Appendix to "Incentives and Stability in Large Two-Sided Matching Markets"

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We first introduce some notation. First, non-strict counterparts of P_s and $\succ_c$ are denoted by R_s and $\succeq_c$, respectively. For any pair of matchings μ and μ' and for any $c \in C$, we write $\mu \succ_c \mu'$ if and only if $\mu(c) \succ_c \mu'(c)$. Similarly, for any $s \in S$, we write $\mu P_s \mu'$ if and only if $\mu(s) P_s \mu'(s)$. Without loss of generality, we assume the set of colleges C are *ordered in decreasing popularity*: if $c' < c$, then $p_{c'} \geq p_c$. With abuse of notation, we write $c = m$, $c > m$ and $c < m$ for $m \in \mathbb{N}$ to mean, respectively, that c is the mth college, c is ordered after the mth college and c is ordered before the m college. We sometimes write p_m, which is the probability associated with the mth college.

A Appendix A: Additional results

A.1 Manipulation via pre-arranged matches

When colleges seek more than one student, there is concern for manipulation not only within the matching mechanism, but also outside the formal process. Sönmez (1999) introduces the idea of manipulation via pre-arranged matches. Suppose that c and s arrange a match before the central matching mechanism is executed. Then s does not participate in the centralized matching mechanism and c participates in the centralized mechanism with the number of positions reduced by one. The SOSM is **manipulable via pre-arranged matches**, or **manipulable via pre-arrangement**, that is, for some market (S, C, P, q), college $c \in C$ and student $s \in S$ we have

$$\phi(S \setminus s, C, P_{-s}, (q_c - 1, q_{-c}))(c) \cup s \succ_c \phi(S, C, P, q)(c), \text{and}$$
$$cR_s \phi(S, C, P, q)(s).$$

In words, both parties that engage in pre-arrangement have incentives to do so: the student is at least as well off in pre-arrangement as when she is matched through the centralized mechanism, and the college strictly prefers s and the assignment of the centralized mechanism to those without pre-arrangement. Sönmez (1999) shows that any stable mechanism is manipulable via pre-arrangement.

In some markets, matching outside the centralized mechanism is discouraged or even legally prohibited. Even so, the student and college can effectively "pre-arrange" a match by listing each other on the top of their preference lists under stable mechanisms such as the SOSM. Thus the scope of manipulation via pre-arrangement is potentially large.

However, we have the following positive result in large markets.

Theorem 3. *Suppose that the sequence of random markets is regular. Then the expected proportion of colleges that can manipulate the SOSM via pre-arranged matches (when other colleges do not manipulate) goes to zero as the number of colleges goes to infinity.*

The intuition is similar to that of Theorem 1. It can be shown that any student involved in pre-arrangement under the SOSM is strictly less preferred by c to any student who would be matched in the absence of the pre-arrangement (Lemma 8). Therefore, in order to profitably manipulate, c should be matched to a better set of students in the central matching. By a similar reasoning to Theorem 1, the probability of being matched to better students in the centralized mechanism is small in a large market for most colleges.

A.2 Manipulation via capacities and pre-arranged matches without sufficient thickness

The next example shows that, when we do not have sufficient thickness, manipulations via capacities or pre-arrangement may be profitable for some colleges even in a large market.

Example 3. Consider the following market $\tilde{\Gamma}_n$ for any n. $|C^n| = |S^n| = n$. $q_{c_1} = 2$ and $q_c = 1$ for each $c \neq c_1$. c_1's preference list is

$$P_{c_1} : s_1, s_2, s_3, s_4, \dots,$$

and $s_1 \succ_{c_1} \{s_2, s_3\}$ (and hence $\{s_1, s_4\} \succ_{c_1} \{s_2, s_3\}$).

c_2's preferences are

$$P_{c_2} : s_3, s_1, s_2, \dots.$$

Further suppose that $p^n_{c_1} = p^n_{c_2} = 1/3$ and $p^n_c = 1/(3(n-2))$ for any n and each $c \neq c_1, c_2$.

With the above setup, with probability $[p^n_{c_1} p^n_{c_2}/(1-p^n_{c_2})] \times [p^n_{c_1} p^n_{c_2}/(1-p^n_{c_1})]^3 = 1/6^4$, students preferences are given by

$$\begin{aligned} P_{s_1} &: c_2, c_1, \dots, \\ P_{s_2} &: c_1, c_2, \dots, \\ P_{s_3} &: c_1, c_2, \dots, \\ P_{s_4} &: c_1, c_2, \dots. \end{aligned}$$

If everyone is truthful, then c_1 is matched to $\{s_2, s_3\}$. Now

(1) Suppose that c_1 reports a quota of one. Then c_1 is matched to s_1, which is preferred to $\{s_2, s_3\}$.

(2) Suppose that c_1 pre-arranges a match with s_4. Then c_1 is matched to $\{s_1, s_4\}$, which is preferred to $\{s_2, s_3\}$.

Since the probability of preference profiles where this occurs is $1/6^4 > 0$ regardless of $n \geq 3$, the opportunity for manipulations via capacities or pre-arrangement for c_1 does not vanish when n becomes large.[1]

A.3 Examples of sufficiently thick markets

The following is a leading example of sufficient thickness.

Example 4 (Nonvanishing proportion of popular colleges)**.** The sequence of random markets is said to have **nonvanishing proportion of popular colleges** if there exists $T \in \mathbb{R}$ and $a \in (0, 1)$ such that for large n

$$p_1^n / p_{[an]}^n \leq T,$$

where $[x]$ denotes the largest integer that does not exceed x. This condition is satisfied if there are not a small number of colleges which are much more popular than all of the other colleges.

There are even sufficiently thick markets where the proportion of popular colleges converges to zero, provided that the convergence is sufficiently slow. We present one such example next.

Example 5. *Consider a sequence of random markets such that there exists $T \in \mathbb{R}$ such that for large n,*

$$p_1^n / p_{[\gamma n / \ln n]}^n \leq T$$

where $\gamma > 0$ is a sufficiently large constant.

Proposition 1. *The sequences of random markets in Examples 4 and 5 are sufficiently thick.*

The intuition for this proposition is the following. By assumption, there are a large number of ex ante popular colleges. With high probability, a substantial part of the positions of these colleges will be vacant. This makes the market thick by having a large number of vacant positions in fairly popular colleges in expectation.

A.4 Equilibrium analysis with incomplete information

The main text of the paper investigated ε-Nash equilibrium under complete information about college preferences. Our result also can be stated in terms of a Bayesian game, in which college preferences are private information.

[1]Manipulation via preference list is also possible in this example. Suppose c_1 reports preferences declaring s_2 and s_3 are unacceptable, such that

$$P'_{c_1} : s_1, s_4, \ldots.$$

Then c_1 is matched to $\{s_1, s_4\}$, which is preferred to $\{s_2, s_3\}$.

The Bayesian game is specified by $(C, S, (\mathcal{U}_c)_{c\in C}, F, k, \mathcal{D})$, where $\mathcal{U}_c$ is the set of possible utility functions for college c, and $F : \prod_{c\in C} \mathcal{U}_c \to [0,1]$ is the distribution of utility types. We assume that each $u_c \in \mathcal{U}_c$ is an additive function $u_c : 2^S \to \mathbb{R}$ on the set of subsets of students. More specifically, we assume that

$$u_c(S') \begin{cases} = \sum_{s\in S'} u_c(s) \text{ if } |S'| \le q_c, \\ < 0 \text{ otherwise,} \end{cases}$$

where $u_c(s) = u_c(\{s\})$. We assume that $sP_cs' \iff u_c(s) > u_c(s')$. If s is acceptable to c, $u_c(s) > 0$. If s is unacceptable, $u_c(s) < 0$. Further, we suppose that $\sup_{n\in\mathbb{N}, s\in S^n, c\in C^n, u_c\in\mathcal{U}_c} u_c(s)$ is finite.

The set of players is C, with von Neumann-Morgenstern expected utility functions in $(\mathcal{U}_c)_{c\in C}$ drawn from distribution F. All the colleges move simultaneously. College c submits a preference list and quota pair (P'_c, q'_c) with $1 \le q'_c \le q_c$ after its realization of utility u_c, but without observing the utilities realized by the other colleges. A strategy for college c is a report $(P'_c(u_c), q'_c(u_c))$ for each possible utility function $u_c \in \mathcal{U}_c$.

After colleges submit a preference profile, random preferences of students are realized according to the given distribution $\mathcal{D}$. The outcome is the assignment resulting from ϕ under reported preferences of colleges and realized students preferences. We assume the distribution of college preferences is independent of the distribution of student preferences, and both distributions are common knowledge. Moreover, a college does not know realizations of student preferences. As in the main text, we assume that students are passive players and always submit their preferences truthfully.

Given $\varepsilon > 0$, a strategy profile $(P^*_c(u_c), q^*_c(u_c))_{c\in C, u_c\in\mathcal{U}_c}$ is an **ε-Bayes Nash equilibrium** if there is no $c \in C$, $u_c \in \mathcal{U}_c$ and (P'_c, q'_c) such that

$$\begin{aligned} &E[u_c(\phi(S, C, (P_S, P'_c, (P^*_{c'}(u_{c'}))_{c'\in C\setminus c}), (q'_c, (q^*_{c'}(u_{c'}))_{c'\in C\setminus c})))] \\ &\qquad\qquad > E[u_c(\phi(S, C, (P_S, (P^*_{c'}(u_{c'}), q^*_{c'}(u_{c'}))_{c'\in C})))] + \varepsilon, \end{aligned}$$

where the expectation is taken with respect to random preference lists of students and distribution F of college preferences.

We say that a strategy $(P'_c(u_c), q'_c(u_c))$ is truth-telling if the college reports the preferences (P_c, q_c) represented by utility function u_c, for each $u_c \in \mathcal{U}_c$. Now, we can restate Theorem 2 for the Bayesian game.

Theorem 4. *Suppose that the sequence of random markets is regular and sufficiently thick. Then for any $\varepsilon > 0$, there exists n_0 such that truth-telling by every college is an ε-Bayes Nash equilibrium for any market in the sequence with more than n_0 colleges.*

Proof. From the proof of Theorem 2, we know that for any $\varepsilon > 0$, there exists n_0 such that for each realization of the utilities of colleges, truth-telling is an ε-Nash equilibrium for any market in the sequence with more than n_0 colleges for that realization of colleges' utilities. Since the result holds for each realization of utilities and we can find n_0 uniformly across utility realizations, truth-telling by all colleges is an ε-Bayesian Nash as well. $\square$

A.5 Weakening distributional assumptions

In the main text, we have focused on a simple case in which student preferences are drawn from the same distribution. This section extends our analysis to cases in which student preferences are drawn from a number of different distributions.

The model is the same as before except for how student preferences are drawn. We now defined a random market as $\tilde{\Gamma}^n = (C^n, S^n, \succ_{C^n}, k^n, (\mathcal{D}^n(r))_{r=1}^{R^n})$, where R^n is a positive integer. Each random market is endowed with R^n different distributions. To represent student preferences, we partition students into R^n regions, where each student is a member of exactly one region.[2] Write $\mathcal{D}^n(r) = (p_c^n(r))_{c\in C^n}$ as the probability distribution on C^n for students in region r. For each student $s \in S^n$ in region r, we construct preferences of s over colleges as described below:

- Step 1: Select a college independently according to $\mathcal{D}^n(r)$. List this college as the top ranked college of student s.

