Equilibrium Selection in Multi-Player Games with Auction Applications*

Paul Milgrom  Joshua Mollner  
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Abstract

We introduce two new equilibrium refinements for finite normal form games, both of which incorporate the intuitive idea that a costless deviation by one player is more likely than a costly deviation by the same or another player. These refinements lead to new restrictions in games with three or more players and select interesting equilibria in some well-known auction games.

1 Introduction

This paper introduces two new refinements of Nash equilibrium. Both incorporate the restriction that a deviation by one player to a best response is more likely than a deviation by the same or another player to a strategy that is not a best response. One incorporates the additional restriction that the same deviation is also more likely than any combination of deviations by multiple players. Neither restriction is implied by Myerson’s (1978) proper equilibrium in games with three or more players, and these restrictions can be consequential. In two well known auction games – the menu auction and the generalized second-price auction – some of the main conclusions that were originally derived using idiosyncratic equilibrium selection criteria can instead be derived using our new refinements.

To motivate our equilibrium refinements, consider the three-player game in Figure 1. Each player has two strategies. The “Geo” player picks the payoff matrix – East or West. Geo’s payoff is always zero, regardless of what

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Figure 1: a three player game

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<thead>
<tr>
<th>West</th>
<th>Left</th>
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<tr>
<td>Up</td>
<td>1,1</td>
<td>1,1</td>
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<tr>
<td>Down</td>
<td>0,1</td>
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<table>
<thead>
<tr>
<th>East</th>
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<tr>
<td>Up</td>
<td>1,0</td>
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<td>Down</td>
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†Row’s payoffs are listed first. Column’s payoffs are listed second. Geo’s payoffs are constant and suppressed.

anybody does, so we omit its payoff from the matrices. For Row, the strategy Up strictly dominates Down: the former always pays one and the latter always pays zero. Column’s decision is the one of interest: its best choice depends on what it believes the other two players will do. The strategy profiles (Up, Left, West) and (Up, Right, West) are both proper equilibria \(^1\) of this game. The first of these equilibria, however, is harder to rationalize if Column believes that the other players are rational, because its choice of Left is optimal only if Row is more likely to deviate to Down, which would be an obvious and costly mistake, than Geo is to deviate to East, which would be no mistake at all.

Our proposed equilibrium refinements both eliminate (Up, Left, West) by insisting that Column must regard Geo’s costless deviation as being more likely than Row’s costly one. One, which we call “extended proper equilibrium,” achieves this by refining proper equilibrium \(^1\). Although extended proper equilibrium is defined to be independent of the utility scales of the payoff matrix, it is informally motivated by the idea that there must exist some scaling of the players’ payoffs such that costly mistakes by any one player are much less likely than less costly mistakes by the same player or by another player. It is the italicized condition that represents the extra restriction in this definition compared with proper equilibrium.

Our other proposed solution, “test-set equilibrium,” incorporates a second idea not implied by any of the tremble-based refinements, namely, that a deviation by just one player to a best response is more likely than any collection of deviations by two or more players. Accordingly, we define each player’s test set to consist of the equilibrium strategy profiles of the other players and the related profiles in which exactly one of them deviates to a different best response. A test-set equilibrium is a Nash equilibrium pro-

\(^1\)Formal proofs of all claims about the game in Figure 1 are in Appendix A.1.
file for which each player’s strategy is weakly undominated in the decision problem in which the possible opposing strategy profiles are just those in the test set.

Both new concepts lead to interesting refinements in two celebrated auction games. Because we wish to highlight the relative ease of applying the test-set refinement, this introduction will focus on applications of that concept.

We begin with the generalized second price auction. This auction and its variants have been used by Google, Yahoo!, and others to sell advertising on search pages. Higher positions on a search page tend to be clicked more frequently by searchers, and so are more valuable to advertisers. Payments by bidders depend on the next-highest bid.

Edelman et al. (2007) (hereafter EOS) have modeled the generalized second-price auction mechanism and applied a particular equilibrium selection. Varian (2007) introduced a very similar model and equilibrium selection. Adopting the EOS model, let us denote the numbers of clicks associated with the \( I \) positions by \( \kappa_1 > \cdots > \kappa_I > 0 \). There are \( N \) bidders, and bidder \( n \) values clicks at \( v_n \) each. A bid, \( b_n \), specifies a maximum price per-click. The highest bid wins the first position; the second highest wins the second position; and so on. The winner of position \( n \) pays a per-click price of \( b_{(n+1)} \), so its total payment is \( \kappa_n b_{(n+1)} \). With \( I = 1 \), this reduces to a second-price auction, and the search ad auction design has become known as the “generalized second price auction.”

This auction game has many Nash equilibria with a variety of outcomes, including ones involving inefficient decisions, or low total payments, or both. Abstracting from ties, let us number the bidders so that the equilibrium bids satisfy \( b_1 > \cdots > b_N \). It is easy to see that for each \( n \leq N \), bidder \( n \) would not strictly prefer to have the price and allocation of any bidder \( m > n \) in a lower position on the page, for the bidder could obtain that outcome by reducing its bid to \( b_n - \epsilon \) for some \( \epsilon > 0 \). EOS propose further limiting the equilibria to ones in which bidder \( n \) also does not envy the price and allocation of bidder \( n - 1 \), who has a higher position on the page. They label an equilibrium with this property as “locally envy-free.” Formally, an equilibrium is locally envy-free if for all \( n \in \{2, \ldots, N\} \), \( \kappa_n (v_n - b_{n+1}) \geq \kappa_{n-1} (v_n - b_n) \).

For generic bidder values, the generalized second price auction has the property that while a single deviation to a different best response can affect the prices paid, it nevertheless leaves the allocation of positions unchanged. It is this property that allows easy application of the test-set criterion.

For these generic values, test-set equilibrium leads to the requirement
that any pure equilibrium is locally envy-free. For suppose that \( b \) is a pure test-set equilibrium of the generalized second price auction game. Consider bidder \( n \)'s reasoning when it thinks about raising its bid slightly to \( b_n + \epsilon \) for some \( \epsilon \in (0, b_{n-1} - b_n) \). “I should test the robustness of my planned bid against some test set, and I will use the set consisting of strategy profiles in which only one other player deviates and that player is still playing a best response. For all strategy profiles in the test set, if I continue to play my equilibrium strategy, I will still win position \( n \). If I were to raise my bid slightly to \( b_n + \epsilon \) and that were to change the outcome (as can happen on the test set), that must mean that bidder \( n - 1 \) has bid less than I expected and that I will win position \( n - 1 \) at a per-click price of about \( b_n \). That change is profitable for me if \( \kappa_n(v_n - b_{n+1}) < \kappa_{n-1}(v_n - b_n - \epsilon) \), and if that inequality holds, then the new bid weakly dominates the equilibrium bid on the test set.” Since test-set equilibrium allows no such dominance, it follows that for all players \( n \) and all \( \epsilon > 0 \) sufficiently small, \( \kappa_n(v_n - b_{n+1}) \geq \kappa_{n-1}(v_n - b_n - \epsilon) \).

Hence, every pure test-set equilibrium of the generalized second price auction is locally envy-free. A similar conclusion can be reached if extended proper equilibrium is applied to a discretized version of the game.

Our second application is the menu auction game of Bernheim and Whinston (1986), which has been applied to study both competition among lobbyists and bidding in combinatorial auctions. In the menu auction game, an auctioneer chooses a decision \( x \) from some finite set \( X \) based on offers it receives. For the auction application, the decision is a resource allocation and the offers are bids. For the political application, the decision may concern legislation or a public good and the offers may be bribes or less direct forms of compensation, the payment of which will be conditional on the decision.

We denote the utility that bidder/lobbyist \( n \) receives from decisions by \( v_n : X \to \mathbb{R} \). Similarly, we denote the utility of the auctioneer by \( v_0 : X \to \mathbb{R} \). Each bidder/lobbyist \( n \) makes a vector of bids \( b_n : X \to \mathbb{R}_+ \) and the auctioneer then chooses \( x^* \in \arg \max_{x \in X} \left[ v_0(x) + \sum_{n \in N} b_n(x) \right] \). Bidder \( n \)'s payoff is \( v_n(x^*) - b_n(x^*) \).

The menu auction game can have very many Nash equilibria, which may involve inefficient decisions, or low total payments, or both. To refine away what they regarded as the implausible Nash equilibria, Bernheim and Whinston consider two approaches. The first restricts attention to equilibria in which bidders adopt what they call “truthful” strategies, such that bidders become indifferent among decisions for which they have made positive bids. Their second approach introduces a new solution concept – coalition-proof equilibrium, which selects only Nash equilibria that are immune to certain coalitional deviations. In any Nash equilibrium of the menu auction satisfy-
ing either of these conditions, the resulting payoff vector is in the core and, among core payoff vectors, it is Pareto optimal for the bidders.

To apply test-set equilibrium to this game with a continuum of bids, we introduce a convenient tie-breaking assumption, namely, that between any two outcomes with the same payoff for a bidder or for the auctioneer, the bidder or auctioneer prefers the outcome with the higher total payoff. This assumption ensures that the auctioneer’s choice is unique and that no bidder is ever indifferent among bids that lead to different outcomes. This implies that, in parallel to generic discrete versions of the auction, the continuous auction is “non-bossy,” meaning that if a bidder changes its bid in a way that affects the outcome, then it is never indifferent about that change. The “non-bossiness” property allows easy application of the test-set criterion to this mechanism.

Test-set equilibrium leads to the requirement that any pure equilibrium leads to payoff vectors that are in the core, although not necessarily bidder-optimal among such outcomes as in [Bernheim and Whinston (1986)]. Suppose that \( b \) is a pure test-set equilibrium of the menu auction game, that the equilibrium decision is \( x^* \), and that each player \( n \)'s payoff is denoted by \( \pi_n = v_n(x^*) - b_n(x^*) \). Let \( x \neq x^* \) and consider bidder \( n \)'s reasoning as it thinks about whether to raise its bid for outcome \( x \) from \( b_n(x) \) to \( b_n(x) + \epsilon \), for some \( \epsilon > 0 \), leaving all its other bids unchanged. “I should test the robustness of my planned bid against some test set, and I will use the set consisting of strategy profiles in which only one other player deviates and that player is still playing a best response. For all strategy profiles in the test set, if I continue to play my equilibrium strategy, the decision will still be \( x \) and my payoff will still be \( \pi_n \). If I were to raise my bid for \( x \) and that were to change the outcome (as can happen on the test set), that must mean that the new outcome would be \( x \) and my new payoff would be \( v_n(x) - b_n(x) - \epsilon \). That change is profitable for me if and only if \( v_n(x) - b_n(x) - \epsilon > \pi_n \), and if that inequality holds, then the new bid weakly dominates the equilibrium bid on the test set.” Since test-set equilibrium allows no such dominance, it follows that for all bidders \( n \) and decisions \( x \), there is an implied lower bound for all losing bids: \( b_n(x) \geq v_n(x) - \pi_n \). Using these bounds, we show that the equilibrium payoffs are in the core.\(^2\)

If we instead apply extended proper equilibrium to a discretized version of the game, the analysis is subtler and the conclusion is weaker: starting

\(^2\)A nearly identical argument, in which extended proper equilibrium implies the same lower bounds for bids and leads to a bidder-optimal core allocation, works as well for the “core-selecting auction” models of [Day and Milgrom (2013)].
from an extended proper equilibrium bid profile, no pair of bidders can change their bids in a way that would lead to significantly higher total value for the coalition consisting of them and the auctioneer.

In the body of the paper, we introduce the two concepts. We begin by defining extended proper equilibrium and showing that, for finite games, such equilibria always exist and are always proper equilibria. For games with two players, the set of extended proper equilibria coincides with the set of proper equilibria. We also characterize extended proper equilibrium in terms of hierarchies of beliefs.

Next, we define test-set equilibrium. Compared to perfect and proper equilibrium, the definition of test-set equilibrium excludes the requirements that all strategies could potentially be played by mistake and that some (more costly) mistakes are much less likely than others. Given this construction, it is perhaps unsurprising that test-set equilibrium does not imply perfect or proper equilibrium. Yet, unlike perfect and proper equilibrium, test-set equilibrium does require that simultaneous mistakes by multiple players must be less likely than any single, costless mistake. As the preceding analyses demonstrate, this restriction can be particularly useful for equilibrium selection in games with three or more players. However, as we show by example, there are games for which no test-set equilibrium exists.

Like test-set equilibrium, extended proper equilibrium also satisfies a kind of “no-domination” condition, which we will describe in detail in the relevant section below. In test-set equilibrium, there are just two important categories of deviations: those that are relatively likely (the test set) and those that are not. In contrast, extended proper equilibrium involves additional categories, particularly categories involving multiple deviations. The likelihood of such deviations in extended proper equilibrium is ambiguous: they may be as likely as some single deviations to best responses or as unlikely as some costly mistakes. This ambiguity implies that, in applications, extended proper equilibrium can be harder to use and sometimes less powerful than test-set equilibrium.

Finally, we return to the applications, using discrete bid spaces to apply the extended proper equilibrium concept and obtaining “approximation” results for that case. For the generalized second-price auction game, we find that every extended proper equilibrium and every test-set equilibrium is (approximately) locally envy-free. For the menu auction game, extended proper equilibrium allocation has a certain “pairwise (approximate) efficiency” property. As observed above, a test-set equilibrium allocation of the same game has a stronger property: it is an (approximate) core allocation.
2 Extended Proper Equilibrium

Extended proper equilibrium is an equilibrium refinement for finite normal form games. Its formal definition and proof of existence are both similar to those for proper equilibrium in [Myerson 1978]. Informally, both concepts restrict players’ “big” (more costly) mistakes to be less likely than their own “small” (less costly) mistakes, but unlike the original proper equilibrium concept, extended proper equilibrium also restricts the relative likelihood of mistakes by different players. It requires that there exists some scaling of the payoff matrix for which each player’s bigger mistakes are less likely than other players’ smaller mistakes. Adapting Myerson’s method, we show that for any finite game and any scaling, such an equilibrium exists. We show that the set of extended proper equilibria is a subset of the set of proper equilibria, and prove that the two sets coincide in games with precisely two players. We then characterize the set of extended proper equilibria in terms of hierarchies of beliefs.

2.1 Definition

A finite $N$-person game in normal form is a $2N$-tuple $G = (S_1, \ldots, S_N; u_1, \ldots, u_N)$, where for each agent $n \in \{1, \ldots, N\}$, $S_n$ is a nonempty, finite set of pure strategies, and $u_n : \times_{n=1}^N S_n \rightarrow \mathbb{R}$ is a utility function. A mixed strategy profile is denoted $\sigma = (\sigma_1, \ldots, \sigma_N) \in \times_{n=1}^N \Delta(S_n)$. We embed $S_n$ in $\Delta(S_n)$ and extend the utility functions $u_n$ to the domain $\times_{n=1}^N \Delta(S_n)$ in the usual way.

Given a mixed strategy profile $\sigma$, we define $L_n(s_n|\sigma)$ as the expected loss for player $n$ from playing $s_n$ instead of a best response when others play $\sigma_{-n}$:

$$L_n(s_n|\sigma) = \max_{\hat{s}_n \in S_n} u_n(\hat{s}_n, \sigma_{-n}) - u_n(s_n, \sigma_{-n}).$$

This quantity is zero for strategies that are best responses to $\sigma_{-n}$ and positive otherwise. Given a vector $\alpha \in \mathbb{R}^N_{++}$, we define an $(\alpha, \epsilon)$-extended proper equilibrium to be a combination of totally mixed strategies in which the probability of strategies with higher scaled losses is at most $\epsilon$ times the probability of strategies with lower scaled losses.

**Definition 1.** Let $\alpha \in \mathbb{R}^N_{++}$ and $\epsilon > 0$. An $(\alpha, \epsilon)$-extended proper equilibrium is a profile of totally mixed strategies $(\sigma_1, \ldots, \sigma_N) \in \times_{n=1}^N \Delta^0(S_n)$ such that

$$\alpha_n L_n(s_n|\sigma) > \alpha_m L_m(s_m|\sigma) \Rightarrow \sigma_n(s_n) \leq \epsilon \cdot \sigma_m(s_m).$$
for all \( n, m \), all \( s_n \in S_n \), and all \( s_m \in S_m \).

We also define an extended proper equilibrium to be a strategy profile that, for some scaling vector \( \alpha \), is a limit of \((\alpha, \epsilon)\)-extended proper equilibria, as \( \epsilon \to 0 \).

**Definition 2.** A strategy profile \( \sigma = (\sigma_1, \ldots, \sigma_N) \in \times_{n=1}^N \Delta(S_n) \) is an extended proper equilibrium if and only if there exist \( \alpha \in \mathbb{R}_+^N \) and sequences \( \{\epsilon_t\}_{t=0}^\infty \) and \( \{\sigma^t\}_{t=0}^\infty \) such that:

(i) each \( \epsilon_t > 0 \) and \( \lim_{t \to \infty} \epsilon_t = 0 \),

(ii) each \( \sigma^t \) is an \((\alpha, \epsilon_t)\)-extended proper equilibrium, and

(iii) \( \lim_{t \to \infty} \sigma^t = \sigma \).

The set of extended proper equilibria is unaffected by affine transformations of the utility functions of the players. The following theorem states that extended proper equilibria exist in every finite normal form game. Its proof and all other proofs are deferred to Appendix A.

**Theorem 1.** Every finite normal form game has at least one extended proper equilibrium.

### 2.2 Lexicographic Characterization

In this section, we employ the framework of Blume et al. (1991) to characterize extended proper equilibrium in terms of hierarchies of the players’ beliefs. Toward that end, let \( \rho = (p^1, \ldots, p^K) \) be a sequence of probability measures on \( \times_{n=1}^N S_n \). Blume et al. (1991) refer to such a sequence as a lexicographic probability system (LPS).

An LPS \( \rho = (p^1, \ldots, p^K) \) on \( \times_{n=1}^N S_n \) gives rise to a partial order on the elements of \( \bigcup_{J \subseteq \{1, \ldots, N\}} \times_{j \in J} S_j \), which is defined as follows. Given some \( J \subseteq \{1, \ldots, N\} \), let \( \rho_J = (p^1_J, \ldots, p^K_J) \) be the marginal on \( \times_{j \in J} S_j \) of \( \rho \).\(^3\) If \( s_J \in \times_{j \in J} S_j \) and \( s_I \in \times_{i \in I} S_i \), then we say that \( s_J >_\rho s_I \), read as “\( s_J \) is infinitely more likely than \( s_I \) according to the LPS \( \rho \),” if \( \min\{k : p^k_J(s_J) > p^k_I(s_I)\} \)

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\(^3\)This notation differs slightly from that used by Blume et al. (1991). First, while they designate players with superscripts and levels of an LPS with subscripts, we do the reverse. In addition, what we denote by \( \rho_{-n} \), they would instead denote by \( \rho^n \). Our purposes differ slightly from theirs, and our notation is a bit more natural for what is presented here.
\(0 \leq \min \{k : p^k_J(s_J) > 0\}\). We write \(s_J \geq \rho \ s_J\) to mean it is not the case that \(s_J > \rho \ s_J\).

We define the best response set of player \(n\) to \(\rho_{-n}\) as follows, where we use the symbol \(\geq \) to represent the lexicographic ordering.\(^4\)

\[
BR_n(\rho_{-n}) = \left\{ s_n \in S_n : \left[ \sum_{s_n \in S_n} p^k_{-n}(s_{-n}) u_n(s_n, s_{-n}) \right]_{k=1}^{K} \geq \sum_{s_n \in S_n} p^k_n(s_{-n}) u_n(s'_n, s_{-n}) \, \forall s'_n \in S_n \right\}
\]

**Definition 3.** A pair \((\rho, \sigma)\) is a lexicographic Nash equilibrium if

(i) for all \(n \in \{1, \ldots, N\}\), \(p^1_n(s_n) > 0\) implies \(s_n \in BR_n(\rho_{-n})\), and

(ii) \(p^1 = \sigma\).

We next introduce two properties that an LPS may possess. The first, “respects within-person preferences,” is equivalent to what Blume et al. (1991) define as “respects preferences.” We use different terminology in order to accentuate the distinction between this property and “respects within-and-across-person preferences,” also defined below.

**Definition 4.** An LPS \(\rho = (p^1, \ldots, p^K)\) on \(\times_{n=1}^N S_n\) respects within-person preferences if for all \(n \in \{1, \ldots, N\}\) and all \(s_n, s'_n \in S_n\) with \(s_n \geq \rho \ s'_n\), it is the case that

\[
\left[ \sum_{s_n \in S_n} p^k_{-n}(s_{-n}) u_n(s_n, s_{-n}) \right]_{k=1}^{K} \geq \sum_{s_n \in S_n} p^k_{-n}(s_{-n}) u_n(s'_n, s_{-n}) \, \forall \ s'_n \in S_n
\]

**Definition 5.** An LPS \(\rho = (p^1, \ldots, p^K)\) on \(\times_{n=1}^N S_n\) respects within-and-across-person preferences if there exists some \(\alpha \in \mathbb{R}^N_+\) such that for all \(n, m \in \{1, \ldots, N\}\), \(s_n^* \in BR_n(\rho_{-n})\), \(s'_m \in BR_m(\rho_{-m})\), and all \(s_n \in S_n\),

\(^4\)Formally, for \(a, b \in \mathbb{R}^K\), \(a \geq_L b\) if and only if whenever \(b^k > a^k\), there exists an \(l < k\) such that \(a^l > b^l\).
$s_m \in S_m$ with $s_n \geq_{\rho} s_m$, it is the case that

\[
\alpha_n \sum_{s-n \in S-n} p_n^k(s-n)[u_n(s_n^*, s-n) - u_n(s_n, s-n)] \geq L \left( \alpha_m \sum_{s-m \in S-m} p_m^k(s^*_m, s-m) - u_m(s_m, s-m) \right)_{K=1}^K.
\]

It is easy to see that Definition 5 is a strengthening of Definition 4, which is a strengthening of Condition (i) of Definition 3. Finally, we state two additional definitions, which are generalizations for LPSs of what it means for a probability measure to have full support or to be a product measure.

**Definition 6.** An LPS $\rho = (p^1, \ldots, p^K)$ on $\times_{n=1}^N S_n$ has full support if for each $s \in \times_{n=1}^N S_n$, $p^k(s) > 0$ for some $k \in \{1, \ldots, K\}$.

**Definition 7.** An LPS $\rho$ on $\times_{n=1}^N S_n$ satisfies strong independence if there is an equivalent $\mathbb{F}$-valued probability measure that is a product measure, where $\mathbb{F}$ is some non-Archimedean ordered field that is a proper extension of $\mathbb{R}$.

For comparison, we incorporate as Propositions 2-4 below the characterizations of Blume et al. (1991). To these, we add the characterization of extended proper equilibrium. These characterizations are useful both because they make some proofs easier (LPSs are easier to work with than sequences of trembles) and because, compared to sequences of trembles, LPSs correspond more closely to intuitive statements about some actions being “infinitely more likely” than others.

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5 An ordered field is non-Archimedean if it contains an element $\epsilon > 0$ such that $\epsilon < \frac{1}{n}$ for all $n \in \mathbb{N}$. Such an element is called an infinitesimal.

