gender, SES tier as well as indicators for free or reduced price lunch, limited English proficiency, special education, and being bilingual. I also assume that there is sufficient noise in the ISEE scores so that conditional on the covariates it is possible to observe an applicant with a given level of English and Math ability under any exam school assignment. Formally, this common support assumption is given by

$$0 < P[Z = z \mid \theta, X] < 1, \ z = 0, 1, 2, 3.$$

Together these two assumptions can be used to identify causal effects of different exam school assignments on either enrollment or achievement, as discussed in Section 2. I make two additional assumptions that allow me to also identify causal effects of enrollment at a given exam school as opposed to a traditional Boston public school for the compliers who enroll at the exam school if they receive an offer and enroll at a traditional Boston public school if they receive no offer. First, I assume that receiving an offer from exam school $s$ as opposed to no offer induces at least some applicants to enroll at exam school $s$ instead of a traditional Boston public school. Second, I assume that this is the only way in which receiving an offer from exam school $s$ as opposed to no offer can affect the enrollment decision of an applicant.$^{25}$ Formally, these first stage and monotonicity

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$^{25}$I also assume in general that receiving an offer from exam school $s$ can only induce an applicant to attend this school as opposed to another school. This rules out, for instance, the case that an applicant is induced to enroll at Latin School as opposed to Latin Academy by receiving an offer from O’Bryant as opposed to no exam school offer.
assumptions are given by

\[ P \left[ S (s) = s, S (0) = 0 \mid \theta, X \right] > 0, \ s = 1, 2, 3 \]
\[ P \left[ S (s) = s', S (0) = s'' \mid \theta, X \right] = 0, \ s' \neq s, s'' \neq 0. \]

In the estimation I approximate the conditional joint distribution of English and Math ability by a bivariate normal distribution,

\[
\begin{bmatrix}
\theta_E \\
\theta_M
\end{bmatrix} \mid X \sim N \left( \begin{bmatrix}
\mu'_{\theta_E} X \\
\mu'_{\theta_M} X
\end{bmatrix}, \begin{bmatrix}
\sigma^2_{\theta_E} & \sigma_{\theta_E \theta_M} \\
\sigma_{\theta_E \theta_M} & \sigma^2_{\theta_M}
\end{bmatrix} \right).
\]

To ensure a valid variance-covariance matrix I use the parametrization

\[
\begin{bmatrix}
\sigma^2_{\theta_E} & \sigma_{\theta_E \theta_M} \\
\sigma_{\theta_E \theta_M} & \sigma^2_{\theta_M}
\end{bmatrix} = \begin{bmatrix}
\omega_{11} & 0 \\
0 & \omega_{22}
\end{bmatrix} \begin{bmatrix}
\omega_{11} & \omega_{21} \\
\omega_{21} & \omega_{22}
\end{bmatrix}.
\]

I also approximate the conditional distributions of the test scores using normal distributions given by

\[
M_k^E \mid \theta, X \sim N \left( \mu_{M_k^E} X + \lambda_{M_k^E} \theta_E, \exp \left( \gamma_{M_k^E} + \delta_{M_k^E} \theta_E \right)^2 \right), \ k = 1, 2, 3
\]
\[
M_k^M \mid \theta, X \sim N \left( \mu_{M_k^M} X + \lambda_{M_k^M} \theta_M, \exp \left( \gamma_{M_k^M} + \delta_{M_k^M} \theta_M \right)^2 \right), \ k = 1, 2, 3,
\]

where \( \mu_{M_k^E} = \mu_{M_k^M} = 0 \) and \( \lambda_{M_k^E} = \lambda_{M_k^M} = 1 \) to pin down the location and scale of the abilities as discussed in Section 2.3. Thus, I restrict the conditional expectations of the measurements to depend linearly on ability and allow for heteroskedasticity in the measurement error with respect to ability. Finally, I restrict test scores to be jointly independent conditional on the abilities and covariates.

Let \( D_s (z) = 1 \left( S (z) = s \right), \ s = 0, 1, 2, 3 \), denote indicators for potential enrollment decisions under different exam school assignments, and let \( Y \left( S (z) \right) \) denote potential outcomes under different exam school assignments. I approximate the conditional expectations of \( D_s (z) \) and \( Y \left( S (z) \right) \) using the linear models

\[
E \left[ D_s (z) \mid \theta, X \right] = \alpha'_{D_s(z)} X + \beta_{D_s(z)}^E \theta_E + \beta_{D_s(z)}^M \theta_M
\]
\[
E \left[ Y \left( S (z) \right) \mid \theta, X \right] = \alpha'_{Y(S(z))} X + \beta_{Y(S(z))}^E \theta_E + \beta_{Y(S(z))}^M \theta_M
\]
where $z = 0, 1, 2, 3$.

The identification of the measurement and latent outcome models specified above follows directly from the nonparametric identification results presented in Sections 2 and 3. I illustrate this in more detail in Appendix C by providing moment equations that identify these particular parametric models.

I estimate the parameters of the measurement model using Maximum Simulated Likelihood (MSL). I use 500 random draws from the conditional joint distribution of $\theta$ given $X$, $f_{\theta|X}$, to evaluate the integral in the conditional joint density of $M$ given $X$, $f_{M|X}$. For a given observation $f_{M|X}$ is given by

$$f_{M|X}(m \mid X; \mu, \lambda, \gamma, \delta, \omega) = \prod_{k=1}^{3} f_{M^E|\theta,X}(m^E_k \mid \theta, X; \mu, \lambda, \gamma, \delta) f_{M^M|\theta,X}(m^M_k \mid \theta, X; \mu, \lambda, \gamma, \delta) f_{\theta|X}(\theta \mid X; \mu, \omega) d\theta$$

where the conditional densities $f_{M^E|\theta,X}$, $f_{M^M|\theta,X}$, $k = 1, 2, 3$, and $f_{\theta|X}$ are as specified above.

I estimate the parameters of the latent outcome models using the Method of Simulated Moments (MSM) based on the moment equations

$$E[D_s \mid M, X, Z] = \alpha_{D_s(Z)}'X + \beta_{D_s(Z)}^E E[\theta_E \mid M, X, Z] + \beta_{D_s(Z)}^M E[\theta_M \mid M, X, Z]$$
$$E[Y \mid M, X, Z] = \alpha_{Y(S(z))}'X + \beta_{Y(S(z))}^E E[\theta_E \mid M, X, Z] + \beta_{Y(S(z))}^M E[\theta_M \mid M, X, Z]$$

for $Z = 0, 1, 2, 3$. The conditional expectations $E[\theta_E \mid M, X, Z]$ and $E[\theta_M \mid M, X, Z]$ are computed using the MSL estimates of the measurement model and 500 random draws from $f_{\theta|X}$. The weighting matrix in the MSM procedure is based on the number of observations in the $(M, X, Z)$ cells. This implies that the parameters of the latent outcome models can be estimated in practice by running a regression of $D_S$ or $Y$ on $X$, $E[\theta_E \mid M, X, Z]$, and $E[\theta_M \mid M, X, Z]$ using observations with $Z = 0, 1, 2, 3$.

The standard errors presented below are based on nonparametric 5-step bootstrap using 50 replications (Davidson and MacKinnon, 1999; Andrews, 2002). I am currently running more bootstrap replications. See website for updated version (http://economics.mit.edu/grad/rokkanen).
MSL procedure after five iterations. I then re-estimate the latent outcome models using these MSL estimates. This provides a computationally attractive approach for taking into account the uncertainty related to both step of the estimation procedure due to the slow speed of convergence of the MSL estimation.

5 Extrapolation Results

5.1 Effects at the Admission Cutoffs

To benchmark the latent factor model-based estimates, I begin with RD estimates of causal effects of exam school attendance for marginal applicants at the admissions cutoffs in the sharp samples. Figures 6a and 6b plot the relationship between the running variables and the probabilities of receiving an offer from and enrolling at a given exam school in windows of ±20 around the admissions cutoffs in the sharp samples. The blue dots show bin averages in windows of width 1. The black solid lines show fits from local linear regressions estimated separately to the left and right of the cutoffs using the edge kernel and a bandwidth computed separately for each exam school using the algorithm by Imbens and Kalyanaraman (2012). Figures 7a and 7b show the same plots for average middle school and high school MCAS composite scores.

Table 2 reports the first stage, reduced form, and Local Average Treatment Effect estimates corresponding to Figures 6 and 7. The estimates are based on local linear regressions using the edge kernel and a bandwidth that is computed separately for each exam school and MCAS outcome using the algorithm by Imbens and Kalyanaraman (2012).\footnote{As noted by Calonico, Cattaneo, and Titiunik (2013), the algorithm by Imbens and Kalyanaraman (2012) may generate too large bandwidths. However, my findings are not sensitive to alternative ways of choosing the bandwidths.} The first stage and reduced form models are given by

\[
D_s = \alpha_{FS} + \beta_{FS}Z_s + \gamma_{FS}R_s + \delta_{FS}Z_s \times R_s + X'\pi_{FS} + \eta
\]

\[
Y = \alpha_{RF} + \beta_{RF}Z_s + \gamma_{RF}R_s + \delta_{RF}Z_s \times R_s + X'\pi_{RF} + \epsilon
\]

where \(D_s\) is an indicator for enrollment at exam school \(s\) in the following school year, \(Y\) is the outcome of interest, \(Z_s\) is an indicator for being at or above the admissions cutoff for exam school \(s\), \(R_s\) is the distance from admissions cutoff for exam school \(s\), and \(X\) is a vector containing indicators for application years and application preferences. The first stage and reduced form estimates are given by \(\beta_{FS}\) and \(\beta_{RF}\), and the Local Average Treatment Effect estimate is given by the ratio \(\frac{\beta_{RF}}{\beta_{FS}}\).
In practice this ratio can be estimated using weighted 2-Stage Least Squares (2SLS).

Figures 6a confirms the sharpness of exam school offers as functions of the running variables in the sharp samples discussed in Section 4.1: the probability of receiving an offer from a given exam school jump from 0 to 1 at the admissions cutoff. However, as can be seen from 6b and the first stage estimates in Table 2, not all applicants receiving an offer from a given exam school choose to enroll there. The enrollment first stages are nevertheless large. An offer from O’Bryant raises the probability of enrollment at O’Bryant from 0 to .78 at the admissions cutoff whereas offers from Latin Academy and Latin School raise the probability of enrollment at these schools from 0 to .95 and .96.

Exam school offers have little effect on the average middle school and high school MCAS composite scores of the marginal applicants, as can be seen from Figures 7a and 7b and the reduced form estimates in Table 2. The only statistically significant effect is found for middle school MCAS composite score at the Latin Academy admissions cutoff: an offer from Latin Academy is estimated to reduce the average score by .181σ. According to the corresponding Local Average Treatment Effect estimate in Table 2, enrollment at Latin Academy leads to a .191σ reduction in the average score among compliers at the admissions cutoff.

Table 3 repeats the estimations separately for applicants whose average 4th grade MCAS scores falls below and above the within-year median. The first stage estimates show large enrollment effects at the admissions cutoffs for both lower-achieving and higher-achieving applicants. These effects are similar in magnitude to the effects estimated for the full sample. The reduced form and Local Average Treatment Effect estimates for applicants with low average 4th grade MCAS scores are relatively noisy due to small sample size, but there is some evidence of treatment effect heterogeneity by prior achievement, a point I return to in Section 5.2. The reduced form estimate suggests that an offer from O’Bryant increases average high school MCAS composite score by .204σ at the admissions cutoff among applicants with low average 4th grade MCAS scores. The corresponding Local Average Treatment Effect estimate suggests that enrollment at O’Bryant increases the average score among the compliers at the admissions cutoff by .275σ. The reduced form and Local Average Treatment Effect estimates for applicants with high average 4th grade MCAS scores are similar to the estimates for the full sample.

It is important to note the incremental nature of the RD estimates reported above. Applicants just below the O’Bryant admissions cutoff do not receive an offer from any exam school, meaning that the counterfactual for these applicants is a traditional Boston public school. On the other
hand, the vast majority of applicants just below the Latin Academy admissions cutoff receive an offer from O'Bryant, and the vast majority of applicants just below the Latin School admissions cutoff receive an offer from Latin Academy. Thus, the reduced form and Local Average Treatment Effect estimates for Latin Academy and Latin School should be interpreted as the effect of receiving an offer from and enrollment at a more selective exam school. I return to this point in Section 5.3.