In general,

- Step $t \leq k$: Select college independently according to $\mathcal{D}^n(r)$ until a college is drawn that has not been previously drawn in steps 1 through $t-1$. List this college as the tth most preferred college of student s.

We will refer to this method of generating student preferences as the **model with heterogenous student preference distributions**. The case with $R^n = 1$ for all n corresponds to our earlier model with one distribution for student preferences.

Our regularity assumptions extend naturally: in addition to conditions in the previous definition, we also require that, for some positive integer $\bar{R}$, $R^n = \bar{R}$ for every n in a regular market. Finally, our assumption of sufficient thickness generalizes easily to the current environment. Let

$$V_T(n) = \{c \in C^n | p_1^n(r)/p_c^n(r) \leq T \quad \text{for all } r, |\{s \in S^n | cP_s s\}| < q_c\},$$
$$Y_T(n) = |V_T(n)|.$$

[2]We frame the heterogeneity of student preferences in terms of multiple regions where students live. Of course alternative interpretations are possible, such as heterogeneity depending on medical specialties, gender, race or academic performance or combinations of these characteristics.

Definition 4. A sequence of random markets is **sufficiently thick** if there exists $T \in \mathbb{R}$ such that

$$E[Y_T(n)] \to \infty,$$

as $n \to \infty$.

Definition 4 is a multi-region generalization of sufficient thickness for one-region setting. The following examples satisfy this version of sufficient thickness.

Example 6 (Two regions with opposite popularity)**.** Fix an arbitrary quota q_c for each college c. There are two regions, $\bar{R} = 2$. $C^n = \{1, 2, \ldots, n\}$ and the probability distributions are:

$$p_c^1(1) = \frac{n - c + 1}{\sum_{c' \in C^n}(n - c' + 1)} = \frac{n - c + 1}{\frac{n(n+1)}{2}},$$
$$p_c^1(2) = \frac{c}{\sum_{c' \in C^n} c'} = \frac{c}{\frac{n(n+1)}{2}}.$$

Students in the first region prefer the first college over the second college and so forth on average, while students in the second region have the opposite preferences. There is an extreme form of differences in preferences in this market.

Example 7 (Multiple regions with within-region symmetry)**.** Fix an arbitrary quota q_c for each college c. Assume there are $\bar{R}$ regions, $\bar{R} \geq 2$. Each college is based in one of the regions. Let $r(c)$ be the region in which college c is. Let $\tilde{p}_m(r)$, $r, m \in \{1, \ldots, \bar{R}\}$ be strictly positive for every r, m. From this, we define the probability $p_c^n(r)$ for any $n \in \mathbb{N}$ as follows:

$$p_c^n(r) = \frac{\tilde{p}_{r(c)}(r)}{\sum_{m \in \bar{R}} \tilde{p}_m(r) \nu_m^n},$$

where $\nu_m^n = |\{c \in C^n | r(c) = m\}|$ denotes the number of colleges in $\tilde{\Gamma}^n$ that is based in region m.

This environment has the following interpretation. Each college is based in one of the regions, and each student lives in one region. Colleges in a given region are equivalent to one another. The "base popularity" of a college in region m for a student living in region r is given by $\tilde{p}_m(r)$. Then we normalize these to obtain $p_c^n(r)$ by the above equation. For any pair of colleges c and c' and region r, we have that

$$p_c^n(r)/p_{c'}^n(r) = \tilde{p}_{r(c)}(r)/\tilde{p}_{r(c')}(r).$$

Such heterogeneous preferences may be present in labor markets or in large urban school districts, where students in the same region have similar preferences while substantial differences are present across regions.

Proposition 2. *Sequences of random markets in Examples 6 and 7 are sufficiently thick.*

These are among the simplest examples incorporating heterogeneity. The two region case shows that directly opposing preferences satisfy sufficient thickness. The multiple region case illustrates that a great deal of heterogeneity in student preferences is allowed.

The equilibrium analysis in the one-region setting (Theorem 2) extends to heterogeneous preference distributions such that the market is sufficiently thick.

Theorem 5. *Consider the model where student preferences are from heterogeneous distributions. Suppose that the sequence of random markets is regular and sufficiently thick. Then for any $\varepsilon > 0$, there exists n_0 such that truth-telling is an ε-Nash equilibrium for any market with more than n_0 colleges.*

A.6 Pre-arrangement in a sufficiently thick market

This section states and proves a result similar in spirit to Theorem 2 for pre-arrangement.

Theorem 6. *Consider the model where student preferences are from heterogeneous distributions. Suppose that $(\tilde{\Gamma}^1, \tilde{\Gamma}^2, \dots)$ is regular and sufficiently thick. Consider the SOSM. For any $\varepsilon > 0$, there exists n_0 such that for any $n > n_0$ and $c \in C^n$, the probability that c can profitably manipulate via pre-arrangement is smaller than ε.*

Proof. In the proof of Theorem 3, we have shown that for each college c, the probability of successful manipulation is at most π_c. By Lemma 10 and sufficient thickness, there exists a T such that

$$\pi_c \leq \frac{4\left[T\bar{q}(2^{\bar{q}}-1)+1\right]}{E[Y_T(n)]}.$$

Given ε, the definition of sufficient thickness implies that $E[Y_T(n)] \to \infty$, so $\pi_c \to 0$. $\square$

A.7 Manipulations by coalitions

The basic model shows that individual colleges have little opportunity to manipulate a large market. One natural question is whether coalitions of colleges can manipulate by coordinating their reports. Formally, a coalition $\bar{C} \subseteq C$ manipulates the market (S, C, P, q) if there exists $(P'_{\bar{C}}, q'_{\bar{C}}) = (P'_c, q'_c)_{c\in\bar{C}}$ such that

$$\phi(S, C, (P'_{\bar{C}}, P_{-\bar{C}}), (q'_{\bar{C}}, q_{-\bar{C}})) \succ_c \phi(S, C, P, q),$$

for some $c \in \bar{C}$.

The notion of coalitional manipulation we consider allows for a broad range of coalitions, for a coalition is said to manipulate even if only some of its members are made strictly better off and others in the coalition are made strictly worse off when they misreport their preferences jointly.

Theorem 7. *Consider the model where student preferences are from heterogeneous distributions. Suppose that the sequence of random markets is regular and sufficiently thick. Consider the SOSM. Then, for any positive integer m and any $\varepsilon > 0$, there exists n_0 such that for any $n > n_0$ and $\bar{C} \subseteq C^n$ with $|\bar{C}| \leq m$, the probability that $\bar{C}$ can profitably manipulate is smaller than ε.*

Our result shows that successful coalitional manipulation is rare: with high probability, not a single college in the coalition is made strictly better off. Thus it is hard for coalitions to manipulate even when monetary transfers are possible among colleges.

A.8 Manipulating the Boston mechanism in large markets

The following example shows that students have incentives to manipulate the Boston mechanism even in large markets.

Example 8. Consider market $\tilde{\Gamma}^n$, where $|S^n| = |C^n| = n$ for each n. $q_c^n = 1$ for every n and $c \in C^n$. Preference lists are common among colleges and given by

$$P_c : s_1, s_2, \ldots, s_n,$$

for every $c \in C^n$.

Let $p_{c_1}^n = (1/2)^{1/n}, p_{c_2}^n = (1 - (1/2)^{1/n})(1/2)^{1/n}$, and $p_c^n = (1 - p_{c_1}^n - p_{c_2}^n)/(n-2)$ for each $c \neq c_1, c_2$. Then, with probability $[p_{c_1}^n p_{c_2}^n/(1 - p_{c_1}^n)]^n = 1/4$, students preferences are

$$P_s : c_1, c_2, \ldots,$$

for each $s \in S^n$. If every student is truth-telling, then s_1 and s_2 are matched to c_1 and c_2, respectively, and other students are matched to their third or less preferred choices. If $s \neq s_1, s_2$ deviates from truth-telling unilaterally and reports preference list

$$P_s : c_2, \ldots,$$

then s is matched to her second choice c_2, which is preferred to the match under truth-telling. This occurs with probability of at least $1/4$, and every student except s_1 and s_2 has an incentive not to be truth-telling.

B Appendix B: Proofs

B.1 Proof of Theorem 1

We prove Theorem 1 through several steps. Specifically, we prove three key lemmas, Lemmas 1, 3 and 7 and then use them to show the theorem. For the proof of Lemmas 1 and 3, we keep the set of students and colleges fixed, and refer to markets in terms of preference profiles only.

Let (P_c, q_c) be a pair of the true preference list and true quota of college c. A report (P'_c, q_c) is said to be a **dropping strategy** if (i) sP_cs' and $sP'_c\emptyset$ imply sP'_cs', and (ii) $\emptyset P_cs$ implies $\emptyset P'_cs$. A dropping strategy does not modify quotas and simply declares some students who are acceptable under P_c as unacceptable. It does not change the relative ordering of acceptable students or declare unacceptable students as acceptable.

B.1.1 Lemma 1: Dropping strategies are exhaustive

Given a stable mechanism φ, denote the matching under φ with respect to reported profile (P, q) by $\varphi(P, q)$. The formal statement of Lemma 1 in the main text is as follows.

Lemma 1. *Consider an arbitrary stable mechanism φ. Fix preference profile (P, q). For some report $(\tilde{P}_c, \tilde{q}_c)$ of c, suppose that $\varphi((\tilde{P}_c, P_{-c}), (\tilde{q}_c, q_{-c})) = \mu$. Then there exists a dropping strategy (P'_c, q_c) such that $\varphi((P'_c, P_{-c}), q) \succeq_c \mu$.*

Proof. Construct dropping strategy (P'_c, q_c) such that P'_c lists all the students it is matched to under μ who are acceptable ($\{s \in \mu(c) | sP_c\emptyset\}$) in the same relative order as in P_c, and reports every other student as unacceptable. Let $(P', q) = (P'_c, P_{-c}, q)$.

We will show $\varphi(P', q)(c)$ is equal to $\{s \in \mu(c) | sP_c\emptyset\}$. This equality implies that $\varphi(P', q) \succeq_c \mu$, since $\varphi(P', q)(c)$ can only differ from $\mu(c)$ in having no unacceptable students under the true preference list P_c. The proof proceeds in two steps.

First, consider the matching μ' obtained from μ by having c keep only the students in $\mu(c)$ that are acceptable under P_c. That is,

$$\mu'(c') = \begin{cases} \{s \in \mu(c) | sP_c\emptyset\} & c' = c, \\ \mu(c') & c' \neq c. \end{cases}$$

Consider the properties of μ' under (P', q). First, μ' is individually rational under (P', q). Second, there is no blocking pair involving c, since $\mu'(c)$ is exactly the set of students acceptable under P'_c. However, there may be a blocking pair involving another college and students who are unmatched. In this case, we can define a procedure which ultimately yields a matching that is stable under (P', q):[3]

[3]This procedure parallels Blum, Roth and Rothblum (1997)'s analogous vacancy-chain procedure for one-to-one matching markets and Cantala (2004)'s algorithm for many-to-one markets.