6 An LPS $\rho = (p^1, \ldots, p^K)$ on $\times_{n=1}^N S_n$ and an $\mathbb{F}$-valued probability measure $\hat{p}$ on $\times_{n=1}^N S_n$, are said to be equivalent if there exists a vector of positive infinitesimals $\epsilon = (\epsilon_1, \ldots, \epsilon_{K-1}) \in \mathbb{F}^{K-1}$ such that $\hat{p}(s) = \epsilon \Box \rho(s)$ for all $s \in \times_{n=1}^N S_n$, where $\epsilon \Box \rho$ is used to denote the probability measure

\[
\epsilon \Box \rho = (1 - \epsilon_1)p^1 + \epsilon_1 [(1 - \epsilon_2)p^2 + \epsilon_2 [(1 - \epsilon_3)p^3 + \epsilon_3 \cdots + \epsilon_{K-2} [(1 - \epsilon_{K-1})p^{K-1} + \epsilon_{K-1}p^K] \cdots]].
\]
Proposition 2. The strategy profile $\sigma \in \times_{n=1}^{N} \Delta(S_n)$ is a Nash equilibrium if and only if there exists some LPS $\rho = (p^1, \ldots, p^K)$ on $\times_{n=1}^{N} S_n$ that satisfies strong independence for which $(\rho, \sigma)$ is a lexicographic Nash equilibrium.

Proposition 3. The strategy profile $\sigma \in \times_{n=1}^{N} \Delta(S_n)$ is a perfect equilibrium if and only if there exists some LPS $\rho = (p^1, \ldots, p^K)$ on $\times_{n=1}^{N} S_n$ that satisfies strong independence and has full support for which $(\rho, \sigma)$ is a lexicographic Nash equilibrium.

Proposition 4. The strategy profile $\sigma \in \times_{n=1}^{N} \Delta(S_n)$ is a proper equilibrium if and only if there exists some LPS $\rho = (p^1, \ldots, p^K)$ on $\times_{n=1}^{N} S_n$ that satisfies strong independence, has full support, and respects within-person preferences for which $(\rho, \sigma)$ is a lexicographic Nash equilibrium.

Proposition 5. The strategy profile $\sigma \in \times_{n=1}^{N} \Delta(S_n)$ is an extended proper equilibrium if and only if there exists some LPS $\rho = (p^1, \ldots, p^K)$ on $\times_{n=1}^{N} S_n$ that satisfies strong independence, has full support, and respects within-and-across-person preferences for which $(\rho, \sigma)$ is a lexicographic Nash equilibrium.

2.3 Relationship to Proper Equilibrium

It is apparent from Propositions 4 and 5 that every extended proper equilibrium is a proper equilibrium. The two sets can be different in games with three or more players, but not for two-player games.

Theorem 6. In two player games, the sets of proper equilibria and extended proper equilibria coincide.

2.4 Undominatedness Property

Next, we establish that extended proper equilibrium strategies must be undominated in a special sense against several specific sets of strategy profiles, which represent plausible individual and joint deviations by others. This attention to possible joint deviations is one factor that distinguishes this solution concept from one we will introduce later: test-set equilibrium.

Proposition 7. If $\sigma$ is an extended proper equilibrium, then there is no pair of players $n$ and $m$ and strategies $\hat{\sigma}_n \in \Delta(S_n)$ and $\hat{s}_m \in BR_m(\sigma_{-m})$ such that (i) $u_n(\hat{\sigma}_n, s_{-n}) > u_n(\sigma_n, s_{-n})$, for some $s_{-n} \in A$, and (ii) $u_n(\hat{\sigma}_n, s_{-n}) > u_n(\sigma_n, s_{-n})$. 

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Theorem: Strategies played with positive probability in any extended proper equilibrium must be undominated in a certain sense.

To intuitively convey the idea behind the statement and the proof, observe that given any strategy profile $\sigma$, any player $n$, any player $m$, and any strategy $\hat{s}_m \in BR_m(\sigma-m)$, the elements of $\times_{i \neq n} S_i$ can be partitioned into four sets: $A$ and $B$ as defined in the statement of Proposition 7, as well as $C$ and $D$, defined below.

$$A = \left\{ (s_m, s_{-nm}) \mid s_m = \hat{s}_m \text{ and } \sigma_i(s_i) > 0 \quad \forall i \notin \{n,m\} \right\}$$

$$B = \left\{ (s_m, s_{-nm}) \mid s_m \in BR_m(\sigma-m) \setminus \hat{s}_m \text{ and } s_i \in BR_i(\sigma-i) \quad \forall i \notin \{n,m\} \right\}$$

$$C = \left\{ (s_m, s_{-nm}) \mid s_m \notin BR_m(\sigma-m) \text{ or } \exists i \notin \{n,m\} \text{ s.t. } s_i \notin BR_i(\sigma-i) \right\}$$

$$D = \left\{ (s_m, s_{-nm}) \mid s_m = \hat{s}_m \text{ and } s_i \in BR_i(\sigma-i) \quad \forall i \notin \{n,m\} \text{ and } \exists i \notin \{n,m\} \text{ s.t. } \sigma_i(s_i) = 0 \right\}$$

The set $A$ contains the strategy profiles in which $m$ plays some best response $\hat{s}_m$ to $\sigma$ and all other opponents of $n$ play strategies that are played with positive probability under $\sigma$. The set $B$ contains the strategy profiles in which $m$ plays some other best response and all other opponents of $n$ also play strategies that are best responses to $\sigma$. The set $C$ contains the strategy profiles in which one opponent of $n$ plays a strategy that is not a best response to $\sigma$. The set $D$ contains the strategy profiles in which $m$ plays $\hat{s}_m$ and some other opponent of $n$ plays a strategy that is not played with positive probability under $\sigma$.

The theorem says that if there is some strategy $\hat{s}_n$ that weakly dominates $\sigma_n$ against $A$ and performs weakly better than $\sigma_n$ against all elements of $B$, then $\sigma$ cannot be extended proper. To show this, we argue that if $\sigma$ is an extended proper equilibrium, then the elements of $A$ must be “infinitely more likely” than the elements of $C$ and $D$ (although the elements of $B$ cannot be ranked against the elements of the other sets.). Consequently, if $\sigma_n$ were dominated in this way, then player $n$ would wish to deviate to $\hat{s}_n$.

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Both below and throughout the paper we use $s_{-nm}$ to indicate a strategy profile for the players $\{1, \ldots, N\} \setminus \{n,m\}$.
3 Test-Set Equilibrium

The second equilibrium refinement we propose is test-set equilibrium. Informally, a Nash equilibrium is a test-set equilibrium if no player uses a strategy that is weakly dominated against his test set, which consists of the equilibrium strategy profiles of the other players and the related profiles in which exactly one player deviates to a different best response. For games with three or more players, test-set equilibrium neither implies nor is implied by any trembles-based refinement. We also show by example that there are games for which no test-set equilibrium exists.

3.1 Definition

Unlike extended proper equilibrium, test-set equilibrium is well-defined for both finite and infinite games.

Definition 8. A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_N)$ is a test-set equilibrium if and only if it is a Nash equilibrium and for all $n$, $\sigma_n$ is not weakly dominated by any $\tilde{\sigma}_n \in \Delta(S_n)$ against

$$T_n(\sigma) = \bigcup_{m \neq n} \{(\tilde{s}_m, \sigma_{-nm}) : \tilde{s}_m \in BR_m(\sigma_{-m})\}.$$ 

We refer to the set $T_n(\sigma)$ in the above definition as the test set of player $n$. A test-set equilibrium is a Nash equilibrium in which all players use strategies that are robust to a certain type of “testing.” Intuitively, each player “tests” his strategy against the set of opponents’ strategy profiles in which at most one opponent deviates, and that deviation is to a best response.

The following result states that the undominatedness property possessed by extended proper equilibrium, as stated in Proposition 7, is in fact implied by the test-set condition.

Proposition 8. If $\sigma$ is a test-set equilibrium, then there is no pair of players $n$ and $m$ and strategies $\tilde{\sigma}_n \in \Delta(S_n)$ and $\tilde{s}_m \in BR_m(\sigma_{-m})$ such that (i) $u_n(\tilde{\sigma}_n, s_{-n}) > u_n(\sigma_n, s_{-n})$, for some $s_{-n} \in A$, and (ii) $u_n(\tilde{\sigma}_n, s_{-n}) \geq$
un(σn, s−n), for all s−n ∈ A ∪ B, where A and B are defined as follows.\footnote{Both below and throughout the paper we use s−nm to indicate a strategy profile for the players \{1, \ldots, N\} \setminus \{n, m\}.}

\begin{align*}
A &= \left\{ (s_m, s_{-nm}) \mid s_m = \hat{s}_m \text{ and } \sigma_i(s_i) > 0 \ \forall i \notin \{n, m\} \right\} \\
B &= \left\{ (s_m, s_{-nm}) \mid s_m \in BR_m(\sigma_{-m}) \setminus \hat{s}_m \text{ and } s_i \in BR_i(\sigma_{-i}) \ \forall i \notin \{n, m\} \right\}
\end{align*}

### 3.2 Existence

Test-set equilibria are not guaranteed to exist in general finite normal-form games. Figure 2 presents an example of a game with no test-set equilibrium.\footnote{We thank Michael Ostrovsky and Markus Baldauf for helpful suggestions that led to the construction of this example.} (Up, Left, West) is the unique Nash equilibrium of this game. However, it is not a test-set equilibrium because East weakly dominates West against Geo’s test set: \{(Up, Left), (Up, Center), (Up, Right), (Up, Down)\}. This game therefore contains no test-set equilibrium, although the unique Nash equilibrium is an extended proper equilibrium. To check that, let \(\alpha = (1, 1, 1)\) and note that for all \(\epsilon > 0\) sufficiently small, \(\sigma^\epsilon\) as defined below is an \((\alpha, \sqrt{\epsilon})\)-extended proper equilibrium.

\begin{align*}
\sigma^\epsilon_{row} &= \left( \frac{1}{1 + \epsilon^3}, \frac{\epsilon^3}{1 + \epsilon^3} \right) \\
\sigma^\epsilon_{col} &= \left( \frac{1}{1 + \epsilon^2 + \epsilon^6}, \frac{\epsilon^6}{1 + \epsilon^2 + \epsilon^6}, \frac{\epsilon^2}{1 + \epsilon^2 + \epsilon^6} \right) \\
\sigma^\epsilon_{geo} &= \left( \frac{1}{1 + \epsilon}, \frac{\epsilon}{1 + \epsilon} \right)
\end{align*}

With these trembles, the joint deviation from equilibrium to (Down, Right) is more likely than the single deviation to Center. Test-set equilibrium, on the other hand, would require the reverse. Moreover, this example illustrates that it may not always be possible to find trembles and an associated equilibrium in which deviations are ordered so that all single deviations to best responses are more likely than any joint deviation to best responses.

While test-set equilibria are not guaranteed to exist in all finite normal form games, there are certain classes of games in which their existence can
be guaranteed. The following result states three conditions, each of which is sufficient to guarantee the existence of a test-set equilibrium.

**Proposition 9.** A finite normal form game has at least one test-set equilibrium if it also satisfies at least one of the following conditions:

(i) it is a two-player game,

(ii) it is a potential game,\(^{10}\) or

(iii) it is a three-player game in which each player has two pure strategies.

### 3.3 Relationship to Other Equilibrium Concepts

The previous example shows that not every extended proper equilibrium is a test-set equilibrium. In the game in Figure 3, it is easily checked that \((\text{Up}, \text{Left})\) is a test-set equilibrium of this game but is not a trembling-hand perfect equilibrium.\(^{11}\) Thus, in games with three or more players, the strongest tremble-based refinement does not imply test-set equilibrium, and test-set equilibrium does not imply the weakest tremble-based refinement: the concepts are logically independent for such games.

For two-player games, the situation is more complex: proper equilibrium implies test-set equilibrium, although trembling hand perfect equilibrium does not.\(^{12}\)

\(^{10}\)Monderer and Shapley (1996).

\(^{11}\)It is not a trembling-hand perfect equilibrium because Row uses Up, a weakly dominated strategy. However, since Row’s test-set contains only Left, Up is not weakly dominated against his test-set.

\(^{12}\)The game depicted in Figure 2 of Myerson (1978) can be used as a counterexample for illustrating the latter fact. As he shows, \((x_2, y_2)\) is a perfect equilibrium of this game. However, it is not a test-set equilibrium.
Figure 3: a two player game†

<table>
<thead>
<tr>
<th></th>
<th>Left</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>1,1</td>
<td>0,0</td>
</tr>
<tr>
<td>Down</td>
<td>1,1</td>
<td>1,0</td>
</tr>
</tbody>
</table>

†Row’s payoffs are listed first. Column’s payoffs are listed second.

**Theorem 10.** In any finite, two-player game, every proper equilibrium is a test-set equilibrium.

### 4 Generalized Second Price Auction

Edelman et al. (2007) study the Generalized Second Price (GSP) auction, and focus on equilibria of the auction that possess a property they call local envy-freeness. Varian (2007) studies the same auction, and makes the same equilibrium selection (he refers to these equilibria as symmetric Nash equilibria). This section introduces the GSP auction, and studies the properties of both extended proper equilibria and test-set equilibria of this auction game. We find that every pure test-set and extended proper equilibrium is locally envy-free, but our refinements also imply additional restrictions, so the existence of such equilibria in pure strategies can depend on the full specification of the game.

#### 4.1 Environment

There are $I$ ad positions and $N$ bidders. The click-rate of the $i$th position is $\kappa_i > 0$, and positions are labeled so that their click rates are in descending order. The value per click of bidder $n$ is $v_n > 0$. Advertiser $n$’s payoff from being in position $i$ is $\kappa_i v_n$ minus his payments to the search engine. In the event that $I < N$, we define $\kappa_{I+1} = \cdots = \kappa_N = 0$ for notational convenience.

The $N$ bidders simultaneously submit bids. A bid is a scalar $b_n \in \mathbb{R}_+$. We denote the set of feasible bids for bidder $n$ by $\mathcal{B}_n$.

Two special cases are of interest. First is the case when the bid spaces are continuous, each taking the form $\mathcal{B}^0 = \mathbb{R}_+$. This is the case studied by Edelman et al. (2007). Second is the case when the bid spaces are discrete, each taking the form $\mathcal{B}^\gamma = \gamma \mathbb{Z} \cap [0, \Gamma]$. Here, $\gamma$ is a parameter controlling
the fineness of the discretization, and \( \Gamma \) controls the upper bound of the bid space. To ensure that \( \Gamma \) is not restrictively small, we assume \( \Gamma \geq \gamma + \max_n v_n \). This bound ensures that no bidder would wish to bid above \( \Gamma \) even if it were possible and additionally, all bids constructed in the proofs of the following results are contained in the relevant bid spaces. While we will apply test-set equilibrium with both continuous and discretized bid spaces, extended proper equilibrium, which is defined only for finite games, will be applied only with discretized bid spaces.

Suppose that the bidders submit bids \( b = (b_1, \ldots, b_N) \). Let \( b^{(i)} \) and \( G(i) \) denote the \( i \)th highest bid and the identities of all advertisers bidding at that level, respectively. For notational convenience, we define \( b^{(N+1)} = 0 \).

The mechanism functions as follows. First, bidders are sorted in order of their bids, where ties are broken uniformly at random. After ties are broken in a particular way, let \( g(i) \) denote the \( i \)th highest bidder. For every \( i \in \{1, 2, \ldots, \min(I, N)\} \), the mechanism allocates position \( i \) to bidder \( g(i) \) at a per-click price of \( b^{(i+1)} \), for a total payment of \( \kappa_i b^{(i+1)} \). If \( N > I \), then bidders \( g(I+1), \ldots, g(N) \) win nothing and pay nothing.

We denote the expected payoff to bidder \( n \) under the bid profile \( b \) by \( \pi_n(b) = E[\kappa_{i_n(b)}(v_n - b^{(i_n(b)+1)})] \), where the expectation is taken over the random variable \( i_n(b) \), the position won by bidder \( n \). A GSP auction then becomes a game \( G = \{B_n\}_{n=1}^N, \{\pi_n(\cdot)\}_{n=1}^N \} \).

[Edelman et al. (2007)] introduce locally envy-free equilibrium as a refinement of Nash equilibrium for this mechanism. In words, an equilibrium of the generalized second price auction is locally envy-free if no bidder would be able to improve his payoff by exchanging bids with the bidder one position above him. The following definition extends their notion to allow for “approximate” local envy-freeness. In the definition below, 0-local envy-freeness is equivalent to local envy-freeness.

**Definition 9.** A pure equilibrium \( b \) of a GSP auction is \( \delta \)-locally envy-free if for any \( i \leq \min\{I+1, N\} \), and for any \( g(i) \in G(i) \), \( \kappa_{i-1} [v_{g(i)} - b^{(i)}] - \kappa_i [v_{g(i)} - b^{(i+1)}] \leq \delta \).

The subsequent results rely on the following assumptions about the environment.

**Assumption 1.** We assume the following:

1. \( \{\kappa_1, \ldots, \kappa_I\} \) is linearly independent over \( \mathbb{Q} \), and
2. no two bidders have the same value per click.
Assumptions (i) and (ii) both hold generically. The first of these includes the condition that no two click-rates are identical.

4.2 Extended Proper Equilibrium and Test-Set Equilibrium

We now present two results. First, we show that, in this game, pure extended proper equilibria are test-set equilibria. Second, we show that pure test-set equilibria (and therefore also pure extended proper equilibria) are approximately locally envy-free. Furthermore, this approximation can be made arbitrarily good by using a sufficiently fine bid space.

Theorem 11. There exists $\bar{\gamma} > 0$ such that for almost every $\gamma \in (0, \bar{\gamma})$, every pure extended proper equilibrium of the GSP auction with bid spaces $B^\gamma$ is a test-set equilibrium.\(^{13}\)

Theorem 12. For all $\delta > 0$ there exists $\bar{\gamma} > 0$ such that for almost every $\gamma \in [0, \bar{\gamma})$ every pure test-set equilibrium of the GSP auction with bid spaces $B^\gamma$ is $\delta$-locally envy-free.\(^{14}\)

Corollary 13. For all $\delta > 0$ there exists $\bar{\gamma} > 0$ such that for almost every $\gamma \in (0, \bar{\gamma})$ every pure extended proper equilibrium of the GSP auction with bid spaces $B^\gamma$ is $\delta$-locally envy-free.\(^{15}\)

Corollary 13 follows from the two preceding theorems. To prove Theorem 12, we show that if some bidder, say bidder $g(i^*)$, envies the bidder one slot above him, bidder $g(i^* - 1)$, then bidder $g(i^*)$’s equilibrium bid is weakly dominated against his test set by a slightly higher alternative bid. Indeed, this alternative performs no worse than the original bid against any element of the test set. Moreover, the alternative performs strictly better against certain downward deviations by bidder $g(i^* - 1)$, which are best responses for bidder $g(i^* - 1)$ and therefore in the test set.

As a consequence of Theorem 12 with sufficiently finely discretized bid spaces, pure extended proper equilibria and pure test-set equilibria inherit the attractive properties of locally envy-free equilibria. In particular, for sufficiently small $\gamma$, they are efficient (that is, for all $i \in \{1, \ldots, \min\{I, N\}\}$, the $i$th highest ad slot is won by the bidder with the $i$th highest value for clicks).

\(^{13}\)Specifically, the result can be proven with $\bar{\gamma} = \min\left\{\frac{\Delta_v}{2}, \frac{\Delta_v \Delta_\kappa}{8\kappa_1}\right\}$, where $\Delta_v = \min_{n \in \{1, \ldots, N-1\}} \{v_n - v_{n+1}\}$ and $\Delta_\kappa = \min_{i \in \{1, \ldots, I\}} \{\kappa_i - \kappa_{i+1}\}$.

\(^{14}\)Specifically, the result can be proven with $\bar{\gamma} = \min\left\{\frac{\Delta_v}{2}, \frac{\Delta_v \Delta_\kappa}{8\kappa_1}, \frac{\delta}{\kappa_i}\right\}$.

\(^{15}\)Specifically, the result can be proven with $\bar{\gamma} = \min\left\{\frac{\Delta_v}{2}, \frac{\Delta_v \Delta_\kappa}{8\kappa_1}, \frac{\delta}{\kappa_i}\right\}$.
Proposition 14. There exists \( \bar{\gamma} > 0 \) such that for almost every \( \gamma \in [0, \bar{\gamma}) \) every pure test-set equilibrium of the GSP auction with bid spaces \( B^\gamma \) is efficient.\(^{16}\)

Corollary 15. There exists \( \bar{\gamma} > 0 \) such that for almost every \( \gamma \in (0, \bar{\gamma}) \) every pure extended proper equilibrium of the GSP auction with bid spaces \( B^\gamma \) is efficient.\(^{17}\)

4.3 Proper Equilibria

Theorem 12 would not remain true if “extended proper equilibrium” were replaced with “proper equilibrium.” Rather, for parameters satisfying Assumption \( \textbf{[3]} \) there exist proper equilibria whose deviations from locally envy-freeness remain bounded away from zero as the fineness of the bid space increases.

Proposition 16. Let \((v_1, v_2, v_3) = (15, 10, 5), \text{ and } (\kappa_1, \kappa_2, \kappa_3) \in [100, 101] \times [3, 4] \times [1, 2].\) For almost every \( \gamma \in (0, 1] \), the bid profile

\[
\begin{align*}
b_3 &= \gamma \left\lfloor \frac{5}{\gamma} \cdot \frac{\kappa_2 - \kappa_3}{\kappa_2} \right\rfloor \\
b_2 &= \gamma \left\lceil \frac{10}{\gamma} \cdot \frac{\kappa_2 - \kappa_3}{\kappa_2} \right\rceil \\
b_1 &= \gamma \left\lceil \frac{1}{\gamma} \left( 15 - \frac{\kappa_2}{\kappa_1} (15 - b_2) \right) \right\rceil
\end{align*}
\]

is a proper equilibrium of the GSP auction with bid spaces \( B^\gamma \) that is not \( \delta \)-locally envy-free for all \( \delta \leq \frac{338}{3}. \)

The intuition behind this example is that if bidder 2 is deciding whether or not to increase his bid from \( b_2 \) to some \( \hat{b}_2 > b_2 \), then he must weigh the relatively probabilities of bidders 1 and 3 deviating to bids in \([b_2, \hat{b}_2]\). If bidder 1 makes such a deviation, then bidder 2 would profit by raising his bid. However, if bidder 3 makes such a deviation, then bidder 2 would lose by raising his bid. Such a deviation is a best response for bidder 1 but not for bidder 3. Consequently in an extended proper equilibrium, bidder 2 must believe that bidder 1 is much more likely to deviate than bidder 3, in which case bidder 2 should wish to deviate from equilibrium by raising his bid.

\(^{16}\)Specifically, the result can be proven with \( \bar{\gamma} = \min \left\{ \frac{\Delta_2}{2}, \frac{\Delta_3}{\kappa_1} \right\}. \)

\(^{17}\)Specifically, the result can be proven with \( \bar{\gamma} = \min \left\{ \frac{\Delta_2}{2}, \frac{\Delta_3}{\kappa_1} \right\}. \)
bid. However, proper equilibrium allows bidder 2 to believe that bidder 3 is much more likely to deviate than bidder 1, in which case bidder 2 would not wish to deviate, leaving the equilibrium intact.

