

# 14.452 Economic Growth: Lecture 4, The Solow Growth Model and the Data

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# Solow Growth Model and the Data

- Use Solow model or extensions to interpret both economic growth over time and cross-country output differences.
- Focus on *proximate causes* of economic growth.

# Growth Accounting I

- Aggregate production function in its general form:

$$Y(t) = F[K(t), L(t), A(t)].$$

- Combined with competitive factor markets, gives Solow (1957) *growth accounting framework*.
- Continuous-time economy and differentiate the aggregate production function with respect to time.
- Dropping time dependence,

$$\frac{\dot{Y}}{Y} = \frac{F_A A}{Y} \frac{\dot{A}}{A} + \frac{F_K K}{Y} \frac{\dot{K}}{K} + \frac{F_L L}{Y} \frac{\dot{L}}{L}. \quad (1)$$

## Growth Accounting II

- Denote growth rates of output, capital stock and labor by  $g \equiv \dot{Y}/Y$ ,  $g_K \equiv \dot{K}/K$  and  $g_L \equiv \dot{L}/L$ .
- Define the contribution of technology to growth as

$$x \equiv \frac{F_A A \dot{A}}{Y A}$$

- Recall with competitive factor markets,  $w = F_L$  and  $R = F_K$ .
- Define factor shares as  $\alpha_K \equiv RK/Y$  and  $\alpha_L \equiv wL/Y$ .
- Putting all these together, (1) the *fundamental growth accounting* equation

$$x = g - \alpha_K g_K - \alpha_L g_L. \quad (2)$$

- Gives estimate of contribution of technological progress, *Total Factor Productivity* (TFP) or Multi Factor Productivity as

$$\hat{x}(t) = g(t) - \alpha_K(t) g_K(t) - \alpha_L(t) g_L(t). \quad (3)$$

- All terms on right-hand side are “estimates” obtained with a range of assumptions from national accounts and other data sources.

## Growth Accounting III

- In continuous time, equation (3) is exact.
- With discrete time, potential problem in using (3): over the time horizon factor shares can change.
- Use beginning-of-period or end-of-period values of  $\alpha_K$  and  $\alpha_L$ ?
  - Either might lead to seriously biased estimates.
  - Best way of avoiding such biases is to use as high-frequency data as possible.
  - Typically use factor shares calculated as the average of the beginning and end of period values.
- In discrete time, the analog of equation (3) becomes

$$\hat{x}_{t,t+1} = g_{t,t+1} - \bar{\alpha}_{K,t,t+1}g_{K,t,t+1} - \bar{\alpha}_{L,t,t+1}g_{L,t,t+1}, \quad (4)$$

- $g_{t,t+1}$  is the growth rate of output between  $t$  and  $t + 1$ ; other growth rates defined analogously.

## Growth Accounting IV

- Moreover,

$$\bar{\alpha}_{K,t,t+1} \equiv \frac{\alpha_K(t) + \alpha_K(t+1)}{2}$$
$$\text{and } \bar{\alpha}_{L,t,t+1} \equiv \frac{\alpha_L(t) + \alpha_L(t+1)}{2}$$

- Equation (4) would be a fairly good approximation to (3) when the difference between  $t$  and  $t+1$  is small and the capital-labor ratio does not change much during this time interval.
- Solow's (1957) applied this framework to US data: a large part of the growth was due to technological progress.
- From early days, however, a number of pitfalls were recognized.
  - Moses Abramovitz (1956): dubbed the  $\hat{\chi}$  term "the measure of our ignorance".
  - If we mismeasure  $g_L$  and  $g_K$  we will arrive at inflated estimates of  $\hat{\chi}$ .