Starting from $\mu'_0 \equiv \mu'$, let S_0 be the set of students that can be part of a blocking pair in μ'_0. Pick some $s \in S_0$. Let college c' be the college that student s prefers the most within the set of colleges that can be part of a blocking pair with s. Construct μ'_1 by assigning student s to c' and if $|\mu'_0(c')| = q_{c'}$, then let c' reject its least preferred student. By construction, μ'_1 is individually rational, college c is matched to the same set of students, college c' strictly prefers $\mu'_1(c')$ over $\mu'_0(c')$ and the assignments of all other colleges are unchanged. As with μ'_0, the only blocking pair of μ'_1 involves unmatched students. If there are blocking pairs of μ'_1, then repeat the same procedure to construct μ'_2, and so on.

At each repetition, the new matching is individually rational, college c is matched to the same set of students, one other college strictly improves and the remaining colleges are not worse off since each blocking pair involves an unassigned student. Since colleges can strictly improve only a finite number of times, this procedure terminates in finite time. The ultimate matching μ'' is stable in (P', q) because it is individually rational and there are no blocking pairs. Moreover, since college c obtains the same matching in each repetition,

$$\mu''(c) = \mu'(c). \tag{1}$$

The second step of the proof utilizes the fact that for any college, the same number of students are matched to it across different stable matchings (Roth 1984a). Since μ'' is stable in (P', q) and $\varphi(P', q)$ is stable in (P', q) by definition, $|\mu''(c)| = |\varphi(P', q)(c)|$. Since there are just $|\mu''(c)|$ acceptable students under P'_c, this implies that

$$\mu''(c) = \varphi(P', q)(c). \tag{2}$$

Equations (1) and (2) together imply that

$$\varphi(P', q)(c) = \mu'(c) = \{s \in \mu(c) | sP_c\emptyset\}.$$

As a result,

$$\varphi(P', q)(c) \succeq_c \mu(c),$$

since $\varphi(P', q)(c)$ can only differ from $\mu(c)$ in having no unacceptable students under the true preference list P_c.

□

B.1.2 Lemma 3: Rejection chains

For preference profile (P, q), let μ be the student-optimal stable matching. Let B^1_c be an arbitrary subset of $\mu(c)$. The **rejection chains** algorithm with input B^1_c is defined as follows.

Algorithm 1. Rejection Chains

(1) Initialization:

(a) μ is the student-optimal stable matching, and B_c^1 is a subset of $\mu(c)$. Let $i = 0$. Let c reject all the students in B_c^1.

(2) Increment i by one.

(a) If $B_c^i = \emptyset$, then terminate the algorithm.

(b) If not, let s be the least preferred student by c among B_c^i, and let $B_c^{i+1} = B_c^i \setminus s$.

(c) Iterate the following steps (call this iteration "Round i").

i. Choosing the applied:

A. If s has already applied to every acceptable college, then finish the iteration and go back to the beginning of Step 2.

B. If not, let c' be the most preferred college of s among those which s has not yet applied while running the SOSM or previously within this algorithm. If $c' = c$, terminate the algorithm.

ii. Acceptance and/or rejection:

A. If c' prefers each of its current mates to s and there is no vacant position, then c' rejects s; go back to the beginning of Step 2c.

B. If c' has a vacant position or it prefers s to one of its current mates, then c' accepts s. Now if c' had no vacant position before accepting s, then c' rejects the least preferred student among those who were matched to c'. Let this rejected student be s and go back to the beginning of Step 2c. If c' had a vacant position, then finish the iteration and go back to the beginning of Step 2.

Algorithm 1 terminates either at Step 2a or at Step 2(c)iB. We say that Algorithm 1 **does not return to c** if it terminates at Step 2a and it **returns to c** if it terminates at Step 2(c)iB.

Lemma 2. *Consider college c and suppose that under (P, q), Algorithm 1 with every possible subset of $\mu(c)$ as an input fails to return to c. For any dropping strategy (P'_c, q_c), let $B_c^1 = \{s \in \mu(c) | \emptyset P'_c s\}$ be non-empty and let μ' be a matching obtained at the end of Algorithm 1 with input B_c^1. Then, under (P'_c, P_{-c}, q),*

(1) *μ' is individually rational,*

(2) *no $c' \neq c$ is a part of a blocking pair of μ', and*

(3) *if (s, c) blocks μ', then $[\arg\min_{P_c} \mu(c)] P_c s$,*[4] *and*

[4]For any binary relation R on X and $X' \subseteq X$, $\arg\min_R X' = \{x \in X' | yRx \text{ for any } y \in X'\}$.

(4) *if* (s,c) *blocks* μ' *and* $\mu'(c)$ *is non-empty, then* $[\arg\min_{P'_c}\mu'(c)]P'_c s$*.*

Proof. Part (1): For $c' \neq c$, c' only accepts students who are acceptable in each step of the SOSM and Algorithm 1. c rejects every student who is unacceptable under P'_c at the outset of Algorithm 1, and accepts no other student by the assumption that Algorithm 1 does not return to c for any subset of $\mu(c)$ as an input. Therefore μ' is individually rational.

Part (2): Suppose that for some $s \in S$ and $c' \in C$ such that $c' \neq c$, $c' P_s \mu'(s)$. Then, by the definition of the SOSM and Algorithm 1, s is rejected by c' either during the SOSM or in Algorithm 1. This implies that $|\mu'(c')| = q_{c'}$ and $[\arg\min_{P_{c'}}\mu'(c)]P_{c'}s$, implying that s and c' do not block μ'.

Part (3): Suppose $cP_s\mu'(s)$ for some $s \in S$. As in Part (2), this implies that s is rejected by c either during the SOSM or in Algorithm 1. Since Algorithm 1 does not return to c by assumption, s is rejected either during the SOSM or at the beginning of Algorithm 1. In the former case, if s is rejected during the SOSM, then $[\arg\min_{P_c}\mu(c)]P_c s$. In the latter case, (s,c) is not a blocking pair because s is declared unacceptable under P'_c.

Part (4): From Part (3), we have that

$$[\arg\min_{P_c}\mu(c)]P_c s. \tag{3}$$

By definition, $\mu'(c) \subseteq \mu(c)$ and expression (3) imply

$$[\arg\min_{P_c}\mu'(c)]P_c s, \tag{4}$$

when $\mu'(c)$ is non-empty.

When $\mu'(c)$ is non-empty, $\mu'(c)$ is constructed by including only acceptable students in $\mu(c)$ according to P'_c, so

$$[\arg\min_{P'_c}\mu'(c)]P'_c\emptyset. \tag{5}$$

Expressions (4) and (5) and the definition of a dropping strategy yield

$$[\arg\min_{P'_c}\mu'(c)]P'_c s.$$

□

Lemma 3 (Rejection chains)**.** *For any market and any* $c \in C$*, if Algorithm 1 does not return to* c *for any non-empty* $B^1_c \subseteq \mu(c)$*, then* c *cannot profitably manipulate by a dropping strategy.*

Proof. Consider an arbitrary dropping strategy (P'_c, q_c) and let $B^1_c = \{s \in \mu(c) | \emptyset P'_c s\}$. When Algorithm 1 does not return to c, we show that $\mu = \phi(P,q)$ is weakly preferred by college c to $\phi(P'_c, P_{-c}, q)$. Let $(P', q) = (P'_c, P_{-c}, q)$.

Let μ' be the matching resulting from Algorithm 1 with input B_c^1. In the first step of the proof, we construct a new matching μ'' by satisfying the blocking pairs involving college c such that college c weakly prefers μ over μ'' according to profile (P, q). The second step of the proof shows that μ'' is weakly preferred to $\phi(P', q)$ by college c. These two steps together will yield our desired conclusion.

First, suppose that c does not block μ' in (P', q). Then let $\mu'' = \mu'$. It is clear that college c weakly prefers μ over μ'' because $\mu''(c) = \mu'(c) \subseteq \mu(c)$. Otherwise, if c blocks μ', construct μ'' as follows: College c admits its most preferred students under P'_c who are willing to be matched possibly leaving the seats occupied by these students at other colleges vacant; that is,

$$\mu''(c') = \begin{cases} \mu'(c) \cup \arg\max_{(P'_c, q_c - |\mu'(c)|)} \{s \in S | cP_s\mu'(s)\} & c' = c, \\ \mu'(c') \setminus \mu''(c) & c' \neq c, \end{cases}$$

where $\arg\max_{(P'_c, q_c - |\mu'(c)|)} X$ denotes at most $q_c - |\mu'(c)|$ students that are most preferred under P'_c in set X. Recall that $\mu'(c) \subseteq \mu(c)$ by definition. Moreover, Part (3) of Lemma 2 shows that, under μ'', $|\mu'(c)|$ positions of c are filled with $\mu'(c)$ and the remaining $|\mu''(c) - \mu'(c)|$ positions are filled with students less preferred to $\arg\min_{P_c} \mu(c)$. Since preferences are responsive, we obtain

$$\mu(c) \succeq_c \mu''(c). \tag{6}$$

In the second step of the proof, we demonstrate that μ'' is weakly preferred to $\phi(P', q)$ by college c. The proof works by comparing μ'' to a stable matching μ''' in (P', q), that we construct below:

Case 1: If μ'' is stable in (P', q), then let $\mu''' = \mu''$. In this case, it is obvious that $\mu''(c)$ is weakly preferred to $\mu'''(c)$ under P'_c.

Case 2: Otherwise, observe the following properties of μ'':

1) Matching μ'' is individually rational.
2) College c is not part of a blocking pair under μ''.
3) The only blocking pairs in μ'' involve colleges who have vacant seats.

Property (1) follows by construction. To establish Property (2), we consider two cases. When $\mu'(c)$ is empty, $\mu''(c)$ is college c's most preferred students according to P'_c who are part of blocking pairs of μ'. College c is not part of a blocking pair under μ''. When $\mu'(c)$ is not empty, suppose college c is preferred by student s to $\mu''(s)$. In this case, Part (4) of Lemma 2 implies that s is less preferred than any student in $\mu'(c)$ under $P_{c'}$ and is less preferred than students in $\arg\max_{(P'_c, q_c - |\mu'(c)|)} \{s \in S | cP_s\mu'(s)\}$ by construction.

Therefore, college c does not form a blocking pair with this student, and, hence, c is not part of any blocking pair under μ''. Finally, Property (3) follows by construction and by Part (2) of Lemma 2.

With these properties in hand, we construct μ'''. Let $\mu''_0 \equiv \mu''$. μ''_0 may have a blocking pair involving a college with a vacant seat. Construct μ''_1 as follows. Let C_0 be the set of colleges that can be part of a blocking pair in μ''_0. Pick some $c' \in C_0$. Let student s be the student that college c' prefers the most within the set of students involved in blocking pairs with c'. Construct μ''_1 by assigning college c' to s, and if $\mu''_0(s) \neq \emptyset$, then s rejects $\mu''_0(s)$, leaving a vacant seat at college $\mu''_0(s)$. Under μ''_1, student s receives a strictly preferred assignment and each other student's assignment is unchanged. For college c, $\mu''_1(c)$ is weakly less preferred under P'_c than the initial matching, $\mu''_0(c)$, because the students are better off in μ''_1. Matching μ''_1 is individually rational, and as with μ''_0, the only blocking pair of μ''_1 involves colleges with vacant seats. If there are blocking pairs of μ''_1, then we repeat the same procedure to construct μ''_2, and so on.