4.4 Converse

The converses of Theorems 11 and 12 are not true. As the following example demonstrates, for parameters satisfying Assumption 1, there exist $\delta$-locally envy-free equilibria that are not test-set equilibria. Similarly, there exist test-set equilibria that are not extended proper.

**Proposition 17.** Let $(v_1, v_2) = (2, 1)$, and $\kappa_1 \in [1, 2]$. For almost every $\gamma \in (0, \frac{1}{3})$, we have the following. For every $\delta \in [0, 1)$ the set of $\delta$-locally envy-free equilibria of the GSP auction with bid spaces $\mathcal{B}^\gamma$ is

\[
E_{LEF} = \left\{(b_1, b_2) \in \mathcal{B}^\gamma \times \mathcal{B}^\gamma \mid b_2 \geq 1 - \frac{\delta}{\kappa_1}, b_2 \leq 2, b_1 > b_2 \right\},
\]

the set of pure test-set equilibria is

\[
E_{TS} = \left\{(b_1, b_2) \in \mathcal{B}^\gamma \times \mathcal{B}^\gamma \mid b_2 \geq \gamma \left\lfloor \frac{1}{\gamma} \right\rfloor, b_2 \leq 2, b_1 > b_2, b_1 \leq \gamma \left\lceil \frac{2}{\gamma} \right\rceil \right\},
\]

and the unique pure extended proper equilibrium is

\[
(b_1, b_2) = \left(\gamma \left\lceil \frac{2}{\gamma} \right\rceil, \gamma \left\lfloor \frac{1}{\gamma} \right\rfloor\right).
\]

In this example, the set of test-set equilibria is a strict subset of the set of locally envy-free equilibria. A rough intuition for why some locally envy-free equilibria fail to be test-set equilibria is the following. Local envy-freeness, in essence, requires players to use bids that are undominated against the set of profiles in which at most one bidder changes his bid, and where that change is an allocation-preserving decrease. In contrast, test-set equilibrium requires players to use bids that are undominated against the set of profiles in which at most one bidder changes his bid, and where that change is either an allocation-preserving decrease or an allocation-preserving increase.

Another feature of this example is that the set of extended proper equilibria is a strict subset of the set of test-set equilibria. This is the case because the test-set condition places no weight on deviations that are not allocation-preserving. In contrast, extended proper equilibrium places some weight on such deviations, albeit ‘infinitely less’ weight than on allocation-preserving deviations.
4.5 Nonexistence

The following example illustrates that for parameters satisfying Assumption 1, pure test-set equilibria (and therefore also pure extended proper equilibria) do not exist.

**Proposition 18.** Let \((v_1, v_2, v_3) = (15, 10, 5)\), and \((\kappa_1, \kappa_2, \kappa_3) \in [100, 101] \times [3, 4] \times [1, 2]\). For almost every \(\gamma \in (0, \frac{5}{808})\), there exists no pure test-set equilibrium of the GSP auction with bid spaces \(B^\gamma\).

To see the intuition for this result, suppose that \(b = (b_1, b_2, b_3)\) were a test-set equilibrium. We show that this requires \(b_1 > b_2 > b_3\). Next, we show that if \(b_2\) is too low, then it is weakly dominated by \(b_2 + \gamma\); by increasing his bid, bidder 2 can take advantage of downward deviations by bidder 1. On the other hand, if \(b_2\) is too high, then it is weakly dominated by \(b_2 - \gamma\); by decreasing his bid, bidder 2 can protect himself against upward deviations by bidder 3. For appropriately chosen parameters, such as those given in the proposition, all possible values of \(b_2\) are either “too low” or “too high.”

5 First Price Menu Auctions

[Bernheim and Whinston (1986)] define the first price menu auction and propose two refinements—truthful equilibrium and coalition-proof equilibrium. The payoffs of any Nash equilibrium of the menu auction satisfying either condition lie on the bidder-optimal frontier of the core. This section introduces the menu auction, and studies the properties of both extended proper equilibria and test-set equilibria of this auction game. We find that the payoffs of every pure test-set equilibrium lie in the core (although not necessarily on the bidder-optimal frontier of the core). Pure extended proper equilibria of this game possess a weaker “pairwise efficiency” property.

5.1 Environment

There is one auctioneer, who selects a decision that affects \(N \geq 2\) bidders, each of whom offer a menu of payments contingent on the decision chosen. Possible choices for the auctioneer are given by a finite set of decisions \(X\). The gross monetary payoffs that bidder \(n\) receives from each decision are described by the function \(v_n : X \rightarrow \mathbb{R}\), and the auctioneer receives gross monetary payoffs described by \(v_0 : X \rightarrow \mathbb{R}\).

The \(N\) bidders simultaneously offer contingent payments to the auctioneer, who subsequently chooses a decision that maximizes his total payoff. A
bid is a function \( b_n : X \to \mathbb{R} \). We denote the set of feasible bids for bidder \( n \) by \( B_n \).

Two special cases are of interest. First is the case when the bid spaces are continuous, each taking the form \( B^0 = \{ b_n : X \to \mathbb{R}_+ \} \). This is the case studied by Bernheim and Whinston (1986). Second is the case when the bid spaces are discrete, each taking the form \( B^γ = \{ b_n : X \to \mathbb{R}_+ \mid b_n(x) \in \gamma \mathbb{Z} \cap [0, \Gamma] \forall x \in X \} \). Here, \( \gamma \) is a parameter controlling the fineness of the discretization, and \( \Gamma \) controls the upper bound of the bid space. To ensure that \( \Gamma \) is not restrictively small, we assume \( \Gamma \geq \sum_{n=0}^{N} [\max_{x \in X} v_n(x) - \min_{x \in X} v_n(x)] \). This bound ensures that no bidder would wish to bid above \( \Gamma \) even if it were possible and additionally, all bids constructed in the proofs of the following results are contained in the relevant bid spaces. While we will apply test-set equilibrium with both continuous and discretized bid spaces, extended proper equilibrium, which is defined only for finite games, will be applied only with discretized bid spaces.

The menu auction game is an extension of the Bertrand pricing game. For such games, adopting tie-breaking rules that favor the efficient outcome can simplify the description and analysis of equilibrium. So, for simplicity, we assume below that between any two outcomes with the same payoff for a bidder or for the auctioneer, the bidder or auctioneer prefers the outcome with the higher total payoff.

The auctioneer chooses a decision that maximizes his total payoff—i.e., given some \( b \in \times_{n=1}^N B_n \), the auctioneer selects an element of the set \( \arg \max_{x \in X} \left[ v_0(x) + \sum_{n=1}^{N} b_n(x) \right] \). We assume that, for the reasons described above, the auctioneer resolves ties in favor of the decision with the higher total payoff when this argmax is not a singleton. Payoff maximization and this tie-breaking rule together induce a decision function \( \pi : \times_{n=1}^N B_n \to X \). Let \( \pi_n(b) = v_n(x(b)) - b_n(x(b)) \), and \( \pi(b) = (\pi_1(b), \ldots, \pi_N(b)) \). A menu auction is then a game \( G = \{ \{ B_n \}_{n=1}^{N}, \{ \pi_n(\cdot) \}_{n=1}^{N} \} \).

We now introduce the following notation. If \( J \subseteq \{1, \ldots, N\} \), let \( J^c = \{1, \ldots, N\} \setminus J \). Let \( B_J(x) = \sum_{n \in J} b_n(x) \) and \( B(x) = \sum_{n=1}^{N} b_n(x) \). Similarly, let \( V_J(x) = \sum_{n \in J} v_n(x) \) and \( V(x) = \sum_{n=1}^{N} v_n(x) \). In addition, if \( \pi \in \mathbb{R}^N \), then let \( \Pi_J = \sum_{n \in J} \pi_n \) and \( \Pi = \sum_{n=1}^{N} \pi_n \).

We also make the following assumption, which holds generically.

**Assumption 2.** For any \( J \subseteq \{1, \ldots, N\} \), both \( V_J(x) \) and \( v_0(x) + V_J(x) \) are injective.
5.2 Test-Set Equilibrium

Introducing additional notation, let $x^J = \arg \max_{x \in X} \{v_0(x) + V_J(x)\}$ be the decision that maximizes the total payoff of the coalition consisting of $J$ together with the auctioneer, and let $x^{opt} = x^{(1,\ldots,N)}$ to be the efficient decision.\(^{18}\) We also define a family of payoff sets. $C^0$ is the set of the core payoffs, and for $\gamma > 0$, $C^\gamma$ is an outer approximation of the set of core payoffs:

$$C^\gamma = \left\{ \pi \in \mathbb{R}^N \left| \forall J \subseteq \{1,\ldots,N\}, \Pi_J \leq [V(x^{opt}) + v_0(x^{opt})] - \left[V_J(x^J) + v_0(x^J)\right] + |J|\gamma \right. \right\}.$$

We also define the bidder-optimal frontier of $C^0$:

$$E^0 = \{ \pi \in \mathbb{R}^N \mid \pi \in C^0 \text{ and } \#\pi' \in C^0 \text{ with } \pi' \geq \pi \}.$$

The following lemma is extremely useful in proving the following results about test-set equilibria of menu auctions, and is also of independent interest. In words, it says that test-set equilibria of menu auctions are those equilibria in which bidders bid ‘sufficiently aggressively’ on all losing decisions.

**Lemma 19.** For all $\gamma \geq 0$, a pure Nash equilibrium $b = (b_1,\ldots,b_N)$ of the menu auction with bid spaces $B^\gamma$ is a test-set equilibrium if and only if for all $n \in \{1,\ldots,N\}$ and all $x \in X$, letting $x^* = x(b)$,

$$v_n(x) - b_n(x) \leq v_n(x^*) - b_n(x^*) + \gamma.$$

To prove this lemma, we show that if some bidder’s Nash equilibrium bid fails this condition for some decision $x$, then it is weakly dominated against his test set by an alternative bid that is slightly higher for that decision. Indeed, this alternative performs no worse than the original bid against any element of the test set, and it performs strictly better in the event that another bidder deviates by raising its bid on $x$ to the highest level consistent with a best response, creating a strategy profile in the test set. Using the lemma, we can prove the following result, which states a sense in which test-set equilibria of menu auctions yield payoffs that are approximately in the core. Furthermore, the payoffs can be made to lie arbitrarily close to the core by making the bid space sufficiently fine.

\(^{18}\)Note that Assumption 2 implies that these decisions are uniquely defined.
Theorem 20. There exists $\bar{\gamma} > 0$ such that for all $\gamma \in [0, \bar{\gamma})$, it is the case that in all pure test-set equilibria of the menu auction with bid spaces $B^\gamma$, the auctioneer implements the efficient decision $x^\text{opt}$ and the bidders receive payoffs in $C^\gamma$.

The following result may be interpreted as a partial converse of Theorem 20. It states that any bidder-optimal core payoff vector can be supported by a test-set equilibrium of certain menu auctions (including the auction with bid spaces $B^0$).

Theorem 21. Let $\pi \in E^0$. Let $b_n(x) = \max\{v_n(x) - \pi_n, 0\}$ be the $\pi$-profit-target strategy of bidder $n$. Then $b = (b_1, \ldots, b_N)$ supports $\pi$ as payoffs of a test-set equilibrium of any menu auction in which $b_n \in B_n \forall n$.

5.3 Extended Proper Equilibrium

Theorem 20 demonstrated that pure test-set equilibria yield (approximate) core payoffs. Whether this is also the case for extended proper equilibria (or even proper equilibria) remains an open question. We are, however, able to demonstrate that extended proper equilibria are “approximately bilaterally efficient,” a weaker property, which states that there is no pair of bidders who could change their bids in a way that would lead to significantly higher total value for the coalition consisting of them and the auctioneer.

Definition 10. A bid profile $b$ is $\delta$-bilaterally efficient if there is no pair of bidders $i$ and $j$ and decision $\hat{x} \in X$ such that, letting $x^* = x(b)$,

$$v_0(\hat{x}) + v_i(\hat{x}) + v_j(\hat{x}) + \sum_{n \neq i,j} b_n(\hat{x}) \geq v_0(x^*) + v_i(x^*) + v_j(x^*) + \sum_{n \neq i,j} b_n(x^*) + \delta.$$

The following theorem demonstrates a sense in which pure extended proper equilibria are approximately bilaterally efficient (i.e. approximately 0-bilaterally efficient). Furthermore, it shows that pure extended proper equilibria can be made arbitrarily close to bilaterally efficient by making the bid space sufficiently fine.

Theorem 22. For all $\delta > 0$, there exists a $\bar{\gamma} > 0$ such that for almost every $\gamma \in (0, \bar{\gamma})$, every pure extended proper equilibrium of the menu auction with bid spaces $B^\gamma$ is $\delta$-bilaterally efficient.\(^\text{20}\)

\(^{19}\)Specifically, the result can be proven with $\bar{\gamma} = \frac{1}{N} \min_{x \neq x^\text{opt}} \{V(x^\text{opt}) + v_0(x^\text{opt}) - [V(x) + v_0(x)]\}$.

\(^{20}\)Specifically, the result can be proven with $\bar{\gamma} = \frac{\delta}{2}$.
To prove this result, we show that if δ-bilateral efficiency fails (with, say, bidders 1 and 2 and decision \( \hat{x} \)), then bidder 1 should prefer to deviate from equilibrium by raising his bid on \( \hat{x} \). While the bidder expects that this deviation will probably not change anything, it would be beneficial if bidder 2 also deviates by raising his bid on \( \hat{x} \). Bidder 1’s deviation might also be costly to him, but only if the other bidders deviate in a way that would change the outcome compared to equilibrium. We show that in the most likely such scenario, some deviator, say bidder 3, must be worse off than if he had played according to equilibrium. We then show that the likelihood of bidder 2’s deviation being costly to him is much smaller than the likelihood of bidder 3’s deviation being costly to him. Thus, the across-person restrictions that extended properness places on trembles require bidder 2’s deviation to be much more likely. Consequently, bidder 1 is much more likely to gain than lose if he deviates from equilibrium by altering his bid in this way.

6 Conclusion

We introduce two new refinements of Nash equilibrium for normal form games – extended proper equilibrium and test-set equilibrium – that discipline the players’ beliefs about the most likely deviations in games, especially games with three or more players. Both incorporate the novel condition that players believe that another player is more likely to deviate to an alternative best response to the equilibrium than the same or another is to deviate to a strategy that is not a best response. Extended proper equilibrium is defined to resemble and refine proper equilibrium. Test-set equilibrium is defined differently: it neither implies nor is implied by any tremble-based concept. Instead, it directly restricts players to act as if the only possible deviations from equilibrium are ones by a single player to a different best response.

We have illustrated the power of these concepts using two examples: the generalized second price auction studied by Edelman et al. (2007) and the first price menu auction studied by Bernheim and Whinston (1986). The former introduced a refinement – “locally envy-free equilibrium” – that is defined only for the GSP mechanism. We show any pure strategy equilibrium satisfying either of our solutions is locally envy-free. In the latter example, the original analysis selects an equilibrium either by restricting attention to equilibria in which players adopt “truthful” strategies or by requiring that the solution be coalition-proof. We show that any test-set equilibrium payoff vector is in the core, even though test-set equilibrium is defined without reference to coalitional games or coalitional deviations and
without “truthfulness” restrictions on bidders’ strategies. We also show that extended proper equilibrium possesses a weaker pairwise efficiency property. These findings illuminate the structure and nature of equilibrium in these two celebrated auction games.
A Omitted Proofs

A.1 Example Games

Proposition 23. For the game in Figure 1, (Up, Left, West) is a proper equilibrium, but is neither an extended proper equilibrium nor a test-set equilibrium.

Proof of Proposition 23. Proper Equilibrium. Let \( \rho = (p^1, p^2, p^3, p^4, p^5) \), where \( p^1 \) puts full weight on (Up, Left, West), \( p^2 \) puts equal weight on (Up, Right, West) and (Down, Left, West), \( p^3 \) puts equal weight on (Down, Right, West) and (Up, Left, East), \( p^4 \) puts equal weight on (Up, Right, East) and (Down, Left, East), and \( p^5 \) puts full weight on (Down, Right, East). Notice that \( \rho \) has full support. We next show that it satisfies strong independence. Let \( \epsilon \in \mathbb{F} \) be a positive infinitesimal, and let \( \delta \) be the following vector of positive infinitesimals

\[
\delta = \left( \frac{\epsilon(2 + 2\epsilon + 2\epsilon^2 + \epsilon^3)}{(1 + \epsilon)^2(1 + \epsilon^2)}, \frac{\epsilon(2 + 2\epsilon + 2\epsilon^2 + \epsilon^3)}{2 + 2\epsilon + 2\epsilon^2 + \epsilon^3}, \frac{\epsilon(2 + \epsilon)}{2 + 2\epsilon + \epsilon^2 + 2 + \epsilon} \right).
\]

Then \( \delta \rho \) is the \( \mathbb{F} \)-valued probability measure defined in Figure 4, which can be expressed as the product of the probability measures \( \hat{p}_{\text{row}} = \left( \frac{1}{1+\epsilon^2}, \frac{\epsilon}{1+\epsilon^2} \right) \), \( \hat{p}_{\text{col}} = \left( \frac{1}{1+\epsilon^2}, \frac{\epsilon}{1+\epsilon^2} \right) \), and \( \hat{p}_{\text{geo}} = \left( \frac{1}{1+\epsilon^2}, \frac{\epsilon^2}{1+\epsilon^2} \right) \).

Figure 4: the probability measure \( \delta \rho \)

Therefore, \( \rho \) satisfies strong independence. We next show that \( \rho \) respects within-person preferences. Row lexicographically prefers Up to Down,\(^{21}\) and indeed we have \( Up >_\rho Down \). Column lexicographically prefers Left to Right,\(^{22}\) and indeed we have \( Left >_\rho Right \). Geo is lexicographically indifferent between East and West, so there is nothing to check for him.

Therefore, (Up, Left, West) is a proper equilibrium by Proposition 4.

\(^{21}\) Up dominates Down against the support of \( p^1_{\text{col,geo}} \).

\(^{22}\) They perform equally well against the support of \( p^4_{\text{row,geo}} \), but Left weakly dominates Right against the support of \( p^2_{\text{col,geo}} \).
**Extended Proper Equilibrium.** We use the notation of Proposition 7 with \( \sigma = (\text{Up}, \text{Left}, \text{West}) \), Column taking the role of \( n \), Right taking the role of \( \hat{\sigma}_n \), Geo taking the role of \( m \), and East taking the role of \( \hat{s}_m \). Then \( A = (\text{Up}, \text{East}) \) and \( B = (\text{Up}, \text{West}) \). Because \( u_{\text{col}}(\text{Up}, \text{Right}, \text{East}) > u_{\text{col}}(\text{Up}, \text{Left}, \text{East}) \) and \( u_{\text{col}}(\text{Up}, \text{Right}, \text{Left}) \geq u_{\text{col}}(\text{Up}, \text{Left}, \text{West}) \), Proposition 7 implies that \( (\text{Up}, \text{Left}, \text{West}) \) is not an extended proper equilibrium.

**Test-Set Equilibrium.** Letting \( \sigma = (\text{Up}, \text{Left}, \text{West}) \), Column’s test set is \( T_{\text{col}}(\sigma) = \{ (\text{Up}, \text{West}), (\text{Up}, \text{East}) \} \).

Right weakly dominates \( \sigma_{\text{col}} = \text{Left} \) against \( T_{\text{col}}(\sigma) \), and so this cannot be a test-set equilibrium.

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**A.2 Extended Proper Equilibrium**

**A.2.1 Technical Lemmas**

The following two lemmas will be useful in proving Proposition 5. The first is a version of Proposition 1 from Blume et al. (1991).

**Lemma 24.** Given any LPS \( \rho = (p^1, \ldots, p^K) \) on \( \times_{n=1}^N S_n \), \( \exists \delta > 0 \) such that if \( r \in (0, \delta)^{K-1} \), then

\[
\sum_{s-n \in S-n} (r\boxtimes p_{-n})(s-n) \left[ u_n(s_n, s-n) - u_n(s'_n, s-n) \right] > 0 \\
\Leftrightarrow \left[ \sum_{s-n \in S-n} p_{-n}(s-n)u_n(s_n, s-n) \right]_k^{K} > L \left[ \sum_{s-n \in S-n} p_{-n}(s-n)u_n(s'_n, s-n) \right]_k^{K}.
\]

\( \forall s_n, s'_n \in S_n \), and \( \forall n \).

**Proof of Lemma 24.** To economize on notation, define for each \( n \in \{1, \ldots, N\} \), the following function

\[
H_n(k, s_n, s'_n) = \sum_{s-n \in S-n} p_{-n}(s-n)[u_n(s_n, s-n) - u_n(s'_n, s-n)].
\]

If \( H_n(k, s_n, s'_n) = 0 \) for all choices of \( n, s_n, s'_n \), and \( k \), then for each \( n \), each strategy \( s_n \in S_n \) must produce identical expected payoffs against every \( p_{-n}^k \).
As a result, the statement holds vacuously with any choice of $\delta$. We therefore assume henceforth that this is not the case. Define the constants

$$W = \max_{n \in \{1, \ldots, N\}} \max_{s_n, s_n' \in S_n, s_n - s_n' \in S_n} |u_n(s_n', s_n - s_n) - u_n(s_n, s_n)|$$

$$B = \min_{n \in \{1, \ldots, N\}} \min_{k \in \{1, \ldots, K\}} \min_{s_n, s_n' \in S_n} \{|H_n(k, s_n, s_n')| : H_n(k, s_n, s_n') \neq 0\}$$

As a result of the previous assumption, $B$ is a well-defined positive number. We also obviously have $W \geq 0$. We then choose $\delta = \frac{B}{B + W} > 0$, and now show that the statement of the lemma holds with this choice of $\delta$. Suppose that for some $n$ and $s_n, s_n' \in S_n$,

$$\sum_{s_n \in S_{-n}} p_{-n}^k(s_n) u_n(s_n, s_n)$$

$$= L \sum_{s_n \in S_{-n}} p_{-n}^k(s_n) u_n(s_n', s_n)$$

Then there is some $k$ such that $H_n(k, s_n, s_n') \geq B > 0$ and $H_n(l, s_n, s_n') = 0$ for all $l < k$. Suppose $r \in (0, \delta)^{K-1}$. Letting $\hat{\rho} = (p^{k+1}, \ldots, p^K)$ and $\hat{r} = (r_{k+1}, \ldots, r_{K-1})$, we have that

$$\sum_{s_n \in S_{-n}} (r \hat{\square} \hat{\rho}) (s_n)[u_n(s_n, s_n) - u_n(s_n', s_n)]$$

$$\geq r_1 \cdots r_{k-1} \left[ (1 - r_k)H_n(k, s_n, s_n') + r_k \sum_{s_n \in S_{-n}} (\hat{r} \hat{\square} \hat{\rho})(s_n)[u_n(s_n, s_n) - u_n(s_n', s_n)] \right]$$

$$\geq r_1 \cdots r_{k-1} \left[ (1 - r_k)B - r_k W \right]$$

$$= 0,$$ 

where the last step follows from the definition of $\delta$. Now suppose instead that for some $n$ and $s_n, s_n' \in S_n$,

$$\sum_{s_n \in S_{-n}} p_{-n}^k(s_n) u_n(s_n, s_n)$$

$$= L \sum_{s_n \in S_{-n}} p_{-n}^k(s_n) u_n(s_n', s_n)$$

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Then $H_n(k, s_n, s'_n) = 0$ for all $k$. If this is the case, then we have

$$\sum_{s_{-n} \in S_{-n}} (r \sqcap \rho_{-n})(s_{-n})[u_n(s_n, s_{-n}) - u_n(s'_n, s_{-n})] = 0.$$  

This completes the proof. \[\square\]

**Lemma 25.** Given any LPS $\rho = (p^1, \ldots , p^K)$ on $\times_{n=1}^N S_n$, any $\alpha \in \mathbb{R}_{++}^N$, and any sequence $r(t) \in (0,1)^{K-1}$ with $r(t) \to 0$, there exists some $T$ such that for all $t \geq T$,

$$\alpha_n \cdot \max \sum_{\hat{s}_n \in S_n} (r(t) \sqcap \rho_{-n})(s_{-n})[u_n(\hat{s}_n, s_{-n}) - u_n(s_n, s_{-n})]$$

$$> \alpha_m \cdot \max \sum_{\hat{s}_m \in S_m} (r(t) \sqcap \rho_{-m})(s_{-m})[u_m(\hat{s}_m, s_{-m}) - u_m(s_m, s_{-m})]$$

$$\iff \left[ \alpha_n \sum_{s_{-n} \in S_{-n}} p^k_{-n}(s_{-n})[u_n(s^*_n, s_{-n}) - u_n(s_n, s_{-n})] \right]_{k=1}^K$$

$$> L \left[ \alpha_m \sum_{s_{-m} \in S_{-m}} p^k_{-m}[u_m(s^*_m, s_{-m}) - u_m(s_m, s_{-m})] \right]_{k=1}^K$$

$\forall s_n \in S_n, s_m \in S_m, s^*_n \in BR_n(\rho_{-n}), s^*_m \in BR_m(\rho_{-m})$, and $\forall n, m$.