# Growth Accounting Results

- Example from Barro and Sala-i-Martin's textbook

Table 10.1  
Growth Accounting for a Sample of Countries

Country	(1) Growth Rate of GDP	(2) Contribution from Capital	(3) Contribution from Labor	(4) TFP Growth Rate
<b>Panel A: OECD Countries, 1947-73</b>				
Canada ( $\alpha = 0.44$ )	0.0517	0.0254 (49%)	0.0088 (17%)	0.0175 (34%)
France <sup>a</sup> ( $\alpha = 0.40$ )	0.0542	0.0225 (42%)	0.0021 (4%)	0.0296 (54%)
Germany <sup>a</sup> ( $\alpha = 0.39$ )	0.0661	0.0269 (41%)	0.0018 (3%)	0.0374 (56%)
Italy <sup>b</sup> ( $\alpha = 0.39$ )	0.0527	0.0180 (34%)	0.0011 (2%)	0.0337 (64%)
Japan <sup>a</sup> ( $\alpha = 0.39$ )	0.0951	0.0328 (35%)	0.0221 (23%)	0.0402 (42%)
Netherlands <sup>c</sup> ( $\alpha = 0.45$ )	0.0536	0.0247 (46%)	0.0042 (8%)	0.0248 (46%)
U.K. <sup>d</sup> ( $\alpha = 0.38$ )	0.0373	0.0176 (47%)	0.0003 (1%)	0.0193 (52%)
U.S. ( $\alpha = 0.40$ )	0.0402	0.0171 (43%)	0.0095 (24%)	0.0135 (34%)
<b>Panel B: OECD Countries, 1960-95</b>				
Canada ( $\alpha = 0.42$ )	0.0369	0.0186 (51%)	0.0123 (33%)	0.0057 (16%)
France ( $\alpha = 0.41$ )	0.0358	0.0180 (53%)	0.0033 (10%)	0.0130 (38%)
Germany ( $\alpha = 0.39$ )	0.0312	0.0177 (56%)	0.0014 (4%)	0.0132 (42%)
Italy ( $\alpha = 0.34$ )	0.0357	0.0182 (51%)	0.0035 (9%)	0.0153 (42%)
Japan ( $\alpha = 0.43$ )	0.0566	0.0178 (31%)	0.0125 (22%)	0.0265 (47%)
U.K. ( $\alpha = 0.37$ )	0.0221	0.0124 (56%)	0.0017 (8%)	0.0080 (36%)
U.S. ( $\alpha = 0.39$ )	0.0318	0.0117 (37%)	0.0127 (40%)	0.0076 (24%)

Table continued

## Growth Accounting Results (continued)

Table 10.1  
(Continued)

Country	(1) Growth Rate of GDP	(2) Contribution from Capital	(3) Contribution from Labor	(4) TFP Growth Rate
<b>Panel C: Latin American Countries, 1940-90</b>				
Argentina ( $\alpha = 0.54$ )	0.0279	0.0128 (46%)	0.0097 (35%)	0.0054 (19%)
Brazil ( $\alpha = 0.45$ )	0.0558	0.0294 (53%)	0.0150 (27%)	0.0114 (20%)
Chile ( $\alpha = 0.52$ )	0.0362	0.0120 (33%)	0.0103 (28%)	0.0138 (38%)
Colombia ( $\alpha = 0.63$ )	0.0454	0.0219 (48%)	0.0152 (33%)	0.0084 (19%)
Mexico ( $\alpha = 0.69$ )	0.0522	0.0259 (50%)	0.0150 (29%)	0.0113 (22%)
Peru ( $\alpha = 0.66$ )	0.0323	0.0252 (78%)	0.0134 (41%)	-0.0062 (-19%)
Venezuela ( $\alpha = 0.55$ )	0.0443	0.0254 (57%)	0.0179 (40%)	0.0011 (2%)
<b>Panel D: East Asian Countries, 1966-90</b>				
Hong Kong <sup>d</sup> ( $\alpha = 0.37$ )	0.073	0.030 (41%)	0.020 (28%)	0.023 (32%)
Singapore ( $\alpha = 0.49$ )	0.087	0.056 (65%)	0.029 (33%)	0.002 (2%)
South Korea ( $\alpha = 0.30$ )	0.103	0.041 (40%)	0.045 (44%)	0.017 (16%)
Taiwan ( $\alpha = 0.26$ )	0.094	0.032 (34%)	0.036 (39%)	0.026 (28%)