At each repetition, the new matching is individually rational and each blocking pair of the new matching involves a college with a vacant seat. As a result, no additional students are rejected when the blocking pair is satisfied, and in each repetition, one student strictly improves and the remaining students do not change their assignment. Since students may improve their assignment a finite number of times, the procedure ends in finite time.

The ultimate matching μ''' is stable in (P', q) because it is individually rational and there are no blocking pairs. Moreover, for college c, each new matching is weakly less preferred under P'_c than the initial matching $\mu''_0(c)$ as students weakly improve in each new matching. Hence, $\mu''(c)$ is weakly preferred to $\mu'''(c)$ under P'_c.

Since the matching produced by SOSM is the least preferred stable matching of every college (attributed to Conway in Knuth (1976), $\phi(P', q)(c)$ is weakly less preferred to the stable matching μ''' by college c under P'_c. This implies that $\phi(P', q)(c)$ is weakly less preferred to $\mu''(c)$ under P'_c, and since P'_c is a dropping strategy of P_c,

$$\mu''(c) \succeq_c \phi(P', q)(c). \tag{7}$$

Equations (6) and (7) together allow us to conclude that

$$\mu(c) \succeq_c \phi(P', q)(c),$$

showing that (P'_c, q_c) is not a profitable strategy when Algorithm 1 does not return to c. $\square$

B.1.3 Lemma 7: Vanishing market power

We are interested in how often Algorithm 1 returns to a particular college c for the case where students draw their preferences from distribution $\mathcal{D}^n$. Let

$$\pi_c = \Pr[\text{Algorithm 1 returns to c for some } B_c^1 \subseteq \mu(c)].$$

Since Algorithm 1 returns to c for some B_c^1 whenever c can manipulate the SOSM (Lemmas 1 and 3), π_c gives an upper bound of the probability that c can manipulate the SOSM when others are truthful conditional on μ being realized as the matching under the SOSM. Here we will show Lemma 7, which bounds π_c for most colleges in large markets.

Consider the following algorithm, which is a stochastic variant of the SOSM.[5]

Algorithm 2. Stochastic Student-Optimal Gale-Shapley Algorithm

(1) Initialization: Let $l = 1$. For every $s \in S$, let $A_s = \emptyset$.

(2) Choosing the applicant:

   (a) If $l \leq |S|$, then let s be the lth student and increment l by one.[6]

   (b) If not, then terminate the algorithm.

(3) Choosing the applied:

   (a) If $|A_s| \geq k$, then return to Step 2.

   (b) If not, select c randomly from distribution $\mathcal{D}^n$ until $c \notin A_s$, and add c to A_s.

(4) Acceptance and/or rejection:

   (a) If c prefers each of its current mates to s and there is no vacant position, then c rejects s. Go back to Step 3.

   (b) If c has a vacant position or it prefers s to one of its current mates, then c accepts s. Now if c had no vacant position before accepting s, then c rejects the least preferred student among those who were matched to c. Let this student be s and go back to Step 3. If c had a vacant position, then go back to Step 2.

[5]To be more precise this is a stochastic version of the algorithm proposed by McVitie and Wilson (1970), which they show produces the same matching as the original SOSM proposed by Gale and Shapley (1962).

[6]Recall that students are ordered in an arbitrarily fixed manner.

A_s records colleges that s has already drawn from $\mathcal{D}^n$. When $|A_s| = k$ is reached, A_s is the set of colleges acceptable to s.

Under the SOSM, a student's application to her t^{th} most preferred college is independent of her preferences after $(t+1)^{\text{th}}$ choice on. Therefore the above algorithm terminates, producing the student-optimal stable matching of any realized preference profile which would follow from completing the draws for random preferences. Let μ be the student-optimal stable matching obtained by the above algorithm.

Suppose that Algorithm 2 is run and the stable matching μ is obtained. Now fix a college $c \in C$ and let B_c^1 be an arbitrary subset of $\mu(c)$. The **stochastic rejection chains** associated with B_c^1 is defined as follows. As the name suggests, this is a stochastic version of Algorithm 1.

Algorithm 3. Stochastic Rejection Chains

(1) Initialization:

   (a) Keep all the preference lists generated in Algorithm 2. Also, for each $s \in S$, let A_s be the set generated at the end of Algorithm 2. Let the student-optimal matching μ be the initial match of the algorithm. Let B_c^1 be a given subset of $\mu(c)$. Let $i = 0$. Let c reject all the students in B_c^1.

(2) Increment i by one.

   (a) If $B_c^i = \emptyset$, then terminate the algorithm.

   (b) If not, let s be the least preferred student by c among B_c^i, and let $B_c^{i+1} = B_c^i \setminus s$.

   (c) Iterate the following steps (call this iteration "Round i".)

      i. Choosing the applied:

         A. If $|A_s| \geq k$, then finish the iteration and go back to the beginning of Step 2.

         B. If not, select c' randomly from distribution $\mathcal{D}^n$ until $c' \notin A_s$, and add c' to A_s. If c is selected, terminate the algorithm.

      ii. Acceptance and/or rejection:

         A. If c' prefers each of its current mates to s and there is no vacant position, then c' rejects s; go back to the beginning of Step 2c.

         B. If c' has a vacant position or it prefers s to one of its current mates, then c' accepts s. Now if c' had no vacant position before accepting s, then c' rejects the least preferred student among those who were matched to c'. Let this rejected student be s and go back to the beginning of Step 2c. If c' had a vacant position, then finish the iteration and go back to the beginning of Step 2.

Algorithm 3 terminates either at Step 2a or at Step 2(c)iB. Similarly to Algorithm 1, we say that Algorithm 1 **returns to c** if it terminates at Step 2(c)iB and **does not return to c** if it terminates at Step 2a.

We are interested in how often the algorithm returns to c, as a student draws c from distribution $\mathcal{D}^n$. It is clear that the probability that Algorithm 1 returns to c is equal to the probability that Algorithm 3 returns to c. That is,

$$\pi_c = \Pr[\text{Algorithm 3 returns to } c \text{ for some } B_c^1 \subseteq \mu(c)].$$

This latter expression is useful since we can investigate the procedure step by step, utilizing conditional probabilities and conditional expectations. Recall that we have ordered the colleges in terms of decreasing popularity. Our notation is that when $c' \leq c$, we mean that c' is more popular than c, or $p_{c'} \geq p_c$ and when we write $c > m$, where m is a natural number, we mean the index of college c is larger than m. Let

$$V_c = \{c' \in C^n | c' \leq c, c' \notin A_s \text{ for every } s \in S^n \text{ at the end of Algorithm 2}\}, \text{ and}$$
$$Y_c = |V_c|.$$

V_c is a random set of colleges that are more popular than c ex ante but listed on no student's preference list at the end of Algorithm 2. Y_c is a random variable indicating the number of such colleges.[7]

Lemma 4. *For any $c > 4k$, we have*

$$E[Y_c] \geq \frac{c}{2} e^{-8\bar{q}nk/c}.$$

Proof. Let $Q^n = \sum_{c=1}^k p_c^n$. Then the probability that c' is not a student's ith choice, denoted $c_{(i)}$, given her first $(i-1)$ choices $c_{(1)}, \ldots, c_{(i-1)}$ is bounded as follows:

$$1 - \frac{p_{c'}^n}{1 - \sum_{j=1}^{i-1} p_{c_{(j)}}^n} \geq 1 - \frac{p_{c'}^n}{1 - Q^n}.$$

Let $E_{c'}$ be the event that $c' \notin A_s$ for every $s \in S$ at the end of Algorithm 2. Since there are at most $\bar{q}nk$ draws from $\mathcal{D}^n$ in Algorithm 2, the above inequality implies that

$$\Pr(E_{c'}) \geq \left(1 - \frac{p_{c'}^n}{1 - Q^n}\right)^{\bar{q}nk}.$$

[7]We abuse notation and denote a random variable and its realization by the same letter when there is no confusion.

If $c' > k$, there are at least $c' - k$ colleges that are at least as popular as c', but not among the k most popular colleges, so we obtain

$$p_{c'}^n \leq \frac{1 - Q^n}{c' - k}.$$

The last two inequalities imply

$$\Pr(E_{c'}) \geq \left(1 - \frac{1}{c' - k}\right)^{\bar{q}nk}. \tag{8}$$

We now show that for any $c' > 2k$,

$$\left(1 - \frac{1}{c' - k}\right)^{\bar{q}nk} \geq e^{-2\bar{q}nk/(c'-k)}. \tag{9}$$

To see this, first note that

$$\left(1 - \frac{1}{c' - k}\right)^{\bar{q}nk} \geq e^{-2\bar{q}nk/(c'-k)} \iff 1 - \frac{1}{c' - k} - e^{-2/(c'-k)} \geq 0.$$

Now, define a function $f(x) = 1 - x - e^{-2x}$. This function f is concave, and $f(0) = 0$ and $f(1/2) = 1/2 - 1/e > 0$. Therefore $f(x) \geq 0$ for any $x \in [0, 1/2]$. Since $c' > 2k$ and k is a positive integer, we have $c' - k > k \geq 1$. Since $c' - k$ is an integer, we thus obtain $c' - k \geq 2$ and hence $1/(c' - k) \in [0, 1/2]$. Therefore $1 - 1/(c' - k) - e^{-2/(c'-k)} = f(1/(c' - k)) \geq 0$, establishing inequality (9).

Moreover, for any $c' > 2k$,

$$e^{-2\bar{q}nk/(c'-k)} \geq e^{-4\bar{q}nk/c'}. \tag{10}$$

Combining inequalities (8), (9), and (10), we obtain:

$$\Pr(E_{c'}) \geq e^{-4\bar{q}nk/c'}.$$

Using the previous inequality, for any $c > 4k$, we have

$$E[Y_c] = \sum_{c'=1}^{c} \Pr(E_{c'}) \geq \sum_{c'=2k}^{c} e^{-4\bar{q}nk/c'} \geq \sum_{c'=c/2}^{c} e^{-8\bar{q}nk/c} = \frac{c}{2} e^{-8\bar{q}nk/c}.$$

$\square$

For $B_c^1 \subseteq \mu(c)$, let

$$\pi_c^{B_c^1} = \Pr[\text{Algorithm 3 with input } B_c^1 \text{ returns to c}|Y_c > E[Y_c]/2, \mu].$$

$\pi_c^{B_c^1}$ gives an upper bound of the probability that c can manipulate the SOSM when others are truthful, conditional on two events: μ is the realized matching under the SOSM with truthful preferences and there are not too small a number of colleges ($Y_c > E[Y_c]/2$) that are more popular than c and appear nowhere on students' preference lists at the end of Algorithm 2.