**Proof of Lemma 25.** Let $\alpha \in \mathbb{R}_{++}^N$. Given $\rho$, select for every $n$ some $s^*_n \in BR_n(\rho_{-n})$. Since all elements of $BR_n(\rho_{-n})$ must produce identical expected payoffs against every $p^k_{-n}$, it will not matter which element is chosen. To economize on notation, define for each $n, m \in \{1, \ldots , N\}$ (possibly equal), the following function

$$H_{nm}(k, s_n, s_m) = \alpha_n \sum_{s_{-n} \in S_{-n}} p^k_{-n}(s_{-n})[u_n(s^*_n, s_{-n}) - u_n(s_n, s_{-n})]$$

$$- \alpha_m \sum_{s_{-m} \in S_{-m}} p^k_{-m}[u_m(s^*_m, s_{-m}) - u_m(s_m, s_{-m})].$$

If $H_{nm}(k, s_n, s_m) = 0$ for all choices of $n, m, s_n, s_m$, and $k$, then for each $n$, each strategy $s_n \in S_n$ must produce identical expected payoffs against every $p^k_{-n}$. As a result, the statement holds vacuously with any choice of $T$. We
therefore assume henceforth that this is not the case. Define the constants

\[ W = \max_{n \in \{1, \ldots, N\}} \left| \sum_{s_n, s_n' \in S_n} \alpha_n \left[ u_n(s_n, s_{-n}) - u_n(s_n', s_{-n}) \right] \right| \]

\[ B = \min_{n,m \in \{1, \ldots, N\}} \left\{ |H_{nm}(k, s_n, s_m)| : H_{nm}(k, s_n, s_m) \neq 0 \right\} \]

As a result of the previous assumption, \( B \) is a well-defined positive number. We also obviously have \( W \geq 0 \). Therefore \( B + 2W > 0 \). In addition, some \( \delta > 0 \) must exist that satisfies the conditions of Lemma 24. Let \( T \) be such that \( r(t) \in \left( 0, \min \left\{ \delta, \frac{B}{B+2W} \right\} \right) \) for all \( t \geq T \). Because \( r(t) \to 0 \), such a \( T \) must exist. We now prove that the statement of this lemma holds with this choice of \( T \). Suppose that for some \( n, m, s_n \in S_n \), and \( s_m \in S_m \),

\[ \left[ \alpha_n \sum_{s_{-n} \in S_{-n}} p_{-n}^k(s_{-n}) \left[ u_n(s_n^*, s_{-n}) - u_n(s_n, s_{-n}) \right] \right]_{k=1}^K \]

\[ > L \left[ \alpha_m \sum_{s_{-m} \in S_{-m}} p_{-m}^k(s_m^*, s_{-m}) - u_m(s_m, s_{-m}) \right]_{k=1}^K \]

Then there is some \( k \) such that \( H_{nm}(k, s_n, s_m) \geq B > 0 \) and \( H_{nm}(l, s_n, s_m) = 0 \) for all \( l < k \). Letting \( \hat{\rho} = (p^{k+1}, \ldots, p^K) \) and \( \hat{r}(t) = (r_{k+1}(t), \ldots, r_{K-1}(t)) \),

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we have that for all $t \geq T$,
\[
\alpha_n \cdot \max_{\hat{s}_n \in S_n} \sum_{s_n \in S_n} (r(t) \square \rho_{-n})(s_n) [u_n(\hat{s}_n, s_n) - u_n(s_n, s_n)] \\
- \alpha_m \cdot \max_{\hat{s}_m \in S_m} \sum_{s_m \in S_m} (r(t) \square \rho_{-m})(s_m) [u_m(\hat{s}_m, s_m) - u_m(s_m, s_m)] \\
= \alpha_n \sum_{s_n \in S_n} (r(t) \square \rho_{-n})(s_n) [u_n(s_n^*, s_n) - u_n(s_n, s_n)] \\
- \alpha_m \sum_{s_m \in S_m} (r(t) \square \rho_{-m})(s_m) [u_m(s_m^*, s_m) - u_m(s_m, s_m)] \\
\geq r_1(t) \cdots r_{k-1}(t) \left[ (1 - r_k(t)) H_{nm}(k, s_n, s_m) + r_k(t) \sum_{s_n \in S_n} (\dot{r}(t) \square \dot{\rho}_{-n})(s_n) \alpha_n [u_n(s_n^*, s_n) - u_n(s_n, s_n)] \\
- r_k(t) \sum_{s_m \in S_m} (\dot{r}(t) \square \dot{\rho}_{-m})(s_m) \alpha_m [u_m(s_m^*, s_m) - u_m(s_m, s_m)] \right] \\
\geq r_1(t) \cdots r_{k-1}(t) \left[ (1 - r_k(t)) B - r_k(t) 2W \right] \\
> r_1(t) \cdots r_{k-1}(t) \left[ B - \frac{B}{B + 2W} (B + 2W) \right] \\
= 0.
\]

The first step in the above follows from Lemma 24, which implies that any maximizers $\hat{s}_n \in S_n$ and $\hat{s}_m \in S_m$ must be elements of $BR_n(\rho_{-n})$ and $BR_m(\rho_{-m})$, respectively. Now suppose instead that for some $n, m$, $s_n \in S_n$, and $s_m \in S_m$
\[
\begin{bmatrix}
\alpha_n \sum_{s_n \in S_n} p_{-n}^k(s_n) [u_n(s_n^*, s_n) - u_n(s_n, s_n)]
\end{bmatrix}_{k=1}^K \\
= L \begin{bmatrix}
\alpha_m \sum_{s_m \in S_m} p_{-m}^k [u_m(s_m^*, s_m) - u_m(s_m, s_m)]
\end{bmatrix}_{k=1}^K .
\]

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Then $H_{nm}(k, s_n, s_m) = 0$ for all $k$. If this is the case, then for all $n,$

\[
\alpha_n \cdot \max_{s_n \in S_n} \sum_{s-n \in S-n}(r(t)\square \rho_{-n})(s-n)[u_n(s_n, s-n) - u_n(s_n, s-n)]
\]

\[
- \alpha_m \cdot \max_{s_m \in S_m} \sum_{s-m \in S-m}(r(t)\square \rho_{-m})(s-m)[u_m(s_m, s-m) - u_m(s_m, s-m)]
\]

\[
= \alpha_n \sum_{s-n \in S-n}(r(t)\square \rho_{-n})(s-n)[u_n(s_n^*, s-n) - u_n(s_n, s-n)]
\]

\[
- \alpha_m \sum_{s-m \in S-m}(r(t)\square \rho_{-m})(s-m)[u_m(s_m^*, s-m) - u_m(s_m, s-m)]
\]

\[
= 0,
\]

where as before, the first step in the above follows from Lemma 24. This completes the proof. □

We now state and prove some properties possessed by LPSs, which will be used in later arguments.

**Lemma 26.** Suppose that an LPS $\rho = (p^1, \ldots, p^K)$ on $\times_{n=1}^N S_n$ is equivalent to an $F$-valued probability measure $\hat{p}$ on $\times_{n=1}^N S_n$. For all $s_J \in \times_{j \in J} S_j$ and all $s_I \in \times_{i \in I} S_i,$ $s_J \geq \rho s_I$ iff $\frac{\hat{p}_J(s_J)}{\hat{p}_I(s_I)} \geq \frac{1}{n}$ for some $n \in \mathbb{N}$.\footnote{If $J \subseteq \{1, \ldots, N\}$, then we use $\hat{p}_J$ to denote the marginal of $\hat{p}$ on $\times_{j \in J} S_j$.}

**Proof of Lemma 26.** Let $m_0 = \min_{k,J} \min\{p^k_J(s_J) : s_J \in \text{supp } p^k\}$. Let $n \in \mathbb{N}$ be such that $m_0 > \frac{1}{n}.$ Because $\rho$ and $\hat{p}$ are equivalent, there exists a vector of infinitesimals $\epsilon = (\epsilon_1, \ldots, \epsilon_{K-1}) \in \mathbb{F}^{K-1}$ such that $\hat{p} = \epsilon \square \rho.$ Suppose $s_J \geq \rho s_I$, so that $k = \min\{l : p^l_J(s_J) > 0\} \leq \min\{l : p^l_I(s_I) > 0\}$. Then $\hat{p}_J(s_J) \geq \epsilon_1 \cdots \epsilon_{k-1}(1 - \epsilon_k)m_0$, and $\hat{p}_I(s_I) \leq \epsilon_1 \cdots \epsilon_{k-1}(1 - \epsilon_k)$. Therefore $\frac{\hat{p}_J(s_J)}{\hat{p}_I(s_I)} \geq m_0 > \frac{1}{n}$.

Suppose instead that $s_J <_{\rho} s_I$, so that $k = \min\{l : p^l_J(s_J) > 0\} > \min\{l : p^l_I(s_I) > 0\}$. Then $\hat{p}_J(s_J) \leq \epsilon_1 \cdots \epsilon_{k-1}(1 - \epsilon_k)$ and $\hat{p}_I(s_I) \geq \epsilon_1 \cdots \epsilon_{k-2}(1 - \epsilon_{k-1})m_0$. Consequently, $\frac{\hat{p}_J(s_J)}{\hat{p}_I(s_I)} \leq \frac{\epsilon_{k-1}(1 - \epsilon_k)}{(1 - \epsilon_{k-1})m_0}$, which is an infinitesimal, and therefore is less than $\frac{1}{n}$ for all $n \in \mathbb{N}$. □

**Lemma 27.** The following are true of an LPS $\rho = (p^1, \ldots, p^K)$ that satisfies strong independence:

(i) Let $I$ and $J$ be disjoint subsets of $\{1, \ldots, N\}$. Then $s_J \geq_{\rho} s'_J$ iff $(s_I, s_J) \geq_{\rho} (s_I, s'_J)$.\footnote{If $J \subseteq \{1, \ldots, N\}$, then we use $\hat{p}_J$ to denote the marginal of $\hat{p}$ on $\times_{j \in J} S_j$.}
(ii) Let $I$ and $J$ be disjoint subsets of $\{1, \ldots, N\}$. Let the same be true of $I'$ and $J'$. Then $s_I >_\rho s'_I$ and $s_J >_\rho s'_J \Rightarrow (s_I, s_J) >_\rho (s'_I, s'_J)$.

Proof of Lemma \ref{lemma:27}. Claim (i). Since $\rho$ satisfies strong independence, there exists an $\mathbb{F}$-valued product measure $\hat{p}$ and a vector of positive infinitesimals $\epsilon = (\epsilon_1, \ldots, \epsilon_K)$ for which $\hat{p} = \epsilon \square \rho$. Then

$$s_J >_\rho s'_J \Leftrightarrow \frac{\hat{p}_J(s_J)}{\hat{p}_J(s'_J)} \geq \frac{1}{n} \text{ for some } n \in \mathbb{N}$$

$$\Leftrightarrow \frac{\hat{p}_{I,J}(s_I, s_J)}{\hat{p}_{I,J}(s_I, s'_J)} \geq \frac{1}{n} \text{ for some } n \in \mathbb{N}$$

$$\Leftrightarrow (s_I, s_J) >_\rho (s_I, s'_J),$$

where the first and third steps follow from Lemma \ref{lemma:26} and the second step follows from the fact that $\hat{p}$ is a product measure.

Claim (ii). Suppose that $s_I >_\rho s'_I$ and $s_J >_\rho s'_J$. Then by Lemma \ref{lemma:26}, $\frac{\hat{p}_J(s'_J)}{\hat{p}_J(s_J)} < \frac{1}{n}$ for some $n \in \mathbb{N}$. Similarly, $\frac{\hat{p}_J(s_I)}{\hat{p}_J(s'_I)} > \frac{1}{n_0}$ for some $n_0 \in \mathbb{N}$, or, equivalently, $\frac{\hat{p}_J(s'_J)}{\hat{p}_J(s_J)} < n_0$. Then since $\hat{p}$ is a product measure,

$$\frac{\hat{p}_{I,J}(s'_I, s'_J)}{\hat{p}_{I,J}(s_I, s_J)} = \frac{\hat{p}_J(s'_J)}{\hat{p}_J(s_J)} \cdot \frac{\hat{p}_I(s_I)}{\hat{p}_I(s'_I)} < \frac{n_0}{n} \text{ for all } n \in \mathbb{N},$$

which implies that $\frac{\hat{p}_{I,J}(s'_I, s'_J)}{\hat{p}_{I,J}(s_I, s_J)} < \frac{1}{n}$ for all $n \in \mathbb{N}$. Another application of Lemma \ref{lemma:26} gives the desired result, $(s_I, s_J) >_\rho (s'_I, s'_J)$. \hfill \Box

A.2.2 Main Results

Proof of Theorem \ref{theorem:1}. \footnote{With a few small adjustments, the argument follows the proof of existence of proper equilibria given by [Myerson 1978].} Let $\alpha \in \mathbb{R}^N_{++}$. We first demonstrate the existence of an $(\alpha, \epsilon)$-extended proper equilibrium for any $\epsilon \in (0, 1)$. Let $M = \max_n |S_n|$, $K = \sum_{n=1}^N |S_n|$, and $\delta = \frac{\epsilon^2}{2M}$. For any player $n$, let $\Delta^*(S_n) = \{\sigma_n \in \Delta(S_n) : \sigma_n(s_n) \geq \delta, \forall s_n \in S_n\}$, which is a nonempty and compact subset of $\Delta^0(S_n)$. Next, we define the correspondence $F : \times_{n=1}^N \Delta^*(S_n) \rightarrow \times_{n=1}^N \Delta^*(S_n)$ by

$$F(\sigma) = \begin{cases} \sigma^* \in \times_{n=1}^N \Delta^*(S_n) & \text{ if } \alpha_n L_n(s_n|\sigma) > \alpha_m L_m(s_m|\sigma) \\
\text{then } \sigma^*_n(s_n) \leq \epsilon \cdot \sigma^*_m(s_m), & \forall n, \forall m, \forall s_n \in S_n, \forall s_m \in S_m \end{cases}.$$
For any $\sigma$, the points in $F(\sigma)$ are those that satisfy a finite collection of linear inequalities, so $F(\sigma)$ is a closed, convex set. We next demonstrate that $F(\sigma)$ is nonempty. Let $\phi_m(s_n)$ be the number of pure strategies $s_m \in S_m$ such that $\alpha_n L_n(s_n|\sigma) < \alpha_m L_m(s_m|\sigma)$. Let $\phi(s_n) = \sum_{m=1}^{N} \phi_m(s_n)$, and let $L_n^0 = \{(s_n \in S_n : L_n(s_n|\sigma) = 0)\}$. Finally, let

$$
\sigma^*_n(s_n) = \begin{cases} 
\frac{\phi(s_n)}{2M} & \text{if } L_n(s_n|\sigma) > 0 \\
\frac{1}{L_n^0} (1 - \sum_{s_n : L_n(s_n|\sigma) > 0} \frac{\phi(s_n)}{2M}) & \text{if } L_n(s_n|\sigma) = 0
\end{cases}
$$

To demonstrate that $\sigma^* \in F(\sigma)$, we show (i) $\sigma^*_n \in \Delta^*(S_n)$, and (ii) if $\alpha_n L_n(s_n|\sigma) > \alpha_m L_m(s_m|\sigma)$, then $\sigma^*_n(s_n) \leq \epsilon \cdot \sigma^*_m(s_m)$. To show (i), we consider two cases. First, if $L_n(s_n|\sigma) > 0$, then $\sigma^*_n(s_n) = \frac{\phi(s_n)}{2M} \geq \frac{\epsilon}{2M} = \delta$. Second, if $L_n(s_n|\sigma) = 0$, then $\sigma^*_n(s_n) = \frac{1}{L_n^0} (1 - \sum_{s_n : L_n(s_n|\sigma) > 0} \frac{\phi(s_n)}{2M}) \geq \frac{1}{L_n^0} (1 - M \frac{1}{2M}) = \frac{1}{L_n^0} \geq \frac{1}{2M} \geq \delta$. To show (ii), we also consider two cases. First, suppose $L_n(s_n|\sigma) > 0$ and $L_m(s_m|\sigma) > 0$. Then $\alpha_n L_n(s_n|\sigma) > \alpha_m L_m(s_m|\sigma)$ implies that $\phi(s_n) \geq \phi(s_m) + 1$, and so $\sigma^*_n(s_n) = \frac{\phi(s_n)}{2M} \leq \frac{\phi(s_n)}{2M} = \epsilon \cdot \sigma^*_m(s_m)$. Second, suppose $L_n(s_n|\sigma) > 0$ and $L_m(s_m|\sigma) = 0$. Then $\phi(s_n) \geq 1$, and so $\sigma^*_n(s_n) = \frac{\phi(s_n)}{2M} \leq \frac{\epsilon}{2M} \leq \epsilon \cdot \sigma^*_m(s_m)$.

Finally, continuity of each $L_n(s_n|\cdot)$ function implies that $F(\cdot)$ must be upper-semicontinuous. Therefore $F$ satisfies all the conditions of the Kakutani Fixed Point Theorem [Kakutani, 1941], so there exists some $\sigma^* \in \times_{n=1}^{N} \Delta^*(S_n)$ such that $\sigma^* \in F(\sigma^*)$. By the definition of $F$, this point is an $(\alpha, \epsilon)$-extended proper equilibrium.

So for any $0 < \epsilon < 1$, there exists an $(\alpha, \epsilon)$-extended proper equilibrium. Since $\times_{n=1}^{N} \Delta(S_n)$ is a compact set, there must exist a convergence subsequence, and an extended proper equilibrium $\sigma = \lim_{\epsilon \to 0} \sigma^\epsilon$.

**Proof of Proposition 2** Sufficiency. Suppose $(\rho, \sigma)$ is a lexicographic Nash equilibrium, where $\rho$ satisfies strong independence. As Blume et al. [1991] observe, strong independence of $\rho$ implies that $p^1$ must be a product measure. Furthermore, if $p^1_n(s_n) > 0$ for a player $n$, then by condition (i) of Definition 3

$$
\sum_{s_n \in S_n} p^1_n(s_n) u_n(s_n, s_n) \geq \sum_{s_n \in S_n} p^1_n(s_n) u_n(s'_n, s_n)
$$

\[25\] The last step uses the fact, proved in the process of showing (i), that $L_m(s_m|\sigma) = 0$ implies that $\sigma^*_n(s_m) \geq \frac{1}{2M}$.

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for all \( s_n' \in S_n \) and all \( n \). Therefore \( p^1 \) satisfies the definition of a Nash equilibrium. Thus, by condition (ii) of Definition 3 \( \sigma = p^1 \) is a Nash equilibrium.

**Necessity.** Suppose that \( \sigma \) is a Nash equilibrium. Taking \( K = 1 \) and \( p^1 = \sigma \) produces an LPS \( \rho \) such that \((\rho, \sigma)\) is a lexicographic Nash equilibrium. Furthermore, \( \sigma \), and therefore \( p^1 \), must be a product measure. Since \( K = 1 \), \( \rho \) therefore satisfies strong independence.