Source: Panel A, columns 5, GDP; column 6, Real Output; Column 7, Investment; Column 8, Investment/GDP; Column 9, Investment/GDP



# Interpreting the Results

- Reasons for mismeasurement:
  - what matters is not labor hours, but effective labor hours
    - important—though difficult—to make adjustments for changes in the *human capital* of workers.
  - measurement of capital inputs:
    - in the theoretical model, capital corresponds to the final good used as input to produce more goods.
    - in practice, capital is machinery, need assumptions about how relative prices of machinery change over time.
    - typical assumption was to use capital expenditures but if machines become cheaper would severely underestimate  $g_K$

# A World of Augmented Solow Economies I

- Mankiw, Romer and Weil (1992) used regression analysis to take the augmented Solow model, with human capital, to data.
- Use the Cobb-Douglas model and envisage a world consisting of  $j = 1, \dots, N$  countries.
- “Each country is an island”: countries do not interact (perhaps except for sharing some common technology growth).
- Country  $j = 1, \dots, N$  has the aggregate production function:

$$Y_j(t) = K_j(t)^\alpha H_j(t)^\beta (A_j(t) L_j(t))^{1-\alpha-\beta}.$$

- Nests the basic Solow model without human capital when  $\alpha = 0$ .
- Countries differ in terms of their saving rates,  $s_{k,j}$  and  $s_{h,j}$ , population growth rates,  $n_j$ , and technology growth rates  $\dot{A}_j(t) / A_j(t) = g_j$ .
- Define  $k_j \equiv K_j / A_j L_j$  and  $h_j \equiv H_j / A_j L_j$ .

## A World of Augmented Solow Economies II

- Focus on a world in which each country is in their steady state
- Assuming that human capital also has depreciation, at the rate  $\delta_h$ , and it is accumulated with the saving rate  $s_h$ , steady state values for country  $j$  would be (to be derived in recitation):

$$k_j^* = \left( \left( \frac{s_{k,j}}{n_j + g_j + \delta_k} \right)^{1-\beta} \left( \frac{s_{h,j}}{n_j + g_j + \delta_h} \right)^\beta \right)^{\frac{1}{1-\alpha-\beta}}$$

$$h_j^* = \left( \left( \frac{s_{k,j}}{n_j + g_j + \delta_k} \right)^\alpha \left( \frac{s_{h,j}}{n_j + g_j + \delta_h} \right)^{1-\alpha} \right)^{\frac{1}{1-\alpha-\beta}}$$

- Consequently:

$$y_j^*(t) \equiv \frac{Y(t)}{L(t)} \tag{5}$$

$$= A_j(t) \left( \frac{s_{k,j}}{n_i + g_i + \delta_k} \right)^{\frac{\alpha}{1-\alpha-\beta}} \left( \frac{s_{h,j}}{n_i + g_i + \delta_h} \right)^{\frac{\beta}{1-\alpha-\beta}}$$

## A World of Augmented Solow Economies II

- Here  $y_j^*(t)$  stands for output per capita of country  $j$  along the balanced growth path.
- Note if  $g_j$ 's are not equal across countries, income per capita will diverge.
- Mankiw, Romer and Weil (1992) make the following assumption:

$$A_j(t) = \bar{A}_j \exp(gt).$$

- Countries differ according to technology *level*, (initial level  $\bar{A}_j$ ) but they share the same common technology growth rate,  $g$ .