Let $c^*(n) = 16\bar{q}nk/\ln(\bar{q}n)$. We will see, in Lemma 7, that $c^*(n)$ is the number of "very popular colleges" in a market with n colleges. Note that $c^*(n)/n$ converges to zero as $n \to \infty$, so the proportion of such colleges goes to zero. Except for these $c^*(n)$ colleges, the following lemma gives an upper bound for manipulability in a large market.

Lemma 5. *Suppose that n is sufficiently large and $c > c^*(n)$. Then we have*

$$\pi_c^{B_c^1} \leq \frac{4\bar{q}}{E[Y_c]},$$

for any $B_c^1 \subseteq \mu(c)$.

Proof. Consider Round 1, beginning with the least preferred student s of $B_c^1 \subseteq \mu(c)$ (if $B_c^1 = \emptyset$, then the inequality is obvious since $\pi_c^{B_c^1} = 0$.). Since $p_{c'}^n \geq p_c^n$ for any $c' \in V_c$, Round 1 ends at Step 2(c)iiB as a student applies to some college with vacant positions, at least with probability $1 - 1/(Y_c + 1) > 1 - 1/(E[Y_c]/2 + 1)$.

Now assume that all Rounds $1, \ldots, i-1$ end at Step 2(c)iiB. Then there are still at least $Y_c - (i-1)$ colleges more popular than c and with a vacant position, since at most $i-1$ colleges in V_c have had their positions filled at Rounds $1, \ldots, i-1$. Therefore Round i initiated by the least preferred student in B_c^i ends at Step 2(c)iiB with probability of at least $1 - 1/(E[Y_c]/2 - (i-1) + 1)$. Note that

$$E[Y_c]/2 - (i-1) + 1 \geq E[Y_c]/4 > 0, \tag{11}$$

for sufficiently large n. To see this, it is sufficient to show

$$E[Y_c]/4 - (i-1) + 1 \geq 0.$$

Since there are at most $\bar{q}$ rounds,

$$E[Y_c]/4 - (i-1) + 1 \geq E[Y_c]/4 - \bar{q} + 2.$$

The definition of Y_c implies that Y_c is weakly increasing in c. Lemma 4 provides a lower bound on $E[Y_c]$, which for $c > c^*(n)$ implies

$$E[Y_c]/4 - \bar{q} + 2 \geq \frac{2\bar{q}kn^{\frac{1}{2}}}{\ln(n)} - \bar{q} + 2,$$

which is positive for sufficiently large n, showing inequality (11) for sufficiently large n.

Since there are at most $\bar{q}$ rounds, Algorithm 3 fails to return to c with probability of at least

$$\prod_{i=1}^{\bar{q}} \left(1 - \frac{1}{E[Y_c]/2 - (i-1) + 1}\right) \geq \left(1 - \frac{1}{E[Y_c]/4}\right)^{\bar{q}}, \tag{12}$$

for sufficiently large n and $c > c^*(n)$, because of inequality (11).

Therefore we have that

$$\pi_c^{B_c^1} \leq 1 - \left(1 - \frac{1}{E[Y_c]/4}\right)^{\bar{q}} \leq \frac{4\bar{q}}{E[Y_c]},$$

where the last inequality holds since $1 - (1-x)^y \leq yx$ for any $x \in (0,1)$ and $y \geq 1$.[8] $\square$

We state without proof the following lemma (this is a straightforward generalization of Lemma 4.4 of Immorlica and Mahdian (2005)).

Lemma 6. *For every c, we have $Var[Y_c] \leq E[Y_c]$.*

Now we are ready to present and prove the last of the three key lemmas.

Lemma 7 (Vanishing market power)**.** *If n is sufficiently large and $c > c^*(n)$, then*

$$\pi_c \leq \frac{[\bar{q}(2^{\bar{q}} - 1) + 1]\ln(\bar{q}n)}{2k\sqrt{\bar{q}n}}.$$

Proof. By the fact that any probability is non-negative and less than or equal to one, the Chebychev inequality, and Lemma 6, we have

$$\begin{aligned}\Pr\left[Y_c \leq \frac{E[Y_c]}{2}\right] &\leq \Pr\left[Y_c \leq \frac{E[Y_c]}{2}\right] + \Pr\left[Y_c \geq \frac{3E[Y_c]}{2}\right] \\ &= \Pr\left[|Y_c - E[Y_c]| \geq \frac{E[Y_c]}{2}\right] \leq \frac{Var[Y_c]}{(E[Y_c]/2)^2} \leq \frac{4}{E[Y_c]}.\end{aligned}$$

[8]Note that conditions for this inequality is satisfied since $4/E[Y_c] \in (0,1)$ for any sufficiently large n and $c > c^*(n)$.

Since the probability of a union of events is at most the sum of the probabilities of individual events (Boole's inequality), Lemma 5 and the fact that there are at most $2^{\bar{q}}-1$ non-empty subsets of $\mu(c)$ imply

$$\begin{aligned}
&\Pr[\text{Algorithm 3 returns to c for some } B_c^1 \subseteq \mu(c) | Y_c \geq E[Y_c]/2, \mu] \\
&\leq \sum_{B_c^1 \subseteq \mu(c)} \pi_c^{B_c^1} \\
&\leq \frac{4\bar{q}(2^{\bar{q}}-1)}{E[Y_c]}.
\end{aligned}$$

This inequality holds for any matching μ. Therefore, we have the same upper bound for probability conditional on $Y_c > E[Y_c]/2$ but not on μ, that is,

$$\Pr\left[\text{Algorithm 3 returns to c for some } B_c^1 \subseteq \mu(c) | Y_c \geq \frac{E[Y_c]}{2}\right] \tag{13}$$

$$\leq \frac{4\bar{q}(2^{\bar{q}}-1)}{E[Y_c]}. \tag{14}$$

By the above inequalities and the fact that probabilities are less than or equal to one,

$$\begin{aligned}
\pi_c &\leq \Pr\left[Y_c \leq \frac{E[Y_c]}{2}\right] + \Pr\left[Y_c > \frac{E[Y_c]}{2}\right] \times \frac{4\bar{q}(2^{\bar{q}}-1)}{E[Y_c]} \\
&\leq \frac{4}{E[Y_c]} + \frac{4\bar{q}(2^{\bar{q}}-1)}{E[Y_c]} \\
&= \frac{4[\bar{q}(2^{\bar{q}}-1)+1]}{E[Y_c]}.
\end{aligned}$$

Applying Lemma 4 and noting that $E[Y_c]$ is increasing in c so $E[Y_{c^*(n)}] \leq E[Y_c]$ for any $c > c^*(n) = 16\bar{q}nk/\ln(\bar{q}k)$, we complete the proof of Lemma 7.[9] □

B.1.4 Theorem 1

Now we prove Theorem 1. Let

$$\begin{aligned}
\alpha(n) = E[\#\{c \in C | \phi(S, C, (P'_c, P_{-c}), (q'_c, q_{-c})) \succ_c \phi(S, C, P, q) \\
\text{for some } (P'_c, q'_c) \text{ in the induced market}\} | \tilde{\Gamma}^n],
\end{aligned}$$

be the expected number of colleges that can manipulate in the market induced by random market $\tilde{\Gamma}^n$ under ϕ when others report preferences truthfully. By Lemma 1, it suffices to consider

[9]Note that Lemma 4 can be applied since for sufficiently large n and $c \geq c^*(n)$, we have $c > 4k$.

dropping strategies. By Lemma 3, the probability that $c \in C$ can successfully manipulate by some dropping strategy is at most π_c. Thus we obtain

$$\begin{aligned}
\alpha(n)/n &= \left[\sum_{c \in C^n} \Pr[c \text{ successfully manipulates}]\right]/n \\
&\leq c^*(n)/n + [\sum_{c \geq c^*(n)}^{n} \pi_c]/n \\
&\leq c^*(n)/n + [\sum_{c \geq c^*(n)}^{n} \frac{[\bar{q}(2^{\bar{q}}-1)+1]\ln(\bar{q}n)}{2k\sqrt{\bar{q}n}}]/n && \text{(by Lemma 7)} \\
&\leq \frac{16\bar{q}k}{\ln(\bar{q}n)} + \frac{(\bar{q}(2^{\bar{q}}-1)+1)\ln(\bar{q}n)}{2\sqrt{\bar{q}}k\sqrt{n}} && \text{(by } c^*(n) = 16\bar{q}nk/\ln(\bar{q}n)\text{)}.
\end{aligned}$$

The first term is proportional to $1/\ln(\bar{q}n)$ and the second term is proportional to $\ln(\bar{q}n)/\sqrt{n}$. Since both expressions approach zero as n approaches infinity, we obtain $\alpha(n)/n \to 0$ as $n \to \infty$, completing the proof.

B.2 Proof of Theorem 3

In this section, we consider the possibility of pre-arrangement, so we reintroduce S and C as arguments to the mechanism to avoid confusion.

B.2.1 Lemma 8

The following lemma says that a student that is involved in pre-arrangement is less preferred by the college to any student who is matched to it without pre-arrangement.

Lemma 8. *If $c \in C$ can manipulate via pre-arrangement with $s \in S$, then*

$$s' P_c s \text{ for every } s' \in \phi(S, C, P, q)(c).$$

Proof. Let $\mu(c) = \phi(S, C, P, q)(c)$. Theorem 2 of Sönmez (1999) implies that, for any stable mechanism, if c can manipulate via pre-arrangement with student s, then either $s \in \mu(c)$ or $s' P_c s$ for every $s' \in \mu(c)$. To show $s \notin \mu(c)$, suppose on the contrary that $s \in \mu(c)$. Consider matching μ' given by

$$\mu'(c') = \begin{cases} \mu(c) \setminus s & \text{if } c' = c, \\ \mu(c') & \text{otherwise.} \end{cases}$$

It is easy to see, from stability of μ in (S, C, P, q), that μ' is stable in $(S \setminus s, C, P_{-s}, q_c - 1, q_{-c})$.

Since the matching under the SOSM is weakly less preferred to any stable matching by colleges and preferences are responsive,

$$\begin{aligned}\phi(S, C, P, q)(c) &= \mu'(c) \cup s \\ &\succeq_c \phi(S \setminus s, C, P_{-s}, q_c - 1, q_{-c})(c) \cup s.\end{aligned}$$

Therefore c cannot manipulate the student-optimal stable mechanism via pre-arrangement. This is a contradiction, completing the proof. $\square$

To profitably manipulate, a college has to pre-arrange a match with a strictly less preferred student. Then the disadvantage of being matched with a less desirable student should be compensated by matching to a better set of students in the centralized matching mechanism after pre-arrangement.

B.2.2 Theorem 3

Now we prove our result on pre-arrangement, Theorem 3. Start with matching $\phi(S, C, P, q)$, and consider what happens when college c reduces capacity by one. There are two cases to consider. First, the capacity reduction may not affect the matching of college c,

$$\phi(S, C, P, q_c - 1, q_{-c})(c) = \phi(S, C, P, q)(c).$$

This happens when $|\phi(S, C, P, q)(c)| < q_c$, or the number of students assigned to college c is less than its total capacity. Because there is an extra seat at college c, no student s would prefer to be assigned to college c over her matching $\phi(S, C, P, q)(s)$ because there would be a blocking pair, which contradicts the stability of $\phi(S, C, P, q)(s)$. Therefore, in the first case, pre-arrangement is not successful.