**Proof of Proposition 3** This is a restatement of Proposition 7 from Blume et al. (1991).

**Proof of Proposition 4** This is a restatement of Proposition 8 from Blume et al. (1991).

**Proof of Proposition 5** **Sufficiency.** Suppose \((\rho, \sigma)\) is a lexicographic Nash equilibrium, where \( \rho = (p^1, \ldots, p^K) \) is an LPS that satisfies the stated conditions. As Blume et al. (1991) point out, \( \rho \) satisfies strong independence if and only if there is a sequence \( r(t) \in (0,1)^{K-1} \) with \( r(t) \to 0 \) such that \( r(t) \sqcup \rho \) is a product measure for all \( t \). For each \( n \), define \( \sigma_n(t) \) as the marginal on \( S_n \) of \( r(t) \sqcup \rho \), and let \( \sigma(t) = (\sigma_1(t), \ldots, \sigma_N(t)) \). Note that \( \lim_{t \to \infty} \sigma(t) = p_1 = \sigma \). Additionally, since \( \rho \) has full support, \( \sigma_n(t) \in \Delta^0(S_n) \) \( \forall n \) and \( \forall t \). Define

\[
m_0 = \min_{n \in \{1, \ldots, N\}} \min_{k \in \{1, \ldots, K\}} \{p^k_n(s_n) : s_n \in \text{supp } p^k_n\}
\]

\[
\epsilon(t) = \max_{k \in \{1, \ldots, K-1\}} \left\{ \frac{r_k(t)}{1 - r_k(t)} m_0 \right\}
\]

Since \( r(t) \to 0 \), we also have \( \epsilon(t) \to 0 \). Let \( T \) be as in the statement of Lemma 25 and let \( \alpha \in \mathbb{R}_+^N \) be as in the definition of “respects within-and-across-person preferences.” We claim that for all \( t \geq T \), \( \sigma(t) \) is an \((\alpha, \epsilon(t))\)-extended proper equilibrium. To see this, suppose \( n, m, s_n \in S_n \), and \( s_m \in S_m \) are such that \( \alpha_n L_n(s_n | \sigma(t)) > \alpha_m L_m(s_m | \sigma(t)) \). Thus,

\[
\alpha_n \cdot \max_{s_n \in S_n} \sum_{s_{-n} \in S_{-n}} \sigma_{-n}(t)(s_{-n}) [u_n(s_n, s_{-n}) - u_n(s_n, s_{-n})] > \alpha_m \cdot \max_{s_m \in S_m} \sum_{s_{-m} \in S_{-m}} \sigma_{-m}(t)(s_{-m}) [u_m(s_m, s_{-m}) - u_m(s_m, s_{-m})].
\]
Since \( r(t) \square \rho \) is a product measure, \((r(t) \square \rho_{-n})(s_{-n}) = \times_{m \neq n} \sigma_m(t)(s_m)\). Therefore the above equation is equivalent to

\[
\alpha_n \cdot \max_{s_n \in S_n} \sum_{s_{-n} \in S_{-n}} (r(t) \square \rho_{-n})(s_{-n})[u_n(s_n, s_{-n}) - u_n(s_n, s_{-n})]
\]

\[
> \alpha_m \cdot \max_{s_m \in S_m} \sum_{s_{-m} \in S_{-m}} (r(t) \square \rho_{-m})(s_{-m})[u_m(s_m, s_{-m}) - u_m(s_m, s_{-m})].
\]

By Lemma 25 we have that for all \( s_n^* \in BR_n(\rho_{-n}) \) and \( s_m^* \in BR_m(\rho_{-m}) \),

\[
\left[ \alpha_n \sum_{s_{-n} \in S_{-n}} p_{n}^K(s_{-n})[u_n(s_n^*, s_{-n}) - u_n(s_n, s_{-n})] \right]_{k=1}^K \]

\[
> L \left[ \alpha_m \sum_{s_{-m} \in S_{-m}} p_{m}^K[u_m(s_m^*, s_{-m}) - u_m(s_m, s_{-m})] \right]_{k=1}^K,
\]

which, because \( \rho \) respects within-and-across-person preferences, implies that \( s_n <_\rho s_m \). Let \( k = \min\{l : p_m^l(s_m) > 0\} \). Because \( \rho \) has full support, such a \( k \) must exist. Furthermore, we must then have \( \sigma_m(t)(s_m) \geq r_1(t) \cdots r_{k-1}(t)[1 - r_k(t)]m_0 \). Because \( s_n <_\rho s_m \), we must also have \( \sigma_n(t)(s_n) \leq r_1(t) \cdots r_k(t) \), and therefore

\[
\epsilon(t) \cdot \sigma_m(t)(s_m) \geq \epsilon(t) \cdot r_1(t) \cdots r_{k-1}(t)[1 - r_k(t)]m_0
\]

\[
\geq \frac{r_k(t)}{1 - r_k(t)} m_0 \cdot r_1(t) \cdots r_{k-1}(t)[1 - r_k(t)]m_0
\]

\[
= r_1(t) \cdots r_k(t)
\]

\[
\geq \sigma_n(t)(s_n).
\]

In review, we have shown that if \( t \geq T \), then \( \alpha_n L_n(s_n|\sigma(t)) > \alpha_m L_m(s_m|\sigma(t)) \) implies that \( \sigma_n(t)(s_n) \leq \epsilon(t) \cdot \sigma_m(t)(s_m) \). Consequently, for all \( t \geq T \), \( \sigma(t) \) is an \((\alpha, \epsilon(t))\)-extended proper equilibrium. Since \( \epsilon(t) \to 0 \) and \( \sigma(t) \to \sigma \), \( \sigma \) is an extended proper equilibrium.

**Necessity.** Suppose \( \sigma \) is an extended proper equilibrium. Then there exists some scaling vector \( \alpha \in \mathbb{R}_+^N \) and some sequence \( \sigma(t) \in \times_{n=1}^N \Delta^0(S_n) \) of \((\alpha, \epsilon(t))\)-extended proper equilibria that converge to \( \sigma \). By Proposition 2 of Blume et al. [1991], there is an LPS \( \rho = (p^1, \ldots, p^K) \) on \( \times_{n=1}^N S_n \) such that a subsequence \( \sigma(\tau) \) of \( \sigma(t) \) can be written as \( \sigma(\tau) = r(\tau) \square \rho \) for a sequence \( r(\tau) \in (0, 1)^{K-1} \) with \( r(\tau) \to 0 \). Note that \( p^1 = \sigma \), since \( \sigma(\tau) \to \sigma \). Also,
since $\sigma(\tau) \in \times_{n=1}^{N} \Delta^0(S_n)$, $\rho$ has full support and satisfies strong independence. Finally, we will show that $\rho$ respects within-and-across-person preferences (with the same choice of $\alpha$). Suppose that for some $n, m, s_n \in S_n, s_m \in S_m, s_n^* \in BR_n(\rho_n),$ and $s_m^* \in BR_m(\rho_m),$

$$\left[ \alpha_n \sum_{s_n \in S_n} p_k^k(s_n)[u_n(s_n^*, s_n) - u_n(s_n, s_n)] \right]^K_{k=1}$$

$$> \left[ \alpha_m \sum_{s_m \in S_m} p_k^k_m[u_m(s_m^*, s_m) - u_m(s_m, s_m)] \right]^K_{k=1}$$

By Lemma [25], there exists some $T$ such that for all $\tau \geq T,$

$$\alpha_n \cdot \max_{\hat{s}_n \in S_n} \sum_{s_n, s_n \in S_n} \left( r(\tau) \rho_n(s_n) \right)[u_n(\hat{s}_n, s_n) - u_n(s_n, s_n)]$$

$$> \alpha_m \cdot \max_{\hat{s}_m \in S_m} \sum_{s_m, s_m \in S_m} \left( r(\tau) \rho_m(s_m) \right)[u_m(\hat{s}_m, s_m) - u_m(s_m, s_m)].$$

That is,

$$\alpha_n \cdot \max_{\hat{s}_n \in S_n} \sum_{s_n, s_n \in S_n} \sigma_n(\tau)(s_n)[u_n(\hat{s}_n, s_n) - u_n(s_n, s_n)]$$

$$> \alpha_m \cdot \max_{\hat{s}_m \in S_m} \sum_{s_m, s_m \in S_m} \sigma_m(\tau)(s_m)[u_m(\hat{s}_m, s_m) - u_m(s_m, s_m)],$$

or $\alpha_n L_n(\sigma(\tau)) > \alpha_m L_m(s_m|\sigma(\tau)).$ Let $k = \min \{ l : p_l^l(s_n) > 0 \}.$ Since $\rho$ has full support, $k$ must exist. To accommodate the case of $k = K,$ the following argument remains valid if we define $r_k(\tau) = 0$ for all $\tau.$ Since $r(\tau) \rightarrow 0$ and $\epsilon(\tau) \rightarrow 0,$ there must exist some $\tau^* \geq T$ for which $\epsilon(\tau^*) < [1 - r_k(\tau^*)]p_k^k(s_n).$ Consequently,

$$\sigma_n(\tau^*)(s_n) \geq r_1(\tau^*) \cdots r_{k-1}(\tau^*)[1 - r_k(\tau^*)]p_n^k(s_n)$$

$$> \epsilon(\tau^*) \cdot r_1(\tau^*) \cdots r_{k-1}(\tau^*).$$

Suppose by way of contradiction that $s_m \leq\rho s_n$, so that $\min \{ l : p_l^l(s_m) > 0 \} \geq k,$ and thus $\sigma_m(\tau^*)(s_m) \leq r_1(\tau^*) \cdots r_{k-1}(\tau^*).$ Then we would have $\sigma_n(\tau^*)(s_n) > \epsilon(\tau^*) \cdot \sigma_m(s_m),$ which, taken together with equation [2], would contradict the fact that $\sigma(\tau^*)$ is an $(\alpha, \epsilon(\tau^*))$-extended proper equilibrium. It must therefore be the case that $s_n <\rho s_m.$ In review, we have shown that equation [1] implies that $s_n <\rho s_m,$ which establishes that $\rho$ respects within-and-across-person preferences. \qed
Proof of Theorem 6. Let $\hat{\sigma}$ be a proper equilibrium. Using the characterization of proper equilibrium in Proposition 4, there must exist some LPS $\rho = (p^1, \ldots, p^K)$ on $S_1 \times S_2$ that satisfies strong independence, has full support, respects within-person preferences, and for which $(\rho, \hat{\sigma})$ is a lexicographic Nash equilibrium. For $n \in \{1, 2\}$ and $k \in \{1, \ldots, K\}$, let
\[
\pi_n^k = \{s_n \in S_n : p_n^k(s_n) > 0 \text{ and } p_n^k(s_n) = 0 \text{ for } l < k\}.
\]
Because $\rho$ has full support, $\{\pi_n^1, \ldots, \pi_n^K\}$ is a partition of $S_n$. Because $\rho$ respects within-person preferences, each $s_n \in \pi_n^k$ is optimal for player $n$ (against $\rho_{-n}$) among strategies in $\cup_{l=k}^{K} \pi_n^l$. We can then use Lemma 24 to construct a number $\delta > 0$ such that for all $n \in \{1, 2\}$, $k \in \{1, \ldots, K\}$, $r_{-n} \in (0, \delta)^{K-1}$, $s_n \in \pi_n^k$, and $s'_n \in \cup_{l=k}^{K} \pi_n^l$,
\[
\sum_{s_n \in S_n} (r_{n} \triangle \rho_{-n})(s_{-n}) [u_n(s_n, s_{-n}) - u_n(s'_n, s_{-n})] \geq 0.
\]
Next, for $n \in \{1, 2\}$, define the following family of sets:
\[
\Delta_{n, \epsilon}^* = \left\{ \sigma_n \in \Delta(S_n) : \sigma_n = r_n \triangle \rho_n \text{ for some } r_n \in \left[ (\epsilon^{2K^2 + K}, \epsilon) \right]^{K-1} \right\}.
\]
In words, $\Delta_{n, \epsilon}^*$ is a set of totally mixed probability distributions that are close to $p_n^1 = \hat{\sigma}_n$ (where the level of closeness is controlled by $\epsilon$). We also define
\[
m_0 = \min_{n \in \{1, 2\}} \min_{k \in \{1, \ldots, K\}} \min \{p_n^k(s_n) : s_n \in \text{supp } p_n^k\}.
\]
We begin by arguing that for any $\alpha \in \mathbb{R}^2_{++}$ and any $\epsilon \in \left( 0, \min \left\{ \delta, \frac{m_0}{m_0 + 1} \right\} \right)$, there exists an $(\alpha, \epsilon)$-extended proper equilibrium in the set $\Delta_{1, \epsilon}^* \times \Delta_{2, \epsilon}^*$. To do so, we define the correspondence $F^\epsilon : \Delta_{1, \epsilon}^* \times \Delta_{2, \epsilon}^* \Rightarrow \Delta_{1, \epsilon}^* \times \Delta_{2, \epsilon}^*$ by
\[
F^\epsilon(\sigma) = \begin{cases} \sigma^* \in \Delta_{1, \epsilon}^* \times \Delta_{2, \epsilon}^* & \text{if } \alpha_n L_n(s_n|\sigma) > \alpha_m L_m(s_m|\sigma) \text{ then } \sigma^*_n(s_n) \leq \epsilon \cdot \sigma^*_m(s_m), \forall n, \forall m, \forall s_n \in S_n, \forall s_m \in S_m \end{cases}.
\]
For any $\sigma$, the points in $F^\epsilon(\sigma)$ are those that satisfy a finite collection of linear inequalities, so $F^\epsilon(\sigma)$ is a closed, convex set.

We next demonstrate that $F^\epsilon(\sigma)$ is nonempty for any $\sigma \in \Delta_{1, \epsilon}^* \times \Delta_{2, \epsilon}^*$. Because $\epsilon < \delta$, we know that $L_n(s_n|\sigma) \leq L_n(s'_n|\sigma)$ for any $s_n \in \pi_n^k$, $s'_n \in \cup_{l=k}^{K} \pi_n^l$. For $k \in \{1, \ldots, K\}$, let
\[
\eta_n^k = \sum_{m=1}^{2} \max \{l : \alpha_n L_n(s_n|\sigma) > \alpha_m L_m(s_m|\sigma) \forall s_n \in \pi_n^k, \forall s_m \in \pi_m^l\}.
\]
Then for $k \in \{1, \ldots, K-1\}$, let
\[
\eta^k_n = \epsilon^{1+(K+1)(\eta^{k+1}_n - \eta^k_n)},
\]
For all $n \in \{1,2\}$ and all $k \in \{1, \ldots, K-1\}$, $\eta^{k+1}_n \geq \eta^k_n$, and so $r^k_n = \epsilon^{1+(K+1)(\eta^{k+1}_n - \eta^k_n)} \leq \epsilon < \delta$. Additionally, $\eta^{k+1}_n \leq 2K - 1$, and so $r^k_n = \epsilon^{1+(K+1)(\eta^{k+1}_n - \eta^k_n)} \geq 2^{2K+K}$. From the previous two inequalities we conclude $r_n \Delta_{n,\epsilon}$. Next, suppose that $\alpha_n L_n(s_n | \sigma) > \alpha_m L_m(s_m | \sigma)$. Let $k$ and $l$ be the indices for which $s_n \in \pi^k_n$ and $s_m \in \pi^l_m$. We must then have $\eta^k_n \geq \eta^l_m + 1$. Consequently,
\[
\begin{align*}
r_n \Delta_{n,\epsilon}(s_n) &\leq r^1_n \cdots r^{k-1}_n \\
&= \epsilon^{k-1+(K+1)\eta^k_n} \\
&\leq \epsilon^{l+(K+1)\eta^l_m} (1 - \epsilon) m_0 \cdot \frac{\epsilon^{k+K-l}}{(1-\epsilon)m_0} \\
&\leq \epsilon^{l+(K+1)\eta^l_m} (1 - \epsilon) m_0 \\
&\leq \epsilon^{l+(K+1)\eta^l_m} (1 - \epsilon) m_0 \cdot (1 - r^l_m)m_0 \\
&= \epsilon \cdot r^1_m \cdots r^{l-1}_m (1 - r^l_m)m_0 \\
&\leq \epsilon \cdot r_m \Delta_{m,\epsilon}(s_m).
\end{align*}
\]
In the above, the fifth step follows from $K \geq l$ and $k \geq 1$; the sixth step follows from $\epsilon < \frac{m_0}{1-m_0}$; and the seventh step follows from $r^l_m \leq \epsilon$. To deal with the case where $l = K$, the above remains correct if we set $r^K_m = 0$.

To review, we have shown (i) $r_n \Delta_{n,\epsilon} \in \Delta_{n,\epsilon}^*$ and (ii) if $\alpha_n L_n(s_n | \sigma) > \alpha_m L_m(s_m | \sigma)$, then $r_n \Delta_{n,\epsilon}(s_n) \leq \epsilon \cdot r_m \Delta_{m,\epsilon}(s_m)$. Thus, $(r_1 \Delta_{1,\epsilon}, r_2 \Delta_{2,\epsilon}) \in \mathcal{F}_\epsilon(\sigma)$, and so $\mathcal{F}_\epsilon(\sigma)$ is nonempty, as desired.

Finally, continuity of each $L_m(s_m | \cdot)$ implies that $\mathcal{F}_\epsilon(\cdot)$ must be upper-semicontinuous. Therefore $\mathcal{F}_\epsilon$ satisfies all the conditions of the Kakutani Fixed Point Theorem [Kakutani1941], so there exists some $\sigma^\epsilon \in \Delta_{1,\epsilon}^* \times \Delta_{2,\epsilon}^*$ such that $\sigma^\epsilon \in \mathcal{F}_\epsilon(\sigma^\epsilon)$. By the definition of $\mathcal{F}_\epsilon$, this point is an $\epsilon$-extended proper equilibrium.

So for all $\epsilon \in \left(0, \min \left\{ \delta, \frac{m_0}{m_0+1} \right\} \right)$, there exists an $(\alpha, \epsilon)$-extended proper equilibrium $\sigma^\epsilon \in \Delta_{1,\epsilon}^* \times \Delta_{2,\epsilon}^*$. Because $\sigma^\epsilon_n \in \Delta_{n,\epsilon}^*$ for all $n \in \{1,2\}$, we must have $|\sigma^\epsilon_n(s_n) - p^\epsilon_n(s_n)| \leq \epsilon \forall s_n \in S_n$, and so $\lim_{\epsilon \to 0} \sigma^\epsilon_n = p^\epsilon_n = \hat{\sigma}_n$. 

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Consequently, this sequence of \((\alpha, \epsilon)\)-extended proper equilibria converges to \(\hat{\sigma}\) as \(\epsilon \to 0\), which establishes that \(\hat{\sigma}\) is extended proper.

\[\square\]

**Proof of Theorem\(^7\)** Suppose that \(\sigma\) is an extended proper equilibrium. By Proposition\(^5\), there must exist some LPS \(\rho\) that satisfies strong independence, has full support, respects within-and-across-person preferences, and for which \((\rho, \sigma)\) is a lexicographic Nash equilibrium.

Let \(n, m \in \{1, \ldots, N\}\) with \(n \neq m\), let \(\hat{\sigma}_n \in \Delta(S_n)\), and let \(\hat{s}_m \in BR_m(\sigma_{-m})\). The elements of \(\times_{i \neq n} S_i\) can be partitioned into the following four sets:

\[
A = \left\{(s_m, s_{-nm}) \mid s_m = \hat{s}_m \text{ and } \sigma_i(s_i) > 0 \forall i \notin \{n, m\}\right\}
\]

\[
B = \left\{(s_m, s_{-nm}) \mid s_m \in BR_m(\sigma_{-m}) \setminus \hat{s}_m \text{ and } \sigma_i(s_i) > 0 \forall i \notin \{n, m\}\right\}
\]

\[
C = \left\{(s_m, s_{-nm}) \mid s_m \notin BR_m(\sigma_{-m}) \text{ or } \exists i \notin \{n, m\} \text{ s.t. } \sigma_i(s_i) = 0\right\}
\]

\[
D = \left\{(s_m, s_{-nm}) \mid s_m = \hat{s}_m \text{ and } \sigma_i(s_i) = 0 \forall i \notin \{n, m\} \text{ s.t. } \sigma_i(s_i) = 0\right\}
\]

Let \(s'_{-n} \in A\) and let \(s''_{-n} \in C\). For some \(j \neq n\), we must have \(s'_j \notin BR_j(\sigma_{-j})\). On the other hand, \(s'_m = \hat{s}_m \in BR_m(\sigma_{-m})\). Therefore, since \(\rho\) respects within-and-across-person preferences, it must be the case that \(s'_m \succ_{\rho} s''_j\). In addition, because \(\sigma_i(s'_i) > 0 \forall i \notin \{n, m\}\), we must have \(p_{-nm}(s'_{-nm}) > 0\), and therefore \(s'_{-nm} \succeq_{\rho} s''_{-nm}\). Then by Lemma\(^27\)(ii), we must have \(s'_{-n} \succ_{\rho} s''_{-n}\).

Let \(s'_{-n} \in A\) and let \(s''_{-n} \in D\). Because \(\sigma_i(s'_i) > 0 \forall i \notin \{n, m\}\), we must have \(p_{-nm}(s'_{-nm}) > 0\). On the other hand, because \(\sigma_i(s''_i) = 0\) for some \(i \notin \{n, m\}\), we must have \(p_{-nm}(s''_{-nm}) = 0\). Therefore \(s'_{-nm} \succ_{\rho} s''_{-nm}\). Finally, because \(s'_m = s''_m = \hat{s}_m\), we can apply Lemma\(^27\)(i) to obtain \(s'_{-n} \succ_{\rho} s''_{-n}\).

As a consequence of the fact that all elements of \(A\) are infinitely more likely under the LPS \(\rho\) than any element of either \(C\) or \(D\), along with the fact that \(\rho\) has full support, the criterion given in the theorem is a necessary condition for \(\sigma_n\) to lexicographically outperform \(\hat{\sigma}_n\) against \(\rho_{-n}\), which is a necessary condition for \((\rho, \sigma)\) to be a lexicographic Nash equilibrium. \[\square\]
A.3 Test-Set Equilibrium

Proof of Proposition 8 Let \( \sigma \) be a strategy profile for which \( \sigma_n \) is weakly dominated by \( \hat{\sigma}_n \) in this way. Therefore,

\[
  u_n(\hat{\sigma}_n, \hat{s}_m, \sigma_{-nm}) > u_n(\sigma_n, \hat{s}_m, \sigma_{-nm}).
\]

Similarly, we also have that for any \( s_m \in \text{BR}_m(\sigma_{-m}) \setminus \hat{s}_m \), \( u_n(\sigma_n, s_m, \sigma_{-nm}) \geq u_n(\sigma_n, \hat{s}_m, \sigma_{-nm}) \). Consequently, we must have that \( \sigma_m(\hat{s}_m) = 0 \), since otherwise these inequalities imply \( u_n(\hat{\sigma}_n, \sigma_{-n}) > u_n(\sigma_n, \sigma_{-n}) \), contradicting Nash equilibrium. Therefore \( (\hat{s}_m, \sigma_{-nm}) \in T_n(\sigma) \subseteq (\hat{s}_m, \sigma_{-nm}) \cup \Delta(B) \).

Consequently, \( \sigma_n \) is weakly dominated by \( \hat{\sigma}_n \) against \( T_n(\sigma) \), and thus \( \sigma \) is not a test-set equilibrium. \( \square \)

Proof of Proposition 9 Claim (i). This follows immediately from Theorem 10 and the existence of proper equilibria in finite normal form games.

Claim (ii). We use the fact that any potential game is strategically equivalent to a game in which \( u_n(s) = P(s) \) \( \forall s \in \times_{n=1}^N S_n \), where \( P \) is referred to as the potential function of the game. We demonstrate that any \( s^* \in \arg \max_{s \in \times_{n=1}^N S_n} P(s) \) is a pure strategy test-set equilibrium. It is well-known that such a strategy profile is a Nash equilibrium, so it only remains to check the test-set condition.

Let \( n \in \{1, \ldots, N\} \) and \( \sigma_{-n} \in T_n(s^*) \). Then \( \exists m \neq n \) and \( \hat{s}_m \in \text{BR}_m(s^*_{-m}) \) for which \( \sigma_{-n} = (\hat{s}_m, s^*_{-nm}) \). Let \( \hat{P} = \max_{s \in \times_{n=1}^N S_n} P(s) \). Since \( \hat{s}_m \in \text{BR}_m(s^*_{-m}) \), we have \( P(s^*_n, \hat{s}_m, s^*_{-nm}) = P(s^*_n, s^*_m, s^*_{-nm}) = \hat{P} \). Moreover, for any \( \hat{\sigma}_n \in \Delta(S_n) \), \( P(\hat{\sigma}_n, \hat{s}_m, \sigma_{-nm}) \leq \hat{P} \). Therefore, \( P(s^*_n, \sigma_{-n}) \geq P(\hat{\sigma}_n, \sigma_{-n}) \), as desired.