## A World of Augmented Solow Economies III

- Using this together with (5) and taking logs, equation for the balanced growth path of income for country  $j = 1, \dots, N$ :

$$\ln y_j^*(t) = \ln \bar{A}_j + gt + \frac{\alpha}{1 - \alpha - \beta} \ln \left( \frac{s_{k,j}}{n_j + g + \delta_k} \right) + \frac{\beta}{1 - \alpha - \beta} \ln \left( \frac{s_{h,j}}{n_j + g + \delta_h} \right). \quad (6)$$

- Mankiw, Romer and Weil (1992) take:
  - $\delta_k = \delta_h = \delta$  and  $\delta + g = 0.05$ .
  - $s_{k,j}$  = average investment rates (investments/GDP).
  - $s_{h,j}$  = fraction of the school-age population that is enrolled in secondary school.

## A World of Augmented Solow Economies IV

- Even with all of these assumptions, (6) can still not be estimated consistently.
- $\ln \bar{A}_j$  is unobserved (at least to the econometrician) and thus will be captured by the error term.
- Most reasonable models would suggest  $\ln \bar{A}_j$ 's should be correlated with investment rates.
- Thus an estimation of (6) would lead to omitted variable bias and inconsistent estimates.
- Implicitly, MRW make another *crucial* assumption, the **orthogonal technology assumption**:

$$\bar{A}_j = \varepsilon_j A, \text{ with } \varepsilon_j \text{ orthogonal to all other variables.}$$

# Cross-Country Income Differences: Regressions I

- MRW first estimate equation (6) without the human capital term for the cross-sectional sample of non-oil producing countries

$$\ln y_j^* = \text{constant} + \frac{\alpha}{1-\alpha} \ln(s_{k,j}) - \frac{\alpha}{1-\alpha} \ln(n_j + g + \delta_k) + \varepsilon_j.$$

# Cross-Country Income Differences: Regressions II

## Estimates of the Basic Solow Model

	MRW 1985	Updated data 1985    2000	
$\ln(s_k)$	1.42 (.14)	1.01 (.11)	1.22 (.13)
$\ln(n + g + \delta)$	-1.97 (.56)	-1.12 (.55)	-1.31 (.36)
Adj $R^2$	.59	.49	.49
Implied $\alpha$	.59	.50	.55
No. of observations	98	98	107



## Cross-Country Income Differences: Regressions III

- Their estimates for  $\alpha / (1 - \alpha)$ , implies that  $\alpha$  must be around  $2/3$ , but should be around  $1/3$ .
- The most natural reason for the high implied values of  $\alpha$  is that  $\varepsilon_j$  is correlated with  $\ln(s_{k,j})$ , either because:
  - 1 the orthogonal technology assumption is not a good approximation to reality or
  - 2 there are also human capital differences correlated with  $\ln(s_{k,j})$ .
- Mankiw, Romer and Weil favor the second interpretation and estimate the augmented model,

$$\ln y_j^* = \text{cst} + \frac{\alpha}{1 - \alpha - \beta} \ln(s_{k,j}) - \frac{\alpha}{1 - \alpha - \beta} \ln(n_j + g + \delta_k) \quad (7)$$

$$+ \frac{\beta}{1 - \alpha - \beta} \ln(s_{h,j}) - \frac{\beta}{1 - \alpha - \beta} \ln(n_j + g + \delta_h) + \varepsilon_j.$$

## Estimates of the Augmented Solow Model

	MRW 1985	Updated data 1985      2000	
$\ln(s_k)$	.69 (.13)	.65 (.11)	.96 (.13)
$\ln(n + g + \delta)$	-1.73 (.41)	-1.02 (.45)	-1.06 (.33)
$\ln(s_h)$	.66 (.07)	.47 (.07)	.70 (.13)
Adj R <sup>2</sup>	.78	.65	.60
Implied $\alpha$	.30	.31	.36
Implied $\beta$	.28	.22	.26
No. of observations	98	98	107

# Cross-Country Income Differences: Regressions IV

- If these regression results are reliable, they give a big boost to the augmented Solow model.
  - Adjusted  $R^2$  suggests that three quarters of income per capita differences across countries can be explained by differences in their physical and human capital investment.
- Immediate implication is technology (TFP) differences have a somewhat limited role.
- But this conclusion should not be accepted without further investigation.