Second, the capacity reduction may affect the matching of college c. In this case, consider the rejection chain algorithm (Algorithm 1) starting with $\phi(S, C, P, q)$ with input $B_c^1 = \arg\min_{P_c} \phi(S, C, P, q)(c)$. We focus on the case where the rejection chain algorithm does not return to college c. Denote the resulting matching by μ'. Under μ', college c obtains $\phi(S, C, P, q)(c) \setminus \arg\min_{P_c} \phi(S, C, P, q)(c)$.

We claim that μ' is stable in $(S, C, P, q_c - 1, q_{-c})$. The ideas follow from Lemma 2.

> First, we claim μ' is individually rational. For $c' \neq c$, c' only accepts students who are acceptable in each step of the SOSM and Algorithm 1. c is matched to $\phi(S, C, P, q)(c) \setminus \arg\min_{P_c} \phi(S, C, P, q)(c)$ at μ', which is clearly individually rational for c under (P_c, q_c-1). Therefore, μ' is individually rational.
>
> Second, there is no blocking pair of μ' involving a college other than c. Suppose that for some $s \in S$ and $c' \in C$ such that $c' \neq c$, $c' P_s \mu'(s)$. Then, by the definition of the SOSM

and Algorithm 1, s is rejected by c' either during the SOSM or in Algorithm 1. This implies that $|\mu'(c')| = q_{c'}$ and $[\arg\min_{P_{c'}} \mu'(c)] P_{c'} s$, implying that s and c' do not block μ'.

Third, there is no blocking pair of μ' involving college c. Suppose $c P_s \mu'(s)$ for some $s \in S$. As in the previous paragraph, this implies that s is rejected by c either during the SOSM or in Algorithm 1. Since Algorithm 1 does not return to c by assumption, s is rejected either during the SOSM or at the beginning of Algorithm 1. In the former case, the fact that s is rejected during the SOSM implies that $[\arg\min_{P_c} \mu(c)] P_c s$, so when college c has capacity $q_c - 1$, s will not form a blocking pair with student s because the other $q_c - 1$ students who are matched to c are more preferred to $[\arg\min_{P_c} \mu(c)]$. In the latter case, (s, c) is not a blocking pair because the student rejected at the beginning of Algorithm 1 is $\arg\min_{P_c} \phi(S, C, P, q)(c)$ by our assumption, the other $q_c - 1$ students who are matched to c are more preferred by college c, and the capacity of college c is $q_c - 1$.

Therefore, μ' is stable in $(S, C, P, q_c - 1, q_{-c})$.

The least preferred stable matching for colleges in $(S, C, P, q_c - 1, q_{-c})$ is equal to $\phi(S, C, P, q_c - 1, q_{-c})(c)$. Therefore, we can conclude that

$$\mu'(c) = \phi(S, C, P, q)(c) \setminus \arg\min_{P_c} \phi(S, C, P, q)(c) \succeq_c \phi(S, C, P, q_c - 1, q_{-c})(c). \tag{15}$$

Since every college is made weakly better off under the SOSM when the set of participating students increases (Gale and Sotomayor (1985)), we obtain

$$\phi(S, C, P, q_c - 1, q_{-c}) \succeq_c \phi(S \setminus s, C, P_{-s}, q_c - 1, q_{-c}). \tag{16}$$

Lemma 7 shows that the rejection chain algorithm (Algorithm 1) does not return with probability at least $1 - \pi_c$. As a result, with probability $1 - \pi_c$,

$$\begin{aligned}
\phi(S, C, P, q)(c) &\succeq_c \phi(S, C, P, q_c - 1, q_{-c})(c) \cup \arg\min_{P_c} \phi(S, C, P, q)(c) \\
&\succeq_c \phi(S \setminus s, C, P_{-s}, q_c - 1, q_{-c})(c) \cup \arg\min_{P_c} \phi(S, C, P, q)(c) \\
&\succeq_c \phi(S \setminus s, C, P_{-s}, q_c - 1, q_{-c})(c) \cup s,
\end{aligned}$$

where the first relation follows from (15) and responsiveness of preferences, the second relation follows from (16) and responsiveness, and the last relationship follows from Lemma 8 and responsiveness.

Therefore the probability that c benefits via pre-arrangement is at most π_c. Finally, by Lemma 7 we complete the proof (this last argument is similar to the one for Theorem 1 and hence omitted).

B.3 Proof of Theorems 2 and 5

Since Theorem 5 is a multi-region generalization of Theorem 2, we prove only the former.

B.3.1 Lemma 10: Uniform vanishing market power

We have a variant of Lemma 7 under the sufficient thickness assumption, which plays a crucial role in the proof of the theorems.

For $B_c^1 \subseteq \mu(c)$, let

$$\pi_c^{B_c^1} = \Pr[\text{Algorithm 3 associated with } B_c^1 \text{ returns to c}|Y_T(n) > E[Y_T(n)]/2, \mu].$$

First we show a variant of Lemma 5.

Lemma 9. *Suppose $(\tilde{\Gamma}^1, \tilde{\Gamma}^2, \dots)$ is regular and sufficiently thick. Let T be such that $E[Y_T(n)] \to \infty$ as $n \to \infty$. Suppose that n is sufficiently large. Then we have*

$$\pi_c^{B_c^1} \leq \frac{4T\bar{q}}{E[Y_T(n)]},$$

for any c and $B_c^1 \subseteq \mu(c)$.

Proof. Consider Round 1, beginning with the least preferred student s of $B_c^1 \subseteq \mu(c)$ (if $B_c^1 = \emptyset$, then the inequality is obvious since $\pi_c^{B_c^1} = 0$.). Since $p_{c'}^n(r) \geq p_c^n(r)/T$ for any $c' \in V_T(n)$ and $r = 1, \dots, R$, Round 1 ends at 2(c)iiB as a student applies to some college with vacant positions, at least with probability $1 - 1/(Y_T(n)/T + 1) > 1 - 1/(E[Y_T(n)]/2T + 1)$.

Now assume that all Rounds $1, \dots, i-1$ end at Step 2(c)iiB. Then there are still at least $Y_T(n) - (i-1)$ colleges more popular than c and with a vacant position, since at most $i-1$ colleges in $V_T(n)$ have had their positions filled at Rounds $1, \dots, i-1$. Therefore Round i initiated by the least preferred student in B_c^i ends at Step 2(c)iiB with probability of at least $1 - 1/(E[Y_T(n)]/2T - (i-1) + 1)$. Since there are at most $\bar{q}$ rounds, Algorithm 3 fails to return to c with probability of at least

$$\begin{aligned}
\prod_{i=1}^{\bar{q}} \left(1 - \frac{1}{E[Y_T(n)]/2T - (i-1) + 1}\right) &\geq \left(1 - \frac{1}{E[Y_T(n)]/2T - \bar{q} + 2}\right)^{\bar{q}} \\
&\geq \left(1 - \frac{1}{E[Y_T(n)]/4T}\right)^{\bar{q}}.
\end{aligned}$$

The first inequality follows since $(\tilde{\Gamma}^1, \tilde{\Gamma}^2, \dots)$ is sufficiently thick, n is sufficiently large and $i \leq \bar{q}$ for each i. The second inequality holds since $E[Y_T(n)]/2 - \bar{q} \geq E[Y_T(n)]/4 > 0$, which follows since $(\tilde{\Gamma}^1, \tilde{\Gamma}^2, \dots)$ is sufficiently thick and n is sufficiently large. Therefore we have that

$$\begin{aligned}\pi_c^{B_c^1} &\leq 1 - \left(1 - \frac{1}{E[Y_T(n)]/4T}\right)^{\bar{q}} \\ &\leq \frac{4T\bar{q}}{E[Y_T(n)]},\end{aligned}$$

where the last inequality holds since $1-(1-x)^y \leq yx$ for any $x \in (0,1)$ and $y \geq 1$. □

Lemma 10 (Uniform vanishing market power)**.** *Suppose that* $(\tilde{\Gamma}^1, \tilde{\Gamma}^2, \dots)$ *is regular and sufficiently thick. For any sufficiently large* n *and any* $c \in C$*, we have*

$$\pi_c \leq \frac{4\left[T\bar{q}(2^{\bar{q}}-1)+1\right]}{E[Y_T(n)]}.$$

Proof. By Lemma 9 and an argument similar that which leads to expression (13) in Lemma 7, we obtain

$$\Pr[\text{Algorithm 3 returns to c}|Y_T(n) > E[Y_T(n)]/2] \leq \frac{4T\bar{q}(2^{\bar{q}}-1)}{E[Y_T(n)]}.$$

Therefore we have

$$\begin{aligned}\pi_c &\leq \Pr\left[Y_T(n) \leq E[Y_T(n)]/2\right] + \Pr\left[Y_T(n) > E[Y_T(n)]/2\right] \times \frac{4T\bar{q}(2^{\bar{q}}-1)}{E[Y_T(n)]} \\ &\leq \frac{4}{E[Y_T(n)]} + \frac{4T\bar{q}(2^{\bar{q}}-1)}{E[Y_T(n)]} \\ &\leq \frac{4\left[T\bar{q}(2^{\bar{q}}-1)+1\right]}{E[Y_T(n)]},\end{aligned}$$

completing the proof. □

B.3.2 Theorems 2 and 5

We only prove Theorem 5, since Theorem 2 is a special case when $R=1$. Suppose that colleges other than c are truth-telling, that is, any $c' \neq c$ reports $(P_{c'}, q_{c'})$. Lemmas 1 and 3 apply here since they do not rely on assumptions about student preferences. These lemmas imply that the probability that c profitably manipulates is at most π_c. By Lemma 10 and sufficient thickness, for any $\varepsilon > 0$, there exists n_0 such that for any market $\tilde{\Gamma}^n$ with $n > n_0$, we have

$$\Pr\left[u(\phi(S, C, P_c', P_{-c}, q)(c)) > u(\phi(S, C, P, q)(c)) \text{ for some } P_c'\right] < \frac{\varepsilon}{\bar{q}\sup_{n\in\mathbb{N}, s\in S^n, c\in C^n} u_c(s)}.$$

Such n_0 can be chosen independent of $c \in C^n$. For any $n > n_0$, for any $c \in C^n$ we have

$$
\begin{aligned}
&Eu_c(\phi(S, C, (P'_c, P_{-c}), (q'_c, q_{-c}))(c)) - Eu_c(\phi(S, C, P, q)) \\
<&\Pr\left[u_c(\phi(S, C, (P'_c, P_{-c}), (q'_c, q_{-c}))(c) > u_c(\phi(S, C, P, q)(c))\right] \bar{q} \sup_{n \in \mathbb{N}, s \in S^n, c \in C^n} u_c(s) \\
<&\varepsilon,
\end{aligned}
$$

which implies that truthful reporting is an ε-Nash equilibrium.