Claim (iii). We first show that for three-player games in which each player has two pure strategies, every extended proper equilibrium is a test-set equilibrium. The result will then follow from Theorem 1

Consider a three player game with strategy sets \( S_n = \{a_n, b_n\} \). Suppose \( \sigma \) is an extended proper equilibrium of this game that fails the test-set condition. Then without loss of generality, there exists \( \hat{\sigma}_1 \in \Delta(S_1) \) that weakly dominates \( \sigma_1 \) against \( T_1(\sigma) \). Also without loss of generality, let \( a_2 \in \text{BR}_2(\sigma_{-2}) \) be such that

\[
  u_1(\hat{\sigma}_1, a_2, \sigma_3) > u_1(\sigma_1, a_2, \sigma_3). \tag{3}
\]

Now if \( \sigma_2(a_2) = 1 \), then (3) is a contradiction to Nash equilibrium. Therefore \( \sigma_2(a_2) < 1 \), so \( b_2 \in \text{BR}_2(\sigma_{-2}) \), and so by the failure of player 1’s test-set
Proof of Theorem 10. Let \( \sigma = (\sigma_1, \sigma_2) \) be a proper equilibrium. Using the characterization of proper equilibrium in Proposition 4 there must exist some LPS \( \rho = (p^1, \ldots, p^K) \) on \( S_1 \times S_2 \) that satisfies strong independence, has full support, respects within-person preferences, and for which \((\rho, \sigma)\) is a lexicographic Nash equilibrium.

Suppose that \( \sigma \) is not a test-set equilibrium. Then without loss of generality, there exists \( \tilde{\sigma}_1 \in \Delta(S_1) \) and \( \tilde{s}_2 \in BR_2(\sigma_1) \) such that (i) \( u_1(\tilde{\sigma}_1, \tilde{s}_2) > u_1(\sigma_1, \tilde{s}_2) \), and (ii) \( u_1(\tilde{\sigma}_1, s_2) \geq u_1(\sigma_1, s_2) \) \( \forall s_2 \in BR_2(\sigma_1) \). Because \( \sigma_1 \in BR_1(\rho_2) \) and because \( \rho \) has full support, there must exist some \( \tilde{s}_2 \geq_{\rho} \hat{s}_2 \) for which \( u_1(\tilde{\sigma}_1, \hat{s}_2) < u_1(\sigma_1, \hat{s}_2) \). By the contrapositive of (ii), \( \hat{s}_2 \not\in BR_2(\sigma_1) \).

Because \( \hat{s}_2 \in BR_2(\sigma_1) \) and because \( \rho \) respects within-person preferences, we obtain \( \hat{s}_2 >_{\rho} \hat{s}_2 \), a contradiction. \( \square \)
A.4 Generalized Second Price Auction

A.4.1 Technical Lemmas

The following lemmas will be useful in proving results about GSP auctions.

Lemma 28 shows that for sufficiently small values of $\gamma > 0$, there will be no ties among the highest $\min\{I + 1, N\}$ bids in any pure equilibrium. Lemma 29 uses Assumption (i) to establish that best responses are allocation-preserving for almost every $\gamma \in \mathbb{R}$.

**Lemma 28.** For all $\gamma \in \left[0, \min\left\{\frac{\Delta_v}{2}, \frac{\Delta_v \Delta \kappa}{8\kappa_1}\right\}\right]$, if $b$ is a pure equilibrium of the GSP auction with bid spaces $B^i$, then for all $i \in \{1, \ldots, \min\{I, N - 1\}\}$, $b(i) > b(i+1)$.

**Proof of Lemma 28.** Let $\gamma \in \left(0, \min\left\{\frac{\Delta_v}{2}, \frac{\Delta_v \Delta \kappa}{8\kappa_1}\right\}\right)$ and $i \in \{1, \ldots, \min\{I, N - 1\}\}$. We show this result for this value of $\gamma$. That it also applies for $\gamma = 0$ can be proven using similar methods. By Assumption (i), $\kappa_i > \kappa_{i+1}$.

Suppose $b(i) = \ldots = b(i+K) = b^*$ is a maximal tie (i.e. there are exactly $K + 1$ bidders who bid $b^*$, where $K \geq 1$). Any bidder with per-click value $v$ who is part of the tie earns the payoff

$$\frac{1}{K+1} \left[ \kappa_{i+K} \left( v - b^{(i+K+1)} \right) + \sum_{k=0}^{K-1} \kappa_{i+k}(v - b^*) \right].$$

If $b^* > b^{(i+K+1)} + \gamma$, then by lowering his bid by $\gamma$ the bidder would earn the payoff

$$\kappa_{i+K} \left( v - b^{(i+K+1)} \right).$$

And by raising his bid by $\gamma$ the bidder would earn a payoff of at least

$$\kappa_i \left( v - b^* - \gamma \right).$$

Let $v' < v''$ be the per-click values of two of the tied bidders. We prove the result by contradiction, considering two cases.

**Case 1:** $b^* \leq b^{(i+K+1)} + \gamma$. We must have $v' \geq b^* - \gamma$, or else this bidder would be better off bidding zero. Therefore $v'' \geq v' + \Delta_v \geq b^* - \gamma + \Delta_v \geq$

---

\(\Delta_v = \min_{n \in \{1, \ldots, N-1\}} \{v_n - v_{n+1}\}\)

\(\Delta \kappa = \min_{i \in \{1, \ldots, I\}} \{\kappa_i - \kappa_{i+1}\}\)

By Assumption (i), $\Delta \kappa > 0$. By Assumption (ii), $\Delta_v > 0$.

---
We then show that the \( b'' \) bidder would benefit by raising his bid to \( b^* + \gamma \).\(^{27}\) The incremental profit is at least

\[
\kappa_i (v'' - b^* - \gamma) - \frac{1}{K+1} \left[ \sum_{k=0}^{K-1} [\kappa_{i+k} (v'' - b^*)] + \kappa_{i+K} \left( v'' - b^{(i+K+1)} \right) \right] \\
\geq \kappa_i (v'' - b^* - \gamma) - \frac{1}{K+1} \kappa_i (v'' - b^*) \\
- \frac{K-1}{K+1} \kappa_{i+1} (v'' - b^*) - \frac{1}{K+1} \kappa_{i+1} (v'' - b^* + \gamma) \\
= \frac{K}{K+1} (\kappa_i - \kappa_{i+1})(v'' - b^*) - \gamma \left( \kappa_i + \frac{1}{K+1} \kappa_{i+1} \right) \\
\geq \frac{1}{2} \Delta \kappa \Delta v^2 - 2\gamma \kappa_i \\
> 0,
\]

where the last step follows because \( \gamma < \frac{\Delta v^2}{8\kappa_1} \).

**Case 2**: \( b^* > b^{(i+K+1)} + \gamma \). Because \( v' \) does not want to reduce his bid to \( b^* - \gamma \),

\[
\frac{1}{K+1} \left[ \sum_{k=0}^{K-1} [\kappa_{i+k} (v' - b^*)] + \kappa_{i+K} \left( v' - b^{(i+K+1)} \right) \right] \geq \kappa_{i+K} \left( v' - b^{(i+K+1)} \right) \\
\implies K \kappa_i (v' - b^*) + \kappa_{i+K} \left( v' - b^{(i+K+1)} \right) \geq (K+1) \kappa_{i+K} \left( v' - b^{(i+K+1)} \right) \\
\implies \kappa_i (v' - b^*) \geq \kappa_{i+K} \left( v' - b^{(i+K+1)} \right) \\
(7)
\]

As a result we conclude

\[
\kappa_i (v' - b^*) \geq \frac{1}{K+1} \left[ \sum_{k=0}^{K-1} [\kappa_{i+k} (v' - b^*)] + \kappa_{i+K} \left( v' - b^{(i+K+1)} \right) \right] .
\]

We next show that the \( v'' \) bidder would profit by increasing his bid to

\(^{27}\)By assumption, \( \Gamma \geq v'' \) and \( \gamma < \frac{\Delta v^2}{8\kappa_1} \). We have also seen that \( v'' \geq b^* + \frac{\Delta v^2}{2} \). Therefore \( b^* + \gamma \leq v'' - \frac{\Delta v^2}{2} + \gamma \leq v'' \leq \Gamma \), and so \( b^* + \gamma \) is an allowable bid.
The incremental profit is at least

\[ \kappa_i \left( v'' - b^* - \gamma \right) - \frac{1}{K+1} \left[ \sum_{k=0}^{K-1} [\kappa_{i+k}(v'' - b^*)] + \kappa_{i+K} \left( v'' - b^{(i+K+1)} \right) \right] \]

\[ = \kappa_i(v'' - v') - \frac{1}{K+1} \left[ \sum_{k=0}^{K-1} [\kappa_{i+k}(v'' - v')] + \kappa_{i+K} \left( v' - b^{(i+K+1)} \right) \right] \]

\[ + \kappa_i(v' - b^*) - \frac{1}{K+1} \left[ \sum_{k=0}^{K-1} [\kappa_{i+k}(v' - b^*)] + \kappa_{i+K} \left( v' - b^{(i+K+1)} \right) \right] \]

\[ \geq \kappa_i(v'' - v') - \frac{1}{K+1} \kappa_i(v'' - v') - \frac{K}{K+1} \kappa_{i+1} (v'' - v') - \kappa_i \gamma \]

\[ \geq \frac{K}{K+1} (\kappa_i - \kappa_{i+1}) (v'' - v') - \kappa_i \gamma \]

\[ \geq \frac{1}{2} \Delta_v \Delta_v - \kappa_i \gamma \]

\[ > 0, \]

where the last step follows from \( \gamma < \frac{\Delta_v \Delta_v}{8 \kappa_1} \).

\[ \square \]

**Lemma 29.** For almost every \( \gamma > 0 \), in the GSP auction with bid spaces \( B^\gamma \) it is the case that for every bidder \( n \) and any bid profile \( b_{-n} \), it is the case that if \( b'_n, b''_n \in BR_n(b_{-n}) \), then \( (b'_n, b_{-n}) \) and \( (b''_n, b_{-n}) \) generate the same lottery over allocations.

**Proof of Lemma 29** There are only countably many numbers of the form \( \frac{v \sum_{i=1}^{I} q_i \kappa_i}{\sum_{i=1}^{I} q_i \kappa_i} \), where \( \{q_i\}_{i=1}^{I} \in \mathbb{Q} \), \( \{z_i\}_{i=1}^{I} \in \mathbb{Z} \), and \( v \in \{v_1, \ldots, v_N\} \). Therefore, almost every \( \gamma > 0 \) cannot be expressed in this form. We prove the statement for those values of \( \gamma \).

Given any \( n \) and any fixed \( b_{-n} \), there are integers \( \{z_i\}_{i=1}^{I} \in \mathbb{Z} \) such that the set of expected payoffs obtainable by bidder \( n \), given \( b_{-n} \), lie in the linear space over \( \mathbb{Q} \) spanned by the set \( \{\kappa_1(v_n - \gamma z_1), \ldots, \kappa_I(v_n - \gamma z_I)\} \). If bidder \( n \) is to be indifferent between two different lotteries over allocations that he may receive, then this set must be linearly dependent over \( \mathbb{Q} \). Equivalently,

\[ b^* + \gamma \geq b^{(i+K+1)} \]

or else the \( v' \) bidder would be better off by bidding zero. Equation (7) then implies that \( v' \geq b^* \). By assumption, \( \Gamma \geq v'' \geq v' + \Delta_v \) and \( \gamma < \frac{\Delta_v}{8 \kappa_1} \). Consequently, \( b^* + \gamma \leq v' + \gamma \leq \Gamma - \Delta_v + \gamma \leq \Gamma \), and so \( b^* + \gamma \) is an allowable bid.

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28We must show that \( b^* + \gamma \) is an allowable bid. To see this, notice that we must have \( v' \geq b^{(i+K+1)} \), or else the \( v' \) bidder would be better off by bidding zero. Equation (7) then implies that \( v' \geq b^* \). By assumption, \( \Gamma \geq v'' \geq v' + \Delta_v \) and \( \gamma < \frac{\Delta_v}{8 \kappa_1} \). Consequently, \( b^* + \gamma \leq v' + \gamma \leq \Gamma - \Delta_v + \gamma \leq \Gamma \), and so \( b^* + \gamma \) is an allowable bid.
there must exist \( \{ q_i \}_{i=1}^I \in \mathbb{Q} \), not all zero, such that
\[
\sum_{i=1}^I q_i \kappa_i (v_n - \gamma z_i) = 0
\]

\[\Rightarrow \gamma \sum_{i=1}^I q_i z_i \kappa_i = v_n \sum_{i=1}^I q_i \kappa_i \tag{8}\]

Because \( v_n > 0 \), \( \{ q_i \}_{i=1}^I \in \mathbb{Q} \) are not all zero, and by Assumption 1(i), \( \{ \kappa_1, \ldots, \kappa_N \} \) is linearly independent over \( \mathbb{Q} \), the right hand side of (8) is nonzero. We must then have \( \sum_{i=1}^I q_i z_i \kappa_i \neq 0 \), and so we can write

\[
\gamma = \frac{v_n \sum_{i=1}^I q_i \kappa_i}{\sum_{i=1}^I q_i z_i \kappa_i},
\]

which is a contradiction to the earlier assumption on \( \gamma \). Therefore, if \( b'_n, b''_n \in BR_n(b_{-n}) \), then \( (b'_n, b_{-n}) \) and \( (b''_n, b_{-n}) \) generate the same lottery over allocations for bidder \( n \). They therefore also generate the same lottery over allocations for all bidders. \( \square \)

**Lemma 30.** For every \( \gamma \in \left[ 0, \min \left\{ \frac{\Delta_v}{2}, \frac{\Delta_v \Delta_{\kappa_1}}{8 \kappa_1} \right\} \right) \) and every \( \delta \in \left[ 0, \Delta_v \Delta_{\kappa_1} \right) \), every \( \delta \)-locally envy-free equilibrium of the GSP auction with bid spaces \( B^\gamma \) is efficient.

**Proof of Lemma 30.** Let \( \gamma \in \left[ 0, \min \left\{ \frac{\Delta_v}{2}, \frac{\Delta_v \Delta_{\kappa_1}}{8 \kappa_1} \right\} \right) \) and \( \delta \in \left[ 0, \Delta_v \Delta_{\kappa_1} \right) \). Let \( b \) be a \( \delta \)-locally envy-free equilibrium. By Lemma 28 for all \( i \in \{ 1, \ldots, \min\{I, N-1\} \} \), \( b^{(i)} > b^{(i+1)} \), and so \( G(i) \) is a singleton with element \( g(i) \). To demonstrate efficiency, it will suffice to show that \( v_{g(i)} \geq v_{g(i+1)} \).

By \( \delta \)-locally envy-freeness,

\[
\kappa_i \left( v_{g(i+1)} - b^{(i+1)} \right) - \kappa_{i+1} \left( v_{g(i+1)} - b^{(i+2)} \right) \leq \delta.
\]

Additionally, because \( g(i) \) does not wish to deviate to \( b^{(i+1)} \),

\[
\kappa_i \left( v_{g(i)} - b^{(i+1)} \right) - \kappa_{i+1} \left( v_{g(i)} - b^{(i+2)} \right) \geq 0.
\]

Subtracting,

\[
(\kappa_i - \kappa_{i+1}) \left( v_{g(i+1)} - v_{g(i)} \right) \leq \delta.
\]

If \( v_{g(i)} < v_{g(i+1)} \), then the left-hand side is greater than or equal to \( \Delta_v \Delta_{\kappa_1} \), while the right-hand side is less than the same quantity, a contradiction. \( \square \)
A.4.2 Main Results

Proof of Theorem 11

Let $\bar{\gamma} = \min \left\{ \frac{\Delta_v}{2}, \frac{\Delta_v \Delta_k}{8c} \right\}$. Let $\gamma \in (0, \bar{\gamma})$ be such that the conclusion of Lemma 29 holds. Suppose $b$ is a pure equilibrium of the GSP auction with bid spaces $\mathcal{B}^*$ that fails the test-set condition. We argue that $b$ is not extended proper. By Lemmas 28 and 29, we can characterize the best response sets of the bidders as follows. Bidder $g(1)$’s best response set consists of all feasible bids exceeding $b^{(2)}$. For $i \in \{2, \ldots, \min\{I, N-1\}\}$, bidder $g(i)$’s best response set consists of all feasible bids in the interval $(b^{(i+1)}, b^{(i-1)})$. For $i \in \{\min\{I+1, N\}, \ldots, N\}$, the best response set of any bidder $g(i) \in G(i)$ consists of all feasible bids below $b^{(\min\{I,N-1\})}$. Because $b$ fails the test-set condition, there exists some bidder, whom we denote $g^*$, for whom some alternate bid $\hat{b}$ weakly dominates $b^{(i^*)}$ against $T_{g(i^*)}(b)$. Obviously, $\hat{b}$ is a best response for $g(i^*)$ and also $\hat{b} \neq b^{(i^*)}$. We then consider three cases.

(i) First, suppose that bidder $g(\min\{i^*-1, I\})$ plays $b^{(\min\{i^*,I+1\})} + \gamma$, and no other opponents deviate. If $g(i^*)$ bids $b^{(i^*)}$, then he wins position $i^*$ at price $b^{(i^*+1)}$. We compare whether bidder $g(i^*)$ is better off by playing $b^{(i^*)}$ or $b^{(\min\{i^*,I+1\})} + \gamma$ against two classes of bid profiles of his opponents:

(i) Second, suppose that bidder $g(\min\{i^*-1, I\})$ plays $b^{(\min\{i^*,I+1\})} + \gamma$, and all other bidders play best responses. We show that $b^{(\min\{i^*,I+1\})} + \gamma$ and $b^{(i^*)}$ achieve the same expected utility in this case. It is not a best response for any bidder $g(i)$, $i < \min\{i^*-1, I\}$ to bid $b^{(\min\{i^*,I+1\})} + \gamma$ or lower. Similarly, it is not a best response for $g(\min\{i^*-1, I\})$ to bid $b^{(\min\{i^*,I+1\})}$ or lower. Thus, in this case, $\min\{i^*-1, I\}$ bidders will bid above $b^{(\min\{i^*,I+1\})} + \gamma$. In addition, if $I > i^*-1$, then we also know that it is not a best response for any bidder $g(i)$, $i > i^*$ to bid $b^{(i^*)}$ or higher, so that $N-i^*$ bidders will bid below $b^{(i^*)}$. Consequently, $g(i^*)$ will receive the same profit whether he bids $b^{(\min\{i^*,I+1\})} + \gamma$ or $b^{(i^*)}$ in this case.

Moreover, $b^{(\min\{i^*,I+1\})} + \gamma$ is a best response for bidder $g(\min\{i^*-1, I\})$. Therefore, by Proposition 7, $b$ is not an extended proper equilibrium.
Case 2: \( \hat{b} < b(i^*) \) and \( i^* < I \). It must then be the case that \( g(i^*) \) prefers winning position \( i^* + 1 \) at price \( b(i^* + 2) \) to winning position \( i^* \) at price \( b(i^*) - \gamma \).

We compare whether bidder \( g(i^*) \) is better off by playing \( b(i^*) \) or \( b(i^*) - \gamma \) against two classes of bid profiles of his opponents:

(i) First, suppose that bidder \( g(i^* + 1) \) plays \( b(i^*) - \gamma \), and no other opponents deviate. If \( g(i^*) \) bids \( b(i^*) \), then he wins position \( i^* \) at price \( b(i^*) - \gamma \). If \( g(i^*) \) bids \( b(i^*) - \gamma \), then he either obtains the former outcome or wins position \( i^* + 1 \) at price \( b(i^* + 2) \), each with probability one half. Therefore \( b(i^*) - \gamma \) achieves strictly higher expected utility than \( b(i^*) \) in this case.

(ii) Second, suppose that bidder \( g(i^* + 1) \) plays a best response other than \( b(i^*) - \gamma \), and all other bidders play best responses. We show that \( b(i^*) - \gamma \) and \( b(i^*) \) achieve the same expected utility in this case. It is not a best response for any bidder \( g(i) \), \( i < i^* \) to bid \( b(i^*) \) or lower. It is not a best response for any bidder \( g(i) \), \( i > i^* + 1 \) to bid \( b(i^*) - \gamma \) or higher. Similarly, it is not a best response for \( g(i^* + 1) \) to bid \( b(i^*) \) or higher. Thus, in this case, \( i^* - 1 \) bidders will bid above \( b(i^*) \), and \( N - i^* \) bidders will bid below \( b(i^*) - \gamma \). Consequently, \( g(i^*) \) will receive the same profit whether he bids \( b(i^*) - \gamma \) or \( b(i^*) \) in this case.

Moreover, \( b(i^*) - \gamma \) is a best response for bidder \( g(i^* + 1) \). Therefore, by Proposition 7, \( b \) cannot be an extended proper equilibrium.

Case 3: \( \hat{b} < b(i^*) \) and \( i^* \geq I \). It must then be the case that \( g(i^*) \) prefers winning position \( i^* + 1 \) at price \( b(i^* + 2) \) to winning position \( i^* \) at price \( b(i^*) - \gamma \).

This implies that \( i^* = I \) and \( i^* < N \). We compare whether bidder \( g(i^*) \) is better off by playing \( b(i^*) \) or \( b(i^*) - \gamma \) against two classes of bid profiles of his opponents:

(i) First, suppose that some bidder \( g(i) \), \( i > i^* \) plays \( b(i^*) - \gamma \), and all other bidders play best responses. If \( g(i^*) \) bids \( b(i^*) \), then he wins position \( i^* \) at price \( b(i^*) - \gamma \). If \( g(i^*) \) bids \( b(i^*) - \gamma \), then he either obtains the former outcome with probability \( \frac{1}{K+1} \) or fails to win a position (since \( i^* = I \)), with probability \( \frac{K}{K+1} \), where \( K \) is the number of bidders \( g(i) \), \( i > i^* \) who play \( b(i^*) - \gamma \). Therefore \( b(i^*) - \gamma \) achieves strictly higher expected utility than \( b(i^*) \) in this case.

(ii) Second, suppose that all bidders play best responses, where no bidders \( g(i) \), \( i > i^* \) play \( b(i^*) - \gamma \). We show that \( b(i^*) - \gamma \) and \( b(i^*) \) achieve the same expected utility in this case. It is not a best response for any
bidder \( g(i), i < i^* \) to bid \( b^{(i^*)} \) or lower. It is not a best response for any bidder \( g(i), i > i^* + 1 \) to bid \( b^{(i^*)} \) or higher. Thus, in this case, \( i^* - 1 \) bidders will bid above \( b^{(i^*)} \), and \( N - i^* \) bidders will bid below \( b^{(i^*)} - \gamma \). Consequently, \( g(i^*) \) will receive the same profit whether he bids \( b^{(i^*)} - \gamma \) or \( b^{(i^*)} \).

Moreover, \( b^{(i^*)} - \gamma \) is a best response for bidder \( g(i^* + 1) \). Therefore, by Proposition 7, \( b \) cannot be an extended proper equilibrium.

Proof of Theorem 12. Fix \( \delta > 0 \). Let \( \bar{\gamma} = \min \left\{ \frac{\Delta_v}{2}, \frac{\Delta_v \Delta_k}{8k_1}, \frac{\delta}{k_1} \right\} \). Let \( \gamma \in (0, \bar{\gamma}) \) be such that the conclusions of Lemma 29 holds. We prove the theorem for this value of \( \gamma \). That it also applies for \( \gamma = 0 \) can be proven using similar methods.

Suppose that \( b \) is a pure extended proper equilibrium of the GSP auction with bid spaces \( B^g \) that is not \( \delta \)-locally envy-free. By Lemmas 28 and 29, we can characterize the best response sets of the bidders as follows. Bidder \( g(1) \)'s best response set consists of all feasible bids exceeding \( b^{(2)} \). For \( i \in \{2, \ldots, \min\{I, N - 1\}\} \), bidder \( g(i) \)'s best response set consists of all feasible bids in the interval \( (b^{(i+1)}, b^{(i-1)}) \). For \( i \in \{\min\{I + 1, N\}, \ldots, N\} \), the best response set of any bidder \( g(i) \in G(i) \) consists of all feasible bids below \( b^{(\min\{I, N-1\})} \). Let \( i^* \) be an index for which the locally envy-free inequality is violated by more than \( \delta \), so that for some member of \( G(i^*) \), whom we denote \( g(i^*) \),

\[
\kappa_{i^* - 1} \left[ v_{g(i^*)} - b^{(i^*)} \right] - \kappa_{i^*} \left[ v_{g(i^*)} - b^{(i^* + 1)} \right] > \delta.
\]

Henceforth, for notational simplicity, we use \( b^* \) to denote \( b^{(i^*)} \). Note that the above inequality requires \( i^* \leq I + 1 \), which means that \( b^* + \gamma \) is a best response for bidder \( g(i^* - 1) \). We now compare whether bidder \( g(i^*) \) is better off by playing \( b^* \) or \( b^* + \gamma \) against two classes of bid profiles by his opponents:

(i) First, suppose that bidder \( g(i^* - 1) \) plays \( b^* + \gamma \), and no other opponents deviate. In this case, bidder \( g(i^*) \) would earn the payoff \( \kappa_{i^*} \left[ v_{g(i^*)} - b^{(i^* + 1)} \right] \) from playing \( b^* \), and he would earn the payoff \( \frac{1}{2} \kappa_{i^* - 1} \left[ v_{g(i^*)} - b^* - \gamma \right] + \frac{1}{2} \kappa_{i^*} \left[ v_{g(i^*)} - b^{(i^* + 1)} \right] \) from playing \( b^* + \gamma \). Therefore the incremental payoff to \( g(i^*) \) of playing \( b^* + \gamma \) over \( b^* \)
in this case is
\[
\frac{1}{2} \kappa_{i^* - 1} \left[ v_{g(i^*)} - b^* \right] - \frac{1}{2} \kappa_{i^*} \left[ v_{g(i^*)} - b^{(i^* + 1)} \right] - \frac{1}{2} \kappa_{i^* - 1} \gamma \\
> \frac{1}{2} \delta - \frac{1}{2} \gamma \kappa_1 \\
> 0
\]

(ii) Second, suppose that bidder \(g(i^* - 1)\) plays a best response other than \(b^* + \gamma\), and no other bidders deviate. In this case, \(g(i^*)\) will earn the same profit whether he bids \(b^*\) or \(b^* + \gamma\).

(iii) Third, suppose that one opponent other than \(g(i^* - 1)\) plays a best response other than his equilibrium bid, and that no other bidders deviate. In this case, \(g(i^*)\) will also earn the same profit whether he bids \(b^*\) or \(b^* + \gamma\).

Therefore, \(b^* + \gamma\) weakly dominates \(b^*\) against \(T_{g(i^*)}(b)\), and so \(b\) is not a test-set equilibrium.

\textbf{Proof of Proposition 14.} Let \(\bar{\gamma} = \min \left\{ \frac{\Delta_v}{2}, \frac{\Delta_v \Delta \kappa}{8 \kappa_1} \right\}\). Let \(\gamma \in [0, \bar{\gamma})\) be such that the conclusion of Theorem 12 applies. Let \(b\) be a test-set equilibrium of the GSP auction with bid spaces \(B^j\). By Lemma 28 for all \(i \in \{1, \ldots, \min\{I, N - 1\}\}, b^{(i)} > b^{(i+1)}\) and so \(G(i)\) is a singleton with element \(g(i)\). To demonstrate efficiency, it will suffice to show that \(v_{g(i)} \geq v_{g(i+1)}\).

By Theorem 12, \(b\) is a \(\frac{\Delta_v \Delta \kappa}{8}\)-locally envy free equilibrium. Therefore,
\[
\kappa_i \left( v_{g(i+1)} - b^{(i+1)} \right) - \kappa_{i+1} \left( v_{g(i+1)} - b^{(i+2)} \right) \leq \frac{\Delta \kappa \Delta_v}{8}.
\]

Additionally, because \(g(i)\) does not wish to deviate to \(b^{(i+1)}\),
\[
\kappa_i \left( v_{g(i)} - b^{(i+1)} \right) - \kappa_{i+1} \left( v_{g(i)} - b^{(i+2)} \right) \geq 0.
\]

Subtracting,
\[
(\kappa_i - \kappa_{i+1}) \left( v_{g(i+1)} - v_{g(i)} \right) \leq \frac{\Delta_v \Delta \kappa}{8}.
\]
If \(v_{g(i)} < v_{g(i+1)}\), then this would imply \(\frac{\Delta_v \Delta \kappa}{8} \geq \Delta_v \Delta \kappa\), a contradiction. \(\Box\)
Proof of Proposition 16. Not locally envy-free. Given the parameter values, it is straightforward to verify that \( b_1 > b_2 > b_3 \). We can also bound the following quantities:

\[
\kappa_1(v_2 - b_2) \geq 100(10 - b_2) > 100 \left( 10 \cdot \frac{K_3}{K_2} - \gamma \right) \geq 100 \left( 10 \cdot \frac{1}{4} - 1 \right) = 150
\]

\[
\kappa_2(v_2 - b_3) \leq 4(10 - b_3) \leq 4 \left( 5 + 5 \cdot \frac{K_3}{K_2} + \gamma \right) \leq 4 \left( 5 + 5 \cdot \frac{2}{3} + 1 \right) = \frac{112}{3}
\]

Therefore, \( \kappa_1(v_2 - b_2) - \kappa_2(v_2 - b_3) > \frac{338}{3} \), and so bidder 2’s locally envy-free inequality is violated by at least \( \frac{338}{3} \).

Properness. For ease of notation, we refer to the equilibrium bid profile as \((b_1^*, b_2^*, b_3^*)\) in this section of the proof. Let \( \gamma \in (0, 1) \) be such that the conclusions of Lemma 29 hold. Let \( M = |B^\gamma| = \left\lceil \frac{\Gamma}{\gamma} \right\rceil + 1 \). For all \( \epsilon \in (0, 1) \), define the following sets:

\[
\Delta_{3, \epsilon} = \left\{ \sigma_3 \in \Delta(B^\gamma) \left| \begin{array}{l}
\sigma_3(b_3) \leq \epsilon \cdot \sigma_3(b_3^*) \text{ if } b_3 \neq b_3^* \\
\sigma_3(b_3) \leq \epsilon \cdot \sigma_3(b_3^*) \text{ if } b_3 \notin (0, b_2^*) \text{ and } b_3^* \in (0, b_2^*) \\
\sigma_3(b_3) \geq \frac{\epsilon M}{2M} \\
\end{array} \right. \right\}
\]

\[
\Delta_{2, \epsilon} = \left\{ \sigma_2 \in \Delta(B^\gamma) \left| \begin{array}{l}
\sigma_2(b_2) \leq \epsilon \cdot \sigma_2(b_2^*) \text{ if } b_2 \neq b_2^* \\
\sigma_2(b_2) \leq \epsilon \cdot \sigma_2(b_2^*) \text{ if } b_2 \notin (b_3^*, b_1^*) \text{ and } b_2^* \in (b_3^*, b_1^*) \\
\sigma_2(b_2) \geq \frac{\epsilon M}{2M} \\
\end{array} \right. \right\}
\]

\[
\Delta_{1, \epsilon} = \left\{ \sigma_1 \in \Delta(B^\gamma) \left| \begin{array}{l}
\sigma_1(b_1) \leq \epsilon \cdot \sigma_1(b_1^*) \text{ if } b_1 \neq b_1^* \\
\sigma_1(b_1) \leq \frac{\epsilon M}{2M} \text{ if } b_1 \neq b_1^* \\
\sigma_1(b_1) \geq \frac{\epsilon M}{2M} \\
\end{array} \right. \right\}
\]

\[\text{It is immediate that } b_2 > b_3. \text{ To see that } b_1 > b_2:\]

\[
b_1 - b_2 \geq 15 - \frac{K_2}{K_1}(15 - b_2) - b_2
\]

\[
= 15 \cdot \left( 1 - \frac{K_2}{K_1} \right) + b_2 \cdot \left( \frac{K_2}{K_1} - 1 \right)
\]

\[
\geq 15 \cdot \left( 1 - \frac{K_2}{K_1} \right) + \left[ 10 \left( 1 - \frac{K_2}{K_2} \right) + \gamma \right] \cdot \left( \frac{K_2}{K_1} - 1 \right)
\]

\[
= 5 - \gamma - \frac{K_2}{K_1}(5 - \gamma) + 10 \cdot 3 \left( \frac{1}{K_2} - \frac{1}{K_1} \right)
\]

\[
\geq 5 - 1 - \frac{4}{100} \cdot 5
\]

\[
> 0.
\]
For all $n$ and all $\epsilon$, $\Delta_{n,\epsilon}^*$ is a nonempty and compact subset of $\Delta^0(B^\gamma)$. Let $V_n(b_n|\sigma) = \pi_n(b_n,\sigma_n)$ and define for each player the correspondence $F^\epsilon_n : \times_{m=1}^3 \Delta_{m,\epsilon}^* \rightarrow \Delta_{n,\epsilon}^*$ by

$$F^\epsilon_n(\sigma) = \left\{ \begin{array}{ll}
\sigma^*_n \in \Delta_{n,\epsilon}^* & \text{if } V_n(b_n|\sigma) < V_n(b'_n|\sigma) \\
\sigma^*_n(b_n) \leq \epsilon \cdot \sigma^*_n(b'_n), & \forall b_n, b'_n \in B^\gamma
\end{array} \right.$$  

For all $n$ and for any $\sigma \in \times_{m=1}^3 \Delta_{m,\epsilon}^*$, the points in $F^\epsilon_n(\sigma)$ are those that satisfy a finite collection of linear inequalities, so $F^\epsilon_n(\sigma)$ is a closed, convex set. We next demonstrate that for sufficiently small values of $\epsilon$, $F^\epsilon_n(\sigma)$ is nonempty. Let $\phi_n(b_n)$ be the number of bids $b'_n \in B^\gamma$ such that $V_n(b_n|\sigma) < V_n(b'_n|\sigma)$, and let $P_n^0 = \{|b_n \in B^\gamma : \phi_n(b_n) = 0\}$. Then let

$$\sigma^*_n(b_n) = \left\{ \begin{array}{ll}
\frac{\epsilon \phi_n(b_n) + (3-n)\epsilon}{1 - \sum_{b'_n: \phi_n(b'_n) > 0} \frac{\epsilon \phi_n(b_n) + (3-n)\epsilon}{2M}} & \text{if } \phi_n(b_n) > 0 \\
0 & \text{if } \phi_n(b_n) = 0
\end{array} \right.$$  

It is straightforward (but tedious) to show that if $\epsilon > 0$ is sufficiently small, then for all $\sigma \in \times_{m=1}^3 \Delta_{m,\epsilon}^*$, we have the following: (i) $V_3(b_3^*|\sigma) > V_3(b_3|\sigma) \ \forall b_3 \neq b_3^*$, (ii) $V_3(b_3^*|\sigma) > V_3(b_3|\sigma) \ \forall b_3 \in (0, b_3^*)$, (iii) $V_2(b_2^*|\sigma) > V_2(b_2|\sigma) \ \forall b_2 \neq b_2^*$, (iv) $V_2(b_2^*|\sigma) > V_2(b_2|\sigma) \ \forall b_2 \in (0, b_2^*)$, (v) $V_1(b_1^*|\sigma) > V_1(b_1|\sigma) \ \forall b_1 \neq b_1^*$. As a consequence of these observations, for all $\epsilon > 0$ sufficiently small, the $\sigma^*_n$ defined above lies in $\Delta_{n,\epsilon}^*$. Moreover, by construction, if $V_n(b_n|\sigma) < V_n(b'_n|\sigma)$, then $\sigma^*_n(b_n) \leq \epsilon \cdot \sigma^*_n(b'_n)$. Consequently, $\sigma^*_n \in F_n(\sigma)$, which demonstrates that $F^\epsilon_n(\sigma)$ is nonempty, as desired.

Let $F^\epsilon(\cdot) = \times_{n=1}^3 F^\epsilon_n(\cdot)$. Then $F^\epsilon$ satisfies all the conditions of the Kakutani Fixed Point Theorem [Kakutani, 1941], so there exists some $\sigma^\epsilon \in \times_{n=1}^3 \Delta_{n,\epsilon}^*$ such that $\sigma^\epsilon \in F^\epsilon(\sigma^\epsilon)$. By the definition of $F^\epsilon$, this point is an $\epsilon$-proper equilibrium.

So for all $\epsilon > 0$ sufficiently small, there exists an $\epsilon$-proper equilibrium in $\sigma^\epsilon \in \times_{n=1}^3 \Delta_{n,\epsilon}^*$. By virtue of the fact that $\sigma_n^\epsilon \in \Delta_{n,\epsilon}^*$, we must have $\sigma_n^\epsilon(b_n^*) \in \left[ \frac{1}{1+(3-n)\epsilon}, 1 \right]$, and so $\lim_{\epsilon \rightarrow 0} \sigma^\epsilon_n(b_n^*) = 1$. Consequently, this sequence of $\epsilon$-proper equilibria converges to $(b_1^*, b_2^*, b_3^*)$ as $\epsilon \rightarrow 0$, which establishes that $(b_1^*, b_2^*, b_3^*)$ is proper. \hfill $\Box$

**Proof of Proposition 17.** Let $\gamma \in (0, \frac{1}{2})$ be such that the conclusions of Theorem 14 and Proposition 14 hold.

$\delta$-locally envy-free equilibrium. Suppose $b$ is a $\delta$-locally envy-free equilibrium of the GSP auction with bid spaces $B^\gamma$ for some $\delta \in [0, 1)$. We must
obviously have \((b_1, b_2) \in B^\gamma \times B^\gamma\). That \(b_1 > b_2\) follows from Lemma 30. Next, suppose that \(b_2 > 2\). Then bidder 1’s payoff is \(\kappa_1(2 - b_2) < 0\), so deviating to \(\hat{b}_1 = 0\) would be profitable. Finally, suppose that \(b_2 < 1 - \frac{\delta}{\gamma}\). Then bidder 2 envies bidder 1 by \(\kappa_1(v_1 - b_2) = \kappa_1(1 - b_2) > \delta\), contradicting \(\delta\)-local envy-freeness. Therefore \(b \in E_{LEF}\).

Now suppose that \(b \in E_{LEF}\). It is easily checked that \(b\) is a Nash equilibrium. The amount by which bidder 2 (who wins the second slot) envies bidder 1 (who wins the first slot) is \(\kappa_1(v_1 - b_2) = \kappa_1(1 - b_2) \leq \delta\). Hence, \(b\) is \(\delta\)-locally envy-free.

Test-set equilibrium. Suppose that \(b\) is a pure test-set equilibrium of the GSP auction with bid spaces \(B^\gamma\). We must obviously have \((b_1, b_2) \in B^\gamma \times B^\gamma\). That \(b_1 > b_2\) follows from Proposition 14. We must have \(b_2 \leq 2\) for the same reasons as above. We cannot have \(b_1 > \gamma \left\lceil \frac{2}{\gamma} \right\rceil\), because such a bid is weakly dominated by \(\hat{b}_1 = \gamma \left\lfloor \frac{2}{\gamma} \right\rfloor\) against \(T_1(b)\). We also cannot have \(b_2 < \gamma \left\lceil \frac{1}{\gamma} \right\rceil\), because such a bid is weakly dominated by \(\hat{b}_2 = \gamma \left\lfloor \frac{1}{\gamma} \right\rfloor\) against \(T_1(b)\).

For the reverse direction, it is easily checked that any \(b \in E_{TS}\) is a test-set equilibrium.

Extended proper equilibrium. Suppose that \(b\) is a pure extended proper equilibrium of the GSP auction with bid spaces \(B^\gamma\). By Theorem 11, \(b \in E_{TS}\). We begin by deriving a contradiction with \(b_1 < \gamma \left\lceil \frac{2}{\gamma} \right\rceil\). First, we cannot have \(b_1 < \gamma \left\lceil \frac{2}{\gamma} \right\rceil\) because it is weakly dominated by \(\hat{b}_1 = \gamma \left\lceil \frac{2}{\gamma} \right\rceil\) (extended proper equilibria are trembling-hand perfect, which in two player games cannot involve weakly dominated strategies.) Second, suppose by way of contradiction that \(b_1 = \gamma \left\lceil \frac{2}{\gamma} \right\rceil\). Consider \(b_1 = \gamma \left\lceil \frac{2}{\gamma} \right\rceil\). Both perform equally well against all actions of bidder 2 except \(b_2 = \gamma \left\lceil \frac{2}{\gamma} \right\rceil\) and \(b''_2 = \gamma \left\lceil \frac{2}{\gamma} \right\rceil\). Against the former, \(\hat{b}_1\) outperforms \(b_1\), but against the latter, \(b_1\) outperforms \(\hat{b}_1\). Notice that \(u_2(b_1, b''_2) = \frac{1}{2} \kappa_1 \left(1 - \gamma \left\lfloor \frac{2}{\gamma} \right\rfloor\right)\) and \(u_2(b_1, b''_2) = \kappa_1 \left(1 - \gamma \left\lfloor \frac{2}{\gamma} \right\rfloor\right)\). Since \(\gamma < 1\), \(u_2(b_1, b''_2) < u_2(b_1, b'')\). Because \(b\) is extended proper, there exists, by Proposition 5, an LPS \(\rho\) on \(B^\gamma \times B^\gamma\) that satisfies strong independence, has full support, respects within-and-across-person preferences, and for which \((\rho, b)\) is a lexicographic Nash equilibrium. Since \(u_2(b_1, b''_2) < u_2(b_1, b'')\), we must have \(b''_2 >_{\rho} b''_2\). Therefore \(\hat{b}_1\) lexicographically outperforms \(b_1\), and so \(b_1 \notin BR_1(\rho_{-1})\), which constitutes a contradiction to \((\rho, b)\) being a lexicographic Nash equilibrium. The only remaining possibility is therefore
\[ b_1 = \gamma \left\lceil \frac{2}{\gamma} \right\rceil. \] A similar argument shows that we must have \( b_2 = \gamma \left\lfloor \frac{1}{\gamma} \right\rfloor \).

For the reverse direction, we must show that \((b_1, b_2) = \left( \gamma \left\lceil \frac{2}{\gamma} \right\rceil, \gamma \left\lfloor \frac{1}{\gamma} \right\rfloor \right)\) is an extended proper equilibrium. This can be shown using methods similar to those in the proof of Proposition 16.

**Proof of Proposition 18.** Let \( \gamma \in \left( 0, \frac{5}{808} \right) \) be such that the conclusions of Lemma 29 and Proposition 14 hold. Suppose that \((b_1, b_2, b_3)\) is a pure test-set equilibrium of the GSP auction with bid spaces \( B^\gamma \). By Proposition 14, \( b_1 > b_2 > b_3 \). By Lemma 29, the best responses of bidder 1 are all feasible bids greater than \( b_2 \), and the best responses of bidder 3 are all feasible bids less than \( b_2 \).

We compare the performance of \( b_2 + \gamma \) to that of \( b_2 \) against the elements of \( T_2(b) \). They perform equally well against all elements of \( T_2(b) \), with the exception of when bidder 1 deviates to \( b_2 + \gamma \). In this case, \( b_2 \) generates the payoff \( \kappa_2(v_2 - b_3) \). On the other hand, \( b_2 + \gamma \) generates the payoff \( \frac{1}{2}\kappa_2(v_2 - b_3) + \frac{1}{2}\kappa_1[v_2 - (b_2 + \gamma)] \). Since \( b \) is a test-set equilibrium, \( b_2 + \gamma \) must not generate a higher expected utility than \( b_2 \). Thus,

\[
0 \geq \frac{1}{2} [\kappa_1(v_2 - b_2 - \gamma) - \kappa_2(v_2 - b_3)] \\
\geq \frac{1}{2} \left[ 100 \left( 10 - b_2 - \frac{5}{808} \right) - 40 \right],
\]

which implies \( b_2 \geq 9.59 \).

We next compare the performance of \( b_2 - \gamma \) to that of \( b_2 \) against the elements of \( T_2(b) \). They perform equally well against all elements of \( T_2(b) \), with the exception of when bidder 3 deviates to \( b_2 - \gamma \). In this case, \( b_2 \) generates the payoff \( \kappa_2[v_2 - (b_2 - \gamma)] \). On the other hand, \( b_2 - \gamma \) generates the payoff \( \frac{1}{2}\kappa_2[v_2 - (b_2 - \gamma)] + \frac{1}{2}\kappa_3v_2 \). Since \( b \) is a test-set equilibrium, \( b_2 - \gamma \) must not generate a higher expected utility than \( b_2 \). Thus,

\[
0 \geq \frac{1}{2} [\kappa_3v_2 - \kappa_2(v_2 - b_2 + \gamma)] \\
\geq \frac{1}{2} \left[ 10 - 4 \left( 10 - b_2 + \frac{5}{808} \right) \right],
\]

which implies \( b_2 \leq 7.51 \), a contradiction. \( \square \)

### A.5 First Price Menu Auction

#### A.5.1 Technical Lemmas

The following three lemmas will be used to prove Theorem 22.
Lemma 31. For all $\gamma \geq 0$, the menu auction with bid spaces $\mathcal{B}^\gamma$ possess the following property. Suppose that $\hat{b}_n(x') = b_n(x')$ and $\hat{b}_n(x'') \leq b_n(x'')$, where $x' \neq x''$. If $x(b_n, b_{-n}) = x'$, then $x(b_n, b_{-n}) \neq x''$.

Proof of Lemma 31 Suppose that $x(b_n, b_{-n}) = x'$ and $x(\hat{b}_n, b_{-n}) = x''$. These imply

\[ v_0(x') + \sum_{m=1}^N b_m(x') \geq v_0(x'') + \sum_{m=1}^N b_m(x'') \]

\[ v_0(x'') + \hat{b}_n(x'') + \sum_{m \neq n} b_m(x'') \geq v_0(x') + \hat{b}_n(x') + \sum_{m \neq n} b_m(x'), \]

and because $x' \neq x''$, consistent tie-breaking by the auctioneer requires that at least one of these be strict. Subtracting the second inequality from the first yields $b_n(x') - \hat{b}_n(x') > b_n(x'') - \hat{b}_n(x'').$ However, because $b_n(x') = \hat{b}_n(x')$ and $b_n(x'') \geq \hat{b}_n(x'')$, this is a contradiction. \qed

Lemma 32. For almost all $\gamma > 0$, the menu auction with bid spaces $\mathcal{B}^\gamma$ possesses the following properties.

(i) If $x(b_n, b_{-n}) = x^*$ and $\hat{b}_n(x^*) \neq b_n(x^*)$, then $\pi(b_n, b_{-n}) \neq \pi(\hat{b}_n, b_{-n})$.

(ii) (Non-bossiness). If $\pi(b_n, b_{-n}) \neq \pi(\hat{b}_n, b_{-n})$, then $\pi_n(b_n, b_{-n}) \neq \pi_n(\hat{b}_n, b_{-n})$.

Proof of Lemma 32 By Assumption 2, $\forall n$, $\forall x, x' \in X$ with $x \neq x'$ we have $v_n(x) - v_n(x') \neq 0$. Thus, for almost all $\gamma > 0$, it is the case that $\forall n$, $\forall x, x' \in X$ with $x \neq x'$ we have $v_n(x) - v_n(x') \notin \gamma\mathbb{Z}$. Fix some such value of $\gamma$.

Claim (i). Suppose $x(b_n, b_{-n}) = x^*$ and $\hat{b}_n(x^*) \neq b_n(x^*)$. We then consider two cases. First, suppose $x(\hat{b}_n, b_{-n}) = x^*$. Then

\[ \pi_n(\hat{b}_n, b_{-n}) = v_n(x^*) - \hat{b}_n(x^*) \neq v_n(x^*) - b_n(x^*) = \pi_n(b_n, b_{-n}). \]

Second, suppose $x(\hat{b}_n, b_{-n}) \neq x^*$. By the assumption on $\gamma$, this then means $\pi_n(b_n, b_{-n}) \neq \pi_n(\hat{b}_n, b_{-n})$.

Claim (ii). Suppose that $\pi_n(b_n, b_{-n}) = \pi_n(\hat{b}_n, b_{-n})$. By the assumption on $\gamma$, this then means $x(b_n, b_{-n}) \neq x(\hat{b}_n, b_{-n})$. Since the bids of all bidders $m \neq n$ remain constant, we obtain $\pi_m(b_n, b_{-n}) \neq \pi_m(\hat{b}_n, b_{-n}) \forall m \neq n$. Thus, $\pi(b_n, b_{-n}) = \pi(\hat{b}_n, b_{-n})$. \qed

Lemma 33. For almost all $\gamma > 0$, the menu auction with bid spaces $\mathcal{B}^\gamma$ possesses the following properties. Suppose $b$ is a Nash equilibrium in which
We prove the result for \( \gamma > 0 \) by using similar methods. Let \( b^* \) be a Nash equilibrium of the menu auction with bid spaces \( B^* \). Let \( x^* = x(b) \).

**Proof of Lemma 19.** We prove the result for \( \gamma > 0 \). That it is also true for \( \gamma = 0 \) can be proven using similar methods. Let \( b \) be a Nash equilibrium of the menu auction with bid spaces \( B^* \). Let \( x^* = x(b) \).

**Necessity.** Recall that between any two outcomes that give a bidder the same payoff, we assume the bidder prefers the outcome with the higher total payoff. Therefore, \( \forall n \) it must be the case that \( b_n \in BR_n(b_{-n}) \) if and only if both \( x(b_n, b_{-n}) = x^* \) and \( b_n(x^*) = b_n(x^*) \).

Suppose there exists some \( n \) and some \( \hat{x} \in X \) for which \( v_n(\hat{x}) - b_n(\hat{x}) > v_n(x^*) - b_n(x^*) + \gamma \). Choose some \( m \neq n \) and let

\[
\begin{align*}
\hat{b}_n(x) &= \begin{cases} 
    b_n(x) & \text{if } x \neq \hat{x} \\
    b_n(x) + \gamma & \text{if } x = \hat{x}
\end{cases} \\
\hat{b}_m(x) &= \begin{cases} 
    b_m(x) & \text{if } x \neq \hat{x} \\
    b_m(x) + \gamma \cdot \left[ \frac{1}{\gamma} \left[ v_0(x^*) + B(x^*) - v_0(x) - B(x) \right] \right] & \text{if } x = \hat{x}
\end{cases}
\end{align*}
\]

It is easily checked that the following are true: (i) if \( \hat{b}_k \in BR_k(b_{-k}) \) for some \( k \), then \( \pi_n(\hat{b}_n, \hat{b}_k, b_{-nk}) \geq \pi_n(b_n, \hat{b}_k, b_{-nk}) \), (ii) \( \pi_n(\hat{b}_n, \hat{b}_m, b_{-nm}) > \pi_n(b_n, b_m, b_{-nm}) \).

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\(^{30}\)If \( \hat{x} = x^* \), then we obtain \( \hat{b}_n(x^*) = b_n(x^*) \) from \( \hat{b}_n(\hat{x}) \in [b_n(\hat{x}), v_n(x^*) - v_n(x^*_n) + b_n(x^*_n)] \). If \( \hat{x} \neq x^* \), then we obtain \( b_n(x^*) = b_n(x^*) \) from the definition of \( b_n \).
Proof of Theorem 20. Let \( \bar{b}_n \) having been a test-set equilibrium. Furthermore, letting \( \hat{b}_n \in BR_{m}(b_{-m}) \), we must have \( \hat{b}_n(x^*) = b_n(x^*) \). Since \( \hat{b}_m \in BR_m(b_{-m}) \), we must have \( x(b_n, \hat{b}_m, b_{-nm}) = x^* \). From the previous two conclusions, combined with \( \pi_n(\hat{b}_n, \hat{b}_m, b_{-nm}) \neq \pi_n(b_n, \hat{b}_m, b_{-nm}) \), we have \( x(\hat{b}_n, \hat{b}_m, b_{-nm}) \neq x^* \). Furthermore, letting \( \hat{x} = x(b_n, \hat{b}_m, b_{-nm}) \), we must have \( b_n(\hat{x}) > b_n(\hat{x}) \), which implies \( \hat{b}_n(\hat{x}) \geq b_n(\hat{x}) + \gamma \). We then have

\[
0 < \pi_n(\hat{b}_n, \hat{b}_m, b_{-nm}) - \pi_n(b_n, \hat{b}_m, b_{-nm}) = v_n(\hat{x}) - b_n(\hat{x}) - [v_n(x^*) - b_n(x^*)] \\
\leq v_n(\hat{x}) - b_n(\hat{x}) - \gamma - [v_n(x^*) - b_n(x^*)]
\]

The first step is because \( \pi_n(\hat{b}_n, \hat{b}_m, b_{-nm}) > \pi_n(b_n, \hat{b}_m, b_{-nm}) \). The second step is by definition. The third step is because \( b_n(\hat{x}) \geq b_n(\hat{x}) + \gamma \). We conclude that \( v_n(\hat{x}) - b_n(\hat{x}) > v_n(x^*) - b_n(x^*) + \gamma \). □

Proof of Theorem 20. Let \( \bar{\gamma} = \frac{1}{N} \min_{x \neq x^{opt}} \{ [V(x^{opt}) + v_0(x^{opt})] - [V(x) + v_0(x)] \} \). By Assumption 2, \( \bar{\gamma} > 0 \). Let \( \gamma \in [0, \bar{\gamma}] \). Assume the auctioneer selects some \( x^* \neq x^{opt} \). Then by Lemma 19, we have for all \( n \in \{1, \ldots, N\} \),

\[
b_n(x^{opt}) \geq v_n(x^{opt}) - [v_n(x^*) - b_n(x^*)] - \gamma
\]

Summing,

\[
B(x^{opt}) \geq V(x^{opt}) - V(x^*) + B(x^*) - N\gamma
\]

so

\[
v_0(x^{opt}) + B(x^{opt}) \geq [V(x^{opt}) + v_0(x^{opt})] - [V(x^*) + v_0(x^*)] + B(x^*) + v_0(x^*) - N\gamma
\]

Since \( \gamma \in [0, \bar{\gamma}] \), \( [V(x^{opt}) + v_0(x^{opt})] - [V(x^*) + v_0(x^*)] > N\gamma \), so

\[
v_0(x^{opt}) + B(x^{opt}) > v_0(x^*) + B(x^*)
\]

a contradiction.

Next, we argue that any such equilibrium generates payoffs \( \pi \in C^{\gamma} \). Let \( J \subseteq \{1, \ldots, N\} \). We know that

\[
B(x^{opt}) + v_0(x^{opt}) \geq B(x^J) + v_0(x^J)
\]
so

\[ B_J(x^{opt}) + B_J(x^{opt}) + v_0(x^{opt}) \geq B_J(x^J) + B_J(x^J) + v_0(x^J) \]
\[ \geq B_J(x^J) + v_0(x^J) \]
\[ \geq V_J(x^J) - V_J(x^{opt}) + B_J(x^{opt}) + v_0(x^J) - |J| \gamma \]

where this last inequality comes from the fact that by Lemma 19, \( \forall n \in J, b_n(x^J) - [v_n(x^{opt}) - b_n(x^{opt})] - \gamma \). Rearranging, we see that the equilibrium yields payoffs

\[ \Pi_J = V_J(x^{opt}) - F_J(x^{opt}) \]
\[ \leq [V(x^{opt}) + v_0(x^{opt})] - [V_J(x^J) + v_0(x^J)] + |J| \gamma \]

as desired. \( \square \)

**Proof of Theorem 21.** By an argument almost identical to that given in Bernheim and Whinston (1986, Theorem 2), \( b \) is a Nash equilibrium resulting in the decision \( x^{opt} \).\footnote{Bernheim and Whinston (1986) include the decision as part of the solution concept, defining a Nash equilibrium as a bid profile and a decision. Because there may be multiple decisions that maximize the auctioneer’s total utility, this technique allows them to resolve the auctioneer’s indifference in whichever way is convenient. Our approach is in the same spirit. A Nash equilibrium, for our purposes, consists only of a bid profile. However, as previously stated, we resolve the auctioneer’s indifference in favor of the efficient outcome, \( x^{opt} \).} Furthermore, \( \forall n \in \{1, \ldots, N\} \) and \( \forall x \in X, \)

\[ b_n(x) \geq v_n(x) - \pi_n = v_n(x) - v_n(x^{opt}) + b_n(x^{opt}). \]

By logic similar to the proof of sufficiency in Lemma 19, this condition implies that \( b \) is a test-set equilibrium. \( \square \)

**Proof of Theorem 22.** Fix \( \delta > 0 \). Let \( \gamma = \frac{\delta}{2} \). Let \( \gamma \in (0, \gamma) \) be such that Lemmas 32 and 33 apply. Suppose that \( b \) is an extended proper equilibrium of the menu auction with bid spaces \( B^\gamma \) that is not \( \delta \)-bilaterally efficient. Let \( x^* = x(b) \) and \( \pi^* = \pi(b) \). Because \( b \) is not \( \delta \)-bilaterally efficient, there exists a decision \( \hat{x} \in X \) such that, after possibly relabeling the bidders,

\[ v_0(\hat{x}) + v_1(\hat{x}) + v_2(\hat{x}) + \sum_{n=3}^{N} b_n(\hat{x}) \geq v_0(x^*) + v_1(x^*) + v_2(x^*) + \sum_{n=3}^{N} b_n(x^*) + \delta. \]
Because $b$ is extended proper, there exists, by Proposition 5, an LPS $\rho = (p^1, \ldots, p^K)$ on $\times_{n=1}^N B_n$ that satisfies strong independence, has full support, and respects within-and-across-person preferences, for which $(\rho, b)$ is a lexicographic Nash equilibrium.

For $n \in \{1, 2\}$, let

$$\hat{b}_n(x) = \begin{cases} b_n(x) + \gamma z_n & \text{if } x = \hat{x} \\ b_n(x) & \text{if } x \neq \hat{x} \end{cases}$$

where $z_n = \left[ \frac{1}{\gamma} \left[ v_n(\hat{x}) - b_n(\hat{x}) - v_n(x^*) + b_n(x^*) \right] \right]$. Furthermore, let $\tilde{b}$ be any “most probable outcome-changing deviation,” that is any strategy profile for which (i) $\pi(b) \neq \pi^*$ and (ii) if $\pi(\tilde{b}') \neq \pi^*$ then $b \geq_\rho \tilde{b}'$. Because $\tilde{b} \neq b$, there must exist some bidder $j$ for whom $\tilde{b}_j \neq b_j$. (It is possible that $j \in \{1, 2\}$.)

We establish the result by proving a series of claims. At several junctures, we use the properties of $\rho$ that follow from Lemma 27 without directly invoking the lemma.

Claim: For all $\tilde{b}_{-j} \geq_\rho b_{-j}$, $\pi_j(b_j, \tilde{b}_{-j}) \geq \pi_j(b_j, b_{-j})$. Furthermore, $\pi_j(b_j, \tilde{b}_{-j}) > \pi_j(b_j, b_{-j})$.

Proof of Claim. We begin by proving the first part of the claim. If it is the case that $\tilde{b}_{-j} = b_{-j}$, then we need only show that $\pi_j(b_j, b_{-j}) \geq \pi_j(b_j, b_{-j})$, which follows directly from the fact that $b$ is an equilibrium. We therefore assume henceforth that $\tilde{b}_{-j} \neq b_{-j}$.

We claim that $b_j(x^*) = \tilde{b}_j(x^*)$. To see this, suppose otherwise. Then by Lemma 22(i), $\pi_j(b_j, \tilde{b}_{-j}) \neq \pi_j(b_j, b_{-j})$. This would imply $(\tilde{b}_j, b_{-j}) \leq_\rho \tilde{b}$, which contradicts $\tilde{b}_{-j} \neq b_{-j}$.

Suppose $\exists \tilde{b}_{-j} \geq_\rho b_{-j}$ for which $\pi_j(b_j, \tilde{b}_{-j}) < \pi_j(b_j, b_{-j})$. Because $\tilde{b}_{-j} \geq_\rho b_{-j}$, $(\tilde{b}_j, \tilde{b}_{-j}) \geq_\rho (b_j, b_{-j}) \geq_\rho \tilde{b}$. Then by the definition of $\tilde{b}$, $\pi_j(b_j, \tilde{b}_{-j}) = \pi_j(b_j, \tilde{b}_{-j})$, so that $\pi_j(b_j, \tilde{b}_{-j}) = v_j(x^*) - b_j(x^*)$. Let $\hat{x} = x(b_j, \tilde{b}_{-j})$. We claim that $\hat{x} \neq x^*$. To see this, suppose otherwise. Then $\pi_j(b_j, \tilde{b}_{-j}) = v_j(x^*) - b_j(x^*) = v_j(x^*) - b_j(x^*)$, which implies the contradiction $\pi_j(b_j, \tilde{b}_{-j}) = \pi_j(b_j, \tilde{b}_{-j})$.

The previous paragraph demonstrates both that $\hat{x} \neq x^*$ and that $\tilde{b}_j(x^*) = b_j(x^*)$. Lemma 31 therefore implies that $\tilde{b}_j(\hat{x}) > b_j(\hat{x})$. Furthermore, because $\pi_j(b_j, \tilde{b}_{-j}) < \pi_j(b_j, b_{-j})$, it must be the case that $\tilde{b}_j(\hat{x}) \leq v_j(\hat{x}) - v_j(x^*) + b_j(x^*)$. Next, define

$$\hat{b}_j(x) = \begin{cases} b_j(x) & \text{if } x = \hat{x} \\ b_j(x) & \text{if } x \neq \hat{x} \end{cases}$$
Observe the following about $\hat{b}_j$:

- $\hat{b}_j(\bar{x}) = \tilde{b}_j(\bar{x}) \in [b_j(\bar{x}), v_j(\bar{x}) - v_j(x^*) + b_j(x^*)]$.

- As previously argued, $x(b_j, \tilde{b}_j') = x^*$.

- Since $\tilde{b}_j(x^*) = b_j(x^*)$, we obtain that $\tilde{b}_j$ and $\hat{b}_j$ agree on $x^*$ as well as $\bar{x}$. Because $x(\hat{b}_j, \tilde{b}_j') = \bar{x} \neq x^*$, Lemma 31 implies that $x(\hat{b}_j, \tilde{b}_j') \neq x^*$.

These three facts, together with Lemma 33(ii) demonstrate that $\pi_j(b_j, \tilde{b}_{-j}) > \pi_j(b_j, \tilde{b}'_{-j})$. Now suppose that $\tilde{b}_{-j}'$ were such that $\pi_j(b_j, \tilde{b}_{-j}') < \pi_j(b_j, \tilde{b}'_{-j})$. By Lemma 33(i) and the first of the above facts, we must have $x(b_j, \tilde{b}_{-j}') \neq x^*$. Then by the definition of $\tilde{b}$, $\pi(b_j, \tilde{b}_{-j}) = 0$. Since $b_j >_\rho \tilde{b}_j$, we must then have $\tilde{b}_{-j} >_\rho \tilde{b}_{-j}'$, and so $\tilde{b}_{-j}' >_\rho \tilde{b}'_{-j}$. This, together with the fact that $\rho$ has full support, implies that $\tilde{b}_j$ lexicographically outperforms $b_j$, and so $b_j \notin BR_j(\rho_{-j})$. This contradicts $(\rho, b)$ being a lexicographic Nash equilibrium.

To prove the second part of the claim, it suffices to show that $\pi_j(b_j, \tilde{b}_{-j}) \neq \pi_j(b_j, \tilde{b}'_{-j})$. By the definition of $\tilde{b}$, $\pi(b_j, \tilde{b}_{-j}) \neq \pi^*$. On the other hand, because $(b_j, \tilde{b}_{-j}) >_\rho (\tilde{b}_j, \tilde{b}_{-j})$, the definition also implies $\pi(b_j, \tilde{b}_{-j}) = \pi^*$. The result then follows from Lemma 32(ii).

**Claim:** For all $\tilde{b}_{-2} \geq_\rho \tilde{b}_{-3}$, $\pi_2(b_2, \tilde{b}_{-2}) \leq \pi_2(b_2, \tilde{b}''_{-2})$.

**Proof of Claim.** Observe the following about $\hat{b}_2$:

- Because $\tilde{b}_{-2} \geq_\rho \tilde{b}_{-3}$ and $b_2 >_\rho \hat{b}_j$, $(b_2, \tilde{b}_{-2}) >_\rho \hat{b}$. Then by the definition of $\hat{b}$, $x(b_2, \tilde{b}_{-2}) = x^*$.

- By the definition of $\tilde{b}_2$, $\tilde{b}_2(\bar{x}) = b_2(\bar{x}) + \gamma z_2 \leq v_2(\bar{x}) - v_2(x^*) + b_2(x^*)$.

- We claim that $\tilde{b}_2(\bar{x}) \geq b_2(\bar{x})$. By the definition of $\tilde{b}_2$, it will suffice to show $b_2(\bar{x}) - b_2(x^*) \leq v_2(\bar{x}) - v_2(x^*)$. Suppose it were instead the case that $b_2(\bar{x}) - b_2(x^*) > v_2(\bar{x}) - v_2(x^*)$. By the definition of $\tilde{b}_1$, $\tilde{b}_1(\bar{x}) - b_1(x^*) \geq v_1(\bar{x}) - v_1(x^*) - \gamma$. In addition we know

$$v_0(\bar{x}) - v_0(x^*) + v_1(\bar{x}) - v_1(x^*) + v_2(\bar{x}) - v_2(x^*) + \sum_{n=3}^{N} |b_n(\bar{x}) - b_n(x^*)| \geq \delta.$$
Combining these three inequalities yields
\[ v_0(\hat{x}) - v_0(x^*) + \hat{b}_1(\hat{x}) - \hat{b}_1(\hat{x}) + \sum_{n=2}^{N} [b_n(\hat{x}) - b_n(x^*)] > \delta - \gamma. \]

Since \( \gamma < \frac{\delta}{2} \), the above inequality implies that \( x(\hat{b}_1, \hat{b}_2) \neq x^* \). Lemma 33(ii) then implies that \( \pi_1(\hat{b}_1, b_{-1}) > \pi_1(b_1, b_{-1}) \), which contradicts \( b \) being a Nash equilibrium.

The claim follows from the above three facts, in conjunction with Lemma 33(ii).

**Claim:** \( \hat{b}_2 > \rho \hat{b}_j \).

**Proof of Claim.** This follows directly from the previous two claims and the fact that \( \rho \) respects within-and-across-person preferences. (This is the step where extended proper equilibrium, as opposed to merely proper equilibrium, is required.)

**Claim:** \( b_1 \notin BR_1(\rho_{-1}) \).

**Proof of Claim.** Observe the following about \( \hat{b}_1 \):

- \( \hat{b}_1(\hat{x}) \in [b_1(\hat{x}), v_1(\hat{x}) - v_1(x^*) + b_1(x^*)] \) for the same reasons that \( \hat{b}_2(\hat{x}) \in [b_2(\hat{x}), v_2(\hat{x}) - v_2(x^*) + b_2(x^*)] \), which was established in the proof of an earlier claim.

- By the definition of \( \hat{b}_1 \) and \( \hat{b}_2 \), \( \hat{b}_1(\hat{x}) - \hat{b}_n(x^*) \geq v_n(\hat{x}) - v_n(x^*) - \gamma \) for \( n \in \{1, 2\} \). In addition we know
  \[ v_0(\hat{x}) - v_0(x^*) + v_1(\hat{x}) - v_1(x^*) + v_2(\hat{x}) - v_2(x^*) + \sum_{n=3}^{N} [b_n(\hat{x}) - b_n(x^*)] \geq \delta. \]

Combining these inequalities yields
\[ v_0(\hat{x}) - v_0(x^*) + \hat{b}_1(\hat{x}) - \hat{b}_1(\hat{x}) + \hat{b}_2(\hat{x}) - \hat{b}_2(\hat{x}) + \sum_{n=3}^{N} [b_n(\hat{x}) - b_n(x^*)] \geq \delta - 2\gamma. \]

Since \( \gamma < \frac{\delta}{2} \), the above inequality implies that \( x(\hat{b}_1, b_{-\{1,2\}}) \neq x^* \).
• We claim that \(x(b_1, \hat{b}_2, b_{-\{1,2\}}) = x^*.\) To see why, suppose otherwise. Lemma 31 would then imply that \(x(b_1, \hat{b}_2, b_{-\{1,2\}}) = \hat{x.}\) Because \(\hat{b}_2(\hat{x}) \in [b_2(\hat{x}), v_2(\hat{x}) - v_2(x^*) + b_2(x^*)],\) Lemma 33(ii) would then imply \(\pi_2(\hat{b}_2, b_{-2}) > \pi_2(b),\) which would contradict \(b\) being a Nash equilibrium.

These facts, together with Lemma 33(ii) demonstrate that \(\pi_1(\hat{b}_1, \hat{b}_2, b_{-\{1,2\}}) > \pi_1(b_1, \hat{b}_2, b_{-\{1,2\}}).\) Now suppose that \(\tilde{b}_1'\) were such that \(\pi_1(\hat{b}_1, \tilde{b}_1') < \pi_1(b_1, \tilde{b}_1').\) By Lemma 33(i) and the first of the above facts, we must have \(x(b_1, \tilde{b}_1') \neq x^*.\) Then by the definition of \(\tilde{b},\) \(\tilde{b} \succeq_\rho (b_1, \tilde{b}_1').\) Consequently \(\tilde{b}_{-1} \succeq_\rho \tilde{b}'_{-1}.\) Additionally, because \(\hat{b}_2 >_\rho \tilde{b}_j, (\hat{b}_2, b_{-\{1,2\}}) >_\rho \tilde{b}_{-1}.\) By transitivity, \((\hat{b}_2, b_{-\{1,2\}}) >_\rho \tilde{b}'_{-1}.\) This, together with the fact that \(\rho\) has full support, implies that \(\hat{b}_1\) lexicographically outperforms \(b_1,\) and so \(b_1 \notin BR_1(\rho_{-1}).\)

The previous claim constitutes a contradiction to \((b, \rho)\) being a lexicographic Nash equilibrium. \(\Box\)

References