# Challenges to Regression Analyses I

- **Technology differences across countries are not orthogonal to all other variables.**
- $\bar{A}_j$  is correlated with measures of  $s_j^h$  and  $s_j^k$  for two reasons.
  - ① *omitted variable bias*: societies with high  $\bar{A}_j$  will be those that have invested more in technology for various reasons; same reasons likely to induce greater investment in physical and human capital as well.
  - ② *reverse causality*: complementarity between technology and physical or human capital imply that countries with high  $\bar{A}_j$  will find it more beneficial to increase their stock of human and physical capital.
- In terms of (7), implies that key right-hand side variables are correlated with the error term,  $\varepsilon_j$ .
- OLS estimates of  $\alpha$  and  $\beta$  and  $R^2$  are biased upwards.

## Challenges to Regression Analyses II

- $\beta$  is too large relative to what we should expect on the basis of microeconomic evidence.
- The working age population enrolled in school ranges from 0.4% to over 12% in the sample of countries.
- Predicted log difference in incomes between these two countries is

$$\frac{\beta}{1 - \alpha - \beta} (\ln 12 - \ln (0.4)) = 0.66 \times (\ln 12 - \ln (0.4)) \approx 2.24.$$

- Thus a country with schooling investment of over 12 should be about  $\exp(2.24) - 1 \approx 8.5$  times richer than one with investment of around 0.4.

## Challenges to Regression Analyses III

- Take Mincer regressions of the form:

$$\ln w_i = \mathbf{X}'_i \boldsymbol{\gamma} + \phi S_i, \quad (8)$$

- Microeconometrics literature suggests that  $\phi$  is between 0.06 and 0.10.
- Can deduce how much richer a country with 12 if we assume:
  - 1 That the micro-level relationship as captured by (8) applies identically to all countries.
  - 2 That there are no *human capital externalities*.
- Then: a country with 12 more years of average schooling should have between  $\exp(0.10 \times 12) \simeq 3.3$  and  $\exp(0.06 \times 12) \simeq 2.05$  times the stock of human capital of a county with fewer years of schooling.

## Challenges to Regression Analyses IV

- Thus holding other factors constant, this country should be about 2-3 times as rich as the country with zero years of average schooling.
- Much less than the 8.5 fold difference implied by the Mankiw-Romer-Weil analysis.
- Thus  $\beta$  in MRW is too high relative to the estimates implied by the microeconomic evidence and thus likely upwardly biased.
- Overestimation of  $\beta$  is, in turn, most likely related to correlation between the error term  $\varepsilon_j$  and the key right-hand side regressors in (7).
- We have so far discussed cross-country “levels” regressions, similar issues apply to “growth regressions” but we have also seen in the first lecture how one might make partial progress here.

# Calibrating Productivity Differences I

- The problems with regression analysis with cross-country data have motivated some macroeconomists to turn to “calibration”-type exercises.
- Suppose each country has access to the Cobb-Douglas aggregate production function:

$$Y_j = K_j^\alpha (A_j H_j)^{1-\alpha}, \quad (9)$$

- Each worker in country  $j$  has  $S_j$  years of schooling.
- Then using the Mincer equation (8) ignoring the other covariates and taking exponents,  $H_j$  can be estimated as

$$H_j = \exp(\phi S_j) L_j,$$

- Does not take into account differences in other “human capital” factors, such as experience.



## Calibrating Productivity Differences II

- Let the rate of return to acquiring the  $S$ th year of schooling be  $\phi(S)$ .
- A better estimate of the stock of human capital can be constructed as

$$H_j = \sum_S \exp\{\phi(S) S\} L_j(S)$$

- $L_j(S)$  now refers to the total employment of workers with  $S$  years of schooling in country  $j$ .
- Series for  $K_j$  can be constructed from Summers-Heston dataset using investment data and the perpetual inventory method.

$$K_j(t+1) = (1 - \delta) K_j(t) + I_j(t),$$

- Assume, following Hall and Jones that  $\delta = 0.06$ .
- With same arguments as before, choose a value of  $1/3$  for  $\alpha$ .