B.4 Proofs of Propositions 1 and 2

B.4.1 Proposition 1

Let a and T satisfy the condition of nonvanishing proportion of popular colleges. Let $c = [an]$. Then it is obvious that $V_c \subseteq V_T(n)$ and hence $Y_c \leq Y_T(n)$. For sufficiently large n, Lemma 4 shows that

$$E[Y_T(n)] \geq E[Y_c] \geq \frac{c}{2} e^{-8\bar{q}nk/c}.$$

$c = [an]$ implies that $\frac{c}{2} e^{-8\bar{q}nk/c} \to \infty$ as $n \to \infty$ with the order $O(n)$. Therefore $E[Y_T(n)] \to \infty$ as $n \to \infty$, completing the proof.

For Example 5, let $c = [\gamma n / \ln(n)]$. Substituting into $\frac{c}{2} e^{-8\bar{q}nk/c}$, we obtain

$$E[Y_T(n)] \geq \frac{c}{2} e^{-8\bar{q}nk/c} \geq \left[\frac{\gamma n / \ln(n) - 1}{2}\right] n^{\frac{-8\bar{q}k}{\gamma}} = \frac{\gamma n^{1 - \frac{8\bar{q}k}{\gamma}}}{2 \ln(n)} - \frac{1}{2} n^{-\frac{8\bar{q}k}{\gamma}}.$$

Therefore, for $\gamma > 8\bar{q}k$, then $n^{1 - \frac{8\bar{q}k}{\gamma}} / \ln(n) \to \infty$ and $n^{-\frac{8\bar{q}k}{\gamma}} \to 0$ as $n \to \infty$. Thus, we conclude that $E[Y_T(n)] \to \infty$ as $n \to \infty$.

B.4.2 Proposition 2

Let

$$V_T^*(n) = \{c \in C^n | p_1^n(r)/p_c^n(r) \leq T \quad \text{for all } r \in \{1, ..., \bar{R}\}\}.$$

Then we have $V_T(n) = \{c \in V_T^*(n) : |\{s \in S^n | cP_s s\}| < q_c\}$. Let $\eta_r(c) = |\{c' \in C^n | p_c^n(r) \leq p_{c'}^n(r)\}|$ be the order of c with respect to popularity in distribution $D^n(r)$. For example, if college c is the most popular among students in region 1 and the least popular among those in region 2, then $\eta_1(c) = 1$ and $\eta_2(c) = n$.

Part (1): Example 6. Let $T = 4$, for example. Then, $V_4^*(n) = \{n/4, n/4 + 1, \ldots, 3n/4\}$. Consider any college $c \in V_4^*(n)$. Let s belong to region $r \in \{1, 2\}$. Since s picks colleges k times according to $D^n(r)$, the probability that c does not appear in the preference list of student s, denoted by $\Pr(F_{c,s})$, is bounded as follows:

$$\Pr(F_{c,s}) \geq \left(1 - \frac{p_c^n(r)}{1 - Q^n(r)}\right)^k,$$

where

$$Q^n(r) = \sum_{c:\eta_r(c) \leq k} p_c^n(r).$$

For any sufficiently large n, we have that $\eta_r(c) > 2k$ for any $c \in V_4^*(n)$ and $r = 1, 2$ since $[n - 2k] > 3n/4 > c > n/4 > 2k$. For such colleges,

$$p_c^n(r) \leq \frac{1 - Q^n(r)}{\eta_r(c) - k} \leq \frac{1 - Q^n(r)}{\eta_r(c)/2}.$$

So

$$\Pr(F_{c,s}) \geq \left(1 - \frac{2}{\eta_r(c)}\right)^k.$$

Since $\eta_r(c) \geq n/4$ for any $c \in V_4^*(n)$ and any $r = 1, 2$, we have

$$\Pr(F_{c,s}) \geq \left(1 - \frac{8}{n}\right)^k.$$

Let E_c be the event that c is not listed by any student. Then, since students draw colleges independently, we have

$$\Pr(E_c) = \prod_{s \in S^n} \Pr(F_{c,s}) \geq (1 - 8/n)^{k\bar{q}n} \to e^{-8k\bar{q}},$$

as $n \to \infty$. Therefore,

$$E[Y_T(n)] = \sum_{c \in V_4^*(n)} \Pr(E_c) \geq \frac{n}{2}(1 - 8/n)^{k\bar{q}n} \to \infty,$$

as $n \to \infty$ (with the order $O(n)$), completing the proof.

Part (2): Example 7. As discussed in Example 7, for any colleges c and c' and region r, we have that

$$p_c^n(r)/p_{c'}^n(r) = \tilde{p}_{r(c)}(r)/\tilde{p}_{r(c')}(r) > 0.$$

Since there are only finite regions, $V_T^*(n) = C^n$ for any sufficiently large T. Fix such T.

As in the proof of Part (1), for any c and s we have

$$\Pr(F_{c,s}) \geq \left[1 - \frac{p_c^n(r(s))}{1 - Q^n(r(s))}\right]^k.$$

Since we have that $p_c^n(r)/p_{c'}^n(r) < T$ for any $c, c' \in C^n$,

$$\begin{aligned}\frac{p_c^n(r)}{1 - Q^n(r)} &\leq \frac{p_c^n(r)}{(n-k)p_c^n(r)/T} \\ &\leq \frac{2T}{n},\end{aligned}$$

for any sufficiently large n. So we have

$$\Pr(E_c) = \prod_{s \in S^n} \Pr(F_{c,s}) \geq \prod_{s \in S^n} \left(1 - \frac{2T}{n}\right)^k \geq (1 - 2T/n)^{k\bar{q}n} \to e^{-2k\bar{q}T},$$

as $n \to \infty$. Therefore

$$E[Y_T(n)] = \sum_{c \in C^n} \Pr(E_c)$$

approaches infinity with the order $O(n)$, completing the proof.

Remark 1. In Examples 4, 6 and 7, the order of convergence of $E[Y_T(n)]$ is $O(n)$. This implies that, by Lemma 10, the order of convergence of the probability of profitable manipulation is $O(1/n)$. This is the same order as in the uniform distribution case, analyzed by Roth and Peranson (1999) and Immorlica and Mahdian (2005).

B.5 Proof of Theorem 7: Coalitional manipulation

The proof is based on a series of arguments similar to those for Theorem 1. First, dropping strategies are exhaustive for manipulations involving a coalition of agents.

Lemma 11 (Dropping strategies are exhaustive for coalitional manipulations)**.** *Consider an arbitrary stable mechanism φ. Fix preference profile (P, q) and let $\bar{C} \subseteq C$ be a coalition of colleges. For some arbitrary report $(\tilde{P}_{\bar{C}}, \tilde{q}_{\bar{C}})$ by this coalition, suppose that $\varphi((\tilde{P}_{\bar{C}}, P_{-\bar{C}}), (\tilde{q}_{\bar{C}}, q_{-\bar{C}})) = \mu$. Then there exists a dropping strategy $(P'_{\bar{C}}, q_{\bar{C}}) = (P'_c, q_c)_{c \in \bar{C}}$ such that $\varphi((P'_{\bar{C}}, P_{-\bar{C}}), q) \succeq_c \mu$ for each $c \in \bar{C}$.*

Proof. For each $c \in \bar{C}$, construct dropping strategy (P'_c, q_c) such that P'_c lists all the students it is matched to under μ who are acceptable ($\{s \in \mu(c) | s P_c \emptyset\}$) in the same relative order as in P_c, and reports every other student as unacceptable. Let $(P', q) = (P'_{\bar{C}}, P_{-\bar{C}}, q)$.

We will show $\varphi(P', q)(c)$ is equal to $\{s \in \mu(c) | sP_c\emptyset\}$ for each $c \in \bar{C}$. This equality implies that $\varphi(P', q) \succeq_c \mu$ for each $c \in \bar{C}$, since $\varphi(P', q)(c)$ can only differ from $\mu(c)$ in having no unacceptable students under the true preference list P_c. The proof proceeds in two steps.

First, consider the matching μ' obtained from μ by having each $c \in \bar{C}$ keep only the students in $\mu(c)$ that are acceptable under P_c. That is,

$$\mu'(c) = \begin{cases} \{s \in \mu(c) | sP_c\emptyset\} & c \in \bar{C}, \\ \mu(c) & c \notin \bar{C}. \end{cases}$$

Consider the properties of μ' under (P', q). First, μ' is individually rational under (P', q). Second, there is no blocking pair involving any $c \in \bar{C}$, since $\mu'(c)$ is exactly the set of students acceptable under P'_c. However, there may be a blocking pair involving another college and students who are unmatched. In this case, we can define a procedure which ultimately yields a matching that is stable under (P', q):

Starting from $\mu'_0 \equiv \mu'$, let S_0 be the set of students that can be part of a blocking pair in μ'_0. Pick some $s \in S_0$. Let college c' be the college that the student s prefers the most within the set of colleges involved in blocking pairs with s. Construct μ'_1 by assigning student s to c' and if $|\mu'_0(c')| = q_{c'}$, then let c' reject its least preferred student. By construction, μ'_1 is individually rational, college c is matched to the same set of students, college c' strictly prefers $\mu'_1(c')$ over $\mu'_0(c')$ and the assignments of all other colleges are unchanged. As with μ'_0, the only blocking pair of μ'_1 involves unmatched students. If there are blocking pairs of μ'_1, then we repeat the same procedure to construct μ'_2, and so on.

At each repetition, the new matching is individually rational, each college $c \in \bar{C}$ is matched to the same set of students, one other college strictly improves and the remaining colleges are not worse off since each blocking pair involves an unassigned student. Since colleges can strictly improve only a finite number of times, this procedure terminates in finite time. The ultimate matching μ'' is stable in (P', q) because it is individually rational and there are no blocking pairs. Moreover, since each college $c \in \bar{C}$ obtains the same matching in each repetition,

$$\mu''(c) = \mu'(c), \text{ for each } c \in \bar{C}. \tag{17}$$

The second step of the proof utilizes the fact that for any college, the same number of students are matched to it across different stable matchings. Since μ'' is stable in (P', q) and $\varphi(P', q)$ is stable in (P', q) by definition, $|\mu''(c)| = |\varphi(P', q)(c)|$, for each $c \in \bar{C}$. Since for each $c \in \bar{C}$, there are just $|\mu''(c)|$ acceptable students under P'_c, this implies that

$$\mu''(c) = \varphi(P', q)(c), \text{ for each } c \in \bar{C}. \tag{18}$$

Equations (17) and (18) together imply that for each $c \in \bar{C}$,

$$\varphi(P', q)(c) = \mu'(c) = \{s \in \mu(c) | sP_c\emptyset\}.$$

As a result, for each $c \in \bar{C}$

$$\varphi(P', q)(c) \succeq_c \mu(c),$$

since $\varphi(P', q)(c)$ can only differ from $\mu(c)$ in having no unacceptable students under the true preference list P_c. $\square$

B.5.1 Lemma 13: Rejection chains with coalitions

For preference profile (P, q), let μ be the student-optimal stable matching. Let B_c^1 be an arbitrary subset of $\mu(c)$. The **rejection chains** algorithm with input $(B_c^1)_{c \in \bar{C}}$ is defined as follows.

Algorithm 4. Rejection Chains with Coalitions

(1) Initialization: μ is the student-optimal stable matching μ, input $(B_c^1)_{c \in \bar{C}}$, $\bar{C}^1 = \bar{C}$, and let $j = 0$.