5.2 Estimates of the Latent Factor Model

Before investigating effects away from the admissions cutoffs I briefly discuss the main estimates of the latent factor model. Figure 9 shows the underlying marginal distributions of English and Math ability in the population of exam school applicants. I construct these using kernel density estimates based on simulations from the estimated measurement model. The marginal distributions look relatively normal which is expected given the joint normality assumption on the conditional distribution of the abilities given covariates. The mean and standard deviation of English ability are 1.165 and .687. The mean and standard deviation of Math ability are 1.121 and .831. Figure 10 shows a scatterplot of English and Math abilities. The relationship between the abilities is relatively linear which is expected given the joint normality assumption on the conditional distribution. The correlation between English and Math ability is .817.

Table 5 reports estimates of the factor loadings on the means and (log) standard deviations of the measures. As discussed in Section 4, I pin down the scales of the abilities by normalizing the factor loadings on the means of 4th grade MCAS scores to 1. The estimated factor loadings on the means of ISEE scores are instead slightly above 1. The estimated factor loadings on the (log) standard deviations of ISEE scores in Reading Comprehension and Verbal Reasoning and 4th grade MCAS score in Math suggest that the variances of these measures are increasing in ability. The estimated factor loadings on the (log) standard deviations of ISEE scores in Mathematical Achievement and Quantitative Reasoning and 4th grade MCAS score in English are small and statistically insignificant.

Table 6 reports estimates of the factor loadings on enrollment (First Stage) and middle school and high school MCAS composite scores (Reduced Form) under a given exam school assignment. For no exam school offer in Column (1) the enrollment outcome is enrollment at a traditional Boston public school. For an offer from a given exam school in Columns (2), (3), and (4) the outcome is enrollment at the exam school in question. The estimated factor loadings on enrollment are largely negative, suggesting that applicants with higher ability are less likely to attend a given
exam school if they receive an offer (or a traditional Boston public school if they receive no offer). However, the opposite is true for Latin School.

Unlike for enrollment, the estimated factor loadings on middle school and high school MCAS composite scores are positive, large in magnitude, and highly statistically significant. This is not surprising: applicants with higher English and Math abilities perform, on average, better in middle school and high school MCAS exams in English and Math irrespective of their exam school assignment. A more interesting finding arising from these estimates is that the factor loadings tend to be larger in magnitude under no exam school offer than under an offer from any given exam school. This is especially true for high school MCAS composite scores. This suggest that applicants with lower English and Math abilities benefit more from access to exam schools. I return to this point below.

5.3 Effects Away from the Admissions Cutoffs

The estimates of the latent factor model can be used to construct empirical counterparts of Figures 1 and 5 that illustrate the extrapolation problem in sharp RD and the latent factor-based approach to solving this problem. Figure 11 plots the latent factor model-based fits and extrapolations of the potential outcomes in the RD experiments for the sharp samples over the full supports of the running variables. The blue dots show bin averages in windows of width 1. The black solid lines show the latent factor model-based fits, and the dashed red lines show the latent factor model-based extrapolations. I smooth the fits and extrapolations with a local linear regression using the edge kernel and a rule of thumb bandwidth (Fan and Gijbels, 1996). The fits and extrapolations are defined as

\[
E \left[ Y \left( S \left( z^{act} \right) \right) \mid R_s = r \right], s = 1, 2, 3
\]

\[
E \left[ Y \left( S \left( z^{cf} \right) \right) \mid R_s = r \right], s = 1, 2, 3
\]

where the counterfactual assignment \( z^{cf} \) is an offer from exam school \( s \) for applicants below the admissions cutoff and an offer from the next most selective exam school for the applicants above the admissions cutoff (no exam school offer in the case of O’Bryant).

Figure 11a plots the fits and extrapolations for middle school MCAS composite scores. For O’Bryant the fits and extrapolations lie on top of each other, suggesting that receiving an offer from O’Bryant has no effect on the average score of either marginal applicants at the admissions cutoff.
or inframarginal applicants away from the admissions cutoff. Similarly, the extrapolations for Latin Academy reveal that the negative effect of receiving an offer from Latin Academy found for marginal applicants at the admissions cutoff in Section 5.1 holds also for inframarginal applicants away from the admissions cutoff. For Latin School the picture arising is markedly different. The extrapolations suggest that receiving an offer from Latin School has no effect on the average score of inframarginal applicants above the admissions cutoff. Inframarginal applicants below the admissions cutoff would instead experience, on average, achievement gains from receiving an offer from Latin School.

Figure 11b plots the fits and extrapolations for high school MCAS composite scores. For all three exam schools the extrapolations suggest little effect from receiving an offer for inframarginal applicants above the admissions cutoffs, with the exception of applicants far above the O’Bryant admissions cutoff for which the effect is negative. For inframarginal applicants below the admissions cutoffs the picture arising is instead markedly different. For all three exam schools the extrapolations suggest a positive effect from receiving an offer for applicants failing to gain access to the exam school in question.

Table 7 reports estimates of the extrapolated first stage and reduced form effects of an offer from a given exam school on enrollment and middle school and high school MCAS composite scores. In addition, the table reports estimates of the extrapolated Local Average Treatment Effects of enrollment at a given exam school on middle school and high school MCAS composite scores for the compliers. The estimates are for the full population of applicants in the sharp samples. The estimates for middle school MCAS composite score show no effect of an offer from or enrollment at O’Bryant. Offer from Latin Academy and Latin school are instead estimated to reduce the average score by .229σ and .214σ. The corresponding Local Average Treatment Effect estimates are 0.236σ for Latin Academy and .0226σ for Latin School. The estimates for high school MCAS composite scores show large positive effects for all three exam schools. The reduced form and Local Average Treatment Effect estimates are .252σ and .293σ for O’Bryant, .279σ and .290σ for Latin Academy, and .199σ and .209σ for Latin School.

Table 8 reports the same estimates separately for applicants below and above the admissions cutoffs. The first stage estimates reveal that the effects of an offer from a given exam school on enrollment at this school are larger among inframarginal applicants below the admissions cutoffs than among inframarginal applicants above the admissions cutoffs. The reduced form and Local Average Treatment Effect estimates for middle school MCAS composite scores show similar negative effects of an offer from and enrollment at Latin Academy as in Table 7 both below and above the
cutoff. The negative Latin School effect reported above is instead entirely driven by inframarginal applicants below the admissions cutoff. The reduced form and Local Average Treatment Effect estimates for high school MCAS composite scores confirm the above findings of large positive effects on the average score of inframarginal applicants below the admissions cutoffs. There is instead little evidence of effects for inframarginal applicants above the admissions cutoffs.

Similar to the RD estimates at the admissions cutoffs discussed in Section 5.1, the above estimates should be interpreted as incremental effects of receiving an offer from or enrollment at a more selective exam school. Thus, these estimates leave unanswered the question of how receiving an offer from or enrollment at a given exam school versus a traditional Boston public school affects achievement. Table 9 addresses this question by reporting the estimates of the extrapolated first stage and reduced form effects of receiving an offer from a given exam school versus no offer from any exam school on enrollment at the exam school in question on middle school and high school MCAS composite scores. In addition, the table reports the extrapolated Local Average Treatment Effect estimates of the effect of enrollment at a given exam school versus a traditional Boston public school on middle school and high school MCAS composite scores for the compliers in the full population of applicants.

According to the estimates for middle school MCAS composite scores, offers from Latin Academy and Latin School versus no exam school offer reduce the average score by $0.275\sigma$ and $0.319\sigma$. The corresponding Local Average Treatment Effects for enrollment at a given exam school versus a traditional Boston public school are $-0.288\sigma$ for Latin Academy, and $-0.330\sigma$ for Latin School. There is instead no evidence of effects for O’Bryant. The estimates for high school MCAS composite scores are small in magnitude and statistically insignificant.

Table 10 reports the same estimates separately for applicant who receive no exam school offer and for applicants who receive an exam school offer. The estimates for middle school MCAS composite scores show similar negative effects of an offer from and enrollment at Latin Academy and Latin School as in Table 9 among both applicant groups. The estimates for high school MCAS scores reveal instead substantial heterogeneity in the treatment effects. The estimates suggest that receiving an offer from a given exam school versus no offer from any exam school has large positive effects among lower-achieving applicants failing to gain access to the exam schools. The reduced from effects are $0.334\sigma$ for O’Bryant, $0.429\sigma$ for Latin Academy, and $0.428\sigma$ for Latin School. The corresponding Local Average Treatment Effects of enrollment at a given exam school versus a traditional Boston public school are $0.376\sigma$ for O’Bryant, $0.435\sigma$ for Latin Academy, and $0.448\sigma$ for
Latin School. The estimates for higher-achieving applicants gaining access to the exam schools are instead negative and large in magnitude. The reduced from effects are $-0.268\sigma$ for O’Bryant, $-0.300\sigma$ for Latin Academy, and $-0.348\sigma$ for Latin School. The corresponding Local Average Treatment Effects are $-0.353\sigma$ for O’Bryant, $-0.325\sigma$ for Latin Academy, and $-0.353\sigma$ for Latin School.

5.4 Placebo Experiments

A natural concern regarding the results in the previous section is that they are just an artifact of extrapolations away from the admissions cutoffs. To address this concern, I study the performance of the model using a set of placebo experiments. I start by dividing the applicants receiving a given exam school assignment $z = 0, 1, 2, 3$ in half based on the within-year median of the running variable distribution for this population.\(^{28}\) I re-estimate the latent outcome models to the left and right of the placebo cutoffs and use the resulting estimates to extrapolate away from these cutoffs. All of the applicants both to the left and to the right of the cutoffs in these placebo RD experiments receive the same exam school assignment. Thus, the extrapolations should show no effects if the identifying assumptions are valid and the empirical specifications provide reasonable approximations of the underlying data generating process.

Figure 12 plots the latent factor model-based fits and extrapolations in the placebo RD experiments.\(^{29}\) Figure 12a plots the estimates for middle school MCAS composite scores, and Figure 12b plots the estimates for high school MCAS composite scores. The blue dots show bin averages in windows of width 1. The black solid lines show the latent factor model-based fits to the data. The dashed red lines show the latent factor model-based extrapolations. I smooth the fits and extrapolations with a local linear regression using the edge kernel and a rule of thumb bandwidth (Fan and Gijbels, 1996). For both outcomes and for each exam school assignment the fits and extrapolations lie on top of each other, thus providing evidence supporting the identifying assumptions and empirical specifications. The only notable exceptions to this can be seen for high school MCAS composite scores far below the placebo cutoff for applicants receiving no offer from any exam school and far above the placebo cutoff for applicants receiving an offer from O’Bryant.

Table 11 reports estimates of the placebo reduced form effects on middle school and high school MCAS composite scores. The estimates are shown for all applicants as well as separately for applicants below and above the placebo cutoffs. The estimated effects are small in magnitude and

\(^{28}\)For applicant receiving no offer I use the average of their exam school-specific running variables.

\(^{29}\)I transform the running variables into percentile ranks within each year in the placebo RD experiments and re-centered them to be 0 at the placebo cutoff for expositional purposes.
statistically insignificant, thus providing further support for the validity of the results presented in Section 5.3.

6 Counterfactual Simulations

6.1 Description of the Admissions Reforms

Estimates of treatment effects away from the exam school admissions cutoffs are useful for predicting effects of reforms that change the exam school assignments of inframarginal applicants. A highly contentious example of this is the use of affirmative action in exam school admissions. I use the estimates of the latent factor model to predict how two particular affirmative action reforms would affect the achievement of exam school applicants.

The first reform reintroduces in the admissions process minority preferences that were in place in the Boston exam schools admissions in 1975-1998. In this counterfactual admissions process 65% of the exam school seats are assigned purely based on achievement. The remaining 35% of the exam school seats are reserved for black and Hispanic applicants and assigned based on achievement. The assignment of seats within each group is based on the DA algorithm discussed in Section 4.1.

The second reform introduces in the admissions process socioeconomic preferences that have been in place in the Chicago exam school admissions since 2010. In this counterfactual admissions process 30% of the exam school seats are assigned purely based on achievement. The remaining 70% of the exam school seats are divided equally across four socioeconomic tiers and assigned within them based on achievement. The assignment of the seats within each group is again based on the DA algorithm.