## Calibrating Productivity Differences III

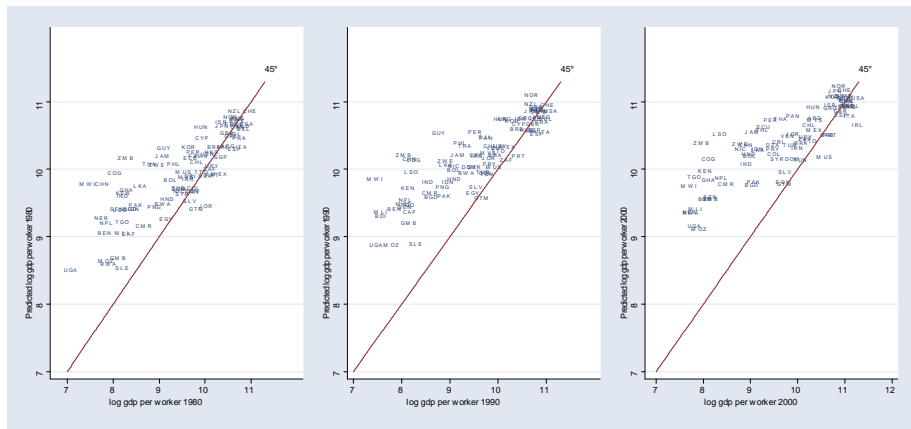
- Given series for  $H_j$  and  $K_j$  and a value for  $\alpha$ , construct “predicted” incomes at a point in time using

$$\hat{Y}_j = K_j^{1/3} (A_{US} H_j)^{2/3}$$

- $A_{US}$  is computed so that  $Y_{US} = K_{US}^{1/3} (A_{US} H_{US})^{2/3}$ .
- Once a series for  $\hat{Y}_j$  has been constructed, it can be compared to the actual output series.
- Gap between the two series represents the contribution of technology.
- Alternatively, could back out country-specific technology terms (relative to the United States) as

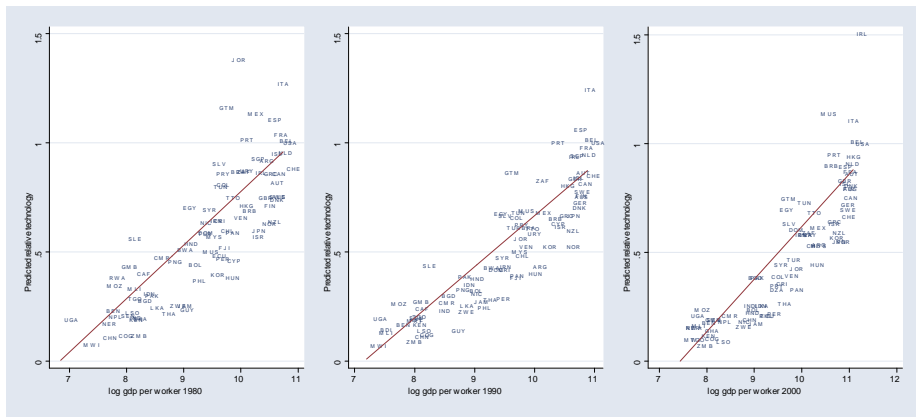
$$\frac{A_j}{A_{US}} = \left( \frac{Y_j}{Y_{US}} \right)^{3/2} \left( \frac{K_{US}}{K_j} \right)^{1/2} \left( \frac{H_{US}}{H_j} \right).$$

# Calibrating Productivity Differences IV



**Figure:** Calibrated technology levels relative to the US technology (from the Solow growth model with human capital) versus log GDP per worker, 1980, 1990 and 2000.

# Calibrating Productivity Differences V



**Figure:** Calibrated technology levels relative to the US technology (from the Solow growth model with human capital) versus log GDP per worker, 1980, 1990 and 2000.