(2) Algorithm:

  (a) Increment j by one (iterate through colleges).

    i. If $\bar{C}^j = \emptyset$, then terminate the algorithm.

    ii. If not, pick some $c \in \bar{C}^j$, and let $\bar{C}^{j+1} = \bar{C}^j \setminus c$ and let $i = 0$.

  (b) Increment i by one (iterate through students)

    i. If $B_c^i = \emptyset$, then go to beginning of Step 2a.

    ii. If not, let s be the least preferred student by c among B_c^i, and let $B_c^{i+1} = B_c^i \setminus s$.

    iii. Iterate the following steps (call this iteration "Round i").

      Choosing the applied:

      A. If s has already applied to every acceptable college, then finish the iteration and go back to the beginning of Step 2b.

      B. If not, let c' be the most preferred college of s among those which s has not yet applied while running the SOSM or previously within this algorithm. If $c' \in \bar{C}$, terminate the algorithm.

      Acceptance and/or rejection:

      A. If c' prefers each of its current mates to s and there is no vacant position, then c' rejects s; go back to the beginning of Step 2(b)iii.

B. If c' has a vacant position or it prefers s to one of its current mates, then c' accepts s. Now if c' had no vacant position before accepting s, then c' rejects the least preferred student among those who were matched to c'. Let this rejected student be s and go back to the beginning of Step 2(b)iii. If c' had a vacant position, then finish the iteration and go back to the beginning of Step 2b.

Algorithm 4 terminates either at Step 2(a)i or at Step 2(b)iiiB. We say that Algorithm 4 **does not return to** $\bar{C}$ if it terminates at Step 2(a)i and it **returns to** $\bar{C}$ if it terminates at Step 2(b)iiiB.

Lemma 12. *Consider coalition $\bar{C}$ and suppose that under (P, q), Algorithm 4 with each possible collection of sets consisting of subsets of $\mu(c)$ for each $c \in \bar{C}$ as an input fails to return to $\bar{C}$. For any dropping strategy $(P'_{\bar{C}}, q_{\bar{C}})$, let $B^1_c = \{s \in \mu(c) | \emptyset P'_c s\}$ for each $c \in \bar{C}$ such $B^1_c \neq \emptyset$ for at least one $c \in \bar{C}$, and let μ' be a matching obtained at the end of Algorithm 4 with input $(B^1_c)_{c\in\bar{C}}$. Then, under $(P'_{\bar{C}}, P_{-\bar{C}}, q)$,*

(1) *μ' is individually rational,*

(2) *no $c \notin \bar{C}$ is a part of a blocking pair of μ', and*

(3) *for $c \in \bar{C}$, if (s, c) blocks μ', then $[\arg\min_{P_c} \mu(c)] P_c s$, and*

(4) *for $c \in \bar{C}$, if (s, c) blocks μ' and $\mu'(c)$ is non-empty, then $[\arg\min_{P'_c} \mu'(c)] P'_c s$.*

Proof. Part (1): For $c \notin \bar{C}$, c only accepts students who are acceptable in each step of the SOSM and Algorithm 4. Each $c \in \bar{C}$ rejects every student who is unacceptable under $P'_{\bar{C}}$ at the outset of Algorithm 4, and accepts no other student by the assumption that Algorithm 4 does not return to $c \in \bar{C}$ for any sequence of sets consisting of subsets of $\mu(c)$ for each $c \in \bar{C}$ as input. Therefore μ' is individually rational.

Part (2): Suppose that for some $s \in S$ and $c \in C \setminus \bar{C}$, $cP_s\mu'(s)$. Then, by the definition of the SOSM and Algorithm 4, s is rejected by c either during the SOSM or in Algorithm 4. This implies that $|\mu'(c)| = q_c$ and $[\arg\min_{P_c} \mu'(c)] P_c s$, implying that s and c do not block μ'.

Part (3): Suppose $cP_s\mu'(s)$ for some $s \in S$ and $c \in \bar{C}$. As in Part (2), this implies that s is rejected by c either during the SOSM or in Algorithm 4. Since Algorithm 4 does not return to c by assumption, s is rejected either during the SOSM or at the beginning of Algorithm 4. In the former case, if s is rejected during the SOSM, then $[\arg\min_{P_c} \mu(c)] P_c s$. In the latter case, (s, c) is not a blocking pair because s is declared unacceptable under P'_c.

Part (4): From Part (3), for each $c \in \bar{C}$, we have that

$$[\arg\min_{P_c} \mu(c)] P_c s. \tag{19}$$

By definition, $\mu'(c) \subseteq \mu(c)$ and expression (19) imply

$$[\arg\min_{P_c} \mu'(c)] P_c s, \tag{20}$$

when $\mu'(c)$ is non-empty.

When $\mu'(c)$ is non-empty, $\mu'(c)$ is constructed by including only acceptable students in $\mu(c)$ according to P'_c, so

$$[\arg\min_{P'_c} \mu'(c)] P'_c \emptyset. \tag{21}$$

Expressions (20) and (21) and the definition of a dropping strategy yield

$$[\arg\min_{P'_c} \mu'(c)] P'_c s.$$

$\square$

Lemma 13 (Rejection chains with coalitions)**.** *For any market and any $\bar{C} \subseteq C$, if Algorithm 4 does not return to $\bar{C}$ with each possible collection of sets $(B^1_c)_{c \in \bar{C}}$, where B^1_c is a subset of $\mu(c)$ for each $c \in \bar{C}$, as input (with at least one B^1_c non-empty), then $\bar{C}$ cannot profitably manipulate by a dropping strategy.*

Proof. Consider an arbitrary dropping strategy $(P'_{\bar{C}}, q_{\bar{C}})$ and let $B^1_c = \{s \in \mu(c) | \emptyset P'_c s\}$ for each $c \in \bar{C}$. When Algorithm 4 does not return to $\bar{C}$, we show that $\mu = \phi(P, q)$ is weakly preferred by each college $c \in \bar{C}$ to $\phi(P'_{\bar{C}}, P_{-\bar{C}}, q)$. Let $(P', q) = (P'_{\bar{C}}, P_{-\bar{C}}, q)$.

Let μ' be the matching resulting from Algorithm 4 with input $(B^1_c)_{c \in \bar{C}}$. The first step of the proof constructs a new matching μ'' by satisfying the blocking pairs in μ' such that μ'' is stable in (P', q)).

We construct μ'' from μ' as follows. Parts (1) and (2) of Lemma 12 imply that μ' is individually rational and no $c \notin \bar{C}$ is part of a blocking pair of μ'. Parts (3) and (4) of Lemma 12 states that if there is a blocking pair involving some $c \in \bar{C}$, then the student involved in the blocking pair is less preferred than any student in $\mu(c)$ under P_c and less preferred than any student in $\mu'(c)$ under P'_c.

Let $\mu''_0 = \mu'$. Begin by selecting some college $c \in \bar{C}$ involved in a blocking pair of μ''_0. Construct matching μ''_1 by assigning college c its most preferred students in blocking pairs up to capacity. The resulting matching is individually rational, at least one student strictly improves over μ''_0, no other student is made worse off, and every blocking pair of μ''_1 involves a college with a vacant seat. Finally, since students are made weakly better off in μ''_1 than in μ''_0, it must be the case that if (s, c) blocks μ''_1, then $\arg\min_{P_c} \mu(c) P_c s$ and $\arg\min_{P'_c} \mu'(c) P'_c s$.

Next, we construct μ''_2 by selecting college c who is involved in a blocking pair of μ''_1. Note that this can either be a college in $\bar{C}$ or not, but the college that is selected must have a vacant seat. As before, we assign c the most preferred students it forms a blocking pair with up to capacity. As before, the resulting matching is individually rational, at least one student strictly improves over μ_1'', no other student is made worse off, and every blocking pair of μ_2'' involves a college with a vacant seat. Finally, since students are made weakly better in μ_2'' than in μ_1'', it must be the case that if (s, c) blocks μ_2'', then $\arg\min_{P_c} \mu(c) P_c s$ and $\arg\min_{P_c'} \mu'(c) P_c s$.

We repeat this process to construct μ_3'', and so on.

In each step of this construction, no additional students are rejected when we satisfy a blocking pair, and in each step, at least one student strictly improves and the remaining students are not made worse off. Since students may improve their assignment only a finite number of times, the procedure ends in finite time.

The ultimate matching μ'' is stable in (P', q) because it is individually rational and there are no blocking pairs. Moreover, for college c, each new student matched to college c is weakly less preferred under P_c' than any student in the initial matching $\mu_0''(c)$ as students obtain a weakly more preferred college in each step. Hence, $\mu'(c)$ is weakly less preferred to $\mu''(c)$ under P_c'. Also, for college c, each new student matched to college c is weakly less preferred under P_c than any student in the matching $\mu(c)$ because students obtain a weakly more preferred college in each step. Thus, $\mu''(c)$ is weakly less preferred to $\mu(c)$ under P_c.

Since the matching produced by SOSM is the least preferred stable matching of every college, $\phi(P', q)(c)$ is weakly less preferred to the stable matching μ'' by each college $c \in \bar{C}$ in (P', q). This will imply that $\phi(P', q)(c)$ is weakly less preferred to $\mu''(c)$ under P_c' for each $c \in \bar{C}$, and since P_c' is a dropping strategy of P_c,

$$\mu''(c) \succeq_c \phi(P', q)(c), \text{ for each } c \in \bar{C}.$$

□

B.5.2 Lemma 14: Uniform vanishing market power for coalitions

For the coalitions, we define

$$\pi_c = \Pr[\text{Algorithm 4 returns to } \bar{C} \text{ for some } (B_c^1)_{c \in \bar{C}} \subseteq \cup_{c \in \bar{C}} \mu(c)].$$

Lemma 14. *Suppose that $(\tilde{\Gamma}^1, \tilde{\Gamma}^2, \dots)$ is regular and sufficiently thick. For any sufficiently large n and coalition $\bar{C} \subseteq C$, we have*

$$\pi_{\bar{C}} \leq \frac{4 \left[T\bar{q} \cdot |\bar{C}| \cdot (2^{\bar{q} \cdot |\bar{C}|} - 1) + 1 \right]}{E[Y_T(n)]}.$$

Proof. The proof follows exactly the steps leading to Lemma 10 with two modifications. The first modification replaces the first instance of $\bar{q}$ in the expression in Lemma 10 with $\bar{q} \cdot |\bar{C}|$

because in the proof that corresponds to Lemma 9 we must allow for the possibility of $\bar{q}$ rounds for each of the $|\bar{C}|$ colleges. The second modification replaces $2^{\bar{q}}$ in the expression in Lemma 10 with $2^{\bar{q}\cdot|\bar{C}|}$ because in the proof that corresponds to Lemma 7, there are at most $2^{\bar{q}\cdot|\bar{C}|}-1$ non-empty subsets of $\cup_{c\in\bar{C}}\mu(c)$. □

Finally, the proof of Theorem 7 follows because in sufficiently thick markets $E[Y_T(n)]\to\infty$ as $n\to\infty$ and $|\bar{C}|\leq m$, so $\pi_{\bar{C}}\to 0$.

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