I generate the socioeconomic tiers by computing for each Census tract in Boston a socioeconomic index that takes into account the following five characteristics: (1) median family income, (2) percent of households occupied by the owner, (3) percent of families headed by a single parent, (4) percent of households where a language other than English is spoken, and (5) an educational attainment score.\textsuperscript{30} The socioeconomic index for a given Census tract is given by the sum of the its percentile ranks in each five characteristics among the Census tracts in Boston (for single-parent and non-English speaking households 1 minus the percentile rank is used). I assign each BPS student a socioeconomic index based on the Census tract they live in and divide the students

\textsuperscript{30}The educational attainment score is calculated based on the educational attainment distribution among individuals over the age of 25: \textit{educational attainment score} = 0.2 \times (\% less than high school diploma) + 0.4 \times (\% high school diploma) + 0.6 \times (\% some college) + 0.8 \times (\% bachelors degree) + 1.0 \times (\% advanced degree).
into socioeconomic tiers based on the quartiles of the socioeconomic index distribution in the BPS population within each year.

To study the effects of the two reforms I reassign the exam school offers based on the counterfactual admissions processes, considering only applicants in the estimation sample described in Section 4.2. I use as the capacity of a given exam school in a given year the number of offers it made to the applicants in the estimation sample in that year. The latent factor model then allows me to predict average middle school and high school MCAS composite scores based on the reassigned exam school offers.\textsuperscript{31}

An important feature of both of the reforms is that they cause substantial changes to the admissions cutoffs of the exam schools. This means that if there is considerable treatment effect heterogeneity in terms of the running variables, predictions of the effects of the reforms based on treatment effects at admissions cutoffs are likely to be misleading. Based on the results in Section 5, this is the case for Boston exam schools. Thus, it is a first-order issue to take this heterogeneity into account when predicting the effects of the reforms.

As with all counterfactuals, there are other dimensions that may change as a result of the reforms. First, the reforms potentially affect the composition of the pool of exam school applicants as some students face a decrease and some students an increase in their ex ante expected probability of being admitted to a given exam school.\textsuperscript{32} Second, the reforms will lead to changes in the composition of applicants who are admitted and consequently enroll at the exam schools. These changes may affect the sorting of teachers across schools (Jackson, 2009) and the way teaching is targeted (Duflo, Dupas, and Kremer, 2011) as well as affect achievement directly through peer effects (Epple and Romano, 2011; Sacerdote, 2011).

However, the above discussion should not be seen as a concern regarding the latent factor-based extrapolation approach per se. It is possible to address the above caveats by building a richer model that incorporates these channels into the latent factor framework. For instance, one can build a model of the exam school application behavior of BPS students along the lines of the work by Walters (2012) on Boston charter schools and incorporate this into the latent factor framework. I leave this and other potential extensions for future research as they are outside the scope of this

\textsuperscript{31}This exercise is closely related to the literature on evaluating the effects of reallocations on the distribution of outcomes. See, for instance, Graham (2011), Graham, Imbens, and Ridder (2010), and Graham, Imbens, and Ridder (2013).

\textsuperscript{32}For instance, Long (2004) and Andrews, Ranchhod, and Sathy (2010) find the college application behavior of high school students in California and Texas to be responsive to affirmative action and other targeted recruiting programs. However, the evidence on this is somewhat mixed (Card and Krueger, 2005; Antonovics and Backes, 2013).
6.2 Simulation Results

The introduction of either minority or socioeconomic preferences substantially affects exam school assignments: 27 – 35% of applicants are affected by the reforms. This can be seen from Table 12, which reports the actual and counterfactual exam school assignments under the two reforms. This is also evident in Table 13 that reports the admissions cutoff faced by different applicant groups under the counterfactual admissions process. The counterfactual admissions cutoffs are expressed as distances from the actual admissions cutoffs. Minority applicants would face substantially lower admissions cutoffs under minority preferences than under the current admissions process whereas the opposite is true for non-minority applicants. Similarly, applicants from lower socioeconomic tiers would face increasingly lower admissions cutoffs under socioeconomic preferences than under the current admissions process.

Table 14 reports descriptive statistics for the exam school applicants based on their counterfactual assignments under minority and socioeconomic preferences. The most notable compositional changes caused by the two reforms can be seen among applicants receiving an offer from Latin School. Under both counterfactual admissions processes, Latin School admits students with considerably lower average 4th grade MCAS scores in English and Math. Similarly, the share of blacks and Hispanics among the admitted students to Latin School would more than double under minority preferences and close to double under socioeconomic preferences. Furthermore, the average 4th grade MCAS scores in Math and English are higher and the shares of blacks and Hispanics lower among the applicants receiving no offer from any exam school under both counterfactual reforms. Changes in the composition of applicants receiving offers from O’Bryant and Latin Academy are instead less marked.

I use the estimated latent factor model to predict potential outcomes for the exam school applicants under the counterfactual admissions processes. These predictions can be used to evaluate whether the changes in exam school assignments caused by the two reforms translate into effects on achievement. To answer this question, Table 15 reports Average Reassignment Effects (ARE) of the reforms on middle school and high school MCAS composite scores. The Average Reassignment Effect is given by the difference in average potential outcomes among the exam school applicants.
under the counterfactual and actual admissions processes:

\[ E \left[ Y^{cf} - Y^{act} \right] = \sum_{z=0}^{3} P \left[ Z^{cf} = z \right] E \left[ Y \left( S (z) \right) \mid Z^{cf} = z \right] \]

\[- \sum_{s=0}^{3} P \left[ Z^{act} = z \right] E \left[ Y \left( S (z) \right) \mid Z^{act} = z \right] \]

where \( Z^{act} \) and \( Z^{cf} \) are an applicant’s actual and counterfactual exam school assignments. The table reports estimates both for the full population of applicants and for applicants whose exam school assignment is affected by the reforms.

The introduction of minority preferences would have no effect on the average middle school MCAS composite score among exam school applicants. However, this masks substantial heterogeneity in the effects across minority and non-minority applicants. The estimates suggest that the reform would reduce the average score among minority applicants by \( .028\sigma \) and increase it among non-minority applicants by \( .043\sigma \). The estimated effects are larger among the affected applicants: \(-.084\sigma \) for minority applicants and \(.113\sigma \) for non-minority applicants. The estimates for high school MCAS composite scores suggest that the introduction of minority preferences would increase the average score by \( .027\sigma \) among all applicants and by \( .062\sigma \) among affected applicants. There is less marked heterogeneity in these effects across minority and non-minority applicants, but the effects are somewhat larger for minority applicants.

The introduction of socioeconomic preferences would have no effect on the average middle school MCAS composite score among exam school applicants. However, there is considerable heterogeneity in the effects across applicants from different socioeconomic tiers. The estimates suggest that the reform would reduce the average score among applicants from the lowest socioeconomic tier by \( .032\sigma \) and increase the average score among applicants from the highest socioeconomic tier by \( .041\sigma \). The estimated effects are larger among the affected applicants: \(-.092\sigma \) for the lowest socioeconomic tier and \(.133\sigma \) for the highest socioeconomic tier. The estimates for high school MCAS composite scores suggest that the introduction of socioeconomic preferences would increase the average score by \( .015\sigma \) among all applicants and by \( .050\sigma \) among affected applicants. There is again considerable heterogeneity in the effects across applicants from different socioeconomic tiers. The reform would increase the average score by \( .068\sigma \) among applicants from the lowest socioeconomic tier and by \( .054\sigma \) among applicants from the highest socioeconomic tier.

There are two mechanisms at work behind these estimates. First, the reforms lower the admis-
sions cutoff faced by minority applicants and applicants from lower socioeconomic tiers. This leads to more lower-achieving applicants, who experience achievement gains from exam school attendance, to gain access to the exam schools. Second, the reforms increase the admissions cutoff faced by non-minority applicants and applicants from higher socioeconomic tiers. This leads to some of the higher-achieving applicants, who experience achievement losses from exam school attendance, to lose their exam school seats.

7 Conclusions

RD design allows for nonparametric identification and estimation of treatment effects for individuals at the cutoff value determining treatment assignment. However, many policies of interest change treatment assignment of individuals away from the cutoff, making knowledge of treatment effects for these individuals of substantial interest. A highly contentious example of this is affirmative action in selective schools that affects admissions cutoffs faced by different applicant groups.

The contributions of this paper are two-fold. First, I develop a new latent factor-based approach to the identification and estimation of treatment effects away from the cutoff in RD. The approach relies on the assumption that sources of omitted variables bias in an RD design can be modeled using unobserved latent factors. My main result is nonparametric identification of treatment effects for all values of the running variable based on the availability of multiple noisy measures of the latent factors. Second, I use the latent factor framework to estimate causal effects of Boston exam school attendance for the full population of applicants and to simulate effects of introducing either minority or socioeconomic preferences in exam school admissions.

My findings highlight the local nature of RD estimates that show little evidence of causal effects for marginal applicants at admissions cutoffs (Abdulkadiroglu, Angrist, and Pathak, forthcoming). The estimates of the latent factor model suggest that achievement gains from exam school attendance are larger among applicants with lower baseline measures of ability. As a result, lower-achieving applicants who currently fail to gain admission to Boston exam schools would experience substantial achievement gains from attending these schools. The simulations predict that the introduction of either minority or socioeconomic preferences in exam school admissions boosts average achievement among applicants. This is largely driven by achievement gains experienced by lower-achieving applicants who gain access to exam schools as a result of the policy change. These findings are of significant policy-relevance given ongoing discussion about the use of affirmative
action in exam school admissions.

I focus in this paper on the heterogeneity in causal effects of exam school attendance based on the running variables used in the admissions process. This is a first-order concern when predicting effects of admissions reforms that widely change the exam school assignments of inframarginal applicants. However, as with all counterfactuals, there are other dimension that may change as a result of these reforms. First, affirmative action might lead to changes in the application behavior of students (Long, 2004; Andrews, Ranchhod, and Sathy, 2010). Second, affirmative action causes changes in student composition that may affect the sorting of teachers across schools (Jackson, 2009) as well as the way teaching is targeted (Duflo, Dupas, and Kremer, 2011). Finally, the changes in student composition may affect achievement directly through peer effects (Epplle and Romano, 2011; Sacerdote, 2011). It is possible to model these channels in the latent factor framework, but this is left for future research.

Boston exam schools, as well as other selective schools, are a natural application for latent factor-based RD extrapolation as admissions are based on noisy measures of applicants’ latent abilities. However, the approach is likely to prove useful also in other educational settings, such as gifted and talented programs (Bui, Craig, and Imberman, forthcoming) and remedial education (Jacob and Lefgren, 2004; Matsudaira, 2008). Moreover, the approach is likely to prove useful in health settings where treatment assignment is based on noisy measures of individuals’ latent health conditions. Such settings include, for instance, the use of birth weight to assign additional medical care for newborns (Almond, Doyle, Kowalski, and Williams, 2010; Bharadwaj, Loken, and Neilson, 2013). As illustrated by the findings for Boston exam schools, local effects identified by RD do not necessarily represent the effects of policy interest. Latent factor-based RD extrapolation provides a framework for investigating external validity in these and other RD designs.

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(b) 4th Grade MCAS Scores

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(a) Middle School MCAS Composite

(b) Middle School MCAS Composite
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(a) Middle School MCAS Composite

(b) High School MCAS Composite
Table 1: Descriptive Statistics for Boston Public School Students and Exam School Applicants

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<th>All Applicants (2)</th>
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<th>Latin School (6)</th>
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<td>0.227</td>
<td>0.043</td>
<td>0.073</td>
<td>0.009</td>
<td>0.006</td>
</tr>
<tr>
<td>English 4</td>
<td>0.000</td>
<td>0.749</td>
<td>0.251</td>
<td>0.870</td>
<td>1.212</td>
</tr>
<tr>
<td>Math 4</td>
<td>0.000</td>
<td>0.776</td>
<td>0.206</td>
<td>0.870</td>
<td>1.275</td>
</tr>
<tr>
<td>N</td>
<td>21,094</td>
<td>5,179</td>
<td>2,791</td>
<td>1,858</td>
<td>843</td>
</tr>
</tbody>
</table>

Notes: This table reports descriptive statistics for 2000-2004. The All BPS column includes all 6th grade students in Boston Public Schools in who do not have missing covariate or 4th grade MCAS information. The All Applicants column includes the subset of students who apply to Boston exam schools. The Assignment columns include the subsets of applicants who receive an offer from a given exam school.
Table 2: RD Estimates for the First Stage, Reduced Form and Local Average Treatment Effects at the Admissions Cutoffs

<table>
<thead>
<tr>
<th></th>
<th>O'Bryant Academy School (1)</th>
<th>Latin School (2)</th>
<th>Latin School (3)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: Middle School MCAS</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>First Stage</td>
<td>0.775***</td>
<td>0.949***</td>
<td>0.962***</td>
</tr>
<tr>
<td></td>
<td>(0.031)</td>
<td>(0.017)</td>
<td>(0.017)</td>
</tr>
<tr>
<td>Reduced Form</td>
<td>-0.084</td>
<td>-0.181***</td>
<td>-0.104</td>
</tr>
<tr>
<td></td>
<td>(0.060)</td>
<td>(0.057)</td>
<td>(0.079)</td>
</tr>
<tr>
<td>LATE</td>
<td>-0.108</td>
<td>-0.191***</td>
<td>-0.108</td>
</tr>
<tr>
<td></td>
<td>(0.078)</td>
<td>(0.060)</td>
<td>(0.082)</td>
</tr>
<tr>
<td>N</td>
<td>1,934</td>
<td>2,328</td>
<td>1,008</td>
</tr>
</tbody>
</table>

| **Panel B: High School MCAS** |                             |                  |                  |
| First Stage             | 0.781***                    | 0.955***         | 0.964***         |
|                        | (0.034)                     | (0.018)          | (0.018)          |
| Reduced Form            | 0.047                       | -0.021           | -0.086           |
|                        | (0.055)                     | (0.044)          | (0.052)          |
| LATE                   | 0.060                       | -0.022           | -0.089           |
|                        | (0.070)                     | (0.046)          | (0.054)          |
| N                      | 1,475                       | 1,999            | 907              |

Notes: This table reports RD estimates of the effect of an exam school offer on exam school enrollment (First Stage), the effect of an exam school offer on MCAS scores (Reduced Form), and the effects of exam school enrollment on MCAS scores (LATE). Heteroskedasticity-robust standard errors shown in parentheses.

* significant at 10%; ** significant at 5%; *** significant at 1%
Table 3: RD Estimates for the First Stage, Reduced Form and Local Average Treatment Effects at the Admissions Cutoffs: Heterogeneity by Average 4th Grade MCAS Scores

<table>
<thead>
<tr>
<th></th>
<th>Low 4th Grade MCAS Composite</th>
<th>High 4th Grade MCAS Composite</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>O'Bryant Academy School</td>
<td>Latin School</td>
</tr>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>First Stage</td>
<td>0.734*** 0.917*** 1.000***</td>
<td>0.780*** 0.956*** 0.961***</td>
</tr>
<tr>
<td>Reduced Form</td>
<td>0.059 0.130 -0.095</td>
<td>-0.122* -0.200*** -0.084</td>
</tr>
<tr>
<td>LATE</td>
<td>0.080 0.142 -0.095</td>
<td>-0.157* -0.209*** -0.088</td>
</tr>
<tr>
<td>N</td>
<td>420 246 681</td>
<td>1,348 1,802 921</td>
</tr>
</tbody>
</table>

Panel A: Middle School MCAS

Panel B: High School MCAS

Notes: This table reports RD estimates of the effect of an exam school offer on exam school enrollment (First Stage), the effect of an exam school offer on MCAS scores (Reduced Form), and the effects of exam school enrollment on MCAS scores (LATE). The estimates are shown separately for applicants whose average 4th grade MCAS scores fall below and above the within-year median. Heteroskedasticity-robust standard errors shown in parentheses.

* significant at 10%, ** significant at 5%, *** significant at 1%
Table 4: Correlations between the ISEE Scores and 4th Grade MCAS Scores

<table>
<thead>
<tr>
<th></th>
<th>ISEE</th>
<th>MCAS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Reading (1) Verbal (2) Math (3) Quantitative (4)</td>
<td>English 4 (5) Math 4 (6)</td>
</tr>
<tr>
<td>Panel A: ISEE</td>
<td>Reading 0.735 0.631 0.621</td>
<td>0.670 0.581</td>
</tr>
<tr>
<td></td>
<td>Verbal 0.735 1 0.619</td>
<td>0.655 0.587</td>
</tr>
<tr>
<td></td>
<td>Math 0.631 0.619 1</td>
<td>0.598 0.740</td>
</tr>
<tr>
<td></td>
<td>Quantitative 0.621 0.617 0.845</td>
<td>0.570 0.718</td>
</tr>
</tbody>
</table>

Panel B: MCAS

<table>
<thead>
<tr>
<th></th>
<th>English 4 (1) Math 4 (2)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Reading 0.670 0.655</td>
</tr>
<tr>
<td></td>
<td>Math 4 0.581 0.587</td>
</tr>
</tbody>
</table>

N 5,179

Notes: This table reports correlations between the ISEE scores and 4th grade MCAS scores.

Table 5: Factor Loadings on the Means and (Log) Standard Deviations of the ISEE and 4th Grade MCAS Scores

<table>
<thead>
<tr>
<th></th>
<th>ISEE</th>
<th>MCAS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Reading (1) Verbal (2) Math (3) Quantitative (4)</td>
<td>English 4 (5) Math 4 (6)</td>
</tr>
<tr>
<td>Panel A: Factor Loading on Mean</td>
<td>θ_E 1.160*** 1.180***</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>(0.029) (0.032)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>θ_M 1.135*** 1.119***</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>(0.025) (0.024)</td>
<td></td>
</tr>
</tbody>
</table>

Panel B: Factor Loading on (Log) Standard Deviation

<table>
<thead>
<tr>
<th></th>
<th>ISEE</th>
<th>MCAS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Reading (1) Verbal (2) Math (3) Quantitative (4)</td>
<td>English 4 (5) Math 4 (6)</td>
</tr>
<tr>
<td></td>
<td>θ_E 0.081*** 0.152***</td>
<td>0.016</td>
</tr>
<tr>
<td></td>
<td>(0.023) (0.015)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>θ_M 0.019 -0.013</td>
<td>0.124***</td>
</tr>
<tr>
<td></td>
<td>(0.020) (0.016)</td>
<td>(0.014)</td>
</tr>
</tbody>
</table>

N 5,179

Notes: This table reports the estimated factor loadings on the means and (log) standard deviations of the ISEE and 4th grade MCAS scores. Standard errors based on nonparametric 5-step bootstrap shown in parentheses.

* significant at 10%, ** significant at 5%, *** significant at 1%
Table 6: Factor Loadings on Enrollment and MCAS Scores Under a Given Exam School Assignment

<table>
<thead>
<tr>
<th></th>
<th>No Offer (1)</th>
<th>O'Bryant Academy (2)</th>
<th>Latin School (3)</th>
<th>Latin School (4)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>θ_E</strong> First Stage</td>
<td>-0.001</td>
<td>-0.073</td>
<td>-0.107**</td>
<td>0.016</td>
</tr>
<tr>
<td></td>
<td>(0.007)</td>
<td>(0.062)</td>
<td>(0.042)</td>
<td>(0.013)</td>
</tr>
<tr>
<td><strong>θ_M</strong> First Stage</td>
<td>-0.008</td>
<td>-0.035</td>
<td>0.011</td>
<td>0.023</td>
</tr>
<tr>
<td></td>
<td>(0.006)</td>
<td>(0.058)</td>
<td>(0.033)</td>
<td>(0.018)</td>
</tr>
<tr>
<td><strong>θ_E</strong> Reduced Form</td>
<td>0.570***</td>
<td>0.476***</td>
<td>0.190**</td>
<td>0.439***</td>
</tr>
<tr>
<td></td>
<td>(0.053)</td>
<td>(0.085)</td>
<td>(0.095)</td>
<td>(0.073)</td>
</tr>
<tr>
<td><strong>θ_M</strong> Reduced Form</td>
<td>0.633***</td>
<td>0.601***</td>
<td>0.742***</td>
<td>0.464***</td>
</tr>
<tr>
<td></td>
<td>(0.056)</td>
<td>(0.070)</td>
<td>(0.069)</td>
<td>(0.061)</td>
</tr>
<tr>
<td><strong>N</strong></td>
<td>2,490</td>
<td>690</td>
<td>728</td>
<td>793</td>
</tr>
</tbody>
</table>

**Panel B: High School MCAS**

<table>
<thead>
<tr>
<th></th>
<th>No Offer (1)</th>
<th>O'Bryant Academy (2)</th>
<th>Latin School (3)</th>
<th>Latin School (4)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>θ_E</strong> First Stage</td>
<td>-0.001</td>
<td>-0.025</td>
<td>-0.102**</td>
<td>0.003</td>
</tr>
<tr>
<td></td>
<td>(0.008)</td>
<td>(0.070)</td>
<td>(0.051)</td>
<td>(0.009)</td>
</tr>
<tr>
<td><strong>θ_M</strong> First Stage</td>
<td>-0.012*</td>
<td>-0.059</td>
<td>0.012</td>
<td>0.027*</td>
</tr>
<tr>
<td></td>
<td>(0.007)</td>
<td>(0.060)</td>
<td>(0.038)</td>
<td>(0.015)</td>
</tr>
<tr>
<td><strong>θ_E</strong> Reduced Form</td>
<td>0.446***</td>
<td>0.282***</td>
<td>0.217***</td>
<td>0.239***</td>
</tr>
<tr>
<td></td>
<td>(0.061)</td>
<td>(0.071)</td>
<td>(0.070)</td>
<td>(0.047)</td>
</tr>
<tr>
<td><strong>θ_M</strong> Reduced Form</td>
<td>0.604***</td>
<td>0.346***</td>
<td>0.267***</td>
<td>0.158***</td>
</tr>
<tr>
<td></td>
<td>(0.060)</td>
<td>(0.059)</td>
<td>(0.064)</td>
<td>(0.044)</td>
</tr>
<tr>
<td><strong>N</strong></td>
<td>1,777</td>
<td>563</td>
<td>625</td>
<td>793</td>
</tr>
</tbody>
</table>

Notes: This table reports the estimated factor loadings on enrollment (First Stage) and MCAS scores (Reduced Form) under a given exam school assignment. First Stage refers to enrollment at a traditional Boston public school in the No Offer column and enrollment at a given exam school in the other columns. Standard errors based on nonparametric 5-step bootstrap shown in parentheses.

* significant at 10%, ** significant at 5%, *** significant at 1%
Table 7: Extrapolated First Stage, Reduced Form, and Local Average Treatment Effects in the Exam School-Specific RD Experiments

<table>
<thead>
<tr>
<th></th>
<th>Latin Academy</th>
<th>Latin School</th>
</tr>
</thead>
<tbody>
<tr>
<td>O'Bryant Academy</td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td><strong>Panel A: Middle School MCAS</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>First Stage</td>
<td>0.869***</td>
<td>0.971***</td>
</tr>
<tr>
<td></td>
<td>(0.035)</td>
<td>(0.011)</td>
</tr>
<tr>
<td>Reduced Form</td>
<td>-0.047</td>
<td>-0.229**</td>
</tr>
<tr>
<td></td>
<td>(0.100)</td>
<td>(0.102)</td>
</tr>
<tr>
<td>LATE</td>
<td>-0.054</td>
<td>-0.236**</td>
</tr>
<tr>
<td></td>
<td>(0.116)</td>
<td>(0.106)</td>
</tr>
<tr>
<td>N</td>
<td>3,029</td>
<td>3,641</td>
</tr>
<tr>
<td><strong>Panel B: High School MCAS</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>First Stage</td>
<td>0.858***</td>
<td>0.962***</td>
</tr>
<tr>
<td></td>
<td>(0.041)</td>
<td>(0.017)</td>
</tr>
<tr>
<td>Reduced Form</td>
<td>0.252***</td>
<td>0.279***</td>
</tr>
<tr>
<td></td>
<td>(0.062)</td>
<td>(0.067)</td>
</tr>
<tr>
<td>LATE</td>
<td>0.293***</td>
<td>0.290***</td>
</tr>
<tr>
<td></td>
<td>(0.077)</td>
<td>(0.071)</td>
</tr>
<tr>
<td>N</td>
<td>2,240</td>
<td>2,760</td>
</tr>
</tbody>
</table>

Notes: This table reports latent factor model based-estimates of the effect of an exam school offer on exam school enrollment (First Stage), the effect of an exam school offer on MCAS scores (Reduced Form), and the effects of exam school enrollment on MCAS scores (LATE) in the RD experiments. Standard errors based on nonparametric 5-step bootstrap shown in parentheses. * significant at 10%, ** significant at 5%, *** significant at 1%
Table 8: Extrapolated First Stage, Reduced Form, and Local Average Treatment Effects in the Exam School-Specific RD Experiments: Heterogeneity by the Running Variables

<table>
<thead>
<tr>
<th></th>
<th>Below Admissions Cutoff</th>
<th></th>
<th>Above Admissions Cutoff</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>O'Bryant Academy School</td>
<td>Latin School</td>
<td>O'Bryant Academy School</td>
<td>Latin School</td>
</tr>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
</tr>
<tr>
<td>First Stage</td>
<td>0.904***</td>
<td>0.992***</td>
<td>0.960***</td>
<td>0.749***</td>
</tr>
<tr>
<td></td>
<td>(0.045)</td>
<td>(0.012)</td>
<td>(0.020)</td>
<td>(0.015)</td>
</tr>
<tr>
<td>Reduced Form</td>
<td>-0.050</td>
<td>-0.234*</td>
<td>-0.245***</td>
<td>-0.038</td>
</tr>
<tr>
<td></td>
<td>(0.128)</td>
<td>(0.123)</td>
<td>(0.081)</td>
<td>(0.037)</td>
</tr>
<tr>
<td>LATE</td>
<td>-0.055</td>
<td>-0.236*</td>
<td>-0.255***</td>
<td>-0.051</td>
</tr>
<tr>
<td></td>
<td>(0.142)</td>
<td>(0.126)</td>
<td>(0.084)</td>
<td>(0.049)</td>
</tr>
<tr>
<td>N</td>
<td>2,339</td>
<td>2,913</td>
<td>3,478</td>
<td>690</td>
</tr>
</tbody>
</table>

Panel A: Middle School MCAS

|                      |                      |                      |                      |                      |
| First Stage          | 0.904***             | 0.992***             | 0.960***              | 0.749***             |
|                      | (0.045)              | (0.012)              | (0.020)               | (0.015)              |
| Reduced Form         | -0.050               | -0.234*              | -0.245***             | -0.038               |
|                      | (0.128)              | (0.123)              | (0.081)               | (0.037)              |
| LATE                 | -0.055               | -0.236*              | -0.255***             | -0.051               |
|                      | (0.142)              | (0.126)              | (0.084)               | (0.049)              |
| N                    | 2,339                 | 2,913                | 3,478                  | 690                  |

Panel B: High School MCAS

|                      |                      |                      |                      |                      |
| First Stage          | 0.904***             | 0.992***             | 0.960***              | 0.749***             |
|                      | (0.045)              | (0.012)              | (0.020)               | (0.015)              |
| Reduced Form         | -0.050               | -0.234*              | -0.245***             | -0.038               |
|                      | (0.128)              | (0.123)              | (0.081)               | (0.037)              |
| LATE                 | -0.055               | -0.236*              | -0.255***             | -0.051               |
|                      | (0.142)              | (0.126)              | (0.084)               | (0.049)              |
| N                    | 2,339                 | 2,913                | 3,478                  | 690                  |

Notes: This table reports latent factor model based-estimates of the effect of an exam school offer on exam school enrollment (First Stage), the effect of an exam school offer on MCAS scores (Reduced Form), and the effects of exam school enrollment on MCAS scores (LATE) in the RD experiments. The estimates are shown separately for applicants whose running variables fall below and above the admissions cutoffs. Standard errors based on nonparametric 5-step bootstrap shown in parentheses.

* significant at 10%, ** significant at 5%, *** significant at 1%
Table 9: Extrapolated First Stage, Reduced Form, and Local Average Treatment Effects for Comparisons between a Given Exam School and Traditional Boston Public Schools

<table>
<thead>
<tr>
<th></th>
<th>O’Bryant Academy (1)</th>
<th>Latin School (2)</th>
<th>Latin School (3)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: Middle School MCAS</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>First Stage</td>
<td>0.767***</td>
<td>0.956***</td>
<td>0.967***</td>
</tr>
<tr>
<td></td>
<td>(0.018)</td>
<td>(0.009)</td>
<td>(0.016)</td>
</tr>
<tr>
<td>Reduced Form</td>
<td>-0.059</td>
<td>-0.275***</td>
<td>-0.319***</td>
</tr>
<tr>
<td></td>
<td>(0.048)</td>
<td>(0.080)</td>
<td>(0.067)</td>
</tr>
<tr>
<td>LATE</td>
<td>-0.077</td>
<td>-0.288***</td>
<td>-0.330***</td>
</tr>
<tr>
<td></td>
<td>(0.063)</td>
<td>(0.085)</td>
<td>(0.069)</td>
</tr>
<tr>
<td>N</td>
<td>4,701</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

|                  |                      |                  |                  |
| **Panel B: High School MCAS** |                      |                  |                  |
| First Stage      | 0.754***             | 0.948***         | 0.968***         |
|                  | (0.024)              | (0.012)          | (0.017)          |
| Reduced Form     | 0.021                | 0.049            | 0.024            |
|                  | (0.041)              | (0.054)          | (0.054)          |
| LATE             | 0.027                | 0.052            | 0.025            |
|                  | (0.054)              | (0.057)          | (0.056)          |
| N                | 3,704                |                  |                  |

Notes: This table reports latent factor model based-estimates of the effect of receiving an offer from a given exam school versus no offer at all on enrollment at this exam school (First Stage), the effect of receiving an offer from a given exam school versus no offer at all on MCAS scores (Reduced Form), and the effect of enrollment at this exam school versus a traditional Boston public school on MCAS scores (LATE). Standard errors based on nonparametric 5-step bootstrap shown in parentheses.
* significant at 10%, ** significant at 5%, *** significant at 1%
Table 10: Extrapolated First Stage, Reduced Form, and Local Average Treatment Effects for Comparisons between a Given Exam School and Traditional Boston Public Schools: Heterogeneity by Exam School Offer Status

<table>
<thead>
<tr>
<th></th>
<th>No Exam School Offer</th>
<th></th>
<th>Exam School Offer</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>O'Bryant Academy</td>
<td>Latin O'Bryant School</td>
<td>O'Bryant Academy</td>
<td>Latin O'Bryant School</td>
</tr>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
<td>(4)</td>
<td>(5)</td>
</tr>
<tr>
<td>First Stage</td>
<td>0.901***</td>
<td>0.994***</td>
<td>0.749***</td>
<td>0.927***</td>
</tr>
<tr>
<td></td>
<td>(0.045)</td>
<td>(0.013)</td>
<td>(0.015)</td>
<td>(0.008)</td>
</tr>
<tr>
<td>Reduced Form</td>
<td>-0.051</td>
<td>-0.246*</td>
<td>-0.069</td>
<td>-0.307***</td>
</tr>
<tr>
<td></td>
<td>(0.126)</td>
<td>(0.139)</td>
<td>(0.080)</td>
<td>(0.049)</td>
</tr>
<tr>
<td>LATE</td>
<td>-0.057</td>
<td>-0.248*</td>
<td>-0.092</td>
<td>-0.332***</td>
</tr>
<tr>
<td></td>
<td>(0.141)</td>
<td>(0.142)</td>
<td>(0.108)</td>
<td>(0.053)</td>
</tr>
<tr>
<td>N</td>
<td>2,490</td>
<td></td>
<td>2,211</td>
<td></td>
</tr>
</tbody>
</table>

Panel A: Middle School MCAS

Panel B: High School MCAS

Notes: This table reports latent factor model based-estimates of the effect of receiving an offer from a given exam school versus no offer at all on enrollment at this exam school (First Stage), the effect of receiving an offer from a given exam school versus no offer at all on MCAS scores (Reduced Form), and the effect of enrollment at this exam school versus a traditional Boston public school on MCAS scores (LATE). The estimates are shown separately for applicants who do not receive an exam school offer and for applicants who receive an exam school offer. Standard errors based on nonparametric 5-step boostrap shown in paretheses.

* significant at 10%, ** significant at 5%, *** significant at 1%
Table 11: Extrapolated Reduced Form Effects in Placebo RD Experiments

<table>
<thead>
<tr>
<th></th>
<th>No Offer</th>
<th>O'Bryant Academy</th>
<th>Latin School</th>
</tr>
</thead>
<tbody>
<tr>
<td>Middle School</td>
<td>0.002</td>
<td>-0.000</td>
<td>0.042</td>
</tr>
<tr>
<td></td>
<td>(0.063)</td>
<td>(0.089)</td>
<td>(0.071)</td>
</tr>
<tr>
<td>MCAS</td>
<td>2,490</td>
<td>690</td>
<td>728</td>
</tr>
<tr>
<td></td>
<td>(0.071)</td>
<td>(0.072)</td>
<td>(0.072)</td>
</tr>
<tr>
<td>High School</td>
<td>-0.057</td>
<td>0.080</td>
<td>-0.041</td>
</tr>
<tr>
<td></td>
<td>(0.067)</td>
<td>(0.071)</td>
<td>(0.058)</td>
</tr>
<tr>
<td>MCAS</td>
<td>1,777</td>
<td>563</td>
<td>625</td>
</tr>
<tr>
<td></td>
<td>(0.072)</td>
<td>(0.051)</td>
<td>(0.051)</td>
</tr>
</tbody>
</table>

Panel A: All Applicants

<table>
<thead>
<tr>
<th></th>
<th>No Offer</th>
<th>O'Bryant Academy</th>
<th>Latin School</th>
</tr>
</thead>
<tbody>
<tr>
<td>Middle School</td>
<td>-0.081</td>
<td>0.015</td>
<td>0.034</td>
</tr>
<tr>
<td></td>
<td>(0.099)</td>
<td>(0.070)</td>
<td>(0.057)</td>
</tr>
<tr>
<td>MCAS</td>
<td>1,244</td>
<td>344</td>
<td>362</td>
</tr>
<tr>
<td></td>
<td>(0.065)</td>
<td>(0.065)</td>
<td>(0.049)</td>
</tr>
<tr>
<td>High School</td>
<td>-0.133</td>
<td>0.027</td>
<td>-0.028</td>
</tr>
<tr>
<td></td>
<td>(0.105)</td>
<td>(0.064)</td>
<td>(0.057)</td>
</tr>
<tr>
<td>MCAS</td>
<td>887</td>
<td>280</td>
<td>311</td>
</tr>
<tr>
<td></td>
<td>(0.049)</td>
<td>(0.049)</td>
<td>(0.049)</td>
</tr>
</tbody>
</table>

Panel B: Below Placebo Cutoff

<table>
<thead>
<tr>
<th></th>
<th>No Offer</th>
<th>O'Bryant Academy</th>
<th>Latin School</th>
</tr>
</thead>
<tbody>
<tr>
<td>Middle School</td>
<td>0.085</td>
<td>-0.015</td>
<td>0.050</td>
</tr>
<tr>
<td></td>
<td>(0.053)</td>
<td>(0.131)</td>
<td>(0.125)</td>
</tr>
<tr>
<td>MCAS</td>
<td>1,246</td>
<td>346</td>
<td>366</td>
</tr>
<tr>
<td></td>
<td>(0.112)</td>
<td>(0.112)</td>
<td>(0.112)</td>
</tr>
<tr>
<td>High School</td>
<td>0.020</td>
<td>0.133</td>
<td>-0.054</td>
</tr>
<tr>
<td></td>
<td>(0.059)</td>
<td>(0.099)</td>
<td>(0.091)</td>
</tr>
<tr>
<td>MCAS</td>
<td>890</td>
<td>283</td>
<td>314</td>
</tr>
<tr>
<td></td>
<td>(0.082)</td>
<td>(0.082)</td>
<td>(0.082)</td>
</tr>
</tbody>
</table>

Panel C: Above Placebo Cutoff

Notes: This table reports latent factor model-based estimates of the effects of placebo offers on MCAS scores. The estimates are shown for all applicants and separately for applicants whose running variables fall below and above the placebo admissions cutoffs. Standard errors based on nonparametric 5-step bootstrap shown in parentheses.

* significant at 10%, ** significant at 5%, *** significant at 1%
Table 12: Actual and Counterfactual Assignments under Minority and Socioeconomic Preferences

<table>
<thead>
<tr>
<th>Counterfactual Assignment</th>
<th>Actual Assignment</th>
<th>No Offer</th>
<th>O’Bryant Academy</th>
<th>Latin School</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: Minority Preferences</td>
<td></td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
</tr>
<tr>
<td>No Offer</td>
<td>2418</td>
<td>221</td>
<td>113</td>
<td>39</td>
</tr>
<tr>
<td>O’Bryant</td>
<td>280</td>
<td>129</td>
<td>133</td>
<td>213</td>
</tr>
<tr>
<td>Latin Academy</td>
<td>88</td>
<td>389</td>
<td>268</td>
<td>45</td>
</tr>
<tr>
<td>Latin School</td>
<td>5</td>
<td>16</td>
<td>276</td>
<td>546</td>
</tr>
<tr>
<td>Panel B: Socioeconomic Preferences</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>No Offer</td>
<td>2579</td>
<td>159</td>
<td>39</td>
<td>14</td>
</tr>
<tr>
<td>O’Bryant</td>
<td>203</td>
<td>319</td>
<td>146</td>
<td>87</td>
</tr>
<tr>
<td>Latin Academy</td>
<td>9</td>
<td>106</td>
<td>403</td>
<td>272</td>
</tr>
<tr>
<td>Latin School</td>
<td>0</td>
<td>171</td>
<td>202</td>
<td>470</td>
</tr>
</tbody>
</table>

Notes: This table reports the actual assignments and the counterfactual assignments under minority and socioeconomic preferences in the exam school admissions.

Table 13: Counterfactual Admissions Cutoffs for Different Applicant Groups under Minority and Socioeconomic Preferences

<table>
<thead>
<tr>
<th></th>
<th>O’Bryant Academy</th>
<th>Latin School</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: Minority Preferences</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Minority</td>
<td>-14.1</td>
<td>-31.9</td>
</tr>
<tr>
<td>Non-Minority</td>
<td>15.8</td>
<td>7.8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Latin Academy</th>
<th>Latin School</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel B: Socioeconomic Preferences</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SES Tier 1</td>
<td>-20.4</td>
<td>-32.9</td>
</tr>
<tr>
<td>SES Tier 2</td>
<td>-6.6</td>
<td>-16.1</td>
</tr>
<tr>
<td>SES Tier 3</td>
<td>-2.5</td>
<td>-17.1</td>
</tr>
<tr>
<td>SES Tier 4</td>
<td>8.0</td>
<td>-5.1</td>
</tr>
</tbody>
</table>

Notes: This table reports the differences between the actual admissions cutoffs and the counterfactual admissions cutoffs under minority and socioeconomic preferences in the exam school admissions.
Table 14: Composition of Applicants by the Counterfactual Assignment under Minority and Socioeconomic Preferences

<table>
<thead>
<tr>
<th></th>
<th>No Offer</th>
<th>O'Bryant Academy School</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>Female</td>
<td>0.502</td>
<td>0.588</td>
</tr>
<tr>
<td>Black</td>
<td>0.430</td>
<td>0.440</td>
</tr>
<tr>
<td>Hispanic</td>
<td>0.182</td>
<td>0.220</td>
</tr>
<tr>
<td>FRPL</td>
<td>0.810</td>
<td>0.771</td>
</tr>
<tr>
<td>LEP</td>
<td>0.116</td>
<td>0.030</td>
</tr>
<tr>
<td>Bilingual</td>
<td>0.386</td>
<td>0.396</td>
</tr>
<tr>
<td>SPED</td>
<td>0.073</td>
<td>0.009</td>
</tr>
<tr>
<td>English 4</td>
<td>0.277</td>
<td>0.981</td>
</tr>
<tr>
<td>Math 4</td>
<td>0.277</td>
<td>0.963</td>
</tr>
</tbody>
</table>

**Panel A: Minority Preferences**

<table>
<thead>
<tr>
<th></th>
<th>No Offer</th>
<th>O'Bryant Academy School</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>Female</td>
<td>0.514</td>
<td>0.572</td>
</tr>
<tr>
<td>Black</td>
<td>0.499</td>
<td>0.360</td>
</tr>
<tr>
<td>Hispanic</td>
<td>0.223</td>
<td>0.191</td>
</tr>
<tr>
<td>FRPL</td>
<td>0.813</td>
<td>0.728</td>
</tr>
<tr>
<td>LEP</td>
<td>0.107</td>
<td>0.058</td>
</tr>
<tr>
<td>Bilingual</td>
<td>0.359</td>
<td>0.423</td>
</tr>
<tr>
<td>SPED</td>
<td>0.073</td>
<td>0.012</td>
</tr>
<tr>
<td>English 4</td>
<td>0.261</td>
<td>1.067</td>
</tr>
<tr>
<td>Math 4</td>
<td>0.217</td>
<td>1.146</td>
</tr>
</tbody>
</table>

**Notes:** This table reports descriptive statistics for the exam school applicants by their counterfactual assignment under minority and socioeconomic preferences in the exam school admissions.
Table 15: Average Reassignment Effects of Introducing Minority or Socioeconomic Preferences into the Boston Exam School Admissions

<table>
<thead>
<tr>
<th></th>
<th>Minority Preferences</th>
<th>Socioeconomic Preferences</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>All Applicants</td>
<td>All Applicants</td>
</tr>
<tr>
<td>Applicant Group</td>
<td>Minority Applicant Group</td>
<td>SES Applicant Group</td>
</tr>
<tr>
<td></td>
<td>All Applicants</td>
<td>SES Applicant Group</td>
</tr>
<tr>
<td></td>
<td>Non-Minority Applicant Group</td>
<td>Tier 1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Tier 2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Tier 3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Tier 4</td>
</tr>
<tr>
<td>Middle</td>
<td>0.001</td>
<td>0.007</td>
</tr>
<tr>
<td>School</td>
<td>(0.005)</td>
<td>(0.006)</td>
</tr>
<tr>
<td>MCAS</td>
<td>4,701</td>
<td>4,701</td>
</tr>
<tr>
<td>High</td>
<td>0.023***</td>
<td>0.015***</td>
</tr>
<tr>
<td>School</td>
<td>(0.004)</td>
<td>(0.004)</td>
</tr>
<tr>
<td>MCAS</td>
<td>3,704</td>
<td>3,704</td>
</tr>
</tbody>
</table>

Panel A: All Applicants

Notes: This table reports the latent factor model-based estimates of the effects of minority and socioeconomic preferences on MCAS scores. The estimates are shown for all applicants and separately for the applicant groups who face different admissions cutoffs after the reforms. The estimates are also shown separately for the affected applicants whose exam school assignment is altered by the reforms. Standard errors based on nonparametric 5-step bootstrap shown in parentheses.

* significant at 10%, ** significant at 5%, *** significant at 1%
Appendix A: Proofs

Proof of Lemma 2  The result follows directly from the Law of Iterated Expectations. ■

Proof of Theorem 1  Notice first that the means of $M_1$ and $M_2$ as well as covariances between $M_1$ and $M_3$ and between $M_2$ and $M_3$ can be written as

\[
\begin{align*}
E[M_1] &= E[\theta] \\
E[M_2] &= \mu_{M_2} + \lambda_{M_2} E[\theta] \\
Cov[M_1, M_3] &= Cov[\theta, W] \\
Cov[M_2, M_3] &= \lambda_{M_2} Cov[\theta, M_3].
\end{align*}
\]

From these equations one can solve for the parameters $\mu_{M_2}$ and $\lambda_{M_2}$ that are given by

\[
\begin{align*}
\lambda_{M_2} &= \frac{Cov[M_2, M_3]}{Cov[\theta, M_3]} \\
\mu_{M_2} &= E[M_2] - \lambda_{M_2} E[M_1].
\end{align*}
\]

Let us now introduce two new random variables, $\tilde{M}_2$ and $\tilde{\nu}_{M_2}$, that are defined as

\[
\begin{align*}
\tilde{M}_2 &= \frac{1}{\lambda_{M_2}} (M_2 - \mu_{M_2}) \\
\tilde{\nu}_{M_2} &= \frac{1}{\lambda_{M}} \nu_{M_2}.
\end{align*}
\]

Thus, $M_1$ and $\tilde{M}_2$ can be written as

\[
\begin{align*}
M_1 &= \theta + \nu_{M_1} \\
\tilde{M}_2 &= \theta + \tilde{\nu}_{M_2}.
\end{align*}
\]

Depending on whether Assumption C.4.a or Assumption C.4.b holds, $M_1$ and $\tilde{M}_2$ satisfy either Assumption A or Assumption B in Evdokimov and White (2012) conditional on $M_3$. In their notation $M = \theta$ and, depending on the assumptions imposed on $\nu_{M_1}$ and $\nu_{M_2}$, either $Y_1 = M_1$, $Y_2 = \tilde{M}_2$, $U_1 = \nu_{M_1}$, and $U_2 = \tilde{\nu}_{M_2}$ or $Y_1 = \tilde{M}_2$, $Y_2 = M_1$, $U_1 = \tilde{\nu}_{M_2}$, and $U_2 = \nu_{M_1}$. The identification of the conditional distributions $f_{\nu_{M_1}|M_3}$, $f_{\tilde{\nu}_{M_2}|M_3}$ and $f_{\theta|M_3}$ follows from either Lemma 1 or Lemma 2 in Evdokimov and White (2012) depending on whether Assumption A or Assumption
B is satisfied. The conditional distribution \( f_{\nu M_2 | M_3} \) and the joint distribution \( f_{\theta, M} \) are given by

\[
\begin{align*}
  f_{\nu M_2 | M_3} (\nu_{M_2} | m_3) &= \frac{1}{\lambda_{M_2}} f_{\tilde{\nu} M_2 | M_3} \left( \frac{1}{\lambda_{M_2}} \nu_{M_2} | m_3 \right) \\
  f_{\theta, M} (\theta, m) &= f_{\nu M_1 | M_3} (m_1 - \theta | m_3) f_{\nu M_2 | M_3} (m_2 - \mu_{M_2} - \lambda_{M_2} \theta | m_3) \\
  &\quad \times f_{\theta | M_3} (\theta | m_3) f_{M_3} (m_3).
\end{align*}
\]

\[\blacksquare\]

**Proof of Theorem 2** Assumptions D.1, D.3, D.4, and D.5 correspond to assumptions 1, 3, 4, and 5 in Hu and Schennach (2008) with \( y = M_3, x = M_1, z = M_2, \) and \( x^* = \theta \) in their notation. Furthermore, as shown in Cunha, Heckman, and Schennach (2010), Assumption D.2 is equivalent to Assumption 2 in Hu and Schennach (2008). The identification of the conditional distributions \( f_{M_1 | \theta}, f_{M_2 | \theta}, \) and \( f_{\theta | M_2} \) then follows from Theorem 1 in Hu and Schennach (2008). The joint distribution \( f_{\theta, M} \) is given by

\[
\begin{align*}
  f_{\theta, M} (\theta, m) &= f_{M_1 | \theta} (m_1 | \theta) f_{M_3 | \theta} (m_3 | \theta) f_{\theta | M_2} (\theta | m_2) f_{M_2} (m_2).
\end{align*}
\]

\[\blacksquare\]

**Proof of Theorem 3** Assumption E.1 allows one to write down the integral equations

\[
\begin{align*}
  E \left[ Y \mid M = m^0, D = 0 \right] &= E \left\{ E \left[ Y (0) \mid \theta \right] \mid M = m^0, D = 0 \right\} \\
  E \left[ Y \mid M = m^1, D = 1 \right] &= E \left\{ E \left[ Y (1) \mid \theta \right] \mid M = m^1, D = 1 \right\}.
\end{align*}
\]

The uniqueness of the solutions to these equations follows directly from Assumption E.3. To see this, suppose that in addition to \( E \left[ Y (0) \mid \theta \right] \) there exists some \( \tilde{E} \left[ Y (0) \mid \theta \right] \) such that

\[
P \left\{ E \left[ Y (0) \mid \theta \right] \neq \tilde{E} \left[ Y (0) \mid \theta \right] \right\} > 0
\]

also satisfying the above equation for all \( m^0 \in M^0 \). Thus,

\[
E \left\{ E \left[ Y (0) \mid \theta \right] - \tilde{E} \left[ Y (0) \mid \theta \right] \mid R = r^0, D = 0 \right\} = 0
\]
for all \( m^0 \in M^0 \), and by Assumption E.3., this implies that \( E [ Y (0) \mid \theta] - \tilde{E} [ Y (0) \mid \theta] = 0 \) for all \( m^0 \in M^0 \), thus leading to a contradiction. An analogous argument can be given for the uniqueness of \( E [ Y (1) \mid \theta] \). Finally, Assumption E.2 quarantees that \( E [ Y (0) \mid \theta] \) and \( E [ Y (1) \mid \theta] \) are determined for all \( \theta \in \Theta \). □

**Proof of Lemma 4** Using Assumptions H.1 and H.2, one can write

\[
E \{ E [ Y (D (1)) - Y (D (0)) \mid \theta] \mid R = r \} \\
= \int E [ Y (D (1)) - Y (D (0)) \mid \theta] f_{\theta \mid R} (\theta \mid r) d\theta \\
= \int E [ Y (1) - Y (0) \mid D (1) > D (0), \theta] P [D (1) > D (0) \mid \theta] f_{\theta \mid R} (\theta \mid r) d\theta \\
= \int E [ Y (1) - Y (0) \mid D (1) > D (0), \theta, R = r] \\
\times P [D (1) > D (0) \mid \theta, R = r] f_{\theta \mid R} (\theta \mid r) d\theta. \tag{5}
\]

Furthermore, using the fact that

\[
P [D (1) > D (0) \mid \theta, R = r] f_{\theta \mid R} (\theta \mid r) \\
= f_{\theta \mid R} (\theta, D (1) > D (0) \mid r) \\
= P [D (1) > D (0) \mid R = r] f_{\theta \mid R, D (0), D (1)} (\theta \mid r, 0, 1),
\]

Equation (5) becomes

\[
E \{ E [ Y (D (1)) - Y (D (0)) \mid \theta] \mid R = r \} \\
= P [D (1) > D (0) \mid R = r] \\
\times \int E [ Y (1) - Y (0) \mid D (1) > D (0), \theta, R = r] f_{\theta \mid R, D (0), D (1)} (\theta \mid r, 0, 1) d\theta \\
= P [D (1) > D (0) \mid R = r] E [ Y (1) - Y (0) \mid D (1) > D (0), R = r]. \tag{6}
\]
Using similar arguments, one can write
\[
E \{ E [D (1) - D (0) \mid \theta] \mid R = r \}
= \int E [D (1) - D (0) \mid \theta] f_{\theta \mid R} (\theta \mid r) d\theta
= \int P [D (1) > D (0) \mid \theta] f_{\theta \mid R} (\theta \mid r) d\theta
= \int P [D (1) > D (0) \mid \theta, R = r] f_{\theta \mid R} (\theta \mid r) d\theta
= P [D (1) > D (0) \mid R = r].
\]
(7)

The result then follows from Equations (6) and (7). ■

Proof of Theorem 4  The proof is analogous to the proof of Theorem 3. ■

Proof of Theorem 5  Notice first that the identification of \( \mu_{M^k_2}, \lambda_{M^k_2}, f_{\nu_{M^k_2} \mid M_3}, f_{\nu_{M^k_2} \mid M_3}, \) and \( f_{\theta \mid M_3} \) follows directly from Assumptions I.1-I.4 using Theorem 1 for all \( k = 1, \ldots, K \). The only remaining issue is the identification of \( f_{\theta \mid M_3} \). Let us start by defining new random variables
\[
\tilde{M}^k_2 = \frac{1}{\lambda_{M^k_2}} \left( M^k_2 - \mu_{M^k_2} \right), \quad k = 1, \ldots, K
\]
\[
\tilde{\nu}_M^k = \frac{1}{\lambda_{M^k_2}} \nu_{M^k_2}, \quad k = 1, \ldots, K.
\]

Thus, \( M_1 = (M^1_1, \ldots, M^K_1) \) and \( \tilde{M}_2 = (\tilde{M}^1_2, \ldots, \tilde{M}^K_2) \) can be written as
\[
M_1 = \theta + \nu_{M_1}
\]
\[
\tilde{M}_2 = \theta + \tilde{\nu}_{M_2}
\]

where \( \theta = (\theta_1, \ldots, \theta_2) \), \( \nu_{M_1} = (\nu^1_{M_1}, \ldots, \nu^K_{M_1}) \), and \( \nu_{M_2} = (\nu^1_{M_2}, \ldots, \nu^K_{M_2}) \).

Now, using Assumption I.5, the conditional characteristic functions of \( M_1 \) and \( \tilde{M}_2 \) given \( M_3 \), \( \phi_{M_1 \mid M_3} \) and \( \phi_{\tilde{M}_2 \mid M_3} \), can be written as
\[
\phi_{M_1 \mid M_3} (m_1 \mid m_3) = \phi_{\theta \mid M_3} (m_1 \mid m_3) \phi_{\nu_{M_1} \mid M_3} (m_1 \mid m_3)
\]
(8)
\[
\phi_{\tilde{M}_2 \mid M_3} (m_2 \mid m_3) = \phi_{\theta \mid M_3} (m_2 \mid m_3) \phi_{\tilde{M}_2 \mid M_3} (m_2 \mid m_3)
\]
(9)
where

\[ \phi_{M_1|M_3}(m_1 | m_3) = \prod_{k=1}^{K} \phi_{M_1|M_3}^k (m_1^k | m_3) \]

\[ \phi_{M_2|M_3}(m_2 | m_3) = \prod_{k=1}^{K} \phi_{M_2|M_3}^k (m_2^k | m_3) \]

where the conditional characteristics functions \( \phi_{M_1|M_3}, \phi_{M_2|M_3} \), and \( \phi_{B}(b) \) are known. Thus, the only unknown in this relationship is the conditional characteristic function of \( \theta \) given \( M_3, \phi_{\theta|M_3} \), that can be obtained from Equations (8) and (9):

\[
\phi_{\theta|M_3}(t | m_3) = \begin{cases} 
\frac{\phi_{M_1|M_3}(t|m_3)}{\phi_{M_2|M_3}(t|m_3)}, & \phi_{\theta|M_3}(t | m_3) \neq 0 \\
\frac{\phi_{M_1|M_3}(t|m_3)}{\phi_{M_2|M_3}(t|m_3)}, & \phi_{\theta|M_3}(t | m_3) \neq 0
\end{cases}
\]

Assumption I.6 guarantees that \( \phi_{\theta|M_3}(t | m_3) \) is identified for all \( t \in \mathbb{R} \).

Finally, the conditional characteristic functions \( \phi_{\theta|M_3}, \phi_{M_1|M_3}, \) and \( \phi_{M_2|M_3} \) uniquely determine the corresponding conditional distributions \( f_{\theta|M_3}, f_{\nu|M_3}, \) and \( f_{\nu,M_3} \). The conditional distribution \( f_{\nu,M_3} \) and the joint distribution \( f_{\theta,M} \) are then given by

\[
f_{\nu,M_3}(\nu,M_3 | m_3) = \prod_{k=1}^{K} \frac{1}{\lambda_{M_3}^k} f_{\nu,M_3}^k (m_1^k | m_3) \frac{1}{\lambda_{M_3}^k} f_{\nu,M_3}^k (m_2^k | m_3)
\]

\[
f_{\theta,M}(\theta, m) = f_{\nu,M_3}(m_1 - \theta | m_3) f_{\nu,M_3}(m_2 - \mu_m - \lambda_m \theta | m_3)
\]

\[
\times f_{\theta|M_3}(\theta | m_3) f_{M_3}(m_3)
\]

where \( \mu_m = (\mu_{M_2}^1, \ldots, \mu_{M_2}^K) \) and \( \lambda_m = diag(\lambda_{M_2}^1, \ldots, \lambda_{M_2}^K) \). \( \blacksquare \)
Appendix B: Deferred Acceptance Algorithm and the Definition of Sharp Samples

The student-proposing Deferred Acceptance (DA) algorithm assigns the exam school offers as follows:

- **Round 1**: Applicants are considered for a seat in their most preferred exam school. Each exam school rejects the lowest-ranking applicants in excess of its capacity. The rest of the applicants are provisionally admitted.

- **Round \( k > 1 \)**: Applicants rejected in Round \( k - 1 \) are considered for a seat in their next most preferred exam school. Each exam school considers these applicants together with the provisionally admitted applicants from Round \( k - 1 \) and rejects the lowest-ranking students in excess of its capacity. The rest of the students are provisionally admitted.

The algorithm terminates once either all applicants are assigned an offer from one of the exam schools or all applicants with no offer are rejected by every exam school in their preference ordering. This produces an admissions cutoff for each exam school that is given by the lowest rank among applicants admitted to the school. By definition none of the applicants with a ranking below this cutoff are admitted to this school. On the other hand, applicants with a rank at or above this cutoff are admitted to either this school or a more preferred exam school depending on their position relative to the admissions cutoffs for these schools.

The DA algorithm-based admissions process implies that only a subset of the applicants to a given exam school that clear the admissions cutoff are admitted to this school. There are three ways in which an applicant can be admitted to exam school \( s \) given the admissions cutoffs:

1. Exam school \( s \) is the applicant’s 1st choice, and she clears the admissions cutoff.

2. The applicant does not clear the admissions cutoff for her 1st choice, exam school \( s \) is her 2nd choice, and she clears the admissions cutoff.

3. The applicant does not clear the admissions cutoff for her 1st or 2nd choice, exam school \( s \) is her 3rd choice, and she clears the admissions cutoff.

However, it is possible to define for each exam school a sharp sample that consist of applicants who are admitted to this school if and only if they clear the admissions cutoff (Abdulkadiroglu, Angrist,
and Pathak, forthcoming). The sharp sample for exam school $s$ is the union of the following three subsets of applicants:

1. Exam school $s$ is the applicant’s 1st choice.

2. The applicant does not clear the admissions cutoff for her 1st choice, and exam school $s$ is her 2nd choice.

3. The applicant does not clear the admissions cutoff for her 1st or 2nd choice, and exam school $s$ is her 3rd choice.

Note that each applicant is included in the sharp sample for at least one exam school (the exam school they listed as their first choice), but an applicant can be included in the sharp sample for more than one exam school. For instance, an applicant who does not clear the admissions cutoff for any of the exam schools is included in the sharp samples for all three schools.
Appendix C: Identification of the Parametric Latent Factor Model

In this appendix I discuss moment conditions that give identification of the parametric measurement and latent outcome models. I ignore the presence of covariates in this discussion as this can be handled straightforwardly by conditioning on them throughout.

Identification of the Measurement Model

Under the parametric measurement model specified in Section 4 the mean and covariances of the measures can be written as

\[
\begin{align*}
E[M^k_1] &= \mu_{\theta_k} \\
E[M^k_2] &= \mu_{M^k_2} + \lambda_{M^k_2} \mu_{\theta_k} \\
E[M^k_3] &= \mu_{M^k_3} + \lambda_{M^k_3} \mu_{\theta_k} \\
Cov[M^k_1, M^k_2] &= \lambda_{M^k_2} \sigma^2_{\theta_k} \\
Cov[M^k_1, M^k_3] &= \lambda_{M^k_3} \sigma^2_{\theta_k} \\
Cov[M^k_2, M^k_3] &= \lambda_{M^k_2} \lambda_{M^k_3} \sigma^2_{\theta_k} \\
Cov[M^E_1, M^M_1] &= \sigma_{\theta_E \theta_M}
\end{align*}
\]

for \(k = E, M\). From these equations one can solve for \(\mu_{\theta_k}, \sigma^2_{\theta_k}, \sigma_{\theta_E \theta_M}, \mu_{M^k_2}\) and \(\lambda_{M^k_2}\) that are given by

\[
\begin{align*}
\mu_{\theta_k} &= E[M^k_1] \\
\lambda_{M^k_2} &= \frac{Cov[M^k_2, M^k_3]}{Cov[M^k_1, M^k_3]} \\
\lambda_{M^k_3} &= \frac{Cov[M^k_3, M^k_2]}{Cov[M^k_1, M^k_2]} \\
\mu_{M^k_2} &= E[M^k_2] - \lambda_{M^k_2} \mu_{\theta_k} \\
\mu_{M^k_3} &= E[M^k_3] - \lambda_{M^k_3} \mu_{\theta_k} \\
\sigma^2_{\theta_k} &= \frac{Cov[M^k_1, M^k_2]}{\lambda_{M^k_2}} \\
\sigma_{\theta_E \theta_M} &= Cov[M^E_1, M^M_1]
\end{align*}
\]

for \(k = E, M\), provided that \(Cov[M^k_1, M^k_2], Cov[M^k_1, M^k_3] \neq 0, k = E, M\).
Furthermore, the conditional means and covariances of the measures can be written as

\[
E \left[ M_1^k \mid M_2^k \right] = E \left[ \theta_k \mid M_2^k \right]
\]

\[
E \left[ M_1^k \mid M_3^k \right] = E \left[ \theta_k \mid M_3^k \right]
\]

\[
Cov \left[ M_1^k, M_3^k \mid M_2^k \right] = \lambda_{M_3^k} Var \left[ \theta_k \mid M_2^k \right]
\]

\[
Cov \left[ M_1^k, M_2^k \mid M_3^k \right] = \lambda_{M_2^k} Var \left[ \theta_k \mid M_3^k \right]
\]

for \( k = E, M \). From these equations one can solve for \( E \left[ \theta_k \mid M_2^k \right] \), \( E \left[ \theta_k \mid M_3^k \right] \), \( Var \left[ \theta_k \mid M_2^k \right] \), and \( Var \left[ \theta_k \mid M_3^k \right] \) that are given by

\[
E \left[ \theta_k \mid M_2^k \right] = E \left[ M_1^k \mid M_2^k \right]
\]

\[
E \left[ \theta_k \mid M_3^k \right] = E \left[ M_1^k \mid M_3^k \right]
\]

\[
Var \left[ \theta_k \mid M_2^k \right] = \frac{Cov \left[ M_1^k, M_3^k \mid M_2^k \right]}{\lambda_{M_2^k}}
\]

\[
Var \left[ \theta_k \mid M_3^k \right] = \frac{Cov \left[ M_1^k, M_2^k \mid M_3^k \right]}{\lambda_{M_2^k}}
\]

for \( k = E, M \).

Finally, the conditional variances of the measures can be written as

\[
Var \left[ M_1^k \mid M_2^k \right] = E \left[ Var \left[ M_1^k \mid \theta_k \right] \mid M_2^k \right] + Var \left[ E \left[ M_1^k \mid \theta_k \right] \mid M_2^k \right]
\]

\[
= E \left[ \exp \left( 2 \left( \gamma_{M_1^k} + \delta_{M_1^k} \right) \theta_k \right) \mid M_2^k \right] + Var \left[ \theta_k \mid M_2^k \right]
\]

\[
= \exp \left( 2 \left( \gamma_{M_1^k} + E \left[ \theta_k \mid M_2^k \right] \delta_{M_1^k} + Var \left[ \theta_k \mid M_2^k \right] \delta_{M_1^k}^2 \right) \right) + Var \left[ \theta_k \mid M_2^k \right]
\]

\[
Var \left[ M_2^k \mid M_3^k \right] = E \left[ Var \left[ M_2^k \mid \theta_k \right] \mid M_3^k \right] + Var \left[ E \left[ M_2^k \mid \theta_k \right] \mid M_3^k \right]
\]

\[
= E \left[ \exp \left( 2 \left( \gamma_{M_2^k} + \delta_{M_2^k} \right) \theta_k \right) \mid M_3^k \right] + \lambda_{M_2^k} Var \left[ \theta_k \mid M_3^k \right]
\]

\[
= \exp \left( 2 \left( \gamma_{M_2^k} + E \left[ \theta_k \mid M_3^k \right] \delta_{M_2^k} + Var \left[ \theta_k \mid M_3^k \right] \delta_{M_2^k}^2 \right) \right) + \lambda_{M_2^k} Var \left[ \theta_k \mid M_2^k \right]
\]

\[
Var \left[ M_3^k \mid M_2^k \right] = E \left[ Var \left[ M_3^k \mid \theta_k \right] \mid M_2^k \right] + Var \left[ E \left[ M_3^k \mid \theta_k \right] \mid M_2^k \right]
\]

\[
= E \left[ \exp \left( 2 \left( \gamma_{M_3^k} + \delta_{M_3^k} \right) \theta_k \right) \mid M_2^k \right] + \lambda_{M_3^k} Var \left[ \theta_k \mid M_2^k \right]
\]

\[
= \exp \left( 2 \left( \gamma_{M_3^k} + E \left[ \theta_k \mid M_2^k \right] \delta_{M_3^k} + Var \left[ \theta_k \mid M_2^k \right] \delta_{M_3^k}^2 \right) \right) + \lambda_{M_3^k} Var \left[ \theta_k \mid M_2^k \right]
\]
which can be further modified as

\[
\begin{align*}
\frac{1}{2} \log \left( \Var \left[ M_1^k \mid M_2^k \right] \right) - \Var \left[ \theta_k \mid M_2^k \right] &= \gamma_{M_1^k} + E \left[ \theta_k \mid M_2^k \right] \delta_{M_1^k} + \Var \left[ \theta_k \mid M_2^k \right] \delta_{M_1^k}^2 \\
\frac{1}{2} \log \left( \Var \left[ M_2^k \mid M_3^k \right] \right) - 2 \Var \left[ \theta_k \mid M_3^k \right] &= \gamma_{M_2^k} + E \left[ \theta_k \mid M_3^k \right] \delta_{M_2^k} + \Var \left[ \theta_k \mid M_3^k \right] \delta_{M_2^k}^2 \\
\frac{1}{2} \log \left( \Var \left[ M_3^k \mid M_2^k \right] \right) - \lambda_{M_2^k}^2 \Var \left[ \theta_k \mid M_3^k \right] &= \gamma_{M_3^k} + E \left[ \theta_k \mid M_3^k \right] \delta_{M_3^k} + \Var \left[ \theta_k \mid M_3^k \right] \delta_{M_3^k}^2
\end{align*}
\]

for \( k = E, M \).

Thus, the parameters \( \gamma_{M_1^k}, \delta_{M_1^k}, \gamma_{M_2^k}, \delta_{M_2^k}, \gamma_{M_3^k}, \text{ and } \delta_{M_3^k} \) can be solved from

\[
\begin{align*}
\left[ \begin{array}{c}
\frac{1}{2} \log \left( \Var \left[ M_1^k \mid M_2^k = m_1 \right] \right) - \Var \left[ \theta_k \mid M_2^k = m_1 \right] \\
\frac{1}{2} \log \left( \Var \left[ M_2^k \mid M_3^k = m_2 \right] \right) - \Var \left[ \theta_k \mid M_3^k = m_2 \right]
\end{array} \right] &= \left[ \begin{array}{c}
\gamma_{M_1^k} + E \left[ \theta_k \mid M_2^k = m_1 \right] \delta_{M_1^k} + \Var \left[ \theta_k \mid M_2^k = m_1 \right] \delta_{M_1^k}^2 \\
\gamma_{M_2^k} + E \left[ \theta_k \mid M_3^k = m_2 \right] \delta_{M_2^k} + \Var \left[ \theta_k \mid M_3^k = m_2 \right] \delta_{M_2^k}^2
\end{array} \right] \\
\left[ \begin{array}{c}
\frac{1}{2} \log \left( \Var \left[ M_2^k \mid M_3^k = m_1 \right] \right) - \lambda_{M_2^k}^2 \Var \left[ \theta_k \mid M_3^k = m_1 \right] \\
\frac{1}{2} \log \left( \Var \left[ M_3^k \mid M_2^k = m_2 \right] \right) - \lambda_{M_3^k}^2 \Var \left[ \theta_k \mid M_2^k = m_2 \right]
\end{array} \right] &= \left[ \begin{array}{c}
\gamma_{M_2^k} + E \left[ \theta_k \mid M_3^k = m_1 \right] \delta_{M_2^k} + \Var \left[ \theta_k \mid M_3^k = m_1 \right] \delta_{M_2^k}^2 \\
\gamma_{M_3^k} + E \left[ \theta_k \mid M_2^k = m_2 \right] \delta_{M_3^k} + \Var \left[ \theta_k \mid M_2^k = m_2 \right] \delta_{M_3^k}^2
\end{array} \right]
\end{align*}
\]

provided that the matrices

\[
\begin{align*}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
E \left[ \theta_k \mid M_2^k = m_1 \right] + 2 \Var \left[ \theta_k \mid M_2^k = m_1 \right] \delta_{M_1^k} & E \left[ \theta_k \mid M_2^k = m_2 \right] + 2 \Var \left[ \theta_k \mid M_2^k = m_2 \right] \delta_{M_1^k} \\
E \left[ \theta_k \mid M_3^k = m_1 \right] + 2 \Var \left[ \theta_k \mid M_3^k = m_1 \right] \delta_{M_2^k} & E \left[ \theta_k \mid M_3^k = m_2 \right] + 2 \Var \left[ \theta_k \mid M_3^k = m_2 \right] \delta_{M_2^k} \\
E \left[ \theta_k \mid M_3^k = m_1 \right] + 2 \Var \left[ \theta_k \mid M_3^k = m_1 \right] \delta_{M_3^k} & E \left[ \theta_k \mid M_3^k = m_2 \right] + 2 \Var \left[ \theta_k \mid M_3^k = m_2 \right] \delta_{M_3^k}
\end{bmatrix}
\end{align*}
\]
are of full rank.

**Identification of the Latent Outcome Models**

Under the parametric latent outcome model specified in Section 4 the conditional expectation of an outcome \( Y \) can be written as

\[
E [ Y \mid M, Z ] = E [ Y (S(Z)) \mid M, Z ]
\]

\[
= \alpha_{Y(S(Z))} + \beta^{E}_{Y(S(Z))} E [ \theta_E \mid M, Z ] + \beta^{M}_{Y(S(Z))} E [ \theta_M \mid M, Z ]
\]

for \( Z = 0, 1, 2, 3 \). Thus, the parameters \( \alpha_{Y(S(Z))}, \beta^{E}_{Y(S(Z))}, \text{ and } \beta^{M}_{Y(S(Z))} \) can be solved from

\[
\begin{bmatrix}
E [ h(Y, S) \mid M = m^Z_1, Z ] \\
E [ h(Y, S) \mid M = m^Z_2, Z ] \\
E [ h(Y, S) \mid M = m^Z_3, Z ]
\end{bmatrix}
= 
\begin{bmatrix}
1 & E [ \theta_E \mid M = m^Z_1, Z ] & E [ \theta_M \mid M = m^Z_1, Z ] \\
1 & E [ \theta_E \mid M = m^Z_2, Z ] & E [ \theta_M \mid M = m^Z_2, Z ] \\
1 & E [ \theta_E \mid M = m^Z_3, Z ] & E [ \theta_M \mid M = m^Z_3, Z ]
\end{bmatrix}
\begin{bmatrix}
\alpha_{Y(S(Z))} \\
\beta^{E}_{Y(S(Z))} \\
\beta^{M}_{Y(S(Z))}
\end{bmatrix}
\]

\[
\Rightarrow 
\begin{bmatrix}
\alpha_{Y(S(Z))} \\
\beta^{E}_{Y(S(Z))} \\
\beta^{M}_{Y(S(Z))}
\end{bmatrix}
= 
\begin{bmatrix}
1 & E [ \theta_E \mid M = m^Z_1, Z ] & E [ \theta_M \mid M = m^Z_1, Z ] \\
1 & E [ \theta_E \mid M = m^Z_2, Z ] & E [ \theta_M \mid M = m^Z_2, Z ] \\
1 & E [ \theta_E \mid M = m^Z_3, Z ] & E [ \theta_M \mid M = m^Z_3, Z ]
\end{bmatrix}^{-1}
\times
\begin{bmatrix}
E [ h(Y, S) \mid M = m^Z_1, Z ] \\
E [ h(Y, S) \mid M = m^Z_2, Z ] \\
E [ h(Y, S) \mid M = m^Z_3, Z ]
\end{bmatrix}
\]

where \( m^Z_1, m^Z_2, m^Z_3 \in \mathcal{M}_Z \), provided that the matrix

\[
\begin{bmatrix}
1 & E [ \theta_E \mid M = m^Z_1, Z ] & E [ \theta_M \mid M = m^Z_1, Z ] \\
1 & E [ \theta_E \mid M = m^Z_2, Z ] & E [ \theta_M \mid M = m^Z_2, Z ] \\
1 & E [ \theta_E \mid M = m^Z_3, Z ] & E [ \theta_M \mid M = m^Z_3, Z ]
\end{bmatrix}
\]

is of full rank. The identification of \( \alpha_{D_s(Z)}, \beta^{E}_{D_s(Z)}, \text{ and } \beta^{M}_{D_s(Z)} \), \( Z = 0, 1, 2, 3 \), is analogous.