# Calibrating Productivity Differences VI

The following features are noteworthy:

- 1 Differences in physical and human capital still matter a lot.
- 2 However, differently from the regression analysis, this exercise also shows significant *technology (productivity) differences*.
- 3 Same pattern visible in the next three figures for the estimates of the technology differences,  $A_j/A_{US}$ , against log GDP per capita in the corresponding year.
- 4 Also interesting is the pattern that the empirical fit of the neoclassical growth model seems to deteriorate over time.

# Challenges to Callibration I

- In addition to the standard assumptions of competitive factor markets, we had to assume :
  - no human capital externalities, a Cobb-Douglas production function, and a range of approximations to measure cross-country differences in the stocks of physical and human capital.
- The calibration approach is in fact a close cousin of the growth-accounting exercise (sometimes referred to as “levels accounting”).
- Imagine that the production function that applies to all countries in the world is

$$F(K_j, H_j, A_j),$$

- Assume countries differ according to their physical and human capital as well as technology—but not according to  $F$ .

## Challenges to Callibration II

- Rank countries in descending order according to their physical capital to human capital ratios,  $K_j/H_j$  Then

$$\hat{x}_{j,j+1} = g_{j,j+1} - \bar{\alpha}_{K,j,j+1}g_{K,j,j+1} - \bar{\alpha}_{L,j,j+1}g_{H,j,j+1}, \quad (10)$$

- where:
  - $g_{j,j+1}$ : proportional difference in output between countries  $j$  and  $j + 1$ ,
  - $g_{K,j,j+1}$ : proportional difference in capital stock between these countries and
  - $g_{H,j,j+1}$ : proportional difference in human capital stocks.
  - $\bar{\alpha}_{K,j,j+1}$  and  $\bar{\alpha}_{L,j,j+1}$ : average capital and labor shares between the two countries.
- The estimate  $\hat{x}_{j,j+1}$  is then the proportional TFP difference between the two countries.

# Challenges to Calibration III

- Levels-accounting faces two challenges.
  - ① Data on capital and labor shares across countries are not widely available. Almost all exercises use the Cobb-Douglas approach (i.e., a constant value of  $\alpha_K$  equal to  $1/3$ ).
  - ② The differences in factor proportions, e.g., differences in  $K_j/H_j$ , across countries are large. An equation like (10) is a good approximation when we consider small (infinitesimal) changes.



# What Does All This Mean?

- There is also a sense perhaps that these are all “weak tests”.
- They impose the structure of the Solow model on the data or exploit the quasi-balanced growth properties.
- These tests do not shed much light on any of the following questions:
  - ① To what extent the equilibrium is efficient or “inside the production possibilities frontier”?
  - ② Is technology driven by market and other incentives or mostly evolving exogenously?
  - ③ Is the way that these neoclassical models frame the effects of technology appropriate?
  - ④ What about recent tectonic shifts?
  - ⑤ And what is a proximate cause and what is a fundamental cause?

# From Correlates to Fundamental Causes

- In this lecture, the focus has been on proximate causes— importance of human capital, physical capital and technology.
- Let us now return to the list of potential fundamental causes discussed in the first lecture:
  - ① luck (or multiple equilibria)
  - ② geographic differences
  - ③ institutional differences
  - ④ cultural differences
- Do we need to worry about the relationship between these fundamental causes and the correlates of growth? In what way? Where is theory useful?

# Conclusions

- Message is somewhat mixed.
  - On the positive side, despite its simplicity, the Solow model has enough substance that we can take it to data in various different forms, including TFP accounting, regression analysis and calibration.
  - On the negative side, however, no single approach is entirely convincing.
- Complete agreement is not possible, but safe to say that consensus favors the interpretation that cross-country differences in income per capita cannot be understood solely on the basis of differences in physical and human capital
- Differences in TFP are not necessarily due to technology in the narrow sense.
- It is also useful and important to think about *fundamental causes*, what lies behind the factors taken as given either Solow